Upper semi-continuity of entropy in non-compact settings

Godofredo Iommi, Mike Todd, and Aníbal Velozo

We prove that the entropy map for countable Markov shifts of finite entropy is upper semi-continuous on the set of ergodic measures. Note that the phase space is non-compact. We also discuss the related problem of existence of measures of maximal entropy.

1. Introduction

The entropy map of the continuous transformation $T : X \to X$ defined on a metric space (X, d) is the map $\mu \mapsto h_\mu(T)$ which is defined on the space of $T-$invariant probability measures $\mathcal{M}_T$, where $h_\mu(T)$ is the entropy of μ (precise definitions can be found in Section 2). The study of the continuity properties of the entropy map, with $\mathcal{M}_T$ endowed with the weak* topology, goes back at least to the work of Bowen [Bo1]. In general it is not a continuous map (see [Wa, p.184]). However, in certain relevant cases it can be shown that the map is upper semi-continuous, that is, if $(\mu_n)_n$ is a sequence in $\mathcal{M}_T$ that converges to μ then $\limsup_{n\to\infty} h(\mu_n) \leq h(\mu)$. For example, if T is an expansive homemorphism of a compact metric space then the entropy map is upper semi-continuous, see [Wa, Theorem 8.2] and [Bo1, De, Mi2]. It is also known that if T is a C^∞ map defined over a smooth compact manifold then, again, the entropy map is upper semi-continuous (see [N, Theorem 4.1] and [Y]). Lyubich [Ly, Corollary 1] proved that the entropy map is upper semi-continuous for rational maps of the Riemann sphere. In all the above examples the phase space is compact: in this article we will drop the compactness assumption on the underlying space.

Markov shifts defined over finite alphabets have been used with remarkable success to study uniformly hyperbolic systems. Indeed, these systems possess finite Markov partitions and are, therefore, semi-conjugated to Markov shifts. See [Bo2] for an example of the wealth of results that can be obtained with this method. Following the 2013 work of Sarig [Sa2], countable Markov partitions have been constructed for a wide range of dynamical systems defined on compact spaces (see [Bu2, LM, LS, O]). The symbolic

coding captures a relevant part, though not all, of the dynamics. We stress that this relevant part is often non-compact. For example, diffeomorphisms defined on compact manifolds have countable Markov partitions that capture all hyperbolic measures. These results have prompted a renewed interest in the ergodic properties of countable Markov shifts.

In this paper we study countable Markov shifts (Σ, σ). More precisely, the shift on sequences of elements of a countable alphabet in which the transitions, allowed and forbidden, are described by a directed graph (see Section 3 for definitions). The space Σ is usually non-compact with respect to its natural topology. If the alphabet is finite, a so-called subshift of finite type, and so the space Σ is compact, it is a classical result that the entropy map is upper semi-continuous [Wa, Theorem 8.2]. In this article we recover upper semi-continuity in the ergodic case for shifts of finite topological entropy defined on non-compact spaces. Our main result is:

Theorem 1.1. *Let (Σ, σ) be a two-sided countable Markov shift of finite topological entropy. If $(\mu_n)_n, \mu$ are ergodic measures in $\mathcal{M}_\sigma$ such that $(\mu_n)_n$ converges in the weak* topology to μ then*

$$\limsup_{n \to \infty} h(\mu_n) \leq h(\mu).$$

That is, the entropy map is upper semi-continuous on the set of ergodic measures.

This has the following corollary:

Corollary 1.2. *Let (Σ^+, σ) be a one-sided countable Markov shift of finite topological entropy. If $(\mu_n)_n, \mu$ are ergodic measures in $\mathcal{M}_\sigma$ such that $(\mu_n)_n$ converges in the weak* topology to μ then*

$$\limsup_{n \to \infty} h(\mu_n) \leq h(\mu).$$

That is, the entropy map is upper semi-continuous on the set of ergodic measures.

The strategy of the proof is the following. First note that if there exists a finite generating partition of the space such that the measure of its boundary is zero for every invariant measure, then the entropy map is upper semi-continuous. This is no longer true for countable generating partitions. Krieger [Kr] constructed a finite generating partition for each ergodic measure. More recently, Hochman [H1, H2] constructed finite partitions that are

generating for every ergodic measure. We can not use the result by Krieger since we need the same partition for every measure and we can not use Hochman's result since his partitions have a large boundary. We overcome this difficulty by constructing finite generating partitions with no boundary.

If $T : X \to X$ is a continuous transformation defined on a compact space for which the entropy map is upper semi-continuous then there exists a measure of maximal entropy. Diffeomorphisms of class C^r, for any $r \in [1, \infty)$ (in the compact setting), with no measure of maximal entropy have been constructed by Misiurewicz [Mi1] and Buzzi [Bu1]. These are examples for which the entropy map is not upper semi-continuous. If the space X is non-compact, even if the entropy map is upper semi-continuous, measures of maximal entropy might not exist. Indeed, in the non-compact case the space of invariant probability measures might also be non-compact and therefore a sequence of measures with entropy converging to the topological entropy may not have a convergent subsequence. For instance, consider the geodesic flow on a non-compact pinched negatively curved manifold. Velozo [V] showed that in that setting the entropy map is always upper semi-continuous, but there are examples for which there is no measure of maximal entropy (see for example [DPPS]). In Section 4 we discuss conditions that guarantee the existence of measures of maximal entropy. We also address the relation between upper semi-continuity of the entropy map and thermodynamic formalism. More precisely, we provide conditions for the existence of equilibrium measures. Finally, in Section 5 we apply our results to suspension flows.

We would expect our results to pass to some systems which are coded by countable Markov shifts, but with the important caveat that it is not always straightforward to compare topologies on the shift versus the coded space, so weak* convergence in one space may not imply weak* convergence in the other.

Acknowledgements

2. Entropy and generators

This section is devoted to recall basic properties and definitions that will be used throughout the article. The reader is referred to [Do, P, Wa] for more details.

2.1. The weak* topology

Let (X, d) be a metric space, we denote by $C_b(X)$ the space bounded continuous functions $\varphi : X \to \mathbb{R}$. Denote by $\mathcal{M}(X)$ the set of Borel probability measures on the metric space (X, d).

Definition 2.1. A sequence of probability measures $(\mu_n)_n$ defined on a metric space (X, d) converges to a measure μ in the weak* topology if for every $\varphi \in C_b(X)$ we have

$$\lim_{n \to \infty} \int \varphi \, d\mu_n = \int \varphi \, d\mu.$$

Remark 2.2. In this notion of convergence we can replace the set of test functions by the space of bounded Lipschitz functions (see [Kl, Theorem 13.16 (ii)]). That is, if for every bounded Lipschitz function $\varphi : X \to \mathbb{R}$ we have

$$\lim_{n \to \infty} \int \varphi \, d\mu_n = \int \varphi \, d\mu.$$

then the sequence $(\mu_n)_n$ converges in the weak* topology to μ.

If the space (X, d) is compact then so is $\mathcal{M}(X)$ with respect to the weak* topology (see [Pa, Theorem 6.4]). For $A \subset X$ denote by int $A, \overline{A}$ and ∂A the interior, the closure and the boundary of the set A, respectively. The following result characterises the weak* convergence (see [Pa, Theorem 6.1]),

Theorem 2.3 (Portmanteau Theorem). *Let (X, d) be a metric space and $(\mu_n)_n, \mu$ measures in $\mathcal{M}(X)$. The following statements are equivalent:*

(a) *The sequence $(\mu_n)_n$ converges in the weak* topology to the measure μ.*

(b) *If $C \subset X$ is a closed set then $\limsup_{n \to \infty} \mu_n(C) \leq \mu(C)$.*

(c) *If $O \subset X$ is an open set then $\liminf_{n \to \infty} \mu_n(O) \geq \mu(O)$.*

(d) *If $A \subset X$ is a set such that $\mu(\partial A) = 0$ then $\lim_{n \to \infty} \mu_n(A) = \mu(A)$.*

Let $T : (X, d) \to (X, d)$ be a continuous dynamical system, denote by $\mathcal{M}_T$ the space of $T-$invariant probability measures.

Proposition 2.4. *Let $T : (X, d) \to (X, d)$ be a continuous dynamical system defined on a metric space, then*

(a) *The space $\mathcal{M}_T$ is closed in the weak* topology ([Wa, Theorem 6.10]).*

(b) *If X is compact then so is $\mathcal{M}_T$ with respect to the weak* topology (see [Wa, Theorem 6.10]).*

(c) *The space $\mathcal{M}_T$ is a convex set for which its extreme points are the ergodic measures (see [Wa, Theorem 6.10]). It is actually a Choquet simplex (each measure is represented in a unique way as a generalized convex combination of the ergodic measures [Wa, p.153]).*

2.2. Entropy of a dynamical system

Let $T : (X, d) \to (X, d)$ be a continuous dynamical system. We recall the definition of entropy of an invariant measure $\mu \in \mathcal{M}_T$ (see [Wa, Chapter 4] for more details).

Definition 2.5. A partition $\mathcal{P}$ of a probability space $(X, \mathcal{B}, \mu)$ is a countable (finite or infinite) collection of pairwise disjoint subset of X whose union has full measure.

Definition 2.6. The *entropy* of the partition $\mathcal{P}$ is defined by

$$H_\mu(\mathcal{P}) := -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P),$$

where $0 \log 0 := 0$.

It is possible that $H_\mu(\mathcal{P}) = \infty$. Given two partitions $\mathcal{P}$ and $\mathcal{Q}$ of X we define the new partition

$$\mathcal{P} \vee \mathcal{Q} := \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$$

Let $\mathcal{P}$ be a partition of X we define the partition $T^{-1}\mathcal{P} := \{T^{-1}P : P \in \mathcal{P}\}$ and for $n \in \mathbb{N}$ we set $\mathcal{P}^n := \bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}$. Recall that a sequence of real numbers $(a_n)_n$ is subadditive if for every $n, m \in \mathbb{N}$ we have $a_{n+m} \le a_n + a_m$. A classical result by Fekete states that if $(a_n)_n$ is a subadditive sequence of non-negative real numbers then the sequence $(a_n/n)_n$ converges to its infimum. Since the measure μ is $T-$invariant, the sequence $H_\mu(\mathcal{P}^n)$ is subadditive.

Definition 2.7. The *entropy* of μ with respect to $\mathcal{P}$ is defined by

$$h_\mu(\mathcal{P}) := \lim_{n \to \infty} \frac{1}{n} H_\mu(\mathcal{P}^n)$$

Definition 2.8. The *entropy* of μ is defined by

$$h_\mu(T) := \sup\left\{h_\mu(\mathcal{P}) : \mathcal{P} \text{ a partition with } H_\mu(\mathcal{P}) < \infty\right\}.$$

If the underlying dynamical system considered is clear we write h_μ instead of $h_\mu(T)$.

2.3. Generators

We now recall the definition and properties of an important concept in ergodic theory, namely generators.

Definition 2.9. Let $(T, X, \mathcal{B}, \mu)$ be a dynamical system. A *one-sided generating partition* $\mathcal{P}$ of $(T, X, \mathcal{B}, \mu)$ is a partition such that $\bigcup_{n=1}^{\infty} \mathcal{P}^n$ generates the sigma-algebra $\mathcal{B}$ up to sets of measure zero. Analogously, if $(T, X, \mathcal{B}, \mu)$ is an invertible dynamical system a *two-sided generating partition* $\mathcal{P}$ of $(T, X, \mathcal{B}, \mu)$ is a partition such that $\bigcup_{n=-\infty}^{\infty} \mathcal{P}^n$ generates the sigma-algebra $\mathcal{B}$ up to sets of measure zero.

A classical result by Kolmogorov and Sinai [Si] states that entropy can be computed with either one- or two-sided generating partitions (see also [Wa, Theorem 4.17 and 4.18] or [Do, Theorem 4.2.2]).

Theorem 2.10 (Kolmogorov-Sinai). *Let $(T, X, \mathcal{B}, \mu)$ be a dynamical system and $\mathcal{P}$ a one-or a two-sided generating partition, then $h_\mu(T) = h(\mu, \mathcal{P})$.*

The existence of generating partitions depends on the acting semi-group. Rohlin [Ro1, Ro2] proved that ergodic (actually aperiodic) invertible systems of finite entropy have countable two-sided generators. This was later improved by Krieger [Kr] (see also [Do, Theorem 4.2.3]).

Theorem 2.11 (Krieger). *Let $(T, X, \mathcal{B}, \mu)$ be an ergodic invertible dynamical system with $h_\mu(T) < \infty$ then there exists a finite two-sided generator $\mathcal{P}$ for μ. The cardinality of $\mathcal{P}$ can be chosen to be any integer larger than $e^{h_\mu(T)}$.*

Note that in Krieger's result the generator depends upon the measure. There are some well known cases in which there exists a finite partition which is a generator for *every* measure. For example, if (Σ, σ) is a transitive two-sided subshift of finite type then the cylinders of length one form a two-sided generating partition for every invariant measure. Hochman proved that a uniform version of Krieger result can be obtained (see [H2, Corollary 1.2]).

Theorem 2.12 (Hochman). *Let $(T, X, \mathcal{B})$ be an invertible dynamical system with no periodic points and of finite entropy. Then there exists a finite two-sided partition $\mathcal{P}$ that is a generator for every ergodic invariant measure.*

2.4. Upper semi-continuity of the entropy

The following well known result (see for example [Do, Lemma 6.6.7]) relates the existence of finite generating partitions with continuity properties of the entropy map.

Proposition 2.13. *Let $(T, X, \mathcal{B})$ be a dynamical system and $\mathcal{P}$ be a finite two-sided generator for every measure in $\mathcal{C} \subset \mathcal{M}_T$. If for every $\mu \in \mathcal{C}$ we have that $\mu(\partial \mathcal{P}) = 0$ then the entropy map is upper semi-continuous in $\mathcal{C}$.*

Proof. Let $\mu \in \mathcal{C}$ and $P \in \mathcal{P}$. Note that since $\mu(\partial P) = 0$ the function $\nu \to \nu(P)$ is continuous at μ. Since the partition $\mathcal{P}$ is finite, the function defined in $\mathcal{C}$ by

$$\nu \to H_\nu(\mathcal{P}) = - \sum_{P \in \mathcal{P}} \nu(P) \log \nu(P),$$

is continuous at μ. Note that $\mu(\partial \mathcal{P}) = 0$ implies that $\mu(\partial \mathcal{P}^n) = 0$. Thus, the function in $\mathcal{C}$ defined by $\nu \to H_\nu(\mathcal{P}^n)$ is also continuous. Since the function defined in $\mathcal{C}$ by

$$\nu \mapsto h_\nu(\mathcal{P}) = \inf_n \frac{1}{n} H_\nu(\mathcal{P}^n)$$

is the infimum of continuous functions at μ we have that the map $\nu \mapsto h_\nu(\mathcal{P})$ is upper semi-continuous at μ. Since $\mathcal{P}$ is a uniform generating partition in $\mathcal{C}$ for every $\nu \in \mathcal{C}$ we have $h_\nu(T) = h_\nu(\mathcal{P})$. Thus, the map

$$\nu \mapsto h_\nu(T)$$

is upper semi-continuous at μ. Since $\mu \in \mathcal{C}$ was arbitrary the result follows. $\square$

Remark 2.14. Note that if in Proposition 2.13 we have $\mathcal{C} = \mathcal{M}_T$ then the entropy map is upper semi-continuous in the space of invariant probability measures.

Remark 2.15. The argument in Proposition 2.13 breaks down if we consider countable (infinite) generating partitions since, in that case, the map $\nu \to H_\nu(\mathcal{P})$ need not to be continuous (see Remark 3.11 or [JMU, p.774]).

Remark 2.16. Expansive maps defined on compact metric spaces are examples of dynamical systems having finite generators as in Proposition 2.13 (see [Wa, Theorem 8.2]). We stress that while Theorem 2.12 provides a finite (uniform) generating partition, in general this does not satisfy the condition of zero measure boundary.

3. Countable Markov shifts

In this section we define countable Markov shifts, both one-and two-sided and prove our main results.

3.1. Two-sided countable Markov shifts

Let (Σ, σ) be a two-sided Markov shift defined over a countable alphabet $\mathcal{A}$. This means that there exists a matrix $S = (s_{ij})_{\mathcal{A} \times \mathcal{A}}$ of zeros and ones such that

$$\Sigma = \left\{ x \in \mathcal{A}^{\mathbb{Z}} : s_{x_i x_{i+1}} = 1 \text{ for every } i \in \mathbb{Z} \right\}.$$

Note that the matrix S induces a directed graph on $\mathcal{A}$. Let $n \in \mathbb{N}$ and $\mathbf{r} := (r_1, \ldots, r_n) \in \mathcal{A}^n$, we say that $\mathbf{r}$ is an *admissible word* if $s_{r_i r_{i+1}} = 1$, for $i \in \{1, \ldots, n-1\}$. We say that the system is transitive if, given $x, y \in \mathcal{A}$, there exists an admissible word starting at x and ending at y. Let $(r_1, \ldots, r_n)$ be an admissible word and $l \in \mathbb{Z}$, we define the corresponding cylinder set by

$$[r_1, \ldots, r_n]_l := \{x \in \Sigma : x_l = r_1, x_{l+1} = r_2, \ldots, x_{l+n-1} = r_n\}.$$

If $l = 0$ then we omit this in the notation, writing $[r_1, \ldots, r_n]$ in place of $[r_1, \ldots, r_n]_0$. We endow Σ with the topology generated by the cylinder sets. Note that, with respect to this topology, the space Σ is non-compact. The *shift map* $\sigma : \Sigma \to \Sigma$ is defined by $(\sigma(x))_i = x_{i+1}$. Let $\mathcal{M}_\sigma$ be the set of σ−invariant probability measures and $\mathcal{E}_\sigma \subset \mathcal{M}_\sigma$ the set of ergodic invariant probability measures.

Each element $a \in \mathcal{A}$ corresponds either to a transitive component of Σ, i.e., a maximal subset Σ' of Σ where there are $x \in \Sigma'$ with $x_i = a$ for some i, and which is transitive (note that (Σ', σ) is also Markov); or, if not, to the wandering set. Since we will be dealing with ergodic measures μ_n converging to some μ, we will take $a \in \mathcal{A}$ such that $\mu([a]) > 0$, and we may assume that $\mu_n([a]) > 0$ for all n. Ergodicity implies that our measures are supported on the transitive component corresponding to a, and thus it is sufficient for our proofs to assume from here on that our system is transitive.

The topological entropy of σ is defined by

$$h_{top}(\sigma) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \chi_{[a]}(x)$$
$$= \sup\{h_\mu(\sigma) : \mu \in \mathcal{M}_\sigma\} = \sup\{h_\mu(\sigma) : \mu \in \mathcal{E}_\sigma\},$$

where $a \in \mathcal{A}$ is an arbitrary symbol and $\chi_{[a]}$ is the characteristic function of the cylinder $[a]$. By transitivity this definition is independent of $a \in \mathcal{A}$. Gurevich formulated this condition in [Gu1, Gu2] and proved that if (Σ, σ) is topologically mixing then in fact the limit exists (see also [DS, Remark 3.2]); he also proved the second and third equalities.[1]

3.2. Proof of Theorem 1.1

Given $a \in \mathcal{A}$ we define

$$\Sigma_a := \{x \in \Sigma : \sigma^k x \in [a] \text{ for infinitely many positive and negative } k \in \mathbb{Z}\}.$$

Observe that Σ_a is a Borel σ-invariant subset of Σ. Therefore, the dynamical system $\sigma : \Sigma_a \to \Sigma_a$ is well defined. Since (Σ, σ) is a finite entropy system, so is (Σ_a, σ).

Remark 3.1. Let $\mu \in \mathcal{E}_\sigma$ be an ergodic σ-invariant probability measure such that $\mu([a]) > 0$, then by the Birkhoff ergodic theorem we have $\mu(\Sigma_a) = 1$. This is the only point in our proof where we use ergodicity of our measures of interest. Moreover, since the system is transitive the set Σ_a is dense in Σ.

The following class of countable Markov shifts that has been studied in [BBG, Ru1, Sa1] will be of importance in what follows.

Definition 3.2. A *loop graph* is a graph made of simple loops which are based at a common vertex and otherwise do not intersect. A *loop system* is the two-sided countable Markov shift defined by a loop graph.

Lemma 3.3. *The system (Σ_a, σ) is topologically conjugate to a loop system $(\overline{\Sigma}, \sigma)$ of finite entropy.*

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[1]If the system were not transitive, we could take the supremum of all these quantities over the transitive components.

Proof. For every $n \in \mathbb{N}$ denote by C_n the set of non-empty cylinders in Σ of the form $[a, x_1, \cdots, x_n, a]$, where $x_i \neq a$ for all $i \in \{1, \ldots, n\}$. Since the entropy of Σ is finite, the number of elements in C_n is finite. Let c_n be the number of elements in C_n and write $C_n = \{A_n^1, \ldots, A_n^{c_n}\}$. Construct a loop graph with exactly c_n loops of length $n+1$, and denote by $\overline{\Sigma}$ the loop system associated to it. For convenience we denote the vertex of the loop graph with the letter a. There is a one to one correspondence between C_n and the non-empty cylinders in $\overline{\Sigma}$ of the form $[axa]$, where the word $\mathbf{x}$ does not contain the letter a and it has length n. Denote by B_n^i to the cylinder in $\overline{\Sigma}$ associated to A_n^i.

Note that every $x \in \Sigma_a$ is of the following form $(\ldots a\mathbf{x_{-1}} a\mathbf{x_0} a\mathbf{x_1} a \ldots)$, where $\mathbf{x_i}$ are admissible words that do not contain the letter a. Observe that for each $\mathbf{x_i}$ there is a unique corresponding cylinder $A_{n_i}^i$. Moreover, for each $A_{n_i}^i$ there is a unique corresponding cylinder $B_{n_i}^i$ in $\overline{\Sigma}$. Finally, for each $B_{n_i}^i$ there is a unique corresponding admissible word $\mathbf{x_i^b}$ in $\overline{\Sigma}$. Following this procedure we can define a bijective map $F : \Sigma_a \to \overline{\Sigma}$ by $F(\ldots a\mathbf{x_{-1}} a\mathbf{x_0} a\mathbf{x_1} a \ldots)$ $= (\ldots a\mathbf{x_{-1}^b} a\mathbf{x_0^b} a\mathbf{x_1^b} a \ldots)$.

We will now prove that F is a homeomorphism. Let U be an open set in $\overline{\Sigma}$. Consider a point $x \in U$ and define $y = F^{-1}(x)$. To prove the continuity of F it is enough to check that y is an interior point of $F^{-1}(U)$. Let $[a\mathbf{x_1^b} a\mathbf{x_2^b} a \cdots a\mathbf{x_m^b} a]_h$, where $h \in \mathbb{Z}$ and no $\mathbf{x_i}$ contains the letter a, be a cylinder contained in U such that $x \in U$. Note that

$$F^{-1}([a\mathbf{x_1^b} a\mathbf{x_2^b} a \cdots a\mathbf{x_m^b} a]_h) = [a\mathbf{x_1} a\mathbf{x_2} a \cdots a\mathbf{x_m} a]_h$$

Note that $y \in [a\mathbf{x_1} a\mathbf{x_2} a \cdots a\mathbf{x_m} a]_h$, $[a\mathbf{x_1} a\mathbf{x_2} a \cdots a\mathbf{x_m} a]_h \subset F^{-1}U$ and the cylinder set $[a\mathbf{x_1} a\mathbf{x_2} a \cdots a\mathbf{x_m} a]_h$ is open. Therefore, F is continuous. A similar argument gives that F^{-1} is also continuous, therefore F is a homeomorphism. By construction we have that $\sigma|_{\overline{\Sigma} \circ F} = F \circ \sigma|_{\Sigma_a}$. Since Σ_a has finite entropy so does $\overline{\Sigma}$. $\qquad \square$

The following result was obtained by Boyle, Buzzi and Gómez [BBG, Lemma 3.7], they established the existence of a *continuous* embedding of a loop system into a subshift of finite type. Let us stress that the relevant part of the result is the continuity. Borel embeddings have been obtained in greater generality (see Hochman [H2, Corollary 1.2] or [H1, Theorem 1.5]).

Theorem 3.4 (Boyle, Buzzi, Gómez). *A loop system of finite topological entropy can be continuously embedded in an invertible subshift of finite type.*

Lemma 3.5. *Let A and B be disjoint open subsets of Σ_a. Suppose A_1 and B_1 are open subsets of Σ such that $A_1 \cap \Sigma_a = A$ and $B_1 \cap \Sigma_a = B$. Then A_1 and B_1 are disjoint.*

Proof. Suppose that A_1 and B_1 are not disjoint. In this case we can find a non-empty open set $U \subset A_1 \cap B_1$. Define $V = U \cap \Sigma_a$ and observe that $V \subset A \cap B$. Since Σ_a is dense in Σ we have that $V = U \cap \Sigma_a$ is non-empty, which contradicts that A and B are disjoint. $\square$

Given an open subset A of Σ_a we define $\hat{A}$ as the largest open subset in Σ such that $\hat{A} \cap \Sigma_a = A$. Similarly, for a closed subset B of Σ_a we define $\check{B}$ as the smallest closed subset of Σ such that $\check{B} \cap \Sigma_a = B$. The existence of both $\hat{A}$ and $\check{B}$ follows from Zorn's Lemma.

Remark 3.6. If $B \subset \Sigma_a$ is closed set, then there exists a closed set $B_1 \subset \Sigma$ such that $B_1 \cap \Sigma_a = B$. Since $B \subset B_1$, we conclude that $\overline{B} \subset B_1$, where $\overline{B}$ is the closure of B in Σ. This implies that $B \subset \overline{B} \cap \Sigma_a \subset B_1 \cap \Sigma_a = B$, and therefore $\overline{B} \cap \Sigma_a = B$. Moreover $\check{B} = \overline{B}$.

Lemma 3.7. *Let A be an open and closed subset of Σ_a. Then $\hat{A} \subset \check{A}$, in particular $\overline{\hat{A}} = \overline{A}$.*

Proof. Let $B := \Sigma_a \smallsetminus A$. Observe that $\Sigma \smallsetminus \check{A}$ is open and that $(\Sigma \smallsetminus \check{A}) \cap \Sigma_a = B$. By the definition of $\hat{B}$ it follows that $(\Sigma \smallsetminus \check{A}) \subset \hat{B}$, or equivalently that $\Sigma \smallsetminus \hat{B} \subset \check{A}$. Observe that A and B are disjoint open subsets of Σ_a, therefore we can use Lemma 3.5 and obtain that $\hat{A} \subset (\Sigma \smallsetminus \hat{B})$. All this together implies that $\hat{A} \subset \check{A}$. $\square$

Lemma 3.8. *Suppose that $\mathcal{R} = \{R_1, \ldots, R_N\}$ is a partition of Σ_a such that the sets R_i, with $i \in \{1, \ldots, N\}$, are open and closed in the topology of Σ_a. Then there exists a finite partition $\hat{\mathcal{R}}$ of Σ which induces the partition $\mathcal{R}$ on Σ_a and $\mu(\partial \hat{\mathcal{R}}) = 0$ for every probability measure on Σ such that $\mu(\Sigma_a) = 1$.*

We observe that there is an important step in the proof of this lemma which uses the density of Σ_a in Σ, which as in Remark 3.1 follows from topological transitivity.

Proof. Let $\hat{\mathcal{R}} = \{\hat{R}_1, \ldots, \hat{R}_N, X\}$, where $X = \Sigma \smallsetminus \bigcup_{i=1}^{N} \hat{R}_i$. The partition is well defined since by Lemma 3.5 the sets $\{\hat{R}_1, \ldots, \hat{R}_N\}$ are disjoint. Observe that the set X is closed and has empty interior (since Σ_a is dense in Σ). Therefore $\mu(\partial X) = \mu(X \smallsetminus \text{int } X) = \mu(X) \leq \mu(\Sigma \smallsetminus \Sigma_a) = 0$. It follows

from Lemma 3.7 that $\mu(\partial \hat{R}_i) = \mu(\overline{\hat{R}_i} \smallsetminus \hat{R}_i) = \mu(\overline{R_i} \smallsetminus \hat{R}_i)$. As observed in Remark 3.6 we have that $\overline{R_i} \cap \Sigma_a = R_i$. Therefore

$$\mu(\partial \hat{R}_i) = \mu(\overline{R_i} \smallsetminus \hat{R}_i) = \mu((\overline{R_i} \cap \Sigma_a) \smallsetminus \hat{R}_i) = \mu(R_i \smallsetminus \hat{R}_i) = 0. \qquad \square$$

Proposition 3.9. *Let $(\mu_n)_n$ be a sequence of invariant probability measures converging in the weak* topology to a measure μ. If $\mu(\Sigma_a) = 1$ and $\mu_n(\Sigma_a) = 1$, for every $n \in \mathbb{N}$, then*

$$\limsup_{n\to\infty} h_{\mu_n}(\sigma) \leq h_\mu(\sigma).$$

Proof. It was shown in Lemma 3.3 that there exists a topological conjugacy $F : \Sigma_a \to \overline{\Sigma}$ between (Σ_a, σ) and $(\overline{\Sigma}, \sigma)$. From Theorem 3.4 there exists a topological embedding $G : \overline{\Sigma} \hookrightarrow \Sigma_0$, where Σ_0 is a subshift of finite type with alphabet $\{1, \ldots, N\}$. The partition $\mathcal{P} = \{[1], [2], \ldots, [N]\}$ is a generating partition of (Σ_0, σ) for every invariant probability measure (see [Wa, Theorem 8.2]). It follows that $\mathcal{Q} = G^{-1}\mathcal{P}$ is a generating partition for every σ-invariant measure in $\overline{\Sigma}$. The continuity of G implies that $G^{-1}([i]))$ is open and closed, in particular

$$\partial G^{-1}([i]) = \overline{G^{-1}([i])} \smallsetminus \text{ int } G^{-1}([i]) = \emptyset.$$

Therefore $\partial \mathcal{Q} = \emptyset$. Let $\mathcal{R} = F^{-1}\mathcal{Q}$. Then $\mathcal{R}$ is a generating partition for (Σ_a, σ), $\partial \mathcal{R} = \emptyset$ and moreover the elements in $\mathcal{R}$ are open and closed. From Lemma 3.8 we construct a partition $\hat{\mathcal{R}}$ of Σ that induces $\mathcal{R}$ when restricted to Σ_a. Since by assumption the measures μ and $(\mu_n)_n$ give full measure to Σ_a, Lemma 3.8 implies that $\mu(\partial \hat{\mathcal{R}}) = 0$. Moreover $h_\mu(\sigma) = h_\mu(\sigma, \hat{\mathcal{R}})$ and $h_{\mu_n}(\sigma) = h_{\mu_n}(\sigma, \hat{\mathcal{R}})$ for every $n \in \mathbb{N}$. Indeed, for every σ-invariant probability measure ν on Σ such that $\nu(\Sigma_a) = 1$ we have that the systems (Σ, σ, ν) and $(\Sigma_a, \sigma|_{\Sigma_a}, \nu|_{\Sigma_a})$ are isomorphic. Therefore

$$h_\nu(\Sigma, \sigma) = h_\nu(\Sigma_a, \sigma) = \lim_{n\to\infty} \frac{1}{n} H_\nu(\mathcal{R}^n),$$

since $\mathcal{R}$ is a generating partition for (Σ_a, σ). Note that for every $n \in \mathbb{N}$ and $\hat{\mathcal{R}}_{i_1, \ldots, 1_n} \in \hat{\mathcal{R}}^n$ we have

$$\nu(\hat{\mathcal{R}}_{i_1, \ldots, 1_n}) = \nu(\hat{\mathcal{R}}_{i_1, \ldots, 1_n} \cap \Sigma_a) = \nu|_{\Sigma_a}(\mathcal{R}_{i_1, \ldots, 1_n}).$$

Thus $h_\nu(\sigma) = h_\nu(\sigma, \hat{\mathcal{R}})$. Therefore, Proposition 2.13 implies that

$$\limsup_{n\to\infty} h_{\mu_n}(\sigma) = \limsup_{n\to\infty} h_{\mu_n}(\sigma, \hat{\mathcal{R}}) \leq h_\mu(\sigma, \hat{\mathcal{R}}) = h_\mu(\sigma). \qquad \square$$

Proof of Theorem 1.1. The above argument proves that if $(\mu_n)_n$ is a sequence of ergodic measures that converges in the weak* topology to the ergodic measure μ then $\lim_{n\to\infty} h_{\mu_n}(\sigma) \le h_\mu(\sigma)$. Indeed, there exists $a \in \mathcal{A}$ such that $\mu([a]) > 0$. Since $\lim_{n\to\infty} \mu_n([a]) = \mu([a]) > 0$, there exists $N \in \mathbb{N}$ such that for every $n > N$ we have that $\mu_n([a]) > 0$. We then use Proposition 3.9 to get the result. $\qquad\square$

3.3. One-sided countable Markov shifts

Let (Σ^+, σ) be a one-sided Markov shift defined over a countable alphabet $\mathcal{A}$. This means that there exists a matrix $S = (s_{ij})_{\mathcal{A} \times \mathcal{A}}$ of zeros and ones such that

$$\Sigma^+ = \left\{ x \in \mathcal{A}^\mathbb{N} : s_{x_i x_{i+1}} = 1 \text{ for every } i \in \mathbb{N} \right\}.$$

The *shift map* $\sigma : \Sigma^+ \to \Sigma^+$ is defined by $(\sigma(x))_i = x_{i+1}$. The entropy of σ is defined by $h_{top} = h_{top}(\sigma) := \sup \{h_\mu(\sigma) : \mu \in \mathcal{E}_\sigma\}$. Note that, as in the two-sided setting, it is possible to give a definition of entropy computing the exponential growth of periodic orbits, but for the purposes of this article the above definition suffices.

Proof of Corollary 1.2. Denote by (Σ, σ) the natural extension of (Σ^+, σ) and by $\mathcal{E}$ and $\mathcal{E}^+$ the corresponding sets of ergodic measures. There exists a bijection $\pi : \mathcal{E}^+ \to \mathcal{E}$ such that for every $\mu \in \mathcal{E}^+$ we have that $h_\mu = h_{\pi(\mu)}$ (see [Do, Fact 4.3.2] and [Sa3, Section 2.3]).

Lemma 3.10. *Let $(\mu_n)_n, \mu \in \mathcal{E}^+$ such that $(\mu_n)_n$ converges weak* to μ. Then $(\pi(\mu_n))_n$ converges weak* to $\pi(\mu)$.*

Proof. Note that in the notion of weak* convergence we can replace the set of test functions by the space of bounded Lipschitz functions (see Remark 2.2). That is, if for every bounded Lipschitz function $\varphi : \Sigma \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \int \varphi \, d\mu_n = \int \varphi \, d\mu,$$

then the sequence $(\mu_n)_n$ converges in the weak* topology to μ. A result by Daon [Da, Theorem 3.1] implies that for every Lipschitz (the result also holds for weakly Hölder and summable variations) function $\varphi : \Sigma \to \mathbb{R}$ there exists a cohomologous Lipschitz function $\psi : \Sigma \to \mathbb{R}$ that depends only on future coordinates. The transfer function can be chosen bounded and uniformly

continuous. The function ψ can be canonically identified with a Lipschitz function $\rho : \Sigma^+ \to \mathbb{R}$. Thus,

$$\int_\Sigma \varphi \, d\pi(\mu_n) = \int_\Sigma \psi \, d\pi(\mu_n) = \int_{\Sigma^+} \rho \, d\mu_n.$$

Therefore,

$$\lim_{n\to\infty} \int_\Sigma \varphi \, d\pi(\mu_n) = \lim_{n\to\infty} \int_{\Sigma^+} \rho \, d\mu_n = \int_{\Sigma^+} \rho \, d\mu = \int_\Sigma \varphi \, d\pi(\mu_n).$$

The result now follows. $\qquad\square$

Let $\mu_n, \mu \in \mathcal{E}_\sigma$ be such that $(\mu_n)_n$ converges weak* to μ. Lemma 3.10 implies that $(\pi(\mu_n))_n$ converges weak* to $\pi(\mu)$. Moreover, for every $n \in \mathbb{N}$ we have that $h_{\pi(\mu_n)} = h_{\mu_n}$ and $h_{\pi(\mu)} = h_\mu$. Since, by Theorem 1.1, the entropy map is upper semi-continuous in (Σ, σ) we obtain the result. $\qquad\square$

Remark 3.11. The finite entropy assumption in Theorem 1.1 and in Corollary 1.2 is essential as the following example shows. Let (Σ^+, σ) be the full shift on a countable alphabet, note that $h_{top}(\sigma) = \infty$. Denote by $\mathcal{P}$ the partition formed by the length one cylinders. This is a generating partition. Let $h \in \mathbb{R}^+$ be a positive real number and $(a_n)_n$ be the sequence defined by $a_n = \frac{h}{\log n}$ for every $n > 1$. Consider the following stochastic vector

$$\vec{p}_n := \left(1 - a_n, \frac{a_n}{n}, \frac{a_n}{n}, \dots, \frac{a_n}{n}, 0, 0, \dots\right),$$

where the term a_n/n appears n times. Let μ_n be the Bernoulli measure defined by $\vec{p}_n$. Note that the sequence $(\mu_n)_n$ converges in the weak* topology to a Dirac measure δ_1 supported on the fixed point at the cylinder C_1. Note that

$$H_{\mu_n}(\mathcal{P}) = -(1 - a_n)\log(1 - a_n) - a_n \log a_n - a_n \log \frac{1}{n}.$$

Therefore,

$$\lim_{n\to\infty} H_{\mu_n}(\mathcal{P}) = h > 0 = H_{\delta_1}(\mathcal{P}).$$

The above example shows that the map $\nu \to H_\nu(\mathcal{P})$ need not to be continuous for countable generating partitions. Moreover, since the measures μ_n are Bernoulli we have that $h_{\mu_n}(\sigma) = H_{\mu_n}(\mathcal{P})$, and therefore the above argument

shows that the entropy map is not upper semi-continuous:

$$\lim_{n \to \infty} h_{\mu_n}(\sigma) = h > 0 = h_{\delta_1}(\sigma).$$

In other words, the entropy map could fail to be upper semi-continuous for general dynamical systems defined on non-compact spaces, even if there exists a uniform (countable) generating partition with no boundary. The construction of the above example is based on examples constructed by Walters ([Wa, p.184]) and by Jenkinson, Mauldin and Urbański [JMU, p.774]. This example also shows that the entropy map is not upper semi-continuous even if we consider a sequence of measures for which their entropy is uniformly bounded.

4. Measures of maximal entropy

A continuous map $T : (X, d) \to (X, d)$ defined on a compact metric space for which the entropy map is upper semi-continuous has a measure of maximal entropy. Indeed, from the variational principle there exists a sequence of ergodic invariant probability measures $(\mu_n)_n$ such that $\lim_{n \to \infty} h_{\mu_n}(T) = h_{top}(T)$. Since the space of invariant measures $\mathcal{M}_T$ is compact, there exists an invariant measure μ which is an accumulation point for $(\mu_n)_n$. It follows from the fact that the entropy map is upper semi-continuous that

$$h_{top}(T) = \lim_{n \to \infty} h_{\mu_n}(T) \le h_\mu(T).$$

Therefore, μ is a measure of maximal entropy. For countable Markov shifts the variational principle holds (see [Gu1, Gu2]) and the entropy map is upper semi-continuous (see Theorem 1.1 and Corollary 1.2), however the space $\mathcal{M}_\sigma$ is no longer compact. Despite this, under a convergence assumption we can prove the existence of measures of maximal entropy. Indeed, Corollary 1.2 provides a new proof of the following result by Gurevich and Savchenko [GS, Theorem 6.3].

Proposition 4.1. *Let (Σ, σ) be a finite entropy countable Markov shift. Let $(\mu_n)_n$ be a sequence of ergodic measures such that $\lim_{n \to \infty} h_{\mu_n}(\sigma) = h_{top}(\sigma)$. If $(\mu_n)_n$ converges in the weak* topology to an ergodic measure μ then $h_\mu(\sigma) = h_{top}(\sigma)$.*

It might happen that there is a sequence $(\mu_n)_n$ with $\lim_{n \to \infty} h_{\mu_n}(\sigma) = h_{top}(\sigma)$, but $(\mu_n)_n$ does not converge in the weak* topology. Examples of

countable Markov shifts with this property have been known for a long time. In [Ru2] a simple construction of a Markov shift of any given entropy with no measure of maximal entropy is provided. In [Gu3, GN] examples are constructed of finite entropy countable Markov shifts having a measure of maximal entropy μ_{max} and sequences of ergodic measures $(\mu_n)_n, (\nu_n)_n$ with $\lim_{n\to\infty} h_{\mu_n}(\sigma) = \lim_{n\to\infty} h_{\nu_n}(\sigma) = h_{top}(\sigma)$ such that $(\mu_n)_n$ converges in the weak* topology to μ_{max} and $(\nu_n)_n$ does not have any accumulation point.

The continuity properties of the entropy map also have consequences in the study of thermodynamic formalism. Let (Σ^+, σ) be a transitive one-sided countable Markov shift of finite entropy and $\varphi : \Sigma^+ \to \mathbb{R}$ a continuous bounded function of summable variations. That is $\sum_{n=1}^{\infty} \mathrm{var}_n(\varphi) < \infty$, where

$$\mathrm{var}_n(\varphi) := \sup \{\varphi(x) - \varphi(y) : x_i = y_i, i \in \{1, \ldots, n\}\}.$$

Sarig (see [Sa3] for a survey on the topic) defined a notion of pressure in this context, the so called Gurevich pressure, that we denote by $P(\varphi)$. He proved the following variational principle

$$P(\varphi) = \sup \left\{ h_\mu(\sigma) + \int \varphi \, d\mu : \mu \in \mathcal{M}_\sigma \right\}$$
$$= \sup \left\{ h_\mu(\sigma) + \int \varphi \, d\mu : \mu \in \mathcal{E}_\sigma \right\}$$

A measure $\mu \in \mathcal{M}_\sigma$ such that $P(\varphi) = h_\mu(\sigma) + \int \varphi \, d\mu$ is called *equilibrium measure*. It directly follows from Corollary 1.2 that

Proposition 4.2. *Let (Σ^+, σ) be a finite entropy countable Markov shift and $\varphi : \Sigma \to \mathbb{R}$ a bounded function of summable variations. Let $(\mu_n)_n$ be a sequence of ergodic measures such that $\lim_{n\to\infty} \left(h_{\mu_n}(\sigma) + \int \varphi \, d\mu_n\right) = P(\varphi)$. If $(\mu_n)_n$ converges in the weak* topology to an ergodic measure μ then $P(\varphi) = h_\mu(\sigma) + \int \varphi \, d\mu$.*

Note that the thermodynamic formalism of a two-sided countable Markov shift can be reduced to the one-sided case (see [Sa3, Section 2.3]).

5. Suspension flows

Let (Σ, σ) be a finite entropy countable Markov shift and let $\tau : \Sigma \to \mathbb{R}^+$ be a locally Hölder potential bounded away from zero. Consider the space $Y = \{(x, t) \in \Sigma \times \mathbb{R} : 0 \le t \le \tau(x)\}$, with the points $(x, \tau(x))$ and $(\sigma(x), 0)$ identified for each $x \in \Sigma$. The *suspension flow* over Σ with *roof function* τ is

the semi-flow $\Phi = (\varphi_t)_{t \in \mathbb{R}}$ on Y defined by $\varphi_t(x,s) = (x, s+t)$ whenever $s + t \in [0, \tau(x)]$. Denote by $\mathcal{M}_\Phi$ the space of flow invariant probability measures. Let

(5.1)
$$\mathcal{M}_\sigma(\tau) := \left\{ \mu \in \mathcal{M}_\sigma : \int \tau \, d\mu < \infty \right\}.$$

It follows directly from results by Ambrose and Kakutani [AK] that the map $R \colon \mathcal{M}_\sigma \to \mathcal{M}_\Phi$, defined by

$$R(\mu) = \frac{(\mu \times \mathrm{Leb})|_Y}{(\mu \times \mathrm{Leb})(Y)},$$

where Leb is the one-dimensional Lebesgue measure, is a bijection. Denote by $\mathcal{E}_\Phi$ the set of ergodic flow invariant measures. Let $F \colon Y \to \mathbb{R}$ be a continuous function. Define $\Delta_F \colon \Sigma \to \mathbb{R}$ by

$$\Delta_F(x) := \int_0^{\tau(x)} F(x,t) \, dt.$$

Kac's Lemma states that if $\nu \in \mathcal{M}_\Phi$ is an invariant measure that can be written as

$$\nu = \frac{\mu \times \mathrm{Leb}}{(\mu \times \mathrm{Leb})(Y)},$$

where $\mu \in \mathcal{M}_\sigma$, then

$$\int_Y F \, d\nu = \frac{\int_\Sigma \Delta_F \, d\mu}{\int_\Sigma \tau \, d\mu}.$$

The following Lemma describes the relation between weak* convergence in $\mathcal{M}_\Phi$ with that in $\mathcal{M}_\sigma$.

Lemma 5.1. *Let* $(\nu_n), \nu \in \mathcal{M}_\Phi$ *be flow invariant probability measures such that*

$$\nu_n = \frac{\mu_n \times Leb}{\int \tau \, d\mu_n} \quad and \quad \nu = \frac{\mu \times Leb}{\int \tau \, d\mu}$$

where $(\mu_n)_n, \mu \in \mathcal{M}_\sigma$ *are shift invariant probability measures. If sequence* $(\nu_n)_n$ *converges in the weak* topology to* ν *then*

$(\mu_n)_n$ *converges in the weak* topology to* μ *and* $\displaystyle\lim_{n \to \infty} \int \tau \, d\mu_n = \int \tau \, d\mu.$

Proof. Assume first that $(\nu_n)_n$ converges in the weak* topology to ν. Let $f : \Sigma \to \mathbb{R}$ be a bounded continuous function. Following Barreira, Radu and Wolf [BRW] there exists a continuous function $F : Y \to \mathbb{R}$ such that

$$(5.2) \qquad f(x) = \Delta_F(x) := \int_0^{\tau(x)} F(x, t) \, dt.$$

Indeed, define $F(x, t) : Y \to \mathbb{R}$ by

$$F(x, t) := \frac{f(x)}{\tau(x)} \psi' \left(\frac{t}{\tau(x)} \right),$$

where $\psi : [0, 1] \to [0, 1]$ is a C^1 function such that $\psi(0) = 0$, $\psi(1) = 1$ and $\psi'(0) = \psi'(1) = 0$. Note that since τ is bounded away from zero, $F(x, t)$ is continuous and bounded. Therefore

$$\lim_{n \to \infty} \int F \, d\nu_n = \int F \, d\nu.$$

By Kac's Lemma we have that

$$(5.3) \qquad \lim_{n \to \infty} \frac{\int \Delta_F \, d\mu_n}{\int \tau \, d\mu_n} = \frac{\int \Delta_F \, d\mu}{\int \tau \, d\mu}.$$

In particular if $f = 1$ is the constant function equal to one, we obtain

$$(5.4) \qquad \lim_{n \to \infty} \frac{\int \tau \, d\mu_n}{\int \tau \, d\mu} = 1.$$

Let $f : \Sigma \to \mathbb{R}$ be a bounded continuous function, then it follows from equations (5.2), (5.3) and (5.4) that

$$\lim_{n \to \infty} \int f \, d\mu_n = \int f \, d\mu.$$

Therefore (μ_n) converges weak* to μ. $\qquad \square$

Proposition 5.2. *If $(\nu_n)_n$ is a sequence of ergodic measures in $\mathcal{E}_\Phi$ converging to an ergodic measure ν, then*

$$\limsup_{n \to \infty} h_{\nu_n}(\Phi) \leq h_\nu(\Phi).$$

Proof. Let $(\nu_n)_n$ be a sequence in $\mathcal{E}_\Phi$ that converges in the weak* topology to $\nu \in \mathcal{E}_\Phi$. Note that there exists a sequence of ergodic measures $(\mu_n)_n \in \mathcal{M}_\sigma$ and $\mu \in \mathcal{M}_\sigma$ ergodic such that

$$\nu_n = \frac{\mu_n \times \mathrm{Leb}}{(\mu_n \times \mathrm{Leb})(Y)} \quad \text{and} \quad \nu = \frac{\mu \times \mathrm{Leb}}{(\mu \times \mathrm{Leb})(Y)}.$$

By Abramov's formula [Ab], for any $\nu' \in \mathcal{M}_\Phi$, with $\nu' = \frac{\mu' \times \mathrm{Leb}}{(\mu' \times \mathrm{Leb})(Y)}$ we have that

$$h_{\nu'}(\Phi) = \frac{h_{\mu'}(\sigma)}{\int \tau \, d\mu'}.$$

Recall that by Lemma 5.1 we have that $\nu_n \to \nu$ in the weak* topology implies that $\mu_n \to \mu$ in the weak* topology and $\lim_{n\to\infty} \int \tau \, d\mu_n = \int \tau \, d\mu$. Since $h_{top}(\sigma) < \infty$ by Theorem 1.1 or Corollary 1.2 we have that

$$\limsup_{n\to\infty} h_{\mu_n}(\sigma) \leq h_\mu(\sigma).$$

Therefore,

$$\limsup_{n\to\infty} h_{\nu_n}(\Phi) = \limsup_{n\to\infty} \frac{h_{\mu_n}(\sigma)}{\int \tau \, d\mu_n} = \frac{\limsup_{n\to\infty} h_{\mu_n}(\sigma)}{\int \tau \, d\mu} \leq \frac{h_\mu(\sigma)}{\int \tau \, d\mu} = h_\nu(\Phi).$$

$\square$

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Facultad de Matemáticas
Pontificia Universidad Católica de Chile (PUC)
Avenida Vicuña Mackenna 4860, Santiago, Chile
E-mail address: giommi@mat.puc.cl

School of Mathematics and Statistics
University of St Andrews, Mathematical Institute
North Haugh, St Andrews, KY16 9SS, Scotland
E-mail address: m.todd@st-andrews.ac.uk

Department of Mathematics, Yale University
New Haven, CT 06511, USA
E-mail address: anibal.velozo@yale.edu