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Log canonical thresholds of Burniat surfaces with $K^2 = 6$

In-Kyun Kim and YongJoo Shin

Let S be a Burniat surface with $K_S^2 = 6$. Then we show that $\mathrm{glct}(S, K_S) = \frac{1}{2}$ by showing that $\mathrm{glct}(S, 2K_S) = \mathrm{lct}(S, E) = \frac{1}{4}$ for some divisor $E \in |2K_S|$. This implies that Tian's conjecture (which fails in general) holds for the polarized pair $(S, 2K_S)$, since the corresponding graded algebra is generated by sections of $H^0(S, 2K_S)$.

Moreover we verify that any divisor $D \in |mK_S|$ such that $\mathrm{glct}(S, K_S) = \mathrm{lct}(S, \frac{1}{m}D)$ for a positive even integer m is invariant under the $\mathbb{Z}_2^2$-action associated to the bicanonical map of S.

1. Introduction

Let X be a normal variety with at worst log canonical singularities and let D be an effective $\mathbb{Q}$-Cartier divisor on it. The discrepancy of the log pair (X, D) is used to measure how singular it is. If (X, D) is not log canonical then the discrepancy of it is $-\infty$. In such a case, it gives no more information than non-log canonicity of (X, D). Then we need the notion of log canonical threshold. The following number is the definition of the log canonical threshold of D along a subvariety Z of X.

$$\mathrm{lct}_Z(X, D) := \sup\{\lambda \in \mathbb{Q} \mid (X, \lambda D) \text{ is log canonical along } Z\}.$$

If $Z = X$ then we write $\mathrm{lct}(X, D) = \mathrm{lct}_Z(X, D)$.

Now we suppose that the variety X is Fano with at worst log canonical singularities, that is, the anti-canonical divisor $-K_X$ of it is ample. Using the definition of the log canonical threshold we can define the global log canonical threshold of X.

Definition 1.1. *The global log canonical threshold of X is the number*

$$\mathrm{glct}(X) := \inf\left\{\mathrm{lct}(X, D) \,\middle|\, \begin{array}{l} D \text{ is an effective } \mathbb{Q}\text{-Cartier divisor on } X \\ \mathbb{Q}\text{-linearly equivalent to } -K_X \end{array}\right\}.$$

Meanwhile, Tian introduced the α-invariant to study the existence of Kähler-Einstein metrics on Fano manifolds. He proved that if the α-invariant of a smooth Fano manifold X is bigger than $\dim X/(\dim X+1)$ then it admits a Kähler-Einstein metric (see [11, 13, 18, 21]). The global log canonical threshold is an algebraic counterpart to Tian's α-invariant [11]. Thus, to prove the existence of a Kähler-Einstein metric of Fano manifolds we can use the global log canonical threshold instead of the α-invariant. Later, Tian generalized the α-invariant for arbitrary polarized pairs (X, L) which is still a counterpart of the global log canonical threshold [11], where X is a smooth variety and L is an ample divisor on it.

Here, we use the global log canonical threshold for a $\mathbb{Q}$-Cartier divisor $\mathcal{L}$ on a normal variety X with at worst log canonical singularity.

Definition 1.2. *Let X be a normal variety with at worst log canonical singularity, and $\mathcal{L}$ be a $\mathbb{Q}$-Cartier divisor on X. The global log canonical threshold of a pair $(X,\mathcal{L})$ is the number*

$$\mathrm{glct}(X,\mathcal{L}) := \inf\left\{ \mathrm{lct}(X,D) \,\middle|\, \begin{array}{l} D \;\; \textit{is an effective } \mathbb{Q}\textit{-Cartier divisor on } X \\ \mathbb{Q}\textit{-linearly equivalent to } \mathcal{L} \end{array} \right\}.$$

At the moment, we know plenty of results about global log canonical thresholds of del Pezzo surfaces and higher dimensional Fano varieties. Thus, it seems interesting to do the same for other varieties. Surfaces of general type (polarized by their canonical divisors) is the most natural choice to start with. However, this is impossible to do for all surfaces of general type, simply because they do not form a bounded family. But one can try to do this for some of them whose geometry is somehow similar to del Pezzo surfaces.

Chen, Chen and Jiang [12] recently established Noether inequality for projective 3-folds with an appendix by Kollár which is the first application of a global log canonical threshold of a surface of general type via its canonical divisor. Kollár in the appendix gave an optimal lower bound of $\mathrm{glct}(X,\mathcal{L})$ in the case when X is a surface of general type with Du Val singularities such that K_X is ample, $K_X^2 = 1$ and $h^0(X, K_X) = 2$, where $\mathcal{L} = K_X$.

In this paper we consider global log canonical thresholds of Burniat surfaces with $K^2 = 6$ via their canonical divisors. Burniat [8] constructed minimal surfaces of general type with $p_g = 0$, and $K^2 = 2, 3, \ldots, 6$ using bidouble coverings on $\mathbb{P}^2$, called by Burniat surfaces with K^2. Peters [20] later reconsidered the surfaces constructed by Burniat. He calculated torsion groups of $H_1(S,\mathbb{Z})$ for Burniat surfaces S with K_S^2. Note the calculation of Burniat surface with $K^2 = 2$ did not correct. Bauer, Catanese [5] and Kulikov [16] pointed out and corrected it. Mendes Lopes and Pardini [17] considered a moduli of Burniat surfaces with $K^2 = 6$. Bauer and Catanese [4–7] also dealt with properties of a moduli of Burniat surfaces with each K^2, and they found extended Burniat surfaces. They clearly verified that surfaces constructed by Inoue [14] is exactly Burniat surfaces in [5]. Inoue [14] constructed the surfaces as quotients of $\mathbb{Z}_2^3$-invariant hypersurfaces of multidegree $(2, 2, 2)$ in a product of three elliptic curves by the action $\mathbb{Z}_2^3$. Alexeev and Pardini [3] compactfied Burniat surfaces with $K^2 = 6$ by adding semi log canonical surfaces with ample K. And Alexeev and Orlov [2] considered derived categories of Burniat surfaces. They constructed an exceptional collection of maximal possible length of Burniat surfaces with $K^2 = 6$, and showed that the complement of the exceptional collection is a quasi-phantom category.

We firstly treat a polarized pair $(S, 2K_S)$ where a smooth surface S is a Burniat surface with $K_S^2 = 6$. The bicanonical map φ of S has an image that is a del Pezzo surface Σ of degree 6 in $\mathbb{P}^6$ blown-up at three points in $\mathbb{P}^2$ in general position. Note the image of a Burniat surface with $K^2 = 5$ is a del Pezzo surface of degree 5 in $\mathbb{P}^5$ blown-up at four points in $\mathbb{P}^2$ in general position. So the case $K^2 = 6$ is easier to treat than other K^2 in some sense. Moreover since Catanese [9] described a bidouble covering using a locally explicit equations (see Section 3), we use it to calculate the global log canonical threshold of S via $2K_S$ (see Proposition 4.2). Then we obtain as follows.

Theorem 1.3 (Theorem 4.3). *Let S be a Burniat surface with $K_S^2 = 6$. Then the global log canonical threshold of the polarized pair (S, K_S) is $\frac{1}{2}$.*

To prove Theorem 1.3 we show that

$$\operatorname{glct}(S, 2K_S) = \operatorname{lct}(S, D) = \frac{1}{4}$$

for some divisor $D \in |2K_S|$.

Since a Burniat surface S with $K_S^2 = 6$ has the $\mathbb{Z}_2^2$-action induced by the bicanonical map φ of S, we ask that the global log canonical threshold of the polarized pair (S, K_S) is given by an invariant global section of mK_S or not by an anti-invariant section of mK_S via the $\mathbb{Z}_2^2$-action for any positive integer m. In Section 5 we give the answer when m is an even integer $m > 0$ as follows.

Theorem 1.4 (Theorem 5.3). *Let S be a Burniat surface with $K_S^2 = 6$ and m be a positive even integer. Then for any divisor $D \in |mK_S|$ such that*

$\mathrm{glct}(S, K_S) = \mathrm{lct}(S, \frac{1}{m}D)$ *it is invariant under the* $\mathbb{Z}_2^2$*-action associated to the bicanonical map* φ *of* S.

2. Preliminary

Let X be a normal variety with at worst log canonical singularities and let A be an ample divisor on it. The following is called the convex lemma.

Lemma 2.1. *Let* B *and* B' *be* $\mathbb{Q}$*-divisors on* X*. Then the log pair*

$$(X, \alpha B + (1-\alpha)B')$$

is log canonical if both (X, B) *and* (X, B') *are log canonical where* $0 \leq \alpha \leq 1$.

There is a useful lemma to obtain a global log canonical threshold. Note that $\sim_{\mathbb{Q}}$ means the $\mathbb{Q}$-linearly equivalent relation.

Lemma 2.2. *Let* $D \sim_{\mathbb{Q}} A$ *be an effective* $\mathbb{Q}$*-Cartier divisor on* X *such that the log pair* (X, D) *is not log canonical at a point* p*. And let* $D_0 \sim_{\mathbb{Q}} A$ *be an effective* $\mathbb{Q}$*-Cartier divisor on* X *such that the log pair* (X, D_0) *is log canonical at the point* p*. Then there is an effective* $\mathbb{Q}$*-Cartier divisor* $D' \sim_{\mathbb{Q}} A$ *on* X *such that at least one component of* D_0 *is not contained in the support of* D' *and the log pair* (X, D') *is not log canonical at the point* p.

Proof. See [11, Remark 2.22]. □

For a non log canonical pair at some smooth point we use the following.

Lemma 2.3 (cf. [15, 8.10 Lemma]). *Let* D *be an effective* $\mathbb{Q}$*-Cartier divisor on* X*. If the log pair* (X, D) *is not log canonical at some smooth point* p *then the inequality*

$$\mathrm{mult}_{\mathsf{p}}(D) > 1$$

holds.

3. Local coordinates of a Burniat surface

Throughout this paper, a smooth surface S is a Burniat surface with $K_S^2 = 6$. Then there is the bicanonical morphism $\varphi\colon S \to \mathbb{P}^6$ given by the linear system $|2K_S|$ of S. Let Σ be the image of φ. Then we have the morphism φ

that is a bidouble covering map of Σ which is a del Pezzo surface of degree 6 (see [17, Proposition 3.1]). Meanwhile, the bidouble covering map φ is constructed by the data of effective divisors B_1, B_2 and B_3 of Σ with the conditions

$$2L_i \sim B_j + B_k$$

for $\{i, j, k\} = \{1, 2, 3\}$, where L_i is a divisor on Σ (see [19]). Indeed, there is the blow-up $\rho\colon \Sigma \to \mathbb{P}^2$ at three points p_1, p_2 and p_3 in general position on $\mathbb{P}^2$ with the exceptional divisors e_1, e_2 and e_3 such that $\rho^{-1}(p_i) = e_i$. Let l_i be the line in $\mathbb{P}^2$ passing through the points p_j and p_k. Denote by e_i' the strict transform of l_i. Then $B_i = e_i + e_i' + m_1^j + m_2^j$ for $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$, where $\varphi(m_1^j)$ and $\varphi(m_2^j)$ are general lines passing through p_j (see [17, Section 3]. Let V be the vector bundle $\bigoplus_{i=1}^3 \mathcal{O}_\Sigma(L_i)$ and denote by w_1, w_2 and w_3 fibre coordinates relative to the three summands. Then there are three equations

$$w_i^2 = f_j f_k$$

where f_* is a defining equation of B_*. The Burniat surface S is a subvariety of V given by

$$w_i^2 = f_j f_k, \qquad f_k w_k = w_i w_j.$$

Let p be the point of S such that $\varphi(p) = e_2' \cap e_3$. Since $f_1(p) \neq 0$ w_1, w_2, w_3 and f_1 are local coordinates of V at p. On S we have the following

$$f_2 = \frac{w_3^2}{f_1}, \qquad f_3 = \frac{w_2^2}{f_1}, \qquad w_1 = \frac{w_2 w_3}{f_1}.$$

It implies that w_2 and w_3 are local coordinates of S at p.

Now we find an effective divisor with a small log canonical threshold. In fact, it is to be the global log canonical threshold of the polarized pair $(S, 2K_S)$.

Proposition 3.1. *There is an effective divisor $E \in |2K_S|$ such that $\operatorname{lct}(S, E) = \frac{1}{4}$ and E is invariant by the $\mathbb{Z}_2^2$-action associated to the bidouble covering map over Σ, the bicanonical map φ of S.*

Proof. Let $L := l_1 + 2l_2$ be the divisor of $\mathbb{P}^2$. Then

$$\rho^*(K_{\mathbb{P}^2} + L) \sim K_\Sigma + e_1' + 2e_2' + e_1 + 2e_3.$$

It implies that $-K_\Sigma \sim e_1' + 2e_2' + e_1 + 2e_3$. We set $e := e_1' + 2e_2' + e_1 + 2e_3$ and $\varphi(p) = e_2' \cap e_3$. Since e is locally isomorphic to the divisor given by $f_2^2 f_3^2$ at $\varphi(p)$, $E := \varphi^*(e) \sim 2K_S$ is given by $\frac{w_2^4 w_3^4}{f_1^4}$ at p. Therefore $\mathrm{lct}(S, E) = \frac{1}{4}$.

Also since E is defined by the pull-back of e it is clear that E is invariant by the $\mathbb{Z}_2^2$-action. □

4. Global log canonical threshold

In this section, we consider the global log canonical threshold (see Theorem 4.3) for a Burniat surface with $K^2 = 6$. For those we prove Proposition 4.2, that is, the global log canonical threshold of the polarized pair $(S, 2K_S)$ is $\frac{1}{4}$ where S is a Burniat surface with $K_S^2 = 6$. To do so, we assume that there is an effective $\mathbb{Q}$-Cartier divisor D such that the log pair $(S, \mu D)$ is not log canonical at some point $\mathsf{p} \in S$ where $\mu = \frac{1}{4}$. Then we obtain a contradiction with Lemma 4.1.

A bidouble covering map ψ from a normal variety X to a smooth variety Y gives the building data, divisors $\mathcal{L}_i$ and effective divisors $\mathcal{B}_i$ on Y for $i = 1, 2, 3$ such that the branch divisor $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3$ of ψ and the relations $2\mathcal{L}_i \sim \mathcal{B}_j + \mathcal{B}_k$ for $\{i, j, k\} = \{1, 2, 3\}$. Conversely the building data with the relations constructs the bidouble covering map ψ. Then $K_X \sim_{\mathbb{Q}} \psi^*(K_Y + \frac{1}{2}\mathcal{B})$. We refer [19] as a reference.

Lemma 4.1 (cf. [15, 3. 16 Proposition and 8.12 Lemma]). *Let $\psi\colon X \to Y$ be a bidouble covering map between a normal variety X and a smooth variety Y branched along an effective divisor $\mathcal{B}$ on Y, and $\mathcal{D}$ be an effective $\mathbb{Q}$-Cartier divisor on X. Then*

$$(X, \mathcal{D}) \textit{ is log canonical if } (Y, \psi(\mathcal{D}) + \frac{1}{2}\mathcal{B}) \textit{ is log canonical.}$$

Proof. Suppose the log pair $(X, \mathcal{D})$ is not log canonical at some point q in S, then so is $(X, \psi^*(\psi(\mathcal{D})))$ because $\psi^*(\psi(\mathcal{D})) \geq \mathcal{D}$. Since $K_X + \psi^*(\psi(\mathcal{D})) \sim_{\mathbb{Q}} \psi^*(K_Y + \psi(\mathcal{D}) + \frac{1}{2}\mathcal{B})$ the pair $(X, \psi^*(\psi(\mathcal{D})))$ is log canonical if and only if the pair $(Y, \psi(\mathcal{D}) + \frac{1}{2}\mathcal{B})$ is log canonical by [15, 3.16 Proposition]. Thus the pair $(Y, \psi(\mathcal{D}) + \frac{1}{2}\mathcal{B})$ is not log canonical at $\psi(\mathsf{q})$. □

We use the notations of Section 3.

Proposition 4.2. *Let S be a Burniat surface with $K_S^2 = 6$. Then the global log canonical threshold of the polarized pair $(S, 2K_S)$ is $\frac{1}{4}$.*

Proof. Suppose that $\mathrm{glct}(S, 2K_S)$ is not $\frac{1}{4}$. Since there is an effective $\mathbb{Q}$-Cartier divisor such that the log canonical threshold of it is $\frac{1}{4}$ (see Proposition 3.1), $\mathrm{glct}(S, 2K_S) < \frac{1}{4}$. Then there is an effective $\mathbb{Q}$-Cartier divisor $D \sim_{\mathbb{Q}} 2K_S$ such that the log pair $(S, \frac{1}{4}D)$ is not log canonical at some point $\mathsf{p} \in S$. By Lemma 2.3, we have

$$\mathrm{mult}_{\mathsf{p}}(D) > 4.$$

We put an effective $\mathbb{Q}$-divisor $d := \varphi(D)$ on Σ. Then

$$(\Sigma, \frac{1}{4}d + B) \text{ is not log canonical at a point } \varphi(\mathsf{p}) \text{ on } \Sigma$$

by Lemma 4.1, where $B := B_1 + B_2 + B_3$.

We consider the case that $\varphi(\mathsf{p}) \notin B_1 \cup B_2 \cup B_3$. Thus $(\Sigma, \frac{1}{4}d)$ is not log canonical at $\varphi(\mathsf{p})$ which implies

$$\mathrm{glct}(\Sigma, d) < \frac{1}{4}.$$

However it contradicts since $d \sim_{\mathbb{Q}} -K_\Sigma$ and $\mathrm{glct}(\Sigma, -K_\Sigma) \geq \frac{1}{2}$ since Σ is a nonsingular del Pezzo surface of degree 6 (see [10, Theorem 1.7]). Thus $\varphi(\mathsf{p}) \in B_1 \cup B_2 \cup B_3$.

Now we treat the case $\mathsf{p} \in (\cup_{i=1}^3 E_i) \cup (\cup_{i=1}^3 E_i') \cup (\cup_{i=1}^2 \cup_{j=1}^3 M_i^j)$, where $2E_i = \varphi^*(e_i)$, $2E_i' = \varphi^*(e_i')$ and $2M_i^j = \varphi^*(m_i^j)$. Note the intersection numbers

$$E_i^2 = E_i'^2 = -1, \; D \cdot E_i = D \cdot E_i' = 2, \; {M_i^j}^2 = 0, \text{ and } D \cdot M_i^j = 4.$$

First we consider the case when $\mathsf{p} \in E_3 \setminus (E_1' \cup E_2')$, that is the log pair $(S, \frac{1}{4}D)$ is not log canonical at p. We write

$$D = \alpha_3 E_3 + \alpha_1' E_1' + \alpha_2' E_2' + \Omega$$

where α_3, α_1' and α_2' are nonnegative rational numbers and $E_3, E_1', E_2' \not\subset \mathrm{Supp}(\Omega)$ with an effective $\mathbb{Q}$-Cartier divisor Ω. If $\alpha_3 = 0$ then we have

$$2 = D \cdot E_3 = \mathrm{mult}_{\mathsf{p}}(D) \cdot \mathrm{mult}_{\mathsf{p}}(E_3) > 4.$$

It is impossible. Thus $\alpha_3 > 0$. Let L be a general curve on S such that $\rho \circ \varphi(L)$ is a line in $\mathbb{P}^2$ passing through p_3. Then $L \not\subset \mathrm{Supp}(D)$. To determine a bound of α_3 we consider the following

$$8 = D \cdot L \geq \alpha_3 E_3 \cdot L = 2\alpha_3.$$

It implies $4 \geq \alpha_3$. On the other hand, since $\mathsf{p} \notin E_1' \cup E_2'$ and $1 \geq \frac{1}{4}\alpha_3$, the log pair $(S, E_3 + \frac{1}{4}\Omega)$ is not log canonical at p. By the inversion of adjunction formula we know that the log pair $(E_3, \frac{1}{4}\Omega|_{E_3})$ is not log canonical at p. It induces

$$\begin{aligned} 2 + \alpha_3 - \alpha_1' - \alpha_2' &= (D - \alpha_3 E_3 - \alpha_1' E_1' - \alpha_2' E_2') \cdot E_3 \\ &= \Omega \cdot E_3 \geq \mathrm{mult}_{\mathsf{p}}(\Omega|_{E_3}) > 4, \end{aligned}$$

and so $\alpha_3 > 2 + \alpha_1' + \alpha_2'$. Meanwhile, we have

$$2 = D \cdot E_1' \geq (\alpha_3 E_3 + \alpha_1' E_1' + \alpha_2' E_2') \cdot E_1' = \alpha_3 - \alpha_1'.$$

It gives $\alpha_2' < 0$ which is impossible. Similarly, for cases that $\mathsf{p} \in E_i \setminus (E_j' \cup E_k')$ and $\mathsf{p} \in E_i' \setminus (E_j \cup E_k)$ with $\{i, j, k\} = \{1, 2, 3\}$ we can obtain contradictions.

We consider the case that $\mathsf{p} \in M_i^j$, with $i = 1, 2$ and $j = 1, 2, 3$. We write

$$D = \alpha_i^j M_i^j + \Omega$$

where $\alpha_i^j \geq 0$ and $M_i^j \not\subset \mathrm{Supp}(\Omega)$ with an effective $\mathbb{Q}$-Cartier divisor Ω. Let L be a general curve such that $\rho \circ \varphi(L)$ is a line in $\mathbb{P}^2$ passing through $p_{j'}$ with $j' \in \{1, 2, 3\} \setminus \{j\}$. Then we have the following intersection numbers

$$D \cdot L = 8, \qquad M_i^j \cdot L = 2.$$

To determine a bound of α_i^j we consider the following

$$8 = D \cdot L \geq \alpha_i^j M_i^j \cdot L = 2\alpha_i^j.$$

It implies $4 \geq \alpha_i^j$. Thus the log pair $(S, M_i^j + \frac{1}{4}\Omega)$ is not log canonical at p, and so

$$4 = (D - \alpha_i^j M_i^j) \cdot M_i^j \geq \mathrm{mult}_{\mathsf{p}}(\Omega|_{M_i^j}) > 4$$

which is impossible.

Hence $\mathsf{p} \in E_i \cap E_j'$ for $i \neq j$ and $i, j = 1, 2, 3$. Without loss of generality $\mathsf{p} \in E_3 \cap E_2'$. Suppose that $E_3 \not\subset \operatorname{Supp}(D)$. Then

$$2 = D \cdot E_3 \geq \operatorname{mult}_{\mathsf{p}}(D) \cdot \operatorname{mult}_{\mathsf{p}}(E_3) = \operatorname{mult}_{\mathsf{p}}(D).$$

It contradicts to $\operatorname{mult}_{\mathsf{p}}(D) > 4$, and so $E_3 \subset \operatorname{Supp}(D)$. Similarly, we can prove that $E_2' \subset \operatorname{Supp}(D)$. It provides

$$D = \alpha_3 E_3 + \alpha_2' E_2' + \Omega$$

where $\alpha_3, \alpha_2' > 0$ and $E_3, E_2' \not\subset \operatorname{Supp}(\Omega)$ with an effective $\mathbb{Q}$-Cartier divisor Ω. Meanwhile, the log pair $(S, \frac{1}{4}(2E_1' + 4E_2' + 2E_1 + 4E_3))$ is log canonical at p. By Lemma 2.2, we can assume that the support of D does not contain at least one component of $2E_1' + 4E_2' + 2E_1 + 4E_3$. We may assume $E_1' \not\subset \operatorname{Supp}(D)$ since we can do a similar procedure when we assume $E_1 \not\subset \operatorname{Supp}(D)$. Then

$$2 = D \cdot E_1' = (\alpha_3 E_3 + \alpha_2' E_2' + \Omega) \cdot E_1' \geq \alpha_3 E_3 \cdot E_1' + \alpha_2' E_2' \cdot E_1' = \alpha_3.$$

It implies that $1 > \frac{1}{2} \geq \frac{1}{4}\alpha_3$. Since the log pair $(S, \frac{1}{4}D)$ is not log canonical at p, the log pair

$$\left(S, E_3 + \frac{\alpha_2'}{4} E_2' + \frac{1}{4}\Omega\right)$$

is not log canonical at p. By the inversion of adjunction formula the log pair

$$\left(E_3, \left(\frac{\alpha_2'}{4} E_2' + \frac{1}{4}\Omega\right)\Big|_{E_3}\right)$$

is not log canonical at p. Thus

$$\operatorname{mult}_{\mathsf{p}}\left(\frac{\alpha_2'}{4} E_2' + \frac{1}{4}\Omega\right) > 1,$$

and so

$$\begin{aligned} 2 + \alpha_3 = (D - \alpha_3 E_3) \cdot E_3 &= 4\left(\frac{\alpha_2'}{4} E_2' + \frac{1}{4}\Omega\right) \cdot E_3 \\ &\geq 4 \operatorname{mult}_{\mathsf{p}}\left(\frac{\alpha_2'}{4} E_2' + \frac{1}{4}\Omega\right) > 4 \end{aligned}$$

which contradicts to $2 \geq \alpha_3$. $\square$

Theorem 4.3. *Let S be a Burniat surface with $K_S^2 = 6$. Then the global log canonical threshold of the polarized pair (S, K_S) is $\frac{1}{2}$.*

Proof. Since $\operatorname{glct}(S, K_S) = 2 \operatorname{glct}(S, 2K_S)$ we obtain the result by Proposition 4.2. □

Remark 4.4. Tian's conjecture (cf. [1, Conjecture 1.1] and [22, Conjecture 5.3]) which fails in general (cf. [1]) holds for a Burniat surface S with $K_S^2 = 6$ via even pluricanonical divisor because $\operatorname{glct}(S, 2K_S) = \operatorname{lct}(S, E)$ for a divisor $E \in |2K_S|$ (cf. Proposition 3.1).

5. Invariant global sections of the pluricanonical divisor mK_S

In this section we follow the notations of Section 4. Let S be a Burniat surface with $K_S^2 = 6$. Then S is constructed by a bidouble covering of the del Pezzo surface Σ of degree 6. There are the divisors

$$B_i = e_i + e_i' + m_1^j + m_2^j$$

and L_i such that

$$2L_i \sim B_j + B_k$$

where $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$ (see Section 3).

For an integer $m \geq 2$ the action of $G := \mathbb{Z}_2 \times \mathbb{Z}_2$ via the bidouble covering map φ, the bicanonical map of S, gives a splitting:

$$H^0(S, mK_S) = \bigoplus_{\chi \in \hat{G}} H_\chi,$$

where G acts on H_χ via the character χ in the character group $\hat{G}$ of G.

For instance, when m is even we have

$$\begin{aligned} H^0(S, mK_S) = \varphi^* & \left(H^0 \left(\Sigma, -\frac{m}{2} K_\Sigma \right) \right) \\ & \bigoplus \bigoplus_{i=1}^{3} H^0 \left(S, \varphi^* \left(-\frac{m}{2} K_\Sigma - L_i \right) + F_i \right), \end{aligned}$$

where $F_i \sim E_j + E_j' + E_k + E_k' + M_1^k + M_2^k + M_1^i + M_2^i$.

We consider the following.

Lemma 5.1. *If m is a positive even integer and $D \in H_\chi$ for $\chi \neq id$ then $D - F_i$ is effective for some $i = 1, 2, 3$.*

Proof. When $m = 2$ we have $H^0(S, 2K_S) = \varphi^*(H^0(\Sigma, -K_\Sigma))$. When $m(> 2)$ is positive even we obtain it since $-\frac{m}{2}K_\Sigma - L_i$ is effective for all $i = 1, 2, 3$. □

In this section, since we have a decomposition of the global sections of mK_S via the $\mathbb{Z}_2^2$-action associated to the bicanonical map φ of S (i.e. $H^0(S, mK_S) = \bigoplus_{\chi \in \hat{G}} H_\chi$) we show that for a positive even integer m, $\mathrm{glct}(S, K_S) = \mathrm{lct}(S, \frac{1}{m}D)$ for some $D \in |mK_S|$ implies $D \in H_{id}$ (called by an invariant global section of mK_S) which is not in H_χ (called by an anti-invariant global section of mK_S) with $\chi \neq id$.

Proposition 5.2. *For every even integer $m > 2$ and $D \in H_\chi$ with $\chi \neq id$, $\mathrm{lct}(S, D) > \frac{1}{2m}$.*

Proof. Suppose that $\mathrm{lct}(S, D) \leq \frac{1}{2m}$. It implies that for every $\epsilon > 0$, the log pair $(S, (\frac{1}{2m} + \epsilon)D)$ is not log canonical at some point p. We set $\frac{1}{\delta} := \frac{1}{2m} + \epsilon$. Then

$$\mathrm{mult}_{\mathsf{p}}(D) > \delta.$$

We put an effective divisor $d := \varphi(D)$ on Σ. Then

$$\left(\Sigma, \frac{1}{\delta}d + \frac{1}{2}B\right) \text{ is not log canonical at a point } \varphi(\mathsf{p}) \text{ on } \Sigma$$

by Lemma 4.1.

We consider the case $\varphi(\mathsf{p}) \notin B_1 \cup B_2 \cup B_3$. Then $(\Sigma, \frac{1}{\delta}d)$ is not log canonical at $\varphi(\mathsf{p})$ which implies

$$\mathrm{glct}(\Sigma, d) < \frac{1}{\delta}.$$

However it contradicts because $d \sim_{\mathbb{Q}} -\frac{m}{2}K_\Sigma$ and $\mathrm{glct}(\Sigma, \Delta) \geq \frac{1}{2}$ for any effective $\mathbb{Q}$-Cartier divisor $\Delta \sim_{\mathbb{Q}} -K_\Sigma$ since Σ is a nonsigular del Pezzo surface of degree 6 (see [10, Theorem 1.7]). Thus $\varphi(\mathsf{p}) \in B_1 \cup B_2 \cup B_3$.

By Lemma 5.1 we have an effective $\mathbb{Q}$-Cartier divisor $D - F_i$ for some $i = 1, 2, 3$. We may deal with $i = 1$.

The case $\mathsf{p} \in E_1 \cap E_2'$
We have

$$D = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3' E_3' + \Omega,$$

where rational numbers $\alpha_1 \geq 0$ and $\alpha_2, \alpha_3' \geq 1$, and $E_1, E_2, E_3' \not\subset \mathrm{Supp}(\Omega)$ with an effective $\mathbb{Q}$-Cartier divisor Ω. Since $\mathsf{p} \notin E_2 \cup E_3'$, the log pair $(S, \frac{1}{\delta}(D - \alpha_2 E_2 - \alpha_3' E_3'))$ is not log canonical at p.

Suppose $\alpha_1 = 0$, and then $m = D \cdot E_1 \geq \mathrm{mult}_{\mathsf{p}}(D)\, \mathrm{mult}_{\mathsf{p}}(E_1) > \delta$. It induces $\frac{1}{m} < \frac{1}{\delta} = \frac{1}{2m} + \epsilon$ which is a contradiction for a small $\epsilon > 0$. So $\alpha_1 \neq 0$.

Since $D - F_1$ is effective $M_1^1 \cdot \Omega \geq M_1^1 \cdot M_1^3 = 1 > 0$. Thus $2m = D \cdot M_1^1 = \alpha_1 + M_1^1 \cdot \Omega$ implies $2m > \alpha_1$. So

$$\frac{\alpha_1}{\delta} = \alpha_1 \left(\frac{1}{2m} + \epsilon \right) < 1 \text{ for a small } \epsilon > 0.$$

We have that a pair $(S, E_1 + \frac{1}{\delta}\Omega)$ is not log canonical at p. By the inversion of adjunction formula

$$\text{the pair } \left(E_1, \frac{1}{\delta}\Omega|_{E_1} \right) \text{ is not log canonical at } \mathsf{p}.$$

It implies that

$$m + \alpha_1 - \alpha_3' = (D - \alpha_1 E_1 - \alpha_2 E_2 - \alpha_3' E_3') \cdot E_1 > \delta.$$

On the other hand, we have

$$m = D \cdot E_3' \geq \alpha_1 + \alpha_2 - \alpha_3'.$$

Then we obtain

$$2m - \alpha_2 > \delta$$

which is a contradiction for a very small $\epsilon > 0$.

The case $\mathsf{p} \in E_2' \cap E_3$
We have

$$D = \alpha_1 E_1 + \alpha_2' E_2' + \alpha_3' E_3' + \Omega,$$

where $\alpha_1 \geq 0$ and $\alpha_2', \alpha_3' \geq 1$, and $E_1, E_2', E_3' \not\subset \mathrm{Supp}(\Omega)$ with an effective $\mathbb{Q}$-Cartier divisor Ω. Then we get

$$m = D \cdot E_1 \geq -\alpha_1 + \alpha_2' + \alpha_3'.$$

On the other hand, by a similar argument of the case $\mathsf{p} \in E_1 \cap E_2'$ the inversion of adjunction formula implies

$$m - \alpha_1 + \alpha_2' = (D - \alpha_1 E_1 - \alpha_2' E_2' - \alpha_3' E_3') \cdot E_2' > \delta.$$

Thus we have

$$2m - \alpha_3' > \delta$$

which is a contradiction for a very small $\epsilon > 0$.

The case $\mathsf{p} \in E_3 \cap E_1'$
We have

$$D = \alpha_1' E_1' + \alpha_2 E_2 + \alpha_3' E_3' + \Omega,$$

where $\alpha_1' \geq 0$ and $\alpha_2, \alpha_3' \geq 1$, and $E_1', E_2, E_3' \not\subset \mathrm{Supp}(\Omega)$ with an effective $\mathbb{Q}$-Cartier divisor Ω. Then we have

$$m = D \cdot E_2 \geq \alpha_1' - \alpha_2 + \alpha_3'.$$

and

$$m + \alpha_1' - \alpha_2 = (D - \alpha_1' E_1' - \alpha_2 E_2 - \alpha_3' E_3') \cdot E_1' > \delta.$$

Thus

$$2m - \alpha_3' > \delta.$$

It gives a contradiction.

The case $\mathsf{p} \in E_1 \setminus (E_2' \cup E_3')$
We have

$$D = \alpha_1 E_1 + \alpha_2' E_2' + \alpha_3' E_3' + \Omega,$$

where $\alpha_1 \geq 0$ and $\alpha_2', \alpha_3' \geq 1$, and $E_1, E_2', E_3' \not\subset \mathrm{Supp}(\Omega)$ with an effective $\mathbb{Q}$-Cartier divisor Ω. Then the inequalities

$$m = D \cdot E_2' \geq \alpha_1 - \alpha_2'$$

and

$$m + \alpha_1 - \alpha_2' - \alpha_3' = (D - \alpha_1 E_1 - \alpha_2' E_2' - \alpha_3' E_3') \cdot E_1 > \delta$$

imply that

$$2m - \alpha_3 > \delta.$$

It is a contradiction.

The case $\mathsf{p} \in E_2' \setminus (E_1 \cup E_3)$, $\mathsf{p} \in E_3 \setminus (E_1' \cup E_2')$ or $\mathsf{p} \in E_1' \setminus (E_2 \cup E_3)$ We can induce a contradiction like the case $\mathsf{p} \in E_1 \setminus (E_2' \cup E_3')$.

The case $\mathsf{p} \in E_1 \cup E_3'$, $\mathsf{p} \in E_2 \cup E_3'$, $\mathsf{p} \in E_1' \cup E_2$, $\mathsf{p} \in E_3' \setminus (E_1 \cup E_2)$ or $\mathsf{p} \in E_2 \setminus (E_1' \cup E_3')$
Since we deal with $i = 1$ we obtain a contradiction directly from the previous cases.

Since we induce a contradiction for every case, we have

$$\mathrm{lct}(S, D) > \frac{1}{2m}. \qquad \square$$

Theorem 5.3. *Let S be a Burniat surface with $K_S^2 = 6$. For any positive even integer m, if a divisor D is in the linear system $|mK_S|$ such that $\mathrm{glct}(S, K_S) = \mathrm{lct}(S, \frac{1}{m}D)$ then the divisor D is invariant under the $\mathbb{Z}_2^2$-action associated to the bicanonical map φ of S.*

Proof. We obtain the result by Theorem 4.3 and Proposition 5.2. $\square$

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References

[1] H. Ahmadinezhad, I. Cheltsov, and J. Schicho, *On a conjecture of Tian*, Math. Z. **288** (2018), no. 1-2, 217–241.

[2] V. Alexeev and D. Orlov, *Derived categories of Burniat surfaces and exceptional collections*, Math. Ann. **357** (2013), no. 2, 743–759.

[3] V. Alexeev and R. Pardini, *Explicit compactifications of moduli spaces of Campedelli and Burniat surfaces*, `arXiv:0901.4431`.

[4] I. Bauer and F. Catanese, *Burniat surfaces. II. Secondary Burniat surfaces form three connected components of the moduli space*, Invent. Math. **180** (2010), no. 3, 559–588.

[5] I. Bauer and F. Catanese, *Burniat surfaces I: fundamental groups and moduli of primary Burniat surfaces*, in: Classification of Algebraic Varieties, EMS Ser. Congr. Rep., 49–76, Eur. Math. Soc., Zürich, (2011).

[6] I. Bauer and F. Catanese, *Burniat surfaces III: deformations of automorphisms and extended Burniat surfaces*, Doc. Math. **18** (2013), 1089–1136.

[7] I. Bauer and F. Catanese, *Erratum to: Burniat surfaces II: secondary Burniat surfaces form three connected components of the moduli space [4]*, Invent. Math. **197** (2014), no. 1, 237–240.

[8] P. Burniat, *Sur les surfaces de genre* $P_{12} > 1$, Ann. Mat. Pura Appl. (4) **71** (1966), 1–24.

[9] F. Catanese, *On the moduli spaces of surfaces of general type*, J. Differential Geom. **19** (1984), no. 2, 483–515.

[10] I. Cheltsov, *Log canonical thresholds of del Pezzo surfaces*, Geom. Funct. Anal. **18** (2008), no. 4, 1118–1144.

[11] I. Cheltsov and C. Shramov, *Log canonical thresholds of smooth Fano threefolds*, Russian Math. Surveys **63** (2008), no. 5, 859–958.

[12] J. A. Chen, M. Chen, and C. Jiang, *The Noether inequality for algebraic threefolds (with an Appendix by János Kollár)*, `arXiv:1803.05553v3`.

[13] J.-P. Demailly and J. Kollár, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 4, 525–556.

[14] M. Inoue, *Some new surfaces of general type*, Tokyo J. Math. **17** (1994), no. 2, 295–319.

[15] J. Kollár, *Singularities of pairs*, in: Algebraic Geometry — Santa Cruz 1995, Vol. 62 of Proc. Sympos. Pure Math., 221–287, Amer. Math. Soc., Providence, RI (1997).

[16] V. S. Kulikov, *Old examples and a new example of surfaces of general type with* $p_g = 0$, Izv. Ross. Akad. Nauk Ser. Mat. **68** (2004), no. 5, 123–170.

[17] M. Mendes Lopes and R. Pardini, *A connected component of the moduli space of surfaces with $p_g = 0$*, Topology **40** (2001), no. 5, 977–991.

[18] A. M. Nadel, *Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature*, Ann. of Math. (2) **132** (1990), no. 3, 549–596.

[19] R. Pardini, *Abelian covers of algebraic varieties*, J. Reine Angew. Math. **417** (1991), 191–213.

[20] C. A. M. Peters, *On certain examples of surfaces with $p_g = 0$ due to Burniat*, Nagoya Math. J. **66** (1977), 109–119.

[21] G. Tian, *On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$*, Invent. Math. **89** (1987), no. 2, 225–246.

[22] G. Tian, *Existence of Einstein metrics on Fano manifolds*, in: Metric and Differential Geometry, Vol. 297 of Progr. Math., pp. 119–159, Birkhäuser/Springer, Basel, (2012).

Research of Institute Mathematics, Seoul National University
1 Gwanak-ro, Gwanak-gu, Seoul 08826, Republic of Korea
E-mail address: soulcraw@gmail.com

School of Mathematics, Korea Institute for Advanced Study
85 Hoegiro, Dongdaemun-gu, Seoul 02455, Republic of Korea
E-mail address: haushin@kias.re.kr

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