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Mapping class group is generated by three involutions

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We prove that the mapping class group of a closed connected orientable surface of genus at least eight is generated by three involutions.

1. Introduction

The mapping class group $\mathrm{Mod}(\Sigma_g)$ of a closed connected orientable surface Σ_g is defined as the group of isotopy classes of orientation–preserving self–diffeomorphisms of Σ_g. We are interested in generating $\mathrm{Mod}(\Sigma_g)$ by the least number of involutions.

The group $\mathrm{Mod}(\Sigma_g)$ cannot be generated by two involutions, because, for example, it contains nonabelian free groups. Thus any generating set consisting of involutions must contain at least three elements. The purpose of this paper is to prove that the mapping class group can be generated generated by three involutions if the genus of the surface is at least eight, answering a question in [1].

Theorem 1. *The mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by three involutions if* $g \geq 8$*, and by four involutions for* $g \geq 3$*.*

After the works of Luo and Brendle-Farb as explained below, Kassabov [11] obtained a generating set consisting of four involutions if $g \geq 7$. He also proved results for lower genus mapping class groups.

For homological reasons as explained in the last section, the groups $\mathrm{Mod}(\Sigma_1)$ and $\mathrm{Mod}(\Sigma_2)$ cannot be generated by involutions. Since there is a surjective homomorphism from $\mathrm{Mod}(\Sigma_g)$ onto the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$, the latter group is also generated by three involutions for $g \geq 8$.

Corollary 2. *The symplectic group* $\mathrm{Sp}(2g, \mathbb{Z})$ *is generated by three involutions for* $g \geq 8$*, and by four involutions for* $g \geq 3$*.*

Here is a brief history of the problem of finding generating sets for $\mathrm{Mod}(\Sigma_g)$ with various properties. Dehn [3] obtained a generating set for $\mathrm{Mod}(\Sigma_g)$ consisting of $2g(g-1)$ Dehn twists. About a quarter century later, Lickorish [10] showed that it can be generated by $3g-1$ Dehn twists, and Humphries [6] reduced this number to $2g+1$. He showed, moreover, that $2g+1$ is minimal: the cardinality of every generating set of $\mathrm{Mod}(\Sigma_g)$ consisting of Dehn twists is at least $2g+1$.

If one does not require generators to be Dehn twists, by using the generating set obtained by Lickorish, it is easy to find a generating set consisting of an element of order g and three Dehn twists. It is also possible to get a generating set with three elements (see Corollary 6 below). Lu [13] obtained a generating set consisting of three elements, two of which are of finite order. A minimal generating set was first obtained by Wajnryb [19] who proved that $\mathrm{Mod}(\Sigma_g)$ can be generated by two elements; one is of order $4g+2$ and the other is a product of a right and a left Dehn twist about disjoint curves. In [12], we showed that $\mathrm{Mod}(\Sigma_g)$ is generated by an element of order $4g+2$ and a Dehn twist, improving Wajnryb's result.

The interest in the problem of finding generating sets for the mapping class group consisting of finite order elements goes back to 1971: Maclachlan [15] proved that $\mathrm{Mod}(\Sigma_g)$ is generated by finitely many elements of orders $2g+2$ and $4g+2$. He deduced from this that the modulus space of Riemann surfaces of genus g is simply-connected. McCarthy and Papadopoulos [17] showed that for $g \geq 3$ the group $\mathrm{Mod}(\Sigma_g)$ is generated normally by a single involution; i.e., it is generated by an involution and its conjugates. Luo [14] observed that every Dehn twist is a product of six involutions for $g \geq 3$. It then follows from the fact that the mapping class group is generated by $2g+1$ Dehn twists, the group $\mathrm{Mod}(\Sigma_g)$ can be generated by $12g+6$ involutions. Luo also asked whether it is possible to generate the mapping class group by torsion elements, where the number of generators is independent of the genus (and boundary components in the case the surface has boundary).

Brendle and Farb [1] answered Luo's question affirmatively by proving that 6 involutions, and also 3 torsion elements, generate the mapping class group for $g \geq 3$. Kassabov [11] improved this result further, proving that $\mathrm{Mod}(\Sigma_g)$ is generated by 4 (resp. 5 and 6) involutions if $g \geq 7$ (resp. $g \geq 5$ and $g \geq 3$). We proved in [12] that the minimal number of torsion generators of $\mathrm{Mod}(\Sigma_g)$ is 2, by showing that the mapping class group $\mathrm{Mod}(\Sigma_g)$ can be generated by two elements of order $4g+2$. See also [16],[4],[8] and [9] for generating the mapping class group by torsion elements of various orders.

In order to prove the main results of their papers, Brendle-Farb and also Kassabov write a Dehn twist as a product of four involutions, and use a generating set consisting of a torsion element and three Dehn twists. Instead of writing a Dehn twist as a product of small number of involutions, it is easier to write a product of two opposite (a right and a left) Dehn twists about disjoint curves as a product of two involutions. Therefore, Wajnryb's generating set in [19], consisting of an element of order $4g+2$ and a product of two opposite Dehn twists, looks like a good candidate to use in order to find a small number of involution generators. However, the element of order $4g+2$ cannot be written as a product of two (orientation–preserving) involutions: otherwise, these two involutions would generate a dihedral group of order $8g+4$ in $\mathrm{Mod}(\Sigma_g)$. But, it follows from [2, Corollary 2.6] that the group $\mathrm{Mod}(\Sigma_g)$ does not contain a dihedral subgroup of order greater than $4g+4$. In order to implement this idea, we find new generating sets for the mapping class group.

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2. Background and results on mapping class groups

In this article we consider only closed surfaces. Let Σ_g be a closed connected oriented surface of genus g embedded in $\mathbb{R}^3$, as illustrated in Figure 1, in such a way that it is invariant under the two rotations ρ_1 and ρ_2; ρ_1 is the rotation by π about the z–axis and ρ_2 is the rotation by π about the line $y = \tan(\pi/g)z$, $x = 0$. The mapping class group $\mathrm{Mod}(\Sigma_g)$ of Σ_g is the group of isotopy classes of orientation–preserving self–diffeomorphisms of Σ_g.

Throughout the paper, diffeomorphisms are considered up to isotopy. Likewise, curves are considered up to isotopy. Simple closed curves are denoted by the lowercase letters a, b, c, with indices, while the right Dehn twists about them by the corresponding capital letters A, B, C. Occasionally, we write t_a for the right Dehn twist about a. For the composition of diffeomorphisms, we use the functional notation: fh means that h is applied first. The notations a_i, b_i, c_i and d_1, d_2 always denote the curves shown in Figure 1.

Let us review the basic relations among Dehn twists that we need below. For the proofs the reader is referred to [5].

Conjugation relation: If $f : \Sigma_g \to \Sigma_g$ is a diffeomorphism and if a is a simple closed curve on Σ_g, then $ft_af^{-1} = t_{f(a)}$.

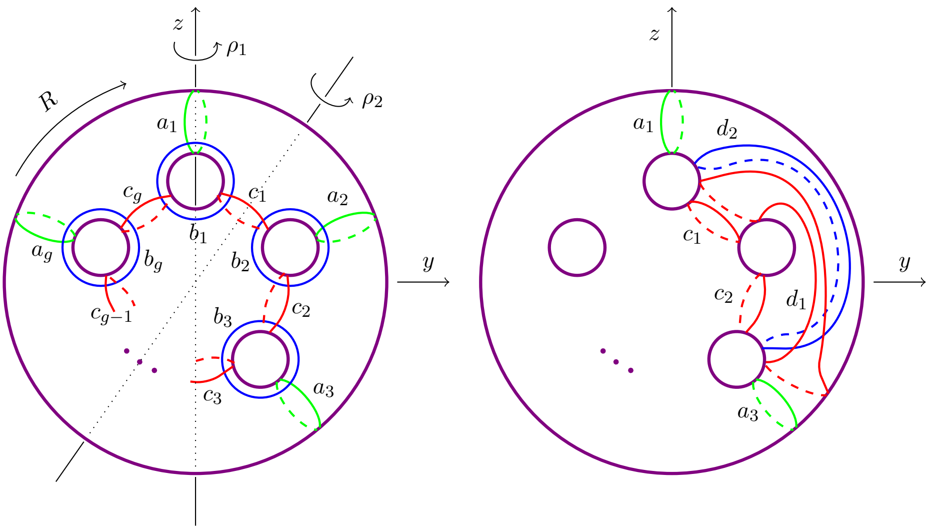

Figure 1: The curves a_i, b_i, c_i, d_i, the rotation R and the involutions ρ_1 and ρ_2 on the surface Σ_g.

Commutativity: If a and b are two disjoint simple closed curves on Σ_g, then $AB = BA$.

Lantern relation: This relation was discovered by Dehn [3] in 1930s, and rediscovered and popularized by Johnson [7] in 1979. Suppose that x_1, x_2, x_3, x_4 are pairwise disjoint simple closed curves on Σ_g bounding a sphere S with four boundary components. Let us choose a point P_i on x_i for each $i = 1, 2, 3, 4$. For $j = 1, 2, 3$, choose a properly embedded arc γ_j on S connecting P_4 and P_j, so that they are disjoint in the interior of S (cf. Figure 2). Suppose that in a small neighborhood of the point P_4, the arcs are read as $\gamma_1, \gamma_2, \gamma_3$ in the clockwise order. Let y_j be the boundary component of a regular neighborhood of $x_4 \cup \gamma_j \cup x_j$ lying on S. Then the Dehn twists about the seven curves $x_1, x_2, x_3, x_4, y_1, y_2, y_3$ satisfy the lantern relation

$$X_1 X_2 X_3 X_4 = Y_1 Y_2 Y_3$$

in the group $\mathrm{Mod}(\Sigma_g)$.

It was proved by Dehn [3] that the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by Dehn twists about finitely many nonseparating simple closed curves. Lickorish also obtained the same result and showed in [10] that it is

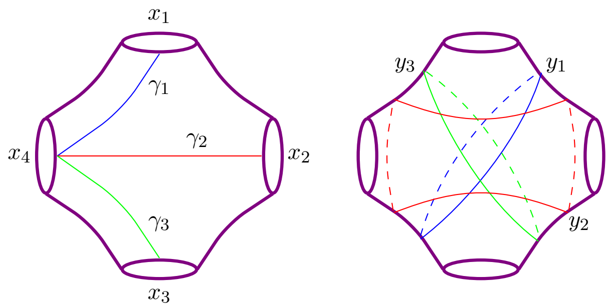

Figure 2: The curves of the lantern relation $X_1X_2X_3X_4 = Y_1Y_2Y_3$ on a sphere with four boundary components.

generated by the Dehn twists

$$A_1, A_2, \ldots, A_g, B_1, B_2, \ldots, B_g, C_1, C_2, \ldots, C_{g-1}$$

about the curves shown in Figure 1. We state this fact as a theorem by adding one more Dehn twist for the sake of symmetry.

Theorem 3. (Dehn-Lickorish) *The mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by the set* $\{A_1, A_2, \ldots, A_g, B_1, B_2, \ldots, B_g, C_1, C_2, \ldots, C_g\}$.

It was shown by Humphries [6] that the set

$$\{A_1, A_2, B_1, B_2 \ldots, B_g, C_1, C_2 \ldots, C_{g-1}\}$$

suffices to generate the group $\mathrm{Mod}(\Sigma_g)$. He also proved that this generating set is minimal, in the sense that $2g$ or fewer Dehn twists cannot generate $\mathrm{Mod}(\Sigma_g)$ if $g \geq 2$.

As a corollary to the Dehn-Lickorish Theorem, it is easy to show that $\mathrm{Mod}(\Sigma_g)$ can be generated by four elements, and also by three elements. We state this as the next corollary. To this end, let R be the rotation by $2\pi/g$ about the x–axis represented in Figure 1, so that $R = \rho_2\rho_1$. Thus, $R(a_k) = a_{k+1}$, $R(b_k) = b_{k+1}$ and $R(c_k) = c_{k+1}$.

Corollary 4. *The mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated*

(i) *by the four elements* R, A_1, B_1, C_1*, and also*

(ii) *by the three elements* $R, A_1, A_1B_1C_1$*.*

Proof. The claim (i) follows from Theorem 3 and the fact that R^k conjugates the Dehn twists A_1, B_1 and C_1 to A_{k+1}, B_{k+1} and C_{k+1}, respectively. The claim (ii) then follows from (i) together with the fact that $A_1B_1C_1(a_1) = b_1$ and $A_1B_1C_1(b_1) = c_1$. □

Although we do not directly use the above corollary, we stated it anyway, because a version of it is proved in the next section and is used in our proof of Theorem 1.

3. Three new generating sets for $\mathrm{Mod}(\Sigma_g)$

In this section, we obtain three new generating sets for the mapping class group. The first two generating sets are variations of the generating sets in Corollary 4. The new generating sets allow us to generate the mapping class group by a small number of involutions.

Recall that R denotes the $2\pi/g$–rotation of Σ_g represented in Figure 1. It is a torsion element of order g in the group $\mathrm{Mod}(\Sigma_g)$. Our first generating set is given in the next theorem. For the proof of it, following the idea of Wajnryb in [19], we employ the lantern relation.

Theorem 5. *If $g \geq 3$, then the mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by the four elements* $R, A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1}$.

Proof. Let G denote the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set

$$\{R, A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1}\}.$$

Let $\mathscr{S}$ denote the set of isotopy classes of nonseparating simple closed curves on the surface Σ_g. We define a subset $\mathscr{G}$ of $\mathscr{S} \times \mathscr{S}$ as

$$\mathscr{G} = \{(a, b) \ : \ AB^{-1} \in G\}.$$

It is clear that

- (symmetry) if $(a, b) \in \mathscr{G}$, then $(b, a) \in \mathscr{G}$,
- (transitivity) if (a, b) and (b, c) are in $\mathscr{G}$, then so is (a, c), and
- (G–invariance) if $(a, b) \in \mathscr{G}$ and $F \in G$, then $(F(a), F(b)) \in \mathscr{G}$.

Thus $\mathscr{G}$ is an equivalence relation on $\mathscr{S}$ and is invariant under the action of G. The last property follows from $Ft_at_b^{-1}F^{-1} = t_{F(a)}t_{F(b)}^{-1}$.

Notice that by the very definition of $\mathscr{G}$, the set $\mathscr{G}$ contains the pairs $(a_1, a_2), (b_1, b_2)$ and (c_1, c_2). Since

$$R^{k-1}(\alpha_1, \alpha_2) = (\alpha_k, \alpha_{k+1})$$

for every $\alpha \in \{a, b, c\}$, the pairs $(a_k, a_{k+1}), (b_k, b_{k+1})$ and (c_k, c_{k+1}) are contained in $\mathscr{G}$. It follows from the transitivity that the pairs $(a_i, a_j), (b_i, b_j)$ and (c_i, c_j) are also contained in $\mathscr{G}$ for all i, j. (Here, by abusing notation we write $f(a, b)$ for $(f(a), f(b))$, and all indices are integers modulo g.)

Since

$$A_1 A_2^{-1} B_1 B_2^{-1}(a_1, a_3) = (b_1, a_3)$$

and

$$B_1 A_3^{-1} C_1 C_2^{-1}(b_1, a_3) = (c_1, a_3),$$

we conclude that the pairs $(a_i, b_j), (a_i, c_j), (b_i, c_j)$ are all contained in the set $\mathscr{G}$ as well. As a result of this, we get that if $X, Y \in \{A_i, B_i, C_i\}$, then the mapping class XY^{-1} is contained in the subgroup G.

We now use the lantern relation to conclude that G contains one, and hence all, of the Dehn twist generators given in Theorem 3.

It is easy to check that the diffeomorphism

$$(B_2 A_1^{-1})(C_1 A_2^{-1})(C_2 A_1^{-1})$$

maps (b_2, a_1) to (d_1, a_1), and the diffeomorphism

$$(B_3 A_1^{-1})(C_2 A_1^{-1})(A_3 A_1^{-1})(B_3 A_1^{-1})$$

maps (d_1, a_1) to (d_2, a_1). Since both of these two diffeomorphisms are contained in the subgroup G, it follows now from the transitivity and the G-invariance of $\mathscr{G}$ that $D_1 A_1^{-1}$ and $D_2 C_1^{-1}$ are contained in G.

Note that a_1, c_1, c_2, a_3 bound a sphere with four boundary components. The Dehn twists about these four curves and about the curves a_2, d_1, d_2 given in Figure 1 satisfy the lantern relation

$$A_1 C_1 C_2 A_3 = A_2 D_1 D_2,$$

which may be rewritten as

$$A_3 = (A_2 C_2^{-1})(D_1 A_1^{-1})(D_2 C_1^{-1}).$$

Since each factor on the right-hand side is contained in the subgroup G, the Dehn twist A_3 is also contained in G. Now from the fact that $A_i A_3^{-1}$, $B_i A_3^{-1}$,

$C_iA_3^{-1}$ are in G, we conclude that G contains all generators A_i, B_i and C_i of $\mathrm{Mod}(\Sigma_g)$ given in Theorem 3. Consequently, $G = \mathrm{Mod}(\Sigma_g)$.

This concludes the proof of the theorem. □

Theorem 6. *If $g \geq 3$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the three elements $R, A_1A_2^{-1}, A_1B_1C_1C_2^{-1}B_3^{-1}A_3^{-1}$.*

Proof. Let us denote by H the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set

$$\{R, A_1A_2^{-1}, A_1B_1C_1C_2^{-1}B_3^{-1}A_3^{-1}\}.$$

It suffices to prove that H contains $B_1B_2^{-1}$ and $C_1C_2^{-1}$.

It follows from

- $R(a_1, a_2) = (a_2, a_3)$,
- $A_1B_1C_1C_2^{-1}B_3^{-1}A_3^{-1}(a_1, a_2) = (b_1, a_2)$,
- $A_1B_1C_1C_2^{-1}B_3^{-1}A_3^{-1}(b_1, a_2) = (c_1, a_2)$,
- $R(b_1, a_2) = (b_2, a_3)$ and
- $R(c_1, a_2) = (c_2, a_3)$

that the elements

- $A_2A_3^{-1}$,
- $B_1A_2^{-1}$,
- $C_1A_2^{-1}$,
- $B_2A_3^{-1}$ and
- $C_2A_3^{-1}$

are contained in H. Now we have

- $B_1B_2^{-1} = B_1A_2^{-1} \cdot A_2A_3^{-1} \cdot A_3B_2^{-1} \in H$ and
- $C_1C_2^{-1} = C_1A_2^{-1} \cdot A_2A_3^{-1} \cdot A_3C_2^{-1} \in H$.

It follows from Theorem 5 that $H = \mathrm{Mod}(\Sigma_g)$, completing the proof. □

Theorem 7. *If $g \geq 8$, then the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by the three elements ρ_1, ρ_2 and $B_1A_2C_3C_4^{-1}A_6^{-1}B_7^{-1}$.*

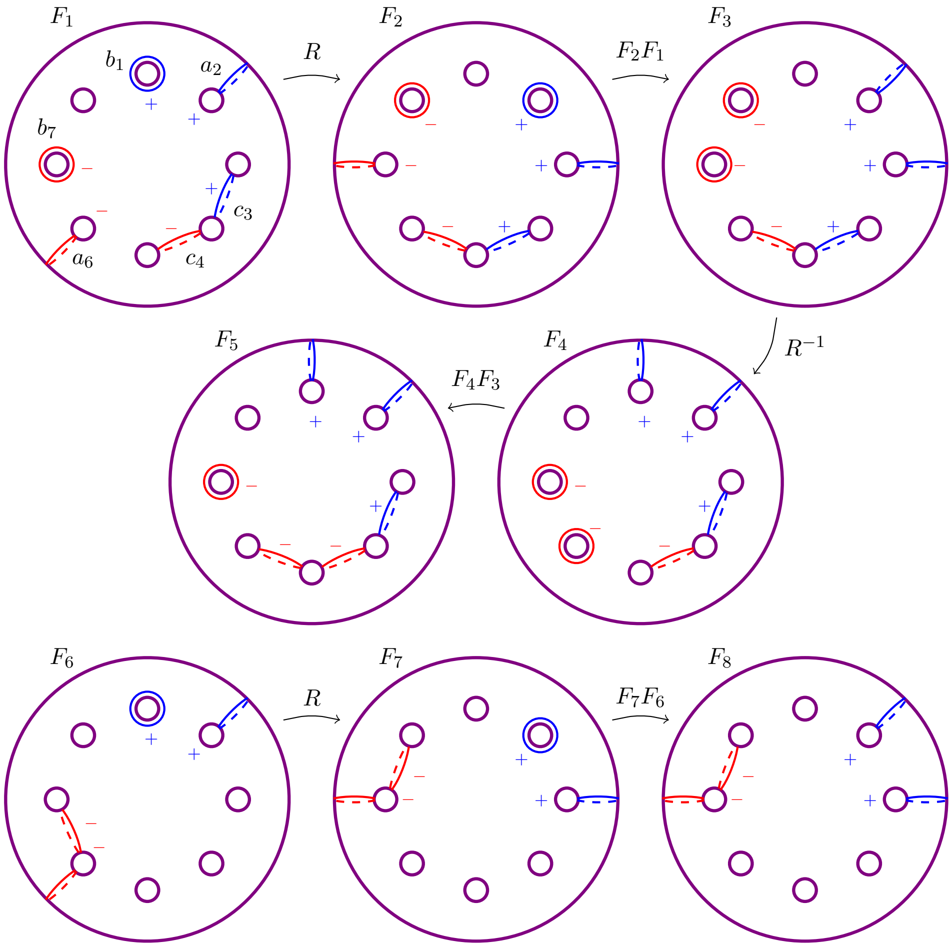

Figure 3: Proof of Theorem 7 for $g = 8$.

Proof. Let $F_1 = B_1A_2C_3C_4^{-1}A_6^{-1}B_7^{-1}$ and let N denote the subgroup of $\mathrm{Mod}(\Sigma_g)$ generated by the set

$$\{\rho_1, \rho_2, F_1\}.$$

Then the rotation $R = \rho_2\rho_1$ is in N. By Theorem 5, it suffices to prove that N contains the elements $A_1A_2^{-1}$, $B_1B_2^{-1}$ and $C_1C_2^{-1}$.

Let F_2 denote the conjugation of F_1 by R:

$$F_2 = RF_1R^{-1} = B_2A_3C_4C_5^{-1}A_7^{-1}B_8^{-1}.$$

It is easy to show that

$$F_2F_1(b_2, a_3, c_4, c_5, a_7, b_8) = (a_2, a_3, c_4, c_5, b_7, b_8),$$

so that $F_3 = A_2A_3C_4C_5^{-1}B_7^{-1}B_8^{-1}$ is contained in N.

Now let

$$F_4 = R^{-1}F_3R = A_1A_2C_3C_4^{-1}B_6^{-1}B_7^{-1}.$$

It can also be shown that

$$F_4F_3(a_1, a_2, c_3, c_4, b_6, b_7) = (a_1, a_2, c_3, c_4, c_5, b_7).$$

Hence, the subgroup N contains the element

$$F_5 = A_1A_2C_3C_4^{-1}C_5^{-1}B_7^{-1}.$$

Therefore,

$$F_4^{-1}F_5 = B_6C_5^{-1} \in N.$$

By conjugating this element by powers of R we see that N contains the elements $B_{i+1}C_i^{-1}$. In particular, $B_2C_1^{-1} \in N$, and hence $\rho_2(B_2C_1^{-1})\rho_2 = B_1C_1^{-1} \in N$. It follows that $B_iC_i^{-1} \in N$ for all i. Now, we have

- $B_1B_2^{-1} = (B_1C_1^{-1})(C_1B_2^{-1}) \in N$, and
- $C_1C_2^{-1} = (C_1B_2^{-1})(B_2C_2^{-1}) \in N$.

It remains to show that $A_1A_2^{-1} \in N$.

Let

$$\begin{aligned} F_6 &= F_1(C_3^{-1}C_4)(B_7C_6^{-1}) \\ &= B_1A_2A_6^{-1}C_6^{-1} \end{aligned}$$

and let

$$\begin{aligned} F_7 &= RF_6R^{-1} \\ &= B_2A_3A_7^{-1}C_7^{-1}. \end{aligned}$$

Since $C_3C_4^{-1} = R^2(C_1C_2^{-1})R^{-2}$ is in N, the elements F_6 and F_7 are also in N.

It can be verified that

$$F_7F_6(b_2, a_3, a_7, c_7) = (a_2, a_3, a_7, c_7)$$

so that N contains the element

$$F_8 = A_2A_3A_7^{-1}C_7^{-1}.$$

Thus we get that

$$F_8F_7^{-1} = A_2B_2^{-1}$$

is in N. By conjugating this with powers of R we see that $A_iB_i^{-1}$ is in N for all i. Therefore

- $A_1A_2^{-1} = (A_1B_1^{-1})(B_1B_2^{-1})(B_2A_2^{-1})$

is contained in N. Consequently, $N = \mathrm{Mod}(\Sigma_g)$.

This completes the proof of the theorem. □

4. Involution generators

Recall that an involution in a group G is an element of order 2. If ρ is an involution in G conjugating x to y, then $\rho x \neq y$ and

$$(\rho xy^{-1})^2 = \rho xy^{-1}\rho xy^{-1} = yx^{-1}xy^{-1} = 1.$$

Thus we have the following elementary but useful lemma.

Lemma 8. *If ρ is an involution in a group G and if x and y are elements in G satisfying $\rho x\rho = y$, then ρxy^{-1} is an involution.*

Consider now the surface Σ_g of genus $g \geq 3$. Since $\rho_2(a_1) = a_2$ and since the involution $\rho_2\rho_1\rho_2$ maps (a_1, b_1, c_1) to (a_3, b_3, c_2), we have

$$\rho_2A_1\rho_2 = A_2 \quad \text{and} \quad (\rho_2\rho_1\rho_2)(A_1B_1C_1)(\rho_2\rho_1\rho_2) = A_3B_3C_2.$$

It now follows from Lemma 8 that

$$\rho_2A_1A_2^{-1} \quad \text{and} \quad \rho_2\rho_1\rho_2A_1B_1C_1C_2^{-1}B_3^{-1}A_3^{-1}$$

are involutions.

Let $\rho_3 = R^3\rho_1 R^{-3}$. For $g \geq 8$, we have

$$\rho_3 B_1 A_2 C_3 \rho_3 = B_7 A_6 C_4,$$

so that $\rho_3 B_1 A_2 C_3 C_4^{-1} A_6^{-1} B_7^{-1}$ is an involution.

Finally, we state and prove our main result.

Theorem 9. *Let Σ_g be the closed connected oriented surface of genus g. Then the mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is generated by the involutions*

(1) ρ_1, ρ_2 *and* $\rho_3 B_1 A_2 C_3 C_4^{-1} A_6^{-1} B_7^{-1}$ *if* $g \geq 8$*, and*

(2) $\rho_1, \rho_2, \rho_2 A_1 A_2^{-1}, \rho_2\rho_1\rho_2 A_1 B_1 C_1 C_2^{-1} B_3^{-1} A_3^{-1}$ *if* $g \geq 3$.

Proof. The proof follows at once from Theorem 7, Theorem 6 and the fact that $R = \rho_2\rho_1$. □

Since the first homology groups of $\mathrm{Mod}(\Sigma_1)$ and $\mathrm{Mod}(\Sigma_2)$ are isomorphic to the cyclic group $\mathbb{Z}_{12}$ and $\mathbb{Z}_{10}$ respectively, these two mapping class groups cannot be generated by involutions. In fact, the group $\mathrm{Mod}(\Sigma_1)$ is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$ and $-I$ is the only element of order two in $\mathrm{SL}(2,\mathbb{Z})$, where I denotes the identity matrix. It was shown by Stukow [18] that the subgroup of $\mathrm{Mod}(\Sigma_2)$ generated by involutions is of index five.

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