Math. Res. Lett.
Volume 27, Number 4, 1129–1163, 2020

On L^1 endpoint Kato-Ponce inequality

Seungly Oh and Xinfeng Wu

We prove that the following endpoint Kato-Ponce inequality holds:

$$\|D^s(fg)\|_{L^{\frac{q}{q+1}}(\mathbb{R}^n)} \lesssim \|D^s f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)} + \|f\|_{L^1(\mathbb{R}^n)}\|D^s g\|_{L^q(\mathbb{R}^n)},$$

for all $1 \le q \le \infty$, provided $s > n/q$ or $s \in 2\mathbb{N}$. Endpoint estimates for several variants of Kato-Ponce inequality in mixed norm Lebesgue spaces are also presented. Our results complement and improve some existing results.

1. Introduction

In [14], Kato and Ponce proved the commutator estimate

$$\|J^s(fg) - f\, J^s g\|_{L^p} \lesssim \|\nabla f\|_{L^\infty} \left\|J^{s-1} g\right\|_{L^p} + \|J^s f\|_{L^p}\|g\|_{L^\infty}$$

for $p \in (1, \infty)$ and $s > 0$, where $J^s := (1 - \Delta)^{s/2}$ is the inhomogeneous fractional differential operator. This commutator estimate plays an important role in the study of Euler and Navier-Stokes equations. Kenig, Ponce, and

Vega [16] proved the following inequality:

$$\|D^s(fg) - fD^s g - gD^s f\|_{L^r} \lesssim \|D^{s_1} f\|_{L^p} \|D^{s_2} g\|_{L^q}$$

for $s_1, s_2, s \in (0,1)$ satisfying $s_1 + s_2 = s$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ for $p, q, r \in (1, \infty)$, where $D^s := (-\Delta)^{s/2}$ is the homogeneous fractional differential operator. Various variants and generalizations of this commutator inequality have been extensively studied in literature (see [3] and the references therein). The following inequalities are known as Kato-Ponce inequalities, also fractional Leibniz rules, which are of fundamentally importance in the study of PDE (see [19, 20] and the references therein):

$$\|D^s(fg)\|_{L^r(\mathbb{R}^n)} \lesssim \|D^s f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)} \|D^s g\|_{L^q(\mathbb{R}^n)}, \tag{1.1}$$

$$\|J^s(fg)\|_{L^r(\mathbb{R}^n)} \lesssim \|J^s f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)} \|J^s g\|_{L^q(\mathbb{R}^n)}, \tag{1.2}$$

where $1 < p, q, r < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Christ and Weinstein [6] proved that (1.1) holds for $s \in (0,1)$, $1 < p, q, r < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Gulisashvili and Kon [12] showed that both (1.1) and (1.2) hold for $s > 0$, $1 < p, q \leq \infty$, $1 < r < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ (see also Grafakos [8]). Bernicot et al [4] extended (1.1) to quasi-Banach regime $r < 1$ under the additional assumption $s > n$. Muscalu and Schlag [19] and Grafakos and the first author [10] independently showed that (1.1) holds for $1 < p, q \leq \infty$, $1/2 < r < \infty$, $s > \max(n/r - n, 0)$ or $s \in 2\mathbb{N}$, and that the constraint condition on s is sharp. Inequality (1.2) was also proved in [10] by the same approach. Note that the proofs of (1.1) in the above mentioned works involve square function estimates, vector-valued maximal inequalities, or Coifman-Meyer multiplier theorem, and hence do not extend to the case where $p, q \in \{1, \infty\}$.

The case $p = q = r = \infty$ was considered by Grafakos, Maldonado and Naibo in [9] and completely settled by Bourgain and Li in [5]. Bourgain-Li's result that both (1.1) and (1.2) hold for $p = q = r = \infty$ is somewhat surprising since the Coifman-Meyer bilinear multipliers (or the more general bilinear Calderón-Zygmund operators) are not bounded from $L^\infty \times L^\infty$ to L^∞ (cf. [11]). The proof in [5] used new summability techniques involving a "low frequency to high frequency switch", thus bypassing the square function estimates, vector-valued maximal inequalities, and Coifman-Meyer multiplier theorem.

The first purpose of this article is to study (1.1) and (1.2) in the L^1 endpoint case $p = 1$ and/or $q = 1$ which were left open by literature. It is well known that general bilinear singular integral operators and bilinear maximal operators fail to be bounded from $L^p \times L^q$ to L^r when $p = 1$ and/or

$q = 1$ (cf. [11, 17]). However, we prove that the L^1 endpoint Kato-Ponce inequalities hold true, which is the content of the following:

Theorem 1.1. *Let $p = 1$, $1 \leq q \leq \infty$ and $\frac{1}{2} \leq r \leq 1$ satisfy $1 + \frac{1}{q} = \frac{1}{r}$. If $s > \frac{n}{r} - n$ or $s \in 2\mathbb{N}$, then both* (1.1) *and* (1.2) *hold.*

When $p = 1$ and $1 \leq q < \infty$, weak-type estimates corresponding to (1.1) and (1.2) were proved by Grafakos and the first author [10]. Hence Theorem 1.1 not only provides the missing $(1, \infty, 1)$ endpoint Kato-Ponce inequalities, but also strengthens the weak-type endpoint estimates in [10].

To prove Theorem 1.1, we shall not only follow the approach of Bourgain-Li [5], but also use techniques developed in [10] to address the summability issue in the quasi-Banach regime. In fact, the endpoint case $(p, q, r) = (1, \infty, 1)$ can still be handled using the method given in [5]. However, this method encounters a road-block when $r < 1$ due to the failure of Young inequality in the quasi-Banach regime. We shall establish new linear and bilinear multiplier estimates on quasi-Banach spaces (see Lemmas 2.2 and 2.3 below), which, combined with the sharp decay estimate in Lemma 2.1, enable us to overcome the difficulty in the quasi-Banach case.

Combining Theorem 1.1 with previous results obtained in [5, 10, 19], we get the Kato-Ponce inequalities for a full range of Lebesgue indices.

Corollary 1.1. *Let $1 \leq p, q \leq \infty$, $\frac{1}{2} \leq r \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $s > \max(0, \frac{n}{r} - n)$ or $s \in 2\mathbb{N}$, then both* (1.1) *and* (1.2) *hold.*

Motivated by applications in time-dependent partial differential equations, Torres and Ward [21] proved that the following Kato-Ponce inequality

$$\|D^s_{(t,x)}(fg)\|_{L^{r'}L^r} \lesssim \|f\|_{L^{p'}L^p}\|D^s_{(t,x)}g\|_{L^{q'}L^q} + \|D^s_{(t,x)}f\|_{L^{p'}L^p}\|g\|_{L^{q'}L^q} \tag{1.3}$$

holds for $1 < r, r', p, p', q, q' < \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $\frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'}$, and $s > 0$ and all $f, g \in \mathcal{S}(\mathbb{R}^{n+1})$. Here $D^s_{(t,x)}$ denotes the fractional derivatives in $(t, x) \in \mathbb{R}^{1+n}$. Recently, Hart, Torres, and the second author extended (1.3) in [13] to $\frac{1}{2} < r, r' < \infty$ provided $s \in 2\mathbb{N}$ or $s > \frac{n+1}{r^*} - (n + 1)$ with $r^* := \min(r, r', 1)$. In the present paper, using the method of Bourgain-Li and the techniques described above, we extend (1.3) to a full range of indices which allow for $p, p', q, q' = 1$ or ∞. Let $x' \in \mathbb{R}^d$ and $x \in \mathbb{R}^n$. Our second main result can be formulated as follows.

Theorem 1.2. *If $1 \le p, p', q, q' \le \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $\frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'}$, and $s \in 2\mathbb{N}$ or $s > \frac{n+d}{r^*} - (n+d)$, then it holds that*

$$\|D^s_{(x',x)}(fg)\|_{L^{r'}_{x'}L^r_x} \lesssim \|f\|_{L^{p'}_{x'}L^p_x}\|D^s_{(x',x)}g\|_{L^{q'}_{x'}L^q_x} + \|D^s_{(x',x)}f\|_{L^{p'}_{x'}L^p_x}\|g\|_{L^{q'}_{x'}L^q_x} \tag{1.4}$$

for $f, g \in \mathcal{S}(\mathbb{R}^{n+d})$. Moreover, the same result holds when $D^s_{(x',x)}$ is replaced by the inhomogeneous fractional derivative $J^s_{(x',x)}$.

Remark 1.3. If $s \notin 2\mathbb{N}$ and $s \le \frac{n+d}{r^*} - (n+d)$, then (1.4) fails. In fact, the counterexample given in [10, Proof of Theorem 2] also works in the context of mixed norm Lebesgue spaces, and shows the sharpness of the constraint condition on s.

Biparameter Kato-Ponce inequality involving partial fractional derivatives was first considered by Muscalu, Pipher, Tao and Thiele in [18], and recently studied by Muscalu and Schlag in [19] and by Grafakos and Oh in [10]. An application of the biparameter Kato-Ponce inequality to KP-I equation was given by Kenig in [15]. Recently, Benea and Muscalu [1, 2] and Di Plinio and Ou [7] considered the following mixed norm version on $\mathbb{R}^{1+1}$:

$$\begin{aligned}\| D^\alpha_{x'} D^\beta_x(fg)\|_{L^{r'}_{x'}L^r_x} \lesssim{}& \|f\|_{L^{p'}_{x'}L^p_x}\| D^\alpha_{x'} D^\beta_x g\|_{L^{q'}_{x'}L^q_x} \\ &+ \| D^\alpha_{x'} f\|_{L^{p'}_{x'}L^p_x}\|D^\beta_x g\|_{L^{q'}_{x'}L^q_x} \\ &+ \| D^\beta_x f\|_{L^{p'}_{x'}L^p_x}\|D^\alpha_{x'} g\|_{L^{q'}_{x'}L^q_x} \\ &+ \| D^\alpha_{x'} D^\beta_x f\|_{L^{p'}_{x'}L^p_x}\|g\|_{L^{q'}_{x'}L^q_x}.\end{aligned} \tag{1.5}$$

Benea and Muscalu [1] showed first that (1.5) holds for $\alpha, \beta > 0$, $1 < p, q, p', q' \le \infty$, $1 \le r, r' < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $\frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'}$. Di Plinio and Ou [7] extended (1.5) to $1/2 < r' < \infty$ provided $\alpha > \frac{1}{r^*} - 1$ and $r \ge 1$. Benea and Muscalu [2] further extended (1.5) to the case $1/2 < r', r < \infty$ provided $\alpha > \frac{1}{r^*} - 1$ and $\beta > \max(\frac{1}{r} - 1, 0)$.

Regarding the endpoint case, certain weak type estimates corresponding to (1.5) were shown in [7] in the case $p' = 1$ and/or $q' = 1$ provided $p, q > 1$, $\alpha > \frac{1}{r^*} - 1$ and $\beta > 0$, but the case $p = 1$ and/or $q = 1$ remains open. Moreover, it is unclear whether (1.5) holds when $r = \infty$ and/or $r' = \infty$.

We answer these questions by showing that inequality (1.5) holds for all $1 \le p, p', q, q' \le \infty$ in all dimensions. Assume $x' \in \mathbb{R}^d$ and $x \in \mathbb{R}^n$ in (1.5) and denote $r^* = \min(r, r', 1)$. The last main result of this article is the following:

Theorem 1.4. *Let* $\frac{1}{2} \le r, r' \le \infty$, $1 \le p, p', q, q' \le \infty$ *satisfy* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ *and* $\frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'}$. *If* $\alpha > \frac{d}{r^*} - d$ *or* $\alpha \in 2\mathbb{N}$ *and* $\beta > \max(0, \frac{n}{r} - n)$ *or* $\beta \in 2\mathbb{N}$, *then* (1.5) *holds.*

Theorem 1.4 not only recovers the results obtained in [2, 7] in the non-endpoint case, but also settles the unsolved endpoint case. Part of the end-point results in Theorem 1.4 is even new in the Lebesgue space context, which complements the bi-parameter inequalities obtained by Muscalu et al [18], Muscalu-Schlag [19] and Grafakos-Oh [10].

When $d = n = 1$, the conditions on α and β in Theorem 1.4 coincide with the ones used in [2] (and [10, 19] in Lebesgue space context). The condition on α seems too strong, but, as pointed out in [2], it is sensible for the mixed derivative variant (1.5), which reflects the non-interchangability of the inside and outside $L^{r'}L^r$ norms.

This article is organized as follows. In Section 2, we introduce necessary notations and prove some lemmas which will be used to show Theorem 1.1. Section 3 is devoted to proving the endpoint Kato-Ponce inequalities in Theorem 1.1. The proofs of Theorems 1.2 and 1.4 are presented in Section 4.

2. Preliminaries and some lemmas

For two positive quantities A and B, we write $A \lesssim B$ if there exists a positive constant C, which does not depend on main parameters, such that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, we write $A \sim B$. $A \gg B$ means $A \ge CB$ for some large constant C. We use $\sum_k$ to denote $\sum_{k\in\mathbb{Z}}$ for simplicity.

For $f \in \mathcal{S}(\mathbb{R}^n)$, define the Fourier transform by $\widehat{f}(\xi) = \widehat{\mathcal{F}[f]}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\, dx$. For $s \in \mathbb{R}$, denote $\langle\xi\rangle^s = (1 + |\xi|^2)^{s/2}$. Define $\widehat{D^s f}(\xi) = |\xi|^s \widehat{f}(\xi)$ and $\widehat{J^s f}(\xi) = \langle\xi\rangle^s \widehat{f}(\xi)$ for $s \ge 0$. Let Φ be a radial non-negative smooth function defined on $\mathbb{R}^n$ which is supported in the ball $\{|x| \le 2\}$ and satisfies $\Phi(x) = 1$ when $|x| \le 1$. Define $\Psi(x) = \Phi(x) - \Phi(2x)$. For each $j \in \mathbb{Z}$, define the *frequency localization operators* Δ_j by $\widehat{\Delta_j f}(\xi) = \Psi(2^{-j}\xi)\widehat{f}(\xi)$, S_j by $\widehat{S_j f}(\xi) = \Phi(2^{-j}\xi)\widehat{f}(\xi)$, and $\widetilde{\Delta}_j$ by $\widetilde{\Delta}_j f = \sum_{k:|k-j|<3} \Delta_k f$.

The following sharp decay estimate was shown by Muscalu-Schlag [19, Chapter 2] and Grafakos-Oh [10, Lemmas 1 and 2]), which will be used to prove our main results.

Lemma 2.1. *[10, 19] Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $s > 0$. Then for any $\gamma \in [0,1]$, there exists a constant $C(n,s,f)$, independent of γ, such that*

$$|(\gamma^2 - \Delta)^{s/2} f(x)| \le C(n,s,f)(1+|x|)^{-n-s}, \qquad \forall\, x \in \mathbb{R}^n.$$

In their proof of L^∞ endpoint Kato-Ponce inequality [5], Bourgain and Li used the fact that convolution operators with L^1 kernels are bounded on L^r for $r \ge 1$ due to the Young inequality. However, the L^r boundedness fails in the quasi-Banach case $r < 1$. In the following two lemmas, we establish new linear and bilinear multiplier inequalities on $L^r, 0 < r \le \infty$, which allow us to overcome this difficulty in the quasi-Banach case.

Lemma 2.2. *Given $r \in (0,\infty]$, let $\sigma = \sigma(\xi)$ be a compactly supported function on $\mathbb{R}^n$ satisfying $|\widehat{\sigma}(x)| \lesssim (1+|x|)^{-\delta}$ for some $\delta > \max(n/r, n)$ and all $x \in \mathbb{R}^n$. Then for any $k \in \mathbb{Z}$,*

$$\left\| \int_{\mathbb{R}^n} \sigma(2^{-k}\xi) \widehat{S_k h}(\xi) e^{i\xi\cdot x}\, d\xi \right\|_{L^r(\mathbb{R}^n)} \lesssim \|S_k h\|_{L^r(\mathbb{R}^n)}.$$

Proof. Suppose that σ is supported on $B(0,M)$ (the ball centered at origin of radius M) in $\mathbb{R}^n$ for some $M \gg 1$. By rescaling, it suffices to prove the lemma in the case $k = 0$. In this case, the integrand from above can be written as

$$\sigma(\xi)\Phi(\xi)\widehat{h}(\xi)e^{i\xi\cdot x}. \tag{2.1}$$

We expand σ into a Fourier series on a cube $[-M,M]^n$:

$$\sigma(\xi) = \chi_{B(0,M)}(\xi) \sum_{\mathbf{m}\in\mathbb{Z}^n} c_{\mathbf{m}} e^{\frac{2\pi}{M} i\mathbf{m}\cdot\xi},$$

where the Fourier coefficients $c_{\mathbf{m}}$ satisfy the decay estimate: $|c_{\mathbf{m}}| \lesssim (1 + |\mathbf{m}|)^{-\delta}$, $\forall\, \mathbf{m} \in \mathbb{Z}^n$ by our hypothesis, and hence the above series is absolutely convergent. The decay estimate on $c_{\mathbf{m}}$ also implies that $\{c_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{Z}^n} \in \ell^{\bar r}(\mathbb{Z}^n)$ with $\bar r := \min\{1, r\}$ since $\delta > n/\bar r$. Plugging the Fourier series of σ into (2.1), we get

$$\begin{aligned}\sigma(\xi)\Phi(\xi)\widehat{h}(\xi)e^{i\xi\cdot x} &= \chi_{B(0,M)}(\xi) \sum_{\mathbf{m}\in\mathbb{Z}^n} c_{\mathbf{m}} e^{\frac{2\pi}{M} i\mathbf{m}\cdot\xi} \Phi(\xi)\widehat{h}(\xi)e^{i\xi\cdot x} \\ &= \sum_{\mathbf{m}\in\mathbb{Z}^n} c_{\mathbf{m}} \widehat{S_0 h}(\xi) e^{i\xi\cdot\left(x + \frac{2\pi}{M}\right)},\end{aligned}$$

where we dropped $\chi_{B(0,M)}$ in front, since it is identically 1 on the support of Φ. Integrating both sides yields

$$\int_{\mathbb{R}^n} \sigma(\xi)\, \widehat{S_0 h}(\xi)\, e^{i\xi\cdot x}\, d\xi = \sum_{\mathbf{m}\in\mathbb{Z}^n} c_{\mathbf{m}}\, S_0 h\left(x + \frac{2\pi}{M}\mathbf{m}\right).$$

From this, using Lemma 4.1 with $r = r'$ and the fact that $\{c_{\mathbf{m}}\}_{\mathbf{m}} \in \ell^{\bar{r}}(\mathbb{Z}^n)$, we obtain

$$\left\| \int_{\mathbb{R}^n} \sigma(\xi)\, \widehat{S_0 h}(\xi)\, e^{i\xi\cdot x}\, d\xi \right\|_{L^r}^{\bar{r}} \leq \sum_{\mathbf{m}\in\mathbb{Z}^n} |c_{\mathbf{m}}|^{\bar{r}} \|S_0 h\|_{L^r}^{\bar{r}} \lesssim \|S_0 h\|_{L^r}^{\bar{r}},$$

which implies the desired estimate. □

Lemma 2.3. *Let $\frac{1}{2} \leq r \leq \infty$ and $1 \leq p, q \leq \infty$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. If $\sigma(\xi)$ is a compactly supported C^∞ function on $\mathbb{R}^n$, then for any $k \in \mathbb{Z}$, and any $N > \frac{n}{2\min(r,1)}$,*

$$\begin{aligned} &\left\| \int_0^1 \int_{\mathbb{R}^{n+n}} \sigma(2^{-k}(t\xi + \eta)) \widehat{S_{k-3} f}(\xi) \widehat{\Delta_k g}(\eta) e^{i(\xi+\eta)\cdot x}\, d\xi\, d\eta\, dt \right\|_{L^r} \\ &\lesssim \left\| (1-\Delta)^N \sigma \right\|_{L^1} \|S_{k-3} f\|_{L^p} \|\Delta_k g\|_{L^q}. \end{aligned}$$

Proof. Suppose supp $\sigma \subset B(0, M)$ in $\mathbb{R}^n$ for some $M \gg 1$. By rescaling, it suffices to show the lemma with $k = 0$. Then the integrand from the left side of the above inequality can be written as

$$\sigma(t\xi + \eta)\chi_{[0,1]}(t)\Phi(2^3\xi)\widehat{f}(\xi)\Psi(\eta)\widehat{g}(\eta)e^{i(\xi+\eta)\cdot x}. \tag{2.2}$$

Using the compact support of σ we can expand σ into an absolutely convergent Fourier series:

$$\sigma(\zeta) = \chi_{B(0,M)}(\zeta) \sum_{\mathbf{m}\in\mathbb{Z}^n} c_{\mathbf{m}} e^{\frac{2\pi}{M} i\mathbf{m}\cdot\zeta},$$

where the Fourier coefficients $c_{\mathbf{m}}$ satisfy

$$|c_{\mathbf{m}}| = \frac{1}{M^n} \left| \widehat{\sigma}\left(\frac{2\pi}{M}\mathbf{m}\right) \right| \lesssim \frac{\left\| (1-\Delta)^N \sigma \right\|_{L^1}}{(1 + |\mathbf{m}|^2)^N}, \qquad \forall\, N \in \mathbb{N}.$$

Denoting $\bar{r} = \min(1, r)$ and choosing $N > \frac{n}{2\bar{r}}$, we have $\{c_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{Z}^n} \in \ell^{\bar{r}}(\mathbb{Z}^n)$ and

$$\|\{c_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{Z}^n}\|_{\ell^{\bar{r}}(\mathbb{Z}^n)} \lesssim \left\| (1-\Delta)^N \sigma \right\|_{L^1}. \tag{2.3}$$

Using the Fourier series expansion of σ, we see that (2.2) is equal to

$$\chi_{[0,1]}(t) \sum_{\mathbf{m} \in \mathbb{Z}^n} c_{\mathbf{m}} e^{\frac{2\pi}{M} i\mathbf{m}\cdot(t\xi+\eta)} \Phi(2^3\xi) \widehat{f}(\xi) \Psi(\eta) \widehat{g}(\eta) e^{i(\xi+\eta)\cdot x},$$

where $\chi_{B(0,M)}(t\xi+\eta)$ was dropped since $\chi_{B(0,M)}(t\xi+\eta) \equiv 1$ on the support of $\chi_{[0,1]}(t)\Phi(2^3\xi)\Psi(\eta)$. It follows that

$$\begin{aligned} &\int_0^1 \int_{\mathbb{R}^{n+n}} \sigma(t\xi+\eta)\, \widehat{S_{-3}f}(\xi)\, \widehat{\Delta_0 g}(\eta)\, e^{i(\xi+\eta)\cdot x}\, d\xi\, d\eta\, dt \\ &= \sum_{\mathbf{m} \in \mathbb{Z}^n} c_{\mathbf{m}} \left[\int_0^1 (S_{-3}f) \left(x + \frac{2\pi}{M} t\mathbf{m} \right) dt \right] \cdot (\Delta_0 g) \left(x + \frac{2\pi}{M} \mathbf{m} \right). \end{aligned}$$

Using Lemma 4.1 with $r' = r$, the Hölder and Minkowski integral inequalities, and (2.3), we obtain

$$\begin{aligned} &\left\| \int_0^1 \int_{\mathbb{R}^{n+n}} \sigma(t\xi+\eta)\, \widehat{S_{-3}f}(\xi)\, \widehat{\Delta_0 g}(\eta)\, e^{i(\xi+\eta)\cdot x}\, d\xi\, d\eta\, dt \right\|_{L^r}^{\bar{r}} \\ &\le \sum_{\mathbf{m} \in \mathbb{Z}^n} |c_{\mathbf{m}}|^{\bar{r}} \cdot \left\| \int_0^1 (S_{-3}f) \left(\cdot + \frac{2\pi}{M} t\mathbf{m} \right) dt \right\|_{L^p}^{\bar{r}} \cdot \left\| (\Delta_0 g) \left(\cdot + \frac{2\pi}{M} \mathbf{m} \right) \right\|_{L^q}^{\bar{r}} \\ &\lesssim \left\| (1-\Delta)^N \sigma \right\|_{L^1}^{\bar{r}} \|S_{-3}f\|_{L^p}^{\bar{r}} \|\Delta_0 g\|_{L^q}^{\bar{r}}. \end{aligned}$$

Hence the proof of Lemma 2.3 is complete. □

The following interpolation inequality will be frequently used in the proof of Theorem 1.1.

Lemma 2.4. *If $a_k \lesssim \min\{2^{ka}A, 2^{-kb}B\}$ for some $a, b, A, B > 0$ and every $k \in \mathbb{Z}$, then, for any $u > 0$, we have $\{a_k\}_{k\in\mathbb{Z}} \in \ell^u(\mathbb{Z})$ and*

$$\|\{a_k\}_{k\in\mathbb{Z}}\|_{\ell^u} \lesssim A^{\frac{b}{a+b}} B^{\frac{a}{a+b}}. \tag{2.4}$$

In particular, if $\|f_k\|_{L^r} \lesssim |a_k|$ for some $0 < r \le \infty$ and every $k \in \mathbb{Z}$, then

$$\left\| \sum_{k\in\mathbb{Z}} f_k \right\|_{L^r} \lesssim A^{\frac{b}{a+b}} B^{\frac{a}{a+b}}.$$

Proof. For any fixed $m \in \mathbb{Z}$ and any $u > 0$,

$$\|\{a_k\}_{k\in\mathbb{Z}}\|_{\ell^u} \lesssim \left(\sum_{k\le m} 2^{auk}A^u + \sum_{k>m} 2^{-buk}B^u\right)^{\frac{1}{u}} \lesssim 2^{am}A + 2^{-bm}B.$$

Choosing m so that $2^m \sim (B/A)^{\frac{1}{a+b}}$ yields (2.4).

Denote $\bar{r} = \min(r, 1)$. If $\|f_k\|_{L^r} \lesssim |a_k|$, then $\|\sum_k f_k\|_{L^r}^{\bar{r}} \le \sum_k \|f_k\|_{L^r}^{\bar{r}} \lesssim |a_k|^{\bar{r}}$. By (2.4),

$$\left\|\sum_k f_k\right\|_{L^r} \lesssim \|\{a_k\}_{k\in\mathbb{Z}}\|_{\ell^{\bar{r}}(\mathbb{Z})} \lesssim A^{\frac{b}{a+b}}B^{\frac{a}{a+b}}.$$

□

3. Proof of Theorem 1.1

We first give

Proof of (1.1). We will actually prove the Kato-Ponce inequality (1.1) for the full range of indices. Let $r \in [\frac{1}{2}, 1]$ and $p, q \in [1, \infty]$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and let $s > \max(n/r - n, 0)$ or $s \in 2\mathbb{N}$. Using a familiar paraproduct decomposition, we may write

$$\begin{aligned} D^s(fg) &= \sum_k D^s(S_{k-3}f\Delta_k g) + \sum_k D^s(\Delta_k f S_{k-3}g) + \sum_k D^s(\Delta_k f\widetilde{\Delta}_k g) \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

We first use the multiplier estimate in Lemma 2.2 to treat the high-high frequency part I_3. Denote $h = \Delta_k f\widetilde{\Delta}_k g$. Noting that supp $\widehat{h} \subset \{|\xi| \le 2^{k+4}\}$ we have

$$\widehat{h}(\xi) = \Phi^2(2^{-k-5}\xi)\widehat{h}(\xi) = \Phi(2^{-k-5}\xi)\widehat{S_{k+5}h}(\xi),$$

and therefore

$$\begin{aligned} D^s(\Delta_k f\widetilde{\Delta}_k g)(x) &= \int_{\mathbb{R}^n} |\xi|^s\Phi(2^{-k-5}\xi)\widehat{S_{k+5}h}(\xi)e^{i\xi\cdot x}\,d\xi \\ &= 2^{ks}\int_{\mathbb{R}^n} \sigma_1(2^{-k}\xi)\widehat{S_{k+5}h}(\xi)e^{i\xi\cdot x}\,d\xi, \end{aligned}$$

where $\sigma_1(\xi) := |\xi|^s\Phi(2^{-5}\xi)$. If $s > \max(n/r - n, 0)$, the decay estimate in Lemma 2.1 with $\gamma = 0$ gives $|\widehat{\sigma_1}(x)| \lesssim (1 + |x|)^{-\delta}$ with $\delta = n + s >$

$\max(n/r, n)$; this estimate is trivial in the case $s \in 2\mathbb{N}$. Applying Lemma 2.2 with $\sigma = \sigma_1$ and Hölder's inequality, we have

$$\begin{aligned}\|D^s(\Delta_k f \widetilde{\Delta}_k g)\|_{L^r} &\lesssim 2^{ks}\|\Delta_k f\, \widetilde{\Delta}_k g\|_{L^r} \\ &\le 2^{ks}\|\Delta_k f\|_{L^p} \left\|\widetilde{\Delta}_k g\right\|_{L^q} \lesssim 2^{ks}\|f\|_{L^p}\|g\|_{L^q}.\end{aligned}$$

From this and Bernstein's inequality (see, e.g., [20, p. 333]), we also have

$$\|D^s(\Delta_k f \widetilde{\Delta}_k g)\|_{L^r} \lesssim 2^{ks}\|\Delta_k f\|_{L^p}\,\|\widetilde{\Delta}_k g\|_{L^q} \lesssim 2^{-ks}\|D^s f\|_{L^p}\|D^s g\|_{L^q}.$$

Applying the interpolation inequality in Lemma 2.4 with $a = b = s$ we obtain

$$\begin{aligned}\|I_3\|_{L^r} &\lesssim \left(\|f\|_{L^p}\|g\|_{L^q}\|D^s f\|_{L^p}\|D^s g\|_{L^q}\right)^{1/2} \\ &\lesssim \|f\|_{L^p}\|D^s g\|_{L^q} + \|D^s f\|_{L^p}\|g\|_{L^q}.\end{aligned}$$

Next, we consider the low-high frequency part I_1. For any two operators A and B, denote by $[A, B] = AB - BA$ the usual commutator. We write

$$\begin{aligned}I_1 &= \sum_k [D^s, S_{k-3} f]\, \Delta_k g + f\, D^s g - \sum_k \Delta_{>k-3} f\, D^s \Delta_k g \\ &=: I_{1.1} + f D^s g + I_{1.2},\end{aligned}$$

where and in what follows $\Delta_{>k-3} := \sum_{j>k-3} \Delta_j$.

$I_{1.2}$ can be estimated in the same way as I_3. Indeed, from the estimate

$$\|\Delta_{>k-3} f\|_{L^p} \le \sum_{j>k-3} 2^{-js}\|\Delta_j D^s f\|_{L^p} \lesssim 2^{-ks}\|D^s f\|_{L^p}, \tag{3.1}$$

we have

$$\|\Delta_{>k-3} f D^s \Delta_k g\|_{L^r} \lesssim 2^{-ks}\|D^s f\|_{L^p}\|D^s g\|_{L^q}.$$

By Bernstein's inequality, we also have

$$\|\Delta_{>k-3} f D^s \Delta_k g\|_{L^r} \lesssim \|\Delta_{>k-3} f\|_{L^p}\,\|D^s \Delta_k g\|_{L^q} \lesssim 2^{ks}\|f\|_{L^p}\|g\|_{L^q}.$$

Hence the desired estimate for $I_{1.2}$ follows by applying Lemma 2.4.

To estimate $I_{1.1}$, the crux is the following commutator estimate. Such estimate was first established by Bourgain and Li [5] in the case $p = q = r = \infty$, based on physical space analysis and Young's inequality. Their arguments extend directly to the Banach range of indices $r \ge 1$, but fail in the non-Banach range $r < 1$. Here we present a new proof relying upon the Fourier analytic techniques and bilinear multiplier theorem (Lemma 2.3), which enables us to obtain the estimate for all $1 \le p, q \le \infty$ and $1/2 \le r \le \infty$.

Proposition 3.1. *Let $1/2 \le r \le \infty$ and $1 \le p, q \le \infty$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then for any $s > 0$ and $k \in \mathbb{Z}$,*

$$\|[D^s, S_{k-3}f]\Delta_k g\|_{L^r} \lesssim 2^{k(s-1)} \|\nabla S_{k-3}f\|_{L^p} \|\Delta_k g\|_{L^q}.$$

Proof. We write

$$\begin{aligned}[D^s, S_{k-3}f]\Delta_k g &= \int_{\mathbb{R}^{n+n}} (|\xi+\eta|^s - |\eta|^s)\, \widehat{S_{k-3}f}(\xi)\widehat{\Delta_k g}(\eta) e^{-i(\xi+\eta)\cdot x}\, d\xi\, d\eta \\ &= \int_{\mathbb{R}^{n+n}} \left(\int_0^1 \frac{d}{dt} |t\xi + \eta|^s\, dt \right) \widehat{S_{k-3}f}(\xi)\widehat{\Delta_k g}(\eta) e^{-i(\xi+\eta)\cdot x}\, d\xi\, d\eta.\end{aligned}$$

Noting that, on the support of the integrand, $|t\xi + \eta| \sim |\eta| \sim 2^k$ uniformly in $t \in [0,1]$ and hence

$$\frac{d}{dt}|t\xi+\eta|^s = s\xi \cdot (t\xi+\eta)|t\xi+\eta|^{s-2}.$$

Moreover, pick $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi(\xi) \equiv 1$ for $\frac{1}{4} \le |\xi| \le 4$ and supp $\varphi \subset \{\frac{1}{8} \le |\xi| \le 8\}$. Since $\varphi(2^{-k}(t\xi+\eta)) \equiv 1$ on the support of the integrand, we may write $[D^s, S_{k-3}f]\Delta_k g$ as

$$\begin{aligned}&\int_0^1 \int_{\mathbb{R}^{n+n}} \xi \cdot (t\xi+\eta)|t\xi+\eta|^{s-2} \varphi(2^{-k}(t\xi+\eta)) \\ &\qquad \times \widehat{S_{k-3}f}(\xi)\widehat{\Delta_k g}(\eta) e^{-i(\xi+\eta)\cdot x}\, d\xi\, d\eta\, dt \\ &= 2^{k(s-1)} \int_0^1 \int_{\mathbb{R}^{n+n}} [2^{-k}(t\xi+\eta)|2^{-k}(t\xi+\eta)|^{s-2}\varphi(2^{-k}(t\xi+\eta))] \\ &\qquad \cdot \widehat{\nabla S_{k-3}f}(\xi)\widehat{\Delta_k g}(\eta) e^{-i(\xi+\eta)\cdot x}\, d\xi\, d\eta\, dt.\end{aligned}$$

Applying Lemma 2.3 with $\sigma(\xi) = \xi_i |\xi|^{s-2}\varphi(\xi), i = 1, \ldots, n$ and f replaced by ∇f, we obtain the desired estimate. □

We continue to estimate $I_{1.1}$. By Bernstein, we get that for $0 < \varepsilon < \min(1, s)$,

$$\begin{aligned}(3.2)\qquad \|\nabla S_{k-3} f\|_{L^p} = \|\nabla D^{-\varepsilon} S_{k-3} D^{\varepsilon} f\|_{L^p} &\lesssim \sum_{j \le k-3} 2^{j(1-\varepsilon)} 2^{j\varepsilon} \|\Delta_j f\|_{L^p} \\ &\lesssim 2^{k(1-\varepsilon)} \sup_{j\in\mathbb{Z}} \|\Delta_j f\|_{L^p}^{\frac{s-\varepsilon}{s}} \|\Delta_j D^s f\|_{L^p}^{\frac{\varepsilon}{s}} \\ &\lesssim 2^{k(1-\varepsilon)} \|f\|_{L^p}^{\frac{s-\varepsilon}{s}} \|D^s f\|_{L^p}^{\frac{\varepsilon}{s}}.\end{aligned}$$

Combining this with Lemma 3.1 yields

$$\|[D^s, S_{k-3} f]\Delta_k g\|_{L^r} \lesssim \min\{2^{ks}\|g\|_{L^q}\|f\|_{L^p},\, 2^{-k\varepsilon}\|D^s g\|_{L^q}\|f\|_{L^p}^{\frac{s-\varepsilon}{s}} \|D^s f\|_{L^p}^{\frac{\varepsilon}{s}}\}.$$

Then applying Lemma 2.4 with $a = s$ and $b = \varepsilon$, we obtain

$$\begin{aligned}\|I_{1.1}\|_{L^r} &\lesssim (\|\Delta_k g\|_{L^q}\|f\|_{L^p})^{\frac{\varepsilon}{s+\varepsilon}} \left(\|D^s \Delta_k g\|_{L^q}\|f\|_{L^p}^{\frac{s-\varepsilon}{s}} \|D^s f\|_{L^p}^{\frac{\varepsilon}{s}}\right)^{\frac{s}{s+\varepsilon}} \\ &= (\|D^s g\|_{L^q}\|f\|_{L^p})^{\frac{s}{s+\varepsilon}} (\|D^s f\|_{L^p}\|g\|_{L^q})^{\frac{\varepsilon}{s+\varepsilon}} \\ &\lesssim \|f\|_{L^p}\|D^s g\|_{L^q} + \|D^s f\|_{L^p}\|g\|_{L^q},\end{aligned}$$

where the last inequality follows from the convexity of natural log:

$$(3.3)\qquad \begin{aligned}&\theta \ln A + (1-\theta)\ln B \le \ln(\theta A + (1-\theta)B), \\ &\text{for } A, B > 0 \text{ and } \theta \in (0,1).\end{aligned}$$

Putting together the estimates obtained above for $I_{1.1}$ and $I_{1.2}$, we get the required bound for I_1. The term I_2 can be handled in the same way. This completes the proof of (1.1).

Now, let us give

Proof of (1.2). We begin with the paraproduct decomposition of $J^s(fg)$:

$$\sum_{k\in\mathbb{N}} J^s(S_{k-3} f \Delta_k g) + \sum_{k\in\mathbb{N}} J^s(\Delta_k f S_{k-3} g) + \sum_{k\in\mathbb{N}} J^s(\Delta_k f\, \widetilde{\Delta_k} g) + J^s(S_0 f\, S_0 g),$$

which we will denote $II_1 + II_2 + II_3 + II_4$.

We deal with II_4 first. Denote $h := S_0 f \cdot S_0 g$ and note that $\widehat{h} = \widehat{S_3^2 h}$. Then

$$II_4 = \int_{\mathbb{R}^n} (1+|\xi|^2)^{\frac{s}{2}}\, \Phi(2^{-3}\xi)\, \widehat{S_3 h}(\xi)\, e^{i\xi\cdot x}\, d\xi.$$

Applying Lemma 2.2 with $k=0$ and $\sigma(\xi) = (1+|\xi|^2)^{\frac{s}{2}}\Phi(2^{-3}\xi)$, we obtain

$$\|II_4\|_{L^r} \lesssim \|S_0 f\|_{L^p}\|S_0 g\|_{L^q} = \left\|J^{-s}S_0 J^s f\right\|_{L^p}\|g\|_{L^q} \lesssim \|J^s f\|_{L^p}\|g\|_{L^q},$$

where the last inequality is due to the fact that $(1+|\xi|^2)^{-s/2}\Phi(\xi) \in \mathcal{S}(\mathbb{R}^n)$, so that $J^{-s}S_0$ is just a convolution operator with an L^1 kernel. This establishes the required estimate for II_4.

We consider next II_3. Denoting $h := \Delta_k f\, \widetilde{\Delta_k} g$ and noting that $\widehat{h} = \widehat{S_{k+5}^2 h}$, we have

$$J^s(\Delta_k f\, \widetilde{\Delta_k} g) = 2^{ks}\int_{\mathbb{R}^n} (2^{-2k} + |2^{-k}\xi|^2)^{\frac{s}{2}}\, \Phi(2^{-k-5}\xi)\, \widehat{S_{k+5}h}(\xi) e^{i\xi\cdot x}\, d\xi.$$

Denote $\sigma_k(\xi) := (2^{-2k} + |\xi|^2)^{\frac{s}{2}}\Phi(2^{-5}\xi)$. If $s > \max(n/r - n, 0)$, Lemma 2.1 implies that $\widehat{\sigma_k}$ satisfy the decay condition of $\widehat{\sigma}$ in Lemma 2.2 uniformly in $k \in \mathbb{N}$; if $s \in 2\mathbb{N}$, direct computations show that for any $N \in \mathbb{N}$, $(1+\Delta)^N \sigma_k \in L^1$ with L^1 norm independent of k, which implies the desired uniform decay estimate in k. Now, applying Lemma 2.2 with $\sigma = \sigma_k$, we get

$$\|J^s(\Delta_k f\, \widetilde{\Delta_k} g)\|_{L^r} \lesssim 2^{ks}\|\Delta_k f\|_{L^p}\|\widetilde{\Delta_k} g\|_{L^q},$$

where the implicit constant is independent of k. Define

$$a_k := 2^{ks}\|\Delta_k f\|_{L^p}\|\widetilde{\Delta_k} g\|_{L^q}$$

if $k \geq 0$ and $a_k := 0$ if $k < 0$. Using Bernstein's inequality and the fact that $\|D^s f\|_{L^p} \lesssim \|J^s f\|_{L^p}$ which holds for $1 \leq p \leq \infty$ (cf. [5, Lemma 2.4]), we have

$$a_k \lesssim \min\{2^{ks}\|f\|_{L^p}\, \|g\|_{L^q},\ 2^{-ks}\|J^s f\|_{L^p}\, \|J^s g\|_{L^q}\}.$$

Applying Lemma 2.4 yields the desired bound for II_3.

It remains to estimate II_1, considering symmetry. We write

$$\begin{aligned} II_1 &= \sum_{k\in\mathbb{N}}[J^s, S_{k-3}f]\Delta_k g + fJ^s\Delta_{>0}g + \sum_{k\in\mathbb{N}}\Delta_{>k-3}fJ^s\Delta_k g \\ &=: II_{1.1} + fJ^s\Delta_{>0}g + II_{1.2}. \end{aligned}$$

$II_{1.2}$ can be treated in the same way as $I_{1.2}$ in the homogeneous case. For $II_{1.1}$, we only prove the following commutator estimate; once this estimate is established, $II_{1.1}$ can be handled in the same way as $I_{1.1}$ in the homogeneous case.

Proposition 3.2. *Let $r \in [\frac{1}{2}, 1]$ and $p, q \in [1, \infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then for any $s > 0$ and $k \in \mathbb{N}$,*

$$\|[J^s, S_{k-3}f]\Delta_k g\|_{L^r} \lesssim 2^{k(s-1)}\|\Delta_k g\|_{L^q}\|\nabla S_{k-3}f\|_{L^p}. \tag{3.4}$$

Proof. We begin by writing

$$\begin{aligned} [J^s, S_{k-3}f]\Delta_k g &= \int_{\mathbb{R}^{2n}} (\langle \xi+\eta\rangle^s - \langle\eta\rangle^s)\, \widehat{S_{k-3}f}(\xi)\widehat{\Delta_k g}(\eta)e^{-i(\xi+\eta)\cdot x}\, d\xi\, d\eta \\ &= \int_{\mathbb{R}^{2n}} \left(\int_0^1 s\xi\cdot(t\xi+\eta)\langle t\xi+\eta\rangle^{s-2}dt\right) \\ &\qquad \times \widehat{S_{k-3}f}(\xi)\widehat{\Delta_k g}(\eta)e^{-i(\xi+\eta)\cdot x}d\xi d\eta. \end{aligned}$$

Noticing that the support of the integrand forces $\langle t\xi+\eta\rangle \sim |\eta| \sim 2^k$ uniformly in $t \in [0,1]$, we may choose $\varphi \in \mathcal{S}(\mathbb{R}^n)$ supported on some annulus so that $[J^s, S_{k-3}f]\Delta_k g$ can be written as

$$\begin{aligned} &\int_0^1\int_{\mathbb{R}^{n+n}} \xi\cdot(t\xi+\eta)\langle t\xi+\eta\rangle^{s-2}\varphi(2^{-k}(t\xi+\eta)) \\ &\qquad\qquad \times \widehat{S_{k-3}f}(\xi)\widehat{\Delta_k g}(\eta)e^{-i(\xi+\eta)\cdot x}\, d\xi\, d\eta\, dt \\ &= 2^{k(s-1)}\int_0^1\int_{\mathbb{R}^{n+n}} [2^{-k}(t\xi+\eta)\, h_k(2^{-k}(t\xi+\eta))\, \varphi(2^{-k}(t\xi+\eta))] \\ &\qquad\qquad \cdot \widehat{\nabla S_{k-3}f}(\xi)\widehat{\Delta_k g}(\eta)e^{-i(\xi+\eta)\cdot x}\, d\xi\, d\eta\, dt, \end{aligned}$$

where $h_k(\xi) = (2^{-2k} + |\xi|^2)^{\frac{s-2}{2}}$. Note that, for each $k \in \mathbb{N}$, the function $\xi h_k(\xi)\varphi(\xi)$ is C^∞ and supported on some annulus. Moreover, direct computations show that h_k and its partial derivatives are bounded uniformly in k within the support of φ, so that the L^1 norm of $\xi h_k(\xi)\varphi(\xi)$ and its partial derivatives are bounded uniformly in k. The desired estimate then follows by applying Lemma 2.3. $\square$

4. Proofs of Theorems 1.2 and 1.4

In this section, we prove variants of the Kato-Ponce inequality in the mixed norm context given in Theorems 1.2 and 1.4. We begin with some preliminary lemmas.

4.1. Preliminary tools in mixed Lebesgue setting

We first introduce notations for the sequel. For $x \in \mathbb{R}^n$ and $x' \in \mathbb{R}^d$, we write $\widetilde{x} = (x', x) \in \mathbb{R}^{n+d}$. We consider in this paper the mixed Lebesgue spaces $L^{p'}_{x'} L^p_x(\mathbb{R}^d \times \mathbb{R}^n)$, or simply $L^{p'}_{x'} L^p_x(\mathbb{R}^{n+d})$, or $L^{p'} L^p$, for $0 < p, p' \leq \infty$, which will be defined by the (quasi-)norms

$$\|f\|_{L^{p'}_{x'} L^p_x(\mathbb{R}^{n+d})} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^n} |f(x', x)|^p dx \right)^{p'/p} dx' \right)^{1/p'}.$$

We shall use S_j, Δ_j, and $\widetilde{\Delta}_j$ to denote the frequency localization operators on $\mathbb{R}^n$ defined in Section 3, and the prime notations S'_j, Δ'_j, and $\widetilde{\Delta}'_j$ to denote the operators on $\mathbb{R}^d$. For any $r, r' > 0$, denote $\bar{r} = \min(1, r)$ and $r^* = \min(r, r', 1)$.

The following lemma provides the subadditivity of $L^{r'}_{x'} L^r_x$ (quasi-)norms.

Lemma 4.1. *[2, Proposition 7] For any $r, r' > 0$, we have*

$$\left\| \sum_k f_k \right\|^{r^*}_{L^{r'}_{x'} L^r_x} \leq \sum_k \|f_k\|^{r^*}_{L^{r'}_{x'} L^r_x}.$$

The following lemma is a mixed norm variant of Lemma 2.2, whose proof is similar to that of Lemma 2.2 and will be omitted.

Lemma 4.2. *Given $r \in (0, \infty]$, let σ be a compactly supported continuous function on $\mathbb{R}^{n+d}$ satisfying $|\widehat{\sigma}(x', x)| \lesssim (1 + |(x', x)|)^{-\delta}$ for some $\delta > \frac{n}{r^*}$. Then for any $k \in \mathbb{Z}$ and any $h \in \mathcal{S}(\mathbb{R}^{n+d})$,*

$$\left\| \int_{\mathbb{R}^{n+d}} \sigma(2^{-k}\xi', 2^{-k}\xi) \widehat{S_k h}(\xi', \xi) e^{i(\xi' \cdot x' + \xi \cdot x)} \, d\xi d\xi' \right\|_{L^{r'}_{x'} L^r_x} \lesssim \|S_k h\|_{L^{r'}_{x'} L^r_x}.$$

Next, we prove a biparameter extension of Lemma 2.2, which will play an important role in the proof of Theorem 1.4.

Lemma 4.3. *Let $r, r' \in (0,\infty]$. Assume that σ_1 and σ_2 are compactly supported functions on $\mathbb{R}^d$ and $\mathbb{R}^n$ respectively satisfying*

$$|\widehat{\sigma_1}(x')| \lesssim (1+|x'|)^{-d-\alpha} \text{ and } |\widehat{\sigma_2}(x)| \lesssim (1+|x|)^{-n-\beta}, \quad \forall\, x' \in \mathbb{R}^d,\ x \in \mathbb{R}^n \tag{4.1}$$

for $\alpha > d/r^* - d$ and $\beta > n/\bar{r} - n$. Then, for any $k, k' \in \mathbb{Z}$ and $\widetilde{h} \in \mathcal{S}(\mathbb{R}^{n+d})$,

$$\left\|\int_{\mathbb{R}^{n+d}} \sigma_1(2^{-k'}\xi')\sigma_2(2^{-k}\xi)\widehat{S'_{k'}S_k\widetilde{h}}(\xi',\xi)e^{i\xi'\cdot x'+i\xi\cdot x}\,d\widetilde{\xi}\right\|_{L^{r'}_{x'}L^r_x} \lesssim \left\|S'_{k'}S_k\widetilde{h}\right\|_{L^{r'}_{x'}L^r_x}, \tag{4.2}$$

$$\left\|\int_{\mathbb{R}^d} \sigma_1(2^{-k'}\xi')\mathcal{F}_1(S'_{k'}\widetilde{h})(\xi',x)e^{i\xi'\cdot x'}\,d\xi'\right\|_{L^{r'}_{x'}L^r_x} \lesssim \left\|S'_{k'}\widetilde{h}\right\|_{L^{r'}_{x'}L^r_x}, \tag{4.3}$$

$$\left\|\int_{\mathbb{R}^n} \sigma_2(2^{-k}\xi)\mathcal{F}_2(S_k\widetilde{h})(x',\xi)e^{i\xi\cdot x}\,d\xi\right\|_{L^{r'}_{x'}L^r_x} \lesssim \left\|S_k\widetilde{h}\right\|_{L^{r'}_{x'}L^r_x}, \tag{4.4}$$

where $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) denotes the partial Fourier transform in x' (resp. x).

Proof. We only prove (4.2); the proofs for (4.3) and (4.4) are similar, the details being omitted. By scaling invariance, it is enough to show (4.2) with $k = k' = 0$. Suppose supp $\sigma_1 \subset [-M', M']^d$ and supp $\sigma_2 \subset [-M, M]^n$. Expanding σ_1 and σ_2 into absolutely convergent Fourier series and arguing as in the proof of Lemma 2.2, we can derive

$$\int_{\mathbb{R}^{n+d}} \sigma_1(\xi')\sigma_2(\xi)\widehat{S'_0S_0\widetilde{h}}(\xi',\xi)e^{i\xi'\cdot x'+i\xi\cdot x}\,d\widetilde{\xi} = \sum_{\mathbf{m}\in\mathbb{Z}^n,\mathbf{m}'\in\mathbb{Z}^d} c_{\mathbf{m}}c'_{\mathbf{m}'}S'_0S_0\widetilde{h}\left(x' + \frac{2\pi}{M'}\mathbf{m}', x + \frac{2\pi}{M}\mathbf{m}\right),$$

where $c'_{\mathbf{m}'}$ and $c_{\mathbf{m}}$ are Fourier coefficients of σ_1 and σ_2 respectively, and satisfy

$$|c'_{\mathbf{m}'}| \lesssim (1+|\mathbf{m}'|)^{-d-\alpha} \quad \text{and} \quad |c_{\mathbf{m}}| \lesssim (1+|\mathbf{m}|)^{-n-\beta} \tag{4.5}$$

with $\alpha > d/r^* - d$ and $\beta > n/\bar{r} - n$ by our hypothesis (4.1). Taking the L^r_x norm from both sides and raising to the $\bar{r}$-th power, we get

$$\begin{aligned}
(4.6)\qquad &\left\|\int_{\mathbb{R}^{n+d}} \sigma_1(\xi')\sigma_2(\xi)\widehat{S'_0S_0\widetilde{h}}(\xi',\xi)e^{i\xi'\cdot x'+i\xi\cdot x}\,d\widetilde{\xi}\right\|^{\bar{r}}_{L^r_x} \\
&\le \sum_{\mathbf{m}\in\mathbb{Z}^n,\mathbf{m}'\in\mathbb{Z}^d} |c_{\mathbf{m}}|^{\bar{r}}|c'_{\mathbf{m}'}|^{\bar{r}} \left\|S'_0S_0\widetilde{h}\left(x'+\frac{2\pi}{M'}\mathbf{m}',\cdot\right)\right\|^{\bar{r}}_{L^r_x} \\
&\lesssim \sum_{\mathbf{m}'\in\mathbb{Z}^d} |c'_{\mathbf{m}'}|^{\bar{r}} \left\|S'_0S_0\widetilde{h}\left(x'+\frac{2\pi}{M'}\mathbf{m}',\cdot\right)\right\|^{\bar{r}}_{L^r_x},
\end{aligned}$$

where we used the fact that $\{c_{\mathbf{m}}\}_{\mathbf{m}} \in \ell^{\bar{r}}(\mathbb{Z}^n)$ due to (4.5).

Note that with our notation, $\underline{(\frac{r'}{\bar{r}})} = \min(\frac{r'}{\bar{r}},1) = \frac{\min(r',\bar{r})}{\bar{r}} = \frac{r^*}{\bar{r}}$, and thus

$$(4.7)\qquad \left\|\sum_k |f_k|\right\|^{\frac{r^*}{\bar{r}}}_{L^{\frac{r'}{\bar{r}}}_{x'}} \le \sum_k \|f_k\|^{\frac{r^*}{\bar{r}}}_{L^{\frac{r'}{\bar{r}}}_{x'}}.$$

Taking $L^{r'/\bar{r}}_{x'}$ (quasi-)norm from both sides of (4.6), raising to the $(r^*/\bar{r})$-th power and applying (4.7) yields

$$\begin{aligned}
&\left\|\int_{\mathbb{R}^{n+d}} \sigma_1(\xi')\sigma_2(\xi)\widehat{S'_0S_0\widetilde{h}}(\xi',\xi)e^{i\xi'\cdot x'+i\xi\cdot x}\,d\widetilde{\xi}\right\|^{r^*}_{L^{r'}_{x'}L^r_x} \\
&\lesssim \left\|\sum_{\mathbf{m}'\in\mathbb{Z}^d} |c'_{\mathbf{m}'}|^{\bar{r}} \left\|S'_0S_0\widetilde{h}\left(x'+\frac{2\pi}{M'}\mathbf{m}',\cdot\right)\right\|^{\bar{r}}_{L^r}\right\|^{\frac{r^*}{\bar{r}}}_{L^{\frac{r'}{\bar{r}}}_{x'}} \\
&\le \sum_{\mathbf{m}'\in\mathbb{Z}^d} |c'_{\mathbf{m}'}|^{r^*}\cdot \left\|\left\|S'_0S_0\widetilde{h}\left(x'+\frac{2\pi}{M'}\mathbf{m}',\cdot\right)\right\|^{\bar{r}}_{L^r}\right\|^{\frac{r^*}{\bar{r}}}_{L^{\frac{r'}{\bar{r}}}_{x'}} \lesssim \left\|S'_0S_0\widetilde{h}\right\|^{r^*}_{L^{r'}_{x'}L^r_x},
\end{aligned}$$

where we used that $\{c'_{\mathbf{m}'}\}_{\mathbf{m}'} \in \ell^{r^*}(\mathbb{Z}^d)$ in view of (4.5). This proves (4.2). □

We now present commutator estimates for full and partial fractional derivatives, which are extensions of Proposition 3.1. The proof is essentially identical to that of Proposition 3.1, the details being omitted.

Proposition 4.1. *Let $r, r' \in [\frac{1}{2}, \infty]$ and $p, q, p', q' \in [1, \infty]$ satisfy $\frac{1}{r'} = \frac{1}{p'} + \frac{1}{q'}$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then for any $s, \alpha, \beta > 0$ and any $f, g \in \mathcal{S}(\mathbb{R}^{n+d})$, we have*

$$\left\| [D_{\widetilde{x}}^s, S_{k-3}^{\widetilde{x}} f] \Delta_k^{\widetilde{x}} g \right\|_{L_{x'}^{r'} L_x^r} \lesssim 2^{k(s-1)} \left\| \Delta_k^{\widetilde{x}} g \right\|_{L_{x'}^{q'} L_x^q} \left\| \nabla_{\widetilde{x}} S_{k-3}^{\widetilde{x}} f \right\|_{L_{x'}^{p'} L_x^p}, \tag{4.8}$$

$$\left\| [D_{x'}^\alpha, S'_{k'-3} f] \Delta'_{k'} g \right\|_{L_{x'}^{r'} L_x^r} \lesssim 2^{k'(\alpha-1)} \left\| \Delta'_{k'} g \right\|_{L_{x'}^{q'} L_x^q} \left\| \nabla_{x'} S'_{k'-3} f \right\|_{L_{x'}^{p'} L_x^p}, \tag{4.9}$$

$$\left\| [D_x^\beta, S_{k-3} f] \Delta_k g \right\|_{L_{x'}^{r'} L_x^r} \lesssim 2^{k(\beta-1)} \| \Delta_k g \|_{L_{x'}^{q'} L_x^q} \| \nabla_x S_{k-3} f \|_{L_{x'}^{p'} L_x^p}, \tag{4.10}$$

where $\widetilde{x} = (x', x) \in \mathbb{R}^{n+d}$, and $D_{\widetilde{x}}^s$, $S_k^{\widetilde{x}}$, $\Delta_k^{\widetilde{x}}$, and $\nabla_{\widetilde{x}}$ denote the operators on $\mathbb{R}^{n+d}$.

To prove Theorem 1.4, we need to use bilinear-biparameter Fourier multipliers T_σ with symbol σ:

$$T_\sigma(f, g)(x', x) = \int_{\mathbb{R}^{2(n+d)}} \sigma(\xi', \xi, \eta', \eta) \widehat{f}(\xi', \xi) \widehat{g}(\eta', \eta) \times e^{i[x' \cdot (\xi' + \eta') + x \cdot (\xi + \eta)]} \, d\xi d\xi' d\eta d\eta'.$$

The following result is a biparameter extension of the commutator estimates in Proposition 3.1.

Proposition 4.2. *Let $\alpha, \beta, p, p', q, q', r, r'$ satisfy the same assumptions as in Proposition 4.1,*

$$\sigma_1(\xi', \xi, \eta', \eta) = \left(|\xi' + \eta'|^\alpha - |\eta'|^\alpha \right) \left(|\xi + \eta|^\beta - |\eta|^\beta \right),$$
$$\sigma_2(\xi', \xi, \eta', \eta) = \left(|\xi' + \eta'|^\alpha - |\xi'|^\alpha \right) \left(|\xi + \eta|^\beta - |\eta|^\beta \right),$$

and T_{σ_j} be the bilinear-biparameter multiplier with symbol σ_j for $j = 1, 2$, respectively. Then for any $k, k' \in \mathbb{Z}$ and $f, g \in \mathcal{S}(\mathbb{R}^{n+d})$,

$$\begin{aligned} &\left\| T_{\sigma_1} \left(S'_{k'-3} S_{k-3} f, \Delta'_{k'} \Delta_k g \right) \right\|_{L_{x'}^{r'} L_x^r} \\ &\lesssim 2^{k'(\alpha-1)+k(\beta-1)} \left\| \nabla_{x'} \nabla_x S'_{k'-3} S_{k-3} f \right\|_{L_{x'}^{p'} L_x^p} \left\| \Delta'_{k'} \Delta_k g \right\|_{L_{x'}^{q'} L_x^q}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} &\left\| T_{\sigma_2} \left(\Delta'_{k'} S_{k-3} f, S'_{k'-3} \Delta_k g \right) \right\|_{L_{x'}^{r'} L_x^r} \\ &\lesssim 2^{k'(\alpha-1)+k(\beta-1)} \left\| \nabla_x \Delta'_{k'} S_{k-3} f \right\|_{L_{x'}^{p'} L_x^p} \left\| \nabla_{x'} S'_{k'-3} \Delta_k g \right\|_{L_{x'}^{q'} L_x^q}. \end{aligned} \tag{4.12}$$

Proof. Since both estimates can be shown in the same manner we just prove (4.12). We express $T_{\sigma_2}\left(S'_{k'-3}S_{k-3}f, \Delta'_{k'}\Delta_k g\right)$ as

$$\begin{aligned}
&\int_{\mathbb{R}^{n+d}}\int_{\mathbb{R}^{n+d}} \left(|\xi'+\eta'|^{\alpha}-|\xi'|^{\alpha}\right)\left(|\xi+\eta|^{\beta}-|\eta|^{\beta}\right)\\
&\qquad\times \widehat{\Delta'_{k'}S_{k-3}}f(\xi',\xi)\widehat{S'_{k'-3}\Delta_k}g(\eta',\eta)e^{-i(\xi'+\eta')\cdot x'-i(\xi+\eta)\cdot x}\,d\xi\,d\xi'\,d\eta\,d\eta'\\
&=\int_{\mathbb{R}^{n+d}}\int_{\mathbb{R}^{n+d}}\left(\eta'\cdot\int_0^1 \mathbf{h}'(\xi'+\tau'\eta')\,d\tau'\right)\left(\xi\cdot\int_0^1 \mathbf{h}(\tau\xi+\eta)\,d\tau\right)\\
&\qquad\times \widehat{\Delta'_{k'}S_{k-3}}f(\xi',\xi)\widehat{S'_{k'-3}\Delta_k}g(\eta',\eta)e^{-i(\xi'+\eta')\cdot x'-i(\xi+\eta)\cdot x}\,d\xi\,d\xi'\,d\eta\,d\eta',
\end{aligned}$$

where $\mathbf{h}(\zeta) := \nabla_{\zeta}|\zeta|^{\beta}$ and $\mathbf{h}'(\zeta') := \nabla_{\zeta'}|\zeta'|^{\alpha}$. Noticing that the support of integrand above forces

$$|\tau\xi+\eta|\sim|\eta|\sim 2^k \quad\text{and}\quad |\xi'+\tau'\eta'|\sim|\xi'|\sim 2^{k'},$$

we may multiply the integrand by $\varphi(2^{-k}(\tau\xi+\eta))\varphi'(2^{-k}(\xi'+\tau'\eta'))$ for suitable smooth functions φ and φ' which are supported on annuli in $\mathbb{R}^n$ and $\mathbb{R}^d$ respectively. Take $L>0$ to be large enough so that supp $\varphi\subset[-L,L]^n$ and supp $\varphi'\subset[-L,L]^d$. Expanding $\varphi\mathbf{h}$ and $\varphi'\mathbf{h}'$ into Fourier series on the cubes $[-L,L]^n$ and $[-L,L]^d$ respectively, we have

$$\begin{aligned}
&T_{\sigma_2}\left(S'_{k'-3}S_{k-3}f,\Delta'_{k'}\Delta_k g\right) = 2^{k'(\alpha-1)}2^{k(\beta-1)}\\
&\quad\times\sum_{\mathbf{m}'\in\mathbb{Z}^d,\mathbf{m}\in\mathbb{Z}^n}\int_0^1 \mathbf{C}'_{\mathbf{m}'}\cdot\nabla_{x'}S'_{k'-3}\Delta_k g\left(x'-\frac{2\pi}{L}\mathbf{m}', x-\frac{2\pi}{L}\tau\mathbf{m}\right)d\tau\\
&\quad\times\int_0^1 \mathbf{C}_{\mathbf{m}}\cdot\nabla_x\Delta'_{k'}S_{k-3}f\left(x'-\frac{2\pi}{L}\tau'\mathbf{m}', x-\frac{2\pi}{L}\mathbf{m}\right)d\tau',
\end{aligned}$$

where $\mathbf{C}_{\mathbf{m}} = (c_{\mathbf{m}}^{(1)},\dots,c_{\mathbf{m}}^{(n)})$ and $\mathbf{C}'_{\mathbf{m}'} = (c'^{(1)}_{\mathbf{m}'},\dots,c'^{(d)}_{\mathbf{m}'})$ denote the Fourier coefficients of $\varphi\mathbf{h}$ and $\varphi'\mathbf{h}'$ respectively. For the component functions of $\varphi\mathbf{h}$ and $\varphi'\mathbf{h}'$ being Schwartz, we see that

$$|\mathbf{C}_{\mathbf{m}}|\lesssim(1+|\mathbf{m}|)^{-M},\qquad |\mathbf{C}'_{\mathbf{m}'}|\lesssim(1+|\mathbf{m}'|)^{-M},\quad\forall\, M>0,$$

and hence $\{|\mathbf{C}_{\mathbf{m}}|\}_{\mathbf{m}}\in\ell^t(\mathbb{Z}^n)$ and $\{|\mathbf{C}'_{\mathbf{m}'}|\}_{\mathbf{m}'}\in\ell^t(\mathbb{Z}^d)$ for all $t>0$. Applying Lemma 4.1, the Hölder and Minkowski inequalities, we obtain the required estimate as in the proof of Lemma 2.3. □

Next, we establish biparameter analogs of (3.2).

Lemma 4.4. *Let $0 < \varepsilon < \min(1, \beta), 0 < \varepsilon' < \min(1, \alpha)$ satisfy $\varepsilon/\beta = \varepsilon'/\alpha$. Then, for any $k \in \mathbb{Z}$ and any $f \in \mathcal{S}(\mathbb{R}^{n+d})$, we have*

$$\|\nabla_{x'} S'_{k'-3} f\|_{L^{p'}L^p} \lesssim 2^{k'(1-\varepsilon')} \|f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}} \|D_{x'}^{\alpha} f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}}, \tag{4.13}$$

$$\|\nabla_x S_{k-3} f\|_{L^{p'}L^p} \lesssim 2^{k(1-\varepsilon)} \|f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}} \|D_x^{\beta} f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}}, \tag{4.14}$$

$$\begin{aligned} \left\|\nabla_{x'}\nabla_x S'_{k'-3} S_{k-3} f\right\|_{L^{p'}L^p} &\lesssim 2^{k'(1-\varepsilon')+k(1-\varepsilon)} \|f\|_{L^{p'}L^p}^{\frac{(\beta-\varepsilon)^2}{\beta^2}} \|D_{x'}^{\alpha} f\|_{L^{p'}L^p}^{\frac{(\beta-\varepsilon)\varepsilon}{\beta^2}} \\ &\quad \times \|D_x^{\beta} f\|_{L^{p'}L^p}^{\frac{(\beta-\varepsilon)\varepsilon}{\beta^2}} \|D_{x'}^{\alpha} D_x^{\beta} f\|_{L^{p'}L^p}^{\frac{\varepsilon^2}{\beta^2}}. \end{aligned} \tag{4.15}$$

Proof. (4.13) and (4.14) can be proven by the same computations as in (3.2).

To show (4.15), we can apply (4.13) with f replaced by $\nabla_x S_{k-3} f$, thereby obtaining

$$\left\|\nabla_{x'} S'_{k'-3} \nabla_x S_{k-3} f\right\|_{L^{p'}L^p} \lesssim 2^{k'(1-\varepsilon')} \|\nabla_x S_{k-3} f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}} \|\nabla_x S_{k-3} D_{x'}^{\alpha} f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}}.$$

Applying (4.14) twice, we see that the right hand side above is bounded by

$$\begin{aligned} C 2^{k'(1-\varepsilon')+k(1-\varepsilon)} &\left(\|f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}} \|D_x^{\beta} f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}} \right)^{\frac{\beta-\varepsilon}{\beta}} \\ &\times \left(\|D_{x'}^{\alpha} f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}} \|D_x^{\beta} D_{x'}^{\alpha} f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}} \right)^{\frac{\varepsilon}{\beta}}, \end{aligned}$$

establishing (4.15). □

We conclude this subsection with a biparameter analog of Lemma 2.4.

Lemma 4.5. *Let $a_j, b_j > 0$ and A_{ij} be positive quantities for $1 \le i, j \le 2$. If*

$$|a_{k,k'}| \lesssim \min\left\{2^{ka_1+k'a'_1} A_{11},\ 2^{-ka_2+k'a'_1} A_{21},\ 2^{ka_1-k'a'_2} A_{12},\ 2^{-ka_2-k'a'_2} A_{22}\right\}$$

for every $k, k' \in \mathbb{Z}$, then $\{a_{k,k'}\}_{k,k' \in \mathbb{Z}} \in \ell^u(\mathbb{Z}^2)$ for all $u > 0$ and

$$\|\{a_{k,k'}\}_{k,k'}\|_{\ell^u(\mathbb{Z}^2)}^{(a_1+a_2)(a'_1+a'_2)} \lesssim A_{11}^{a_2 a'_2} A_{12}^{a_2 a'_1} A_{21}^{a_1 a'_2} A_{22}^{a_1 a'_1}. \tag{4.16}$$

If for some $r, r' \in (0, \infty]$, $\|f_{k,k'}\|_{L_{x'}^{r'} L_x^r} \lesssim |a_{k,k'}|, \forall\, k, k' \in \mathbb{Z}$, then

$$\left\| \sum_{k,k'} f_{k,k'} \right\|_{L_{x'}^{r'} L_x^r}^{(a_1+a_2)(a'_1+a'_2)} \lesssim A_{11}^{a_2 a'_2} A_{12}^{a_2 a'_1} A_{21}^{a_1 a'_2} A_{22}^{a_1 a'_1}.$$

Proof. To prove (4.16), we may assume $u < \infty$. For any $m, m_1', m_2' \in \mathbb{Z}$, we have

$$\begin{aligned}
&\|\{a_{k,k'}\}_{k,k'}\|_{\ell^u(\mathbb{Z}^2)} \\
&\lesssim \Bigg(\sum_{k\le m, k'\le m_1'} 2^{ua_1k+ua_1'k'} A_{11}^u + \sum_{k\le m, k'>m_1'} 2^{ua_1uk-ua_2'k'} A_{12}^u \\
&\quad + \sum_{k>m, k'\le m_2'} 2^{-ua_2k+ua_1'k'} A_{21}^u + \sum_{k>m, k'>m_2'} 2^{-ua_2k-ua_2'k'} A_{22}^u \Bigg)^{\frac{1}{u}} \\
&\lesssim 2^{a_1m}(2^{a_1'm_1'}A_{11} + 2^{-a_2'm_1'}A_{12}) + 2^{-a_2m}(2^{a_1'm_2'}A_{21} + 2^{-a_2'm_2'}A_{22}).
\end{aligned}$$

Choosing m_1' and m_2' such that

$$2^{m_1'} \sim (A_{12}/A_{11})^{\frac{1}{a_1'+a_2'}} \quad \text{and} \quad 2^{m_2'} \sim (A_{22}/A_{21})^{\frac{1}{a_1'+a_2'}}$$

gives

$$\|\{a_{k,k'}\}_{k,k'}\|_{\ell^u(\mathbb{Z}^2)} \lesssim 2^{ma_1} A_{11}^{\frac{a_2'}{a_1'+a_2'}} A_{12}^{\frac{a_1'}{a_1'+a_2'}} + 2^{-ma_2} A_{21}^{\frac{a_2'}{a_1'+a_2'}} A_{22}^{\frac{a_1'}{a_1'+a_2'}}, \quad \forall\, m \in \mathbb{Z}.$$

Now, optimizing the above inequality in m yields (4.16).

If $\|f_{k,k'}\|_{L_{x'}^{r'}L_x^r} \lesssim |a_{k,k'}|$, then by Lemma 4.1 and (4.16),

$$\Bigg\|\sum_{k,k'} f_{k,k'}\Bigg\|_{L_{x'}^{r'}L_x^r} \lesssim \|\{a_{k,k'}\}_{k,k'}\|_{\ell^{r^*}(\mathbb{Z}^2)} \lesssim (A_{11}^{a_2a_2'} A_{12}^{a_2a_1'} A_{21}^{a_1a_2'} A_{22}^{a_1a_1'})^{\frac{1}{(a_1+a_2)(a_1'+a_2')}},$$

concluding the proof. □

4.2. Proof of Theorem 1.2

Using Lemma 4.2 and (4.8), Theorem 1.2 can be proven by the same arguments given in the proof of (1.1) aside from the norms involved (L^p norms replaced by $L^{p'}L^p$ norms and triangle inequality replaced by Lemma 2.4). We omit the details. □

4.3. Proof of Theorem 1.4

By the paraproduct decomposition we write:

$$\begin{aligned}
D_{x'}^{\alpha} D_x^{\beta}(fg) =& \sum_{k',k} D_{x'}^{\alpha} D_x^{\beta}(S'_{k'-3} S_{k-3} f \Delta'_{k'} \Delta_k g) \\
&+ \sum_{k',k} D_{x'}^{\alpha} D_x^{\beta}(\Delta'_{k'} S_{k-3} f S'_{k'-3} \Delta_k g) \\
&+ \sum_{k',k} D_{x'}^{\alpha} D_x^{\beta}(\Delta'_{k'} S_{k-3} f \widetilde{\Delta'_{k'}} \Delta_k g) \\
&+ \sum_{k',k} D_{x'}^{\alpha} D_x^{\beta}(S'_{k'-3} \Delta_k f \Delta'_{k'} S_{k-3} g) \\
&+ \sum_{k',k} D_{x'}^{\alpha} D_x^{\beta}(\Delta'_{k'} \Delta_k f S'_{k'-3} S_{k-3} g) \\
&+ \sum_{k',k} D_{x'}^{\alpha} D_x^{\beta}(\Delta'_{k'} \Delta_k f \widetilde{\Delta'_{k'}} S_{k-3} g) \\
&+ \sum_{k',k} D_{x'}^{\alpha} D_x^{\beta}(S'_{k'-3} \Delta_k f \Delta'_{k'} \widetilde{\Delta_k} g) \\
&+ \sum_{k',k} D_{x'}^{\alpha} D_x^{\beta}(\Delta'_{k'} \Delta_k f S'_{k'-3} \widetilde{\Delta_k} g) \\
&+ \sum_{k',k} D_{x'}^{\alpha} D_x^{\beta}(\Delta'_{k'} \Delta_k f \widetilde{\Delta'_{k'}} \widetilde{\Delta_k} g) \\
=:& I_1 + \cdots + I_9.
\end{aligned}$$

We will deal with I_1, I_2, I_3, I_7, and I_9 separately, and leave out I_4, I_5, I_6, I_8 by symmetry.

4.3.1. Estimate for I_9. Assume that α, β satisfy $\alpha > d/r^* - d$ and $\beta > \max(0, n/r - n)$. We shall apply inequality (4.2) with $\widetilde{h} := \Delta'_{k'} \Delta_k f \, \widetilde{\Delta'_{k'}} \widetilde{\Delta_k} g$ to estimate I_9. Noting that supp $\widehat{\widetilde{h}}(\xi', \xi) \subset \{|\xi'| \leq 2^{k'+4}, \ |\xi| \leq 2^{k+4}\}$, we may write

$$\widehat{\widetilde{h}}(\xi', \xi) = \Phi'(2^{-k'-5}\xi')\Phi(2^{-k-5}\xi)\widehat{S'_{k'+5} S_{k+5} \widetilde{h}}(\xi', \xi).$$

Therefore $D_{x'}^{\alpha} D_x^{\beta} \widetilde{h}(x', x)$ can be written as

$$\begin{aligned}
&\int_{\mathbb{R}^{d+n}} \left(|\xi'|^{\alpha} \Phi'(2^{-k'-5}\xi')\right) \left(|\xi|^{\beta} \Phi(2^{-k-5}\xi)\right) \widehat{S'_{k'+5} S_{k+5} \widetilde{h}}(\xi', \xi) e^{i\xi' \cdot x' + i\xi \cdot x} \, d\xi' \, d\xi \\
&= 2^{\alpha k' + \beta k} \int_{\mathbb{R}^{d+n}} \sigma_1(2^{-k'}\xi')\sigma_2(2^{-k}\xi) \widehat{S'_{k'+5} S_{k+5} \widetilde{h}}(\xi', \xi) e^{i\xi' \cdot x' + i\xi \cdot x} \, d\xi' \, d\xi,
\end{aligned}$$

where $\sigma_1(\xi') = |\xi'|^\alpha \Phi'(2^{-5}\xi')$ and $\sigma_2(\xi) = |\xi|^\beta \Phi(2^{-5}\xi)$. By Lemma 2.1,

$$|\widehat{\sigma_1}(x')| \lesssim (1+|x'|)^{-d-\alpha}, \qquad |\widehat{\sigma_2}(x)| \lesssim (1+|x|)^{-n-\beta}.$$

Note that these estimates hold trivially when $\alpha \in 2\mathbb{N}$ and $\beta \in 2\mathbb{N}$. Then by (4.2) and Hölder's inequality,

$$\begin{aligned} &\left\| D^\alpha_{x'} D^\beta_x \left(\Delta'_{k'} \Delta_k f \, \widetilde{\Delta'_{k'}} \widetilde{\Delta}_k g \right) \right\|_{L^{r'}_{x'} L^r_x} \\ &\lesssim 2^{\alpha k' + \beta k} \left\| \Delta'_{k'} \Delta_k f \, \widetilde{\Delta'_{k'}} \widetilde{\Delta}_k g \right\|_{L^{r'}_{x'} L^r_x} \\ &\leq 2^{\alpha k' + \beta k} \left\| \Delta'_{k'} \Delta_k f \right\|_{L^{p'}_{x'} L^p_x} \left\| \widetilde{\Delta'_{k'}} \widetilde{\Delta}_k g \right\|_{L^{q'}_{x'} L^q_x} =: a_{k,k'}. \end{aligned}$$

The Bernstein inequality gives

$$\begin{aligned} a_{k,k'} \lesssim \min \Big\{ & 2^{\alpha k' + \beta k} \|f\|_{L^{p'} L^p} \|g\|_{L^{q'} L^q}, \; 2^{-\alpha k' + \beta k} \|D^\alpha_{x'} f\|_{L^{p'} L^p} \|D^\alpha_{x'} g\|_{L^{q'} L^q}, \\ & 2^{\alpha k' - \beta k} \left\| D^\beta_x f \right\|_{L^{p'} L^p} \left\| D^\beta_x g \right\|_{L^{q'} L^q}, \\ & 2^{-\alpha k' - \beta k} \left\| D^\alpha_{x'} D^\beta_x f \right\|_{L^{p'} L^p} \left\| D^\alpha_{x'} D^\beta_x g \right\|_{L^{q'} L^q} \Big\}. \end{aligned}$$

Now, applying Lemma 4.5 with $a_1 = a_2 = \beta$ and $a'_1 = a'_2 = \alpha$, we obtain

$$\begin{aligned} \|I_9\|_{L^{r'}_{x'} L^r_x} \lesssim \Big(& \|f\|_{L^{p'} L^p} \|g\|_{L^{q'} L^q} \|D^\alpha_{x'} f\|_{L^{p'} L^p} \|D^\alpha_{x'} g\|_{L^{q'} L^q} \left\| D^\beta_x f \right\|_{L^{p'} L^p} \\ & \times \left\| D^\beta_x g \right\|_{L^{q'} L^q} \left\| D^\alpha_{x'} D^\beta_x f \right\|_{L^{p'} L^p} \left\| D^\alpha_{x'} D^\beta_x g \right\|_{L^{q'} L^q} \Big)^{\frac{1}{4}}, \end{aligned}$$

from which the desired bound follows.

4.3.2. Estimate for I_1. To simplify the presentation, it will be beneficial to argue using Fourier symbols. For $F, G \in \mathcal{S}(\mathbb{R}^{n+d})$, we write $D^\alpha_{x'} D^\beta_x (F \cdot G)$ as a bilinear-biparameter Fourier multiplier with symbol $|\xi' + \eta'|^\alpha |\xi + \eta|^\beta$:

$$\begin{aligned} D^\alpha_{x'} D^\beta_x (FG)(x', x) = \int_{\mathbb{R}^{2(n+d)}} & |\xi' + \eta'|^\alpha |\xi + \eta|^\beta \widehat{F}(\xi', \xi) \widehat{G}(\eta', \eta) \\ & \times e^{i[x' \cdot (\xi' + \eta') + x \cdot (\xi + \eta)]} d\xi' d\xi d\eta' d\eta. \end{aligned} \tag{4.17}$$

We decompose the symbol $|\xi' + \eta'|^\alpha |\xi + \eta|^\beta$ as

$$\begin{aligned} |\xi' + \eta'|^\alpha \, |\xi + \eta|^\beta = \sigma_1 &+ |\eta'|^\alpha (|\xi + \eta|^\beta - |\eta|^\beta) \\ &+ (|\xi' + \eta'|^\alpha - |\eta'|^\alpha) |\eta|^\beta + |\eta'|^\alpha \, |\eta|^\beta, \end{aligned}$$

where $\sigma_1 := (|\xi' + \eta'|^\alpha - |\eta'|^\alpha)(|\xi + \eta|^\beta - |\eta|^\beta)$. This leads to a decomposition of the operator:

$$D^\alpha_{x'} D^\beta_x (FG) = T_{\sigma_1}(F, G) + [D^\beta_x, F] D^\alpha_{x'} G + [D^\alpha_{x'}, F] D^\beta_x G + F D^\alpha_{x'} D^\beta_x G,$$

where T_{σ_1} denotes the bilinear-biparameter multiplier with symbol σ_1. Applying this decomposition with $F = S'_{k'-3} S_{k-3} f$ and $G = \Delta'_{k'} \Delta_k g$, and using $S_k = 1 - \Delta_{>k}$ and $\sum_k \Delta_k = 1$, we get

$$\begin{aligned} I_1 &= \sum_{k,k'} T_{\sigma_1}\left(S'_{k'-3} S_{k-3} f, \Delta'_k \Delta_k g\right) \\ &\quad + \left(\sum_k [D^\beta_x, S_{k-3} f] D^\alpha_{x'} \Delta_k g - \sum_{k,k'} [D^\beta_x, \Delta_{>k'-3} S_{k-3} f] D^\alpha_{x'} \Delta'_k \Delta_k g \right) \\ &\quad + \left(\sum_{k'} [D^\alpha_{x'}, S'_{k'-3} f] D^\beta_x \Delta'_k g - \sum_{k,k'} [D^\alpha_{x'}, S'_{k'-3} \Delta_{>k-3} f] D^\beta_x \Delta'_k \Delta_k g \right) \\ &\quad + \sum_{k,k'} S'_{k'-3} S_{k-3} f D^\alpha_{x'} D^\beta_x \Delta'_{k'} \Delta_k g \\ &=: I_1^A + (I_1^B - I_1^C) + (I_1^D - I_1^E) + I_1^F. \end{aligned}$$

We shall treat $I_1^A, \ldots, I_1^F$ separately. We start with I_1^F that does not contain any commutators. Using $S_k = 1 - \Delta_{>k}$ again, we further decompose

$$\begin{aligned} I_1^F &= f \cdot D^\alpha_{x'} D^\beta_x g - \sum_{k'} \Delta'_{>k'-3} f \cdot D^\alpha_{x'} D^\beta_x \Delta'_{k'} g - \sum_k \Delta_{>k-3} f \cdot D^\alpha_{x'} D^\beta_x \Delta_k g \\ &\quad + \sum_{k',k} \Delta'_{>k'-3} \Delta_{>k-3} f \ \cdot D^\alpha_{x'} D^\beta_x \Delta'_{k'} \Delta_k g =: I_1^{F_1} - I_1^{F_2} - I_1^{F_3} + I_1^{F_4}. \end{aligned}$$

The estimate for $I_1^{F_1}$ is trivial. Using Bernstein's inequality and (3.1), $I_1^{F_4}$ can be handled in the same way as I_9 and the same bound can be derived. For $I_1^{F_3}$, applying the Hölder and Bernstein inequalities, we obtain

$$\left\| \Delta_{>k-3} f D^\alpha_{x'} D^\beta_x \Delta_k g \right\|_{L^{r'} L^r} \lesssim 2^{\beta k} \|\Delta_{>k-3} f\|_{L^{p'} L^p} \|\Delta_k D^\alpha_{x'} g\|_{L^{q'} L^q} =: a_k.$$

It follows from Bernstein's inequality and (3.1) that

$$a_k \lesssim \min\left\{ 2^{\beta k} \|f\|_{L^{p'} L^p} \|D^\alpha_{x'} g\|_{L^{q'} L^q},\ 2^{-\beta k} \left\| D^\beta_x f \right\|_{L^{p'} L^p} \left\| D^\alpha_{x'} D^\beta_x g \right\|_{L^{q'} L^q} \right\}.$$

Applying Lemma 2.4 with $a = b = \beta$, we obtain the desired bound. $I_1^{F_2}$ can be treated similarly.

We tackle next I_1^B which closely resembles estimates from Section 3. By Proposition 4.1,

$$\left\|[D_x^\beta, S_{k-3}f]D_{x'}^\alpha \Delta_k g\right\|_{L^{r'}L^r} \lesssim 2^{k(\beta-1)}\|\Delta_k g\|_{L^{q'}L^q}\|\nabla_x S_{k-3}f\|_{L^{p'}L^p} =: b_k.$$

For $\varepsilon \in (0,\beta)$, it follows from (4.14) and Bernstein's inequality that

$$b_k \lesssim \min\Big\{2^{k\beta}\|D_{x'}^\alpha g\|_{L^{q'}L^q}\|f\|_{L^{p'}L^p},\\ 2^{-k\varepsilon}\left\|D_{x'}^\alpha D_x^\beta g\right\|_{L^{q'}L^q}\|f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}}\left\|D_x^\beta f\right\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}}\Big\}.$$

Then applying Lemma 2.4 with $a = \beta$ and $b = \varepsilon$ yields the desired estimate for I_1^B. I_1^D can be treated similarly, the details being omitted.

Let us now estimate I_1^A, I_1^C and I_1^E, which contain double summations. To estimate I_1^A, we use inequality (4.11) to get

$$\begin{aligned}&\|T_{\sigma_1}(S'_{k'-3}S_{k-3}f, \Delta'_k\Delta_k g)\|_{L^{r'}L^r}\\ &\lesssim 2^{k'(\alpha-1)+k(\beta-1)}\left\|\nabla_{x'}\nabla_x S'_{k'-3}S_{k-3}f\right\|_{L^{p'}L^p}\left\|\Delta'_{k'}\Delta_k g\right\|_{L^{q'}L^q}\\ &=: \widetilde{a}_{k,k'}.\end{aligned}$$

Utilizing (4.15) and Bernstein's inequality, $\widetilde{a}_{k,k'}$ is majorized by a constant multiple of

$$\begin{aligned}\min\Big\{&2^{k\beta+k'\alpha}\|f\|_{L^{p'}L^p}\|g\|_{L^{q'}L^q},\ 2^{k\beta-k'\varepsilon'}\|D_{x'}^\alpha g\|_{L^{q'}L^q}\|f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}}\|D_{x'}^\alpha f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}},\\ &2^{-k\varepsilon+k'\alpha}\|D_x^\beta g\|_{L^{q'}L^q}\|f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}}\|D_x^\beta f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}},\\ &2^{-k'\varepsilon'-k\varepsilon}\|D_x^\beta D_{x'}^\alpha g\|_{L^{q'}L^q}\|f\|_{L^{p'}L^p}^{(\frac{\beta-\varepsilon}{\beta})^2}\|D_{x'}^\alpha f\|_{L^{p'}L^p}^{\frac{(\beta-\varepsilon)\varepsilon}{\beta^2}}\\ &\quad\times\|D_x^\beta f\|_{L^{p'}L^p}^{\frac{(\beta-\varepsilon)\varepsilon}{\beta^2}}\|D_{x'}^\alpha D_x^\beta f\|_{L^{p'}L^p}^{\frac{\varepsilon^2}{\beta^2}}\Big\}.\end{aligned}$$

Applying Lemma 4.5 with $(a_1,a_2)=(\beta,\varepsilon)$ and $(a_1',a_2')=(\alpha,\varepsilon')$ and recalling $\frac{\varepsilon'}{\alpha}=\frac{\varepsilon}{\beta}$, we get

$$\begin{aligned}
&\left\|I_1^A\right\|_{L^{r'}L^r}^{(\beta+\varepsilon)(\alpha+\varepsilon')}\\
&\lesssim (\|f\|_{L^{p'}L^p}\|g\|_{L^{q'}L^q})^{\varepsilon\varepsilon'}\left(\|D_{x'}^\alpha g\|_{L^{q'}L^q}\|f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}}\|D_{x'}^\alpha f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}}\right)^{\alpha\varepsilon}\\
&\quad\times\left(\|D_x^\beta g\|_{L^{q'}L^q}\|f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}}\|D_x^\beta f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}}\right)^{\beta\varepsilon'}\\
&\quad\times\left(\|D_x^\beta D_{x'}^\alpha g\|_{L^{q'}L^q}\|f\|_{L^{p'}L^p}^{(\frac{\beta-\varepsilon}{\beta})^2}\|D_{x'}^\alpha f\|_{L^{p'}L^p}^{\frac{(\beta-\varepsilon)\varepsilon}{\beta^2}}\|D_x^\beta f\|_{L^{p'}L^p}^{\frac{(\beta-\varepsilon)\varepsilon}{\beta^2}}\|D_{x'}^\alpha D_x^\beta f\|_{L^{p'}L^p}^{\frac{\varepsilon^2}{\beta^2}}\right)^{\alpha\beta}\\
&=\|f\|_{L^{p'}L^p}^{\alpha\beta}\|g\|_{L^{q'}L^q}^{\varepsilon\varepsilon'}\|D_{x'}^\alpha g\|_{L^{q'}L^q}^{\alpha\varepsilon}\|D_{x'}^\alpha f\|_{L^{p'}L^p}^{\alpha\varepsilon}\|D_x^\beta g\|_{L^{q'}L^q}^{\alpha\varepsilon}\\
&\quad\times\|D_x^\beta f\|_{L^{p'}L^p}^{\alpha\varepsilon}\|D_x^\beta D_{x'}^\alpha g\|_{L^{q'}L^q}^{\alpha\beta}\|D_{x'}^\alpha D_x^\beta f\|_{L^{p'}L^p}^{\varepsilon\varepsilon'}.
\end{aligned}$$

Therefore $\left\|I_1^A\right\|_{L^{r'}L^r}$ is bounded by a constant multiple of

$$\begin{aligned}
&\left(\|D_{x'}^\alpha D_x^\beta f\|_{L^{p'}L^p}\|g\|_{L^{q'}L^q}\right)^{\frac{\varepsilon\varepsilon'}{(\beta+\varepsilon)(\alpha+\varepsilon')}}\left(\|f\|_{L^{p'}L^p}\|D_x^\beta D_{x'}^\alpha g\|_{L^{q'}L^q}\right)^{\frac{\alpha\beta}{(\beta+\varepsilon)(\alpha+\varepsilon')}}\\
&\times\left(\|D_{x'}^\alpha g\|_{L^{q'}L^q}\|D_x^\beta f\|_{L^{p'}L^p}\right)^{\frac{\alpha\varepsilon}{(\beta+\varepsilon)(\alpha+\varepsilon')}}\left(\|D_{x'}^\alpha f\|_{L^{p'}L^p}\|D_x^\beta g\|_{L^{q'}L^q}\right)^{\frac{\beta\varepsilon'}{(\beta+\varepsilon)(\alpha+\varepsilon')}}.
\end{aligned}$$

Using the convexity of natural log in (3.3) (noticing that the sum of the four exponents is equal to 1), the required estimate for I_1^A follows.

We now estimate I_1^C. Applying (4.10) with f replaced by $\Delta_{>k'-3}f$ and g by $D_{x'}^\alpha\Delta_k' g$ and using Bernstein's inequality, we see that

$$\|[D_x^\beta,\Delta_{>k'-3}S_{k-3}f]D_{x'}^\alpha\Delta_k'\Delta_k g\|_{L_{x'}^{r'}L_x^r}$$

is bounded by a constant multiple of

$$2^{k'\alpha+k(\beta-1)}\left\|\Delta_{k'}'\Delta_k g\right\|_{L^{q'}L^q}\left\|\nabla_x S_{k-3}\Delta_{>k'-3}'f\right\|_{L^{p'}L^p}=:c_{k,k'}.$$

It follows from (4.14) and Bernstein's inequality that

$$\begin{aligned}
c_{k,k'}\lesssim\min\Big\{&2^{k\beta+k'\alpha}\|f\|_{L^{p'}L^p}\|g\|_{L^{q'}L^q},\ 2^{k\beta-k'\alpha}\|D_{x'}^\alpha f\|_{L^{p'}L^p}\|D_{x'}^\alpha g\|_{L^{q'}L^q},\\
&2^{-k\varepsilon+k'\alpha}\|D_x^\beta g\|_{L^{q'}L^q}\|f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}}\|D_x^\beta f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}},\\
&2^{-k\varepsilon-k'\alpha}\|D_{x'}^\alpha D_x^\beta g\|_{L^{q'}L^q}\|D_{x'}^\alpha f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}}\|D_{x'}^\alpha D_x^\beta f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}}\Big\}.
\end{aligned}$$

Applying Lemma 4.5 with $(a_1, a_2) = (\beta, \varepsilon)$ and $(a_1', a_2') = (\alpha, \alpha)$ we obtain

$$\begin{aligned}
\left\|I_1^C\right\|_{L^{r'}L^r}^{2\alpha(\beta+\varepsilon)} &\lesssim (\|f\|_{L^{p'}L^p}\|g\|_{L^{q'}L^q})^{\alpha\varepsilon}(\|D_{x'}^\alpha f\|_{L^{p'}L^p}\|D_{x'}^\alpha g\|_{L^{q'}L^q})^{\alpha\varepsilon}\\
&\quad\times\left(\|D_x^\beta g\|_{L^{q'}L^q}\|f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}}\|D_x^\beta f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}}\right)^{\alpha\beta}\\
&\quad\times\left(\|D_{x'}^\alpha D_x^\beta g\|_{L^{q'}L^q}\|D_{x'}^\alpha f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}}\|D_{x'}^\alpha D_x^\beta f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}}\right)^{\alpha\beta}\\
&= \|f\|_{L^{p'}L^p}^{\alpha\beta}\|g\|_{L^{q'}L^q}^{\alpha\varepsilon}\|D_{x'}^\alpha g\|_{L^{q'}L^q}^{\alpha\varepsilon}\|D_{x'}^\alpha f\|_{L^{p'}L^p}^{\alpha\beta}\|D_x^\beta g\|_{L^{q'}L^q}^{\alpha\beta}\\
&\quad\times\|D_x^\beta f\|_{L^{p'}L^p}^{\alpha\varepsilon}\|D_x^\beta D_{x'}^\alpha g\|_{L^{q'}L^q}^{\alpha\beta}\|D_{x'}^\alpha D_x^\beta f\|_{L^{p'}L^p}^{\alpha\varepsilon}.
\end{aligned}$$

Consequently

$$\begin{aligned}
\left\|I_1^C\right\|_{L^{r'}L^r} &\lesssim \left(\|D_{x'}^\alpha D_x^\beta f\|_{L^{p'}L^p}\|g\|_{L^{q'}L^q}\right)^{\frac{\varepsilon}{2(\beta+\varepsilon)}}\left(\|f\|_{L^{p'}L^p}\|D_x^\beta D_{x'}^\alpha g\|_{L^{q'}L^q}\right)^{\frac{\beta}{2(\beta+\varepsilon)}}\\
&\quad\times\left(\|D_{x'}^\alpha g\|_{L^{q'}L^q}\|D_x^\beta f\|_{L^{p'}L^p}\right)^{\frac{\varepsilon}{2(\beta+\varepsilon)}}\\
&\quad\times\left(\|D_{x'}^\alpha f\|_{L^{p'}L^p}\|D_x^\beta g\|_{L^{q'}L^q}\right)^{\frac{\beta}{2(\beta+\varepsilon)}},
\end{aligned}$$

from which the desired bound follows by applying (3.3).

This concludes all estimates for I_1.

4.3.3. Estimate for I_2. We write $D_{x'}^\alpha D_x^\beta(FG)$ as a bilinear-biparameter multiplier in (4.17) and decompose its symbol as

$$\begin{aligned}
|\xi'+\eta'|^\alpha|\xi+\eta|^\beta &= \sigma_2 + |\xi'|^\alpha(|\xi+\eta|^\beta - |\eta|^\beta)\\
&\quad + |\eta|^\beta(|\xi'+\eta'|^\alpha - |\xi'|^\alpha) + |\xi'|^\alpha|\eta|^\beta,
\end{aligned}$$

where $\sigma_2 := (|\xi'+\eta'|^\alpha - |\xi'|^\alpha)(|\xi+\eta|^\beta - |\eta|^\beta)$. This leads to a decomposition of the multiplier:

$$D_{x'}^\alpha D_x^\beta(FG) = T_{\sigma_2}(F,G) + [D_x^\beta, D_{x'}^\alpha F]G + [D_{x'}^\alpha, D_x^\beta G]F + D_{x'}^\alpha F\, D_x^\beta G,$$

where T_{σ_2} denotes the bilinear Fourier multiplier with symbol σ_2. Applying this identity with $F = \Delta_{k'}' S_{k-3} f$ and $G = S_{k'-3}'\Delta_k g$ and using $S_k = 1 - \Delta_{>k}$

and $\sum_k \Delta_k = 1$, we get

$$\begin{aligned}
I_2 &= \sum_{k',k} T_{\sigma_2}(\Delta'_{k'} S_{k-3} f, S'_{k'-3}\Delta_k g) + \sum_k [D_x^\beta, D_{x'}^\alpha S_{k-3} f]\Delta_k g \\
&\quad - \sum_{k',k} [D_x^\beta, D_{x'}^\alpha \Delta'_{k'} S_{k-3} f]\Delta'_{>k'-3}\Delta_k g + \sum_{k'} [D_{x'}^\alpha, D_x^\beta S'_{k'-3} g]\Delta'_{k'} f \\
&\quad - \sum_{k',k} [D_{x'}^\alpha, D_x^\beta S'_{k'-3}\Delta_k g]\Delta'_{k'}\Delta_{>k-3} f \\
&\quad + \sum_{k',k} D_{x'}^\alpha \Delta'_{k'} S_{k-3} f \cdot D_x^\beta S'_{k'-3}\Delta_k g \\
&=: I_2^A + I_2^B - I_2^C + I_2^D - I_2^E + I_2^F.
\end{aligned}$$

The terms I_2^B and I_2^D involve single summations and can easily be estimated as in Section 3, by using Proposition 4.1 ((4.9) and (4.10)), Lemmas 4.4 and 2.4. We leave out I_2^E by symmetry, and treat I_2^A, I_2^C, and I_2^F separately.

We begin with the first term I_2^A. By inequality (4.12), we get

$$\begin{aligned}
&\left\| T_{\sigma_2}(\Delta'_{k'} S_{k-3} f, S'_{k'-3}\Delta_k g)\right\|_{L^{r'}L^r} \\
&\lesssim 2^{k'(\alpha-1)+k(\beta-1)} \left\|\nabla_x \Delta'_{k'} S_{k-3} f\right\|_{L^{p'}L^p} \left\|\nabla_{x'} S'_{k'-3}\Delta_k g\right\|_{L^{q'}L^q} \\
&=: \overline{a}_{k,k'}.
\end{aligned}$$

By iterative applications of Lemma 4.4 and Bernstein's inequality, we derive

$$\begin{aligned}
\overline{a}_{k,k'} \lesssim \min\Big\{ & 2^{k\beta+k'\alpha}\|f\|_{L^{p'}L^p}\|g\|_{L^{q'}L^q}, \\
& 2^{k\beta-k'\varepsilon'}\|D_{x'}^\alpha f\|_{L^{p'}L^p}\|g\|_{L^{q'}L^q}^{\frac{\alpha-\varepsilon'}{\alpha}}\|D_{x'}^\alpha g\|_{L^{q'}L^q}^{\frac{\varepsilon'}{\alpha}}, \\
& 2^{-k\varepsilon+k'\alpha}\|D_x^\beta g\|_{L^{q'}L^q}\|f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}}\|D_x^\beta f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}}, \\
& 2^{-k'\varepsilon'-k\varepsilon}\|D_x^\beta D_{x'}^\alpha f\|_{L^{p'}L^p}^{\frac{\varepsilon'}{\beta}}\|D_{x'}^\alpha f\|_{L^{p'}L^p}^{\frac{\beta'-\varepsilon}{\beta}} \\
& \quad \times \|D_{x'}^\alpha D_x^\beta g\|_{L^{q'}L^q}^{\frac{\varepsilon}{\alpha}}\|D_x^\beta g\|_{L^{q'}L^q}^{\frac{\alpha-\varepsilon}{\alpha}} \Big\},
\end{aligned}$$

where ε and ε' satisfy $0<\varepsilon<\min(1,\beta)$, $0<\varepsilon'<\min(1,\alpha)$ and $\frac{\varepsilon}{\beta}=\frac{\varepsilon'}{\alpha}$. Applying Lemma 4.5 with $(a_1,a_2)=(\beta,\varepsilon)$ and $(a_1',a_2')=(\alpha,\varepsilon')$, we have

$$\begin{aligned}
&\left\|I_2^A\right\|_{L^{r'}L^r}^{(\beta+\varepsilon)(\alpha+\varepsilon')}\\
&\lesssim (\|f\|_{L^{p'}L^p}\|g\|_{L^{q'}L^q})^{\varepsilon\varepsilon'}\left(\|D_{x'}^\alpha f\|_{L^{p'}L^p}\|g\|_{L^{q'}L^q}^{\frac{\alpha-\varepsilon'}{\alpha}}\|D_{x'}^\alpha g\|_{L^{q'}L^q}^{\frac{\varepsilon'}{\alpha}}\right)^{\alpha\varepsilon}\\
&\quad\times\left(\|D_x^\beta g\|_{L^{q'}L^q}\|f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}}\|D_x^\beta f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}}\right)^{\beta\varepsilon'}\\
&\quad\times\left(\|D_x^\beta D_{x'}^\alpha f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}}\|D_{x'}^\alpha f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}}\|D_{x'}^\alpha D_x^\beta g\|_{L^{q'}L^q}^{\frac{\varepsilon'}{\alpha}}\|D_x^\beta g\|_{L^{q'}L^q}^{\frac{\alpha-\varepsilon'}{\alpha}}\right)^{\alpha\beta}\\
&=\|f\|_{L^{p'}L^p}^{\alpha\varepsilon}\|g\|_{L^{q'}L^q}^{\alpha\varepsilon}\|D_{x'}^\alpha g\|_{L^{q'}L^q}^{\varepsilon\varepsilon'}\|D_{x'}^\alpha f\|_{L^{p'}L^p}^{\alpha\beta}\\
&\quad\times\|D_x^\beta g\|_{L^{q'}L^q}^{\alpha\beta}\|D_x^\beta f\|_{L^{p'}L^p}^{\varepsilon\varepsilon'}\|D_x^\beta D_{x'}^\alpha g\|_{L^{q'}L^q}^{\alpha\varepsilon}\|D_{x'}^\alpha D_x^\beta f\|_{L^{p'}L^p}^{\alpha\varepsilon}.
\end{aligned}$$

Hence $\left\|I_2^A\right\|_{L_{x'}^{r'}L_x^r}$ is controlled by a constant multiple of

$$\begin{aligned}
&\left(\|f\|_{L^{p'}L^p}\|D_{x'}^\alpha D_x^\beta g\|_{L^{q'}L^q}\right)^{\frac{\alpha\varepsilon}{(\beta+\varepsilon)(\alpha+\varepsilon')}}\left(\|D_x^\beta D_{x'}^\alpha f\|_{L^{p'}L^p}\|g\|_{L^{q'}L^q}\right)^{\frac{\beta\varepsilon'}{(\beta+\varepsilon)(\alpha+\varepsilon')}}\\
&\times\left(\|D_{x'}^\alpha g\|_{L^{q'}L^q}\|D_x^\beta f\|_{L^{p'}L^p}\right)^{\frac{\varepsilon\varepsilon'}{(\beta+\varepsilon)(\alpha+\varepsilon')}}\left(\|D_{x'}^\alpha f\|_{L^{p'}L^p}\|D_x^\beta g\|_{L^{q'}L^q}\right)^{\frac{\alpha\beta}{(\beta+\varepsilon)(\alpha+\varepsilon)}},
\end{aligned}$$

which gives the desired bound for I_2^A, in view of (3.3).

Next, let us consider I_2^C. Applying inequality (4.10) with f replaced by $D_{x'}^\alpha\Delta_{k'}'f$ and g replaced by $\Delta_{>k'-3}'g$, we get

$$\begin{aligned}
&\|[D_x^\beta, D_{x'}^\alpha\Delta_{k'}'S_{k-3}f]\Delta_{>k'-3}'\Delta_k g\|_{L^{r'}L^r}\\
&\lesssim 2^{\alpha k'+(\beta-1)k}\left\|\nabla_x S_{k-3}\Delta_{k'}'f\right\|_{L^{p'}L^p}\left\|\Delta_{>k'-3}'\Delta_k g\right\|_{L^{q'}L^q}=:\overline{c}_{k,k'}.
\end{aligned}$$

By Lemma 4.4 and Bernstein's inequality, we see that $\overline{c}_{k,k'}$ is bounded by

$$\begin{aligned}
C\min\Big\{&2^{k\beta+k'\alpha}\|f\|_{L^{p'}L^p}\|g\|_{L^{q'}L^q},\ 2^{k\beta-k'\alpha}\|D_{x'}^\alpha f\|_{L^{p'}L^p}\|D_{x'}^\alpha g\|_{L^{q'}L^q},\\
&2^{-k\varepsilon+k'\alpha}\|D_x^\beta g\|_{L^{q'}L^q}\|f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}}\|D_x^\beta f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}},\\
&2^{-k\varepsilon-k'\alpha}\|D_{x'}^\alpha D_x^\beta g\|_{L^{q'}L^q}\|D_{x'}^\alpha f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}}\|D_{x'}^\alpha D_x^\beta f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}}\Big\},
\end{aligned}$$

which is exactly the same majorizing term appearing in the estimate for I_1^C. Thus the required bound for I_2^C follows along the same lines.

Finally, we treat I_2^F. Using $S_{k-3} = 1 - \Delta_{>k-3}$, we may write

$$\begin{aligned} I_2^F &= D_{x'}^\alpha f\, D_x^\beta g - \sum_{k'} D_{x'}^\alpha \Delta'_{k'} f\, D_x^\beta \Delta'_{>k'-3} g - \sum_k D_{x'}^\alpha \Delta_{>k-3} f\, D_x^\beta \Delta_k g \\ &\quad + \sum_{k',k} D_{x'}^\alpha \Delta'_{k'} \Delta_{>k-3} f\, D_x^\beta \Delta'_{>k'-3} \Delta_k g \\ &=: I_2^{F_1} - I_2^{F_2} - I_2^{F_3} + I_2^{F_4}. \end{aligned}$$

The estimate for $I_2^{F_1}$ is trivial. Since the terms $I_2^{F_2}$, $I_2^{F_3}$ and $I_2^{F_4}$ can be similarly estimated we just treat $I_2^{F_4}$. By Bernstein's inequality,

$$\begin{aligned} &\|D_{x'}^\alpha \Delta'_{k'} \Delta_{>k-3} f\, D_x^\beta \Delta'_{>k'-3} \Delta_k g\|_{L^{r'} L^r} \\ &\lesssim 2^{k'\alpha + k\beta} \|\Delta'_{k'} \Delta_{>k-3} f\|_{L^{p'} L^p} \|\Delta'_{>k'-3} \Delta_k g\|_{L^{q'} L^q} =: \overline{f}_{k,k'}. \end{aligned}$$

Using Bernstein's inequality again and a mixed norm version of (3.1) we get

$$\begin{aligned} \overline{f}_{k,k'} \lesssim \min \Big(& 2^{k'\alpha + k\beta} \|f\|_{L^{p'} L^p} \|g\|_{L^{q'} L^q}, \\ & 2^{-k'\alpha + k\beta} \|D_{x'}^\alpha f\|_{L^{p'} L^p} \|D_{x'}^\alpha g\|_{L^{q'} L^q}, \\ & 2^{k'\alpha - k\beta} \|D_x^\beta f\|_{L^{p'} L^p} \|D_x^\beta g\|_{L^{q'} L^q}, \\ & 2^{-k'\alpha - k\beta} \|D_{x'}^\alpha D_x^\beta f\|_{L^{p'} L^p} \|D_{x'}^\alpha D_x^\beta g\|_{L^{q'} L^q} \Big), \end{aligned}$$

from which the required bound follows from Lemma 4.5 and (3.3). This concludes all estimates for I_2.

4.3.4. Estimates for I_3 and I_7. To estimates I_3, we assume that $\alpha > d/r^* - d$ and $\beta > \max(0, n/r - n)$. Write

$$\begin{aligned} \sum_{k',k} D_{x'}^\alpha D_x^\beta (\Delta'_{k'} S_{k-3} f \cdot \widetilde{\Delta'_{k'}} \Delta_k g) &= \sum_{k',k} D_{x'}^\alpha [D_x^\beta, \Delta'_{k'} S_{k-3} f] \widetilde{\Delta'_{k'}} \Delta_k g \\ &\quad + \sum_{k',k} D_{x'}^\alpha \left(\Delta'_{k'} S_{k-3} f \cdot D_x^\beta \widetilde{\Delta'_{k'}} \Delta_k g \right) \\ &=: I_3^A + I_3^B. \end{aligned}$$

To estimate I_3^A, we establish the following lemma.

Lemma 4.6. *Let α, β, p, p', q, q', r, r' satisfy the conditions of Theorem 1.4. Then*

$$\begin{aligned}&\left\| D^{\alpha}_{x'}[D^{\beta}_{x}, \Delta'_{k'} S_{k-3} f] \widetilde{\Delta'_{k'}} \Delta_k g \right\|_{L^{r'} L^r} \\ &\lesssim 2^{\alpha k' + (\beta-1)k} \left\| \widetilde{\Delta'_{k'}} \Delta_k g \right\|_{L^{q'} L^q} \left\| \Delta'_{k'} \nabla_x S_{k-3} f \right\|_{L^{p'} L^p}.\end{aligned}$$

Proof. This is a corollary of inequalities (4.2) and (4.10). Indeed, denote $f_{k,k'} = \Delta'_{k'} S_{k-3} f$, $g_{k,k'} = \widetilde{\Delta'_{k'}} \Delta_k g$, and $h = h_{k,k'} = [D^{\beta}_x, f_{k,k'}] g_{k,k'}$. Then

$$\widehat{h}(\xi', \xi) = \int_{\mathbb{R}^{n+d}} (|\xi|^{\beta} - |\eta|^{\beta}) \widehat{f_{k,k'}}(\xi' - \eta', \xi - \eta) \widehat{g_{k,k'}}(\eta', \eta) \, d\eta d\eta'.$$

By the support properties of $\widehat{f_{k,k'}}$ and $\widehat{g_{k,k'}}$, we may write

$$\widehat{h}(\xi', \xi) = \Phi'(2^{-k'-5}\xi') \Phi(2^{-k-5}\xi) \widehat{S'_{k'+5} S_{k+5} h}(\xi', \xi),$$

and hence $D^{\alpha}_{x'} h(x', x)$ can be expressed as

$$2^{\alpha k'} \int_{\mathbb{R}^{n+d}} |2^{-k'}\xi'|^{\alpha} \Phi'(2^{-k'-5}\xi') \Phi(2^{-k-5}\xi) \widehat{S'_{k'+5} S_{k+5} h}(\xi', \xi) e^{i(x\cdot\xi + x'\cdot\xi')} \, d\xi d\xi'.$$

Applying (4.2) with $\sigma_1(\xi') = |\xi'|^{\alpha} \Phi'(2^{-5}\xi')$ and $\sigma_2(\xi) = \Phi(2^{-5}\xi)$, we get

$$\left\| D^{\alpha}_{x'}[D^{\beta}_{x}, \Delta'_{k'} S_{k-3} f] \widetilde{\Delta'_{k'}} \Delta_k g \right\|_{L^{r'} L^r} \lesssim 2^{\alpha k'} \left\| [D^{\beta}_{x}, \Delta'_{k'} S_{k-3} f] \widetilde{\Delta'_{k'}} \Delta_k g \right\|_{L^{r'} L^r}.$$

The desired estimate then follows by utilizing (4.10) with f replaced by $\Delta'_{k'} f$ and g replaced by $\widetilde{\Delta'_{k'}} g$. □

Let us return to estimate I_3^A. By Lemma 4.6,

$$\begin{aligned}&\| D^{\alpha}_{x'}[D^{\beta}_{x}, \Delta'_{k'} S_{k-3} f] \widetilde{\Delta'_{k'}} \Delta_k g \|_{L^{r'} L^r} \\ &\lesssim 2^{\alpha k' + (\beta-1)k} \left\| \widetilde{\Delta'_{k'}} \Delta_k g \right\|_{L^{q'} L^q} \left\| \Delta'_{k'} \nabla S_{k-3} f \right\|_{L^{p'} L^p}.\end{aligned}$$

We denote the right hand side by $\widetilde{\widetilde{a}}_{k,k'}$. Using Bernstein's inequality and (4.14), we deduce that

$$\begin{aligned}\widetilde{\widetilde{a}}_{k,k'} \lesssim \min\Bigg(& 2^{\alpha k'+\beta k}\|g\|_{L^{q'}L^q}\|f\|_{L^{p'}L^p}, \\ & 2^{\alpha k'-\varepsilon k}\Big\|D_x^\beta g\Big\|_{L^{q'}L^q}\|f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}}\|D_x^\beta f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}}, \\ & 2^{-\alpha k'+\beta k}\|D_{x'}^\alpha f\|_{L^{p'}L^p}\|D_{x'}^\alpha g\|_{L^{q'}L^q}, \\ & 2^{-\alpha k'-\varepsilon k}\Big\|D_x^\beta D_{x'}^\alpha g\Big\|_{L^{q'}L^q}\|D_{x'}^\alpha f\|_{L^{p'}L^p}^{\frac{\beta-\varepsilon}{\beta}}\|D_x^\beta D_{x'}^\alpha f\|_{L^{p'}L^p}^{\frac{\varepsilon}{\beta}}\Bigg).\end{aligned}$$

The required estimate for I_3^A then follows by Lemma 4.5 and the convexity of natural log.

For I_3^B, we decompose it as

$$\begin{aligned}I_3^B &= \sum_{k'} D_{x'}^\alpha\left(\Delta'_{k'}f\cdot D_x^\beta\widetilde{\Delta'_{k'}}g\right) - \sum_{k',k} D_{x'}^\alpha\left(\Delta'_{k'}\Delta_{>k-3}f\cdot D_x^\beta\widetilde{\Delta'_{k'}}\Delta_k g\right)\\ &=: I_3^{B_1}+I_3^{B_2}.\end{aligned}$$

We consider $I_3^{B_2}$ only as $I_3^{B_1}$ can be treated similarly. Denote $\widetilde{h} = \Delta'_{k'}\Delta_{>k-3}f\cdot D_x^\beta\widetilde{\Delta'_{k'}}\Delta_k g$. By the support property of $\mathcal{F}_1\widetilde{h}$, we can write

$$\mathcal{F}_1\widetilde{h}(\xi',x) = \Phi'(2^{-k'-5}\xi')\mathcal{F}_1(S'_{k'+5}\widetilde{h})(\xi',x),$$

and therefore

$$\begin{aligned}&D_{x'}^\alpha(\Delta'_{k'}\Delta_{>k-3}fD_x^\beta\widetilde{\Delta'_{k'}}\Delta_k g)\\ &= 2^{k'\alpha}\int_{\mathbb{R}^d}|2^{-k'}\xi'|^\alpha\Phi'(2^{-k'-5}\xi')\mathcal{F}_1(S'_{k'+5}\widetilde{h})(\xi',x)e^{i\xi'\cdot t}d\xi'.\end{aligned}$$

Applying (4.3) with $\sigma_1 = |\cdot|^\alpha\Phi'(2^{-5}\cdot)$, the Hölder and Bernstein inequalities, we get

$$\begin{aligned}&\|D_{x'}^\alpha\left(\Delta'_{k'}\Delta_{>k-3}f\cdot D_x^\beta\widetilde{\Delta'_{k'}}\Delta_k g\right)\|_{L^{r'}L^r}\\ &\lesssim 2^{k'\alpha+k\beta}\|\Delta'_{k'}\Delta_{>k-3}f\|_{L^{p'}L^p}\|\widetilde{\Delta'_{k'}}\Delta_k g\|_{L^{q'}L^q}.\end{aligned}$$

Note that the right hand side satisfies the same estimate as $\overline{f}_{k,k'}$ (which was previously used in estimating $I_2^{F_4}$). The required estimate for $I_3^{B_2}$ then follows along the same lines. I_7 can be treated similarly.

This concludes the proof of Theorem 1.4. □

Acknowledgements

The authors would like to express their sincere appreciation to the referee for his/her percipient comments on their work. Both authors are in debt to Loukas Grafakos, Jarod Hart and Rodolfo H. Torres for inspiring discussions leading to the present work. The second author also thank Virginia Naibo for valuable discussions. The second author is supported by NNSF of China (Nos. 11671397, 12071473).

References

[1] C. Benea and C. Muscalu, *Multiple Vector Valued Inequalities via the Helicoidal Method*, arXiv e-prints (2015).

[2] C. Benea and C. Muscalu, *Quasi-Banach valued inequalities via the helicoidal method*, J. Funct. Anal. **273** (2017), no. 4, 1295–1353.

[3] A. Bényi and T. Oh, *Smoothing of commutators for a Hörmander class of bilinear pseudodifferential operators*, J. Fourier Anal. Appl. **20** (2014), no. 2, 282–300.

[4] F. Bernicot, D. Maldonado, K. Moen, and V. Naibo, *Bilinear Sobolev-Poincaré inequalities and Leibniz-type rules*, J. Geom. Anal. **24** (2014), no. 2, 1144–1180.

[5] J. Bourgain and D. Li, *On an endpoint Kato-Ponce inequality*, Differential Integral Equations **27** (2014), no. 11-12, 1037–1072.

[6] F. M. Christ and M. I. Weinstein, *Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation*, J. Funct. Anal. **100** (1991), no. 1, 87–109.

[7] F. Di Plinio and Y. Ou, *Banach-valued multilinear singular integrals*, Indiana Univ. Math. J. **67** (2018), no. 5, 1711–1763.

[8] L. Grafakos, *Multilinear operators in harmonic analysis and partial differential equations*, in: Harmonic Analysis and Nonlinear Partial Differential Equations, RIMS Kôkyûroku Bessatsu, B33, 11–27, Res. Inst. Math. Sci. (RIMS), Kyoto (2012).

[9] L. Grafakos, D. Maldonado, and V. Naibo, *A remark on an endpoint Kato-Ponce inequality*, Differential Integral Equations **27** (2014), no. 5-6, 415–424.

[10] L. Grafakos and S. Oh, *The Kato-Ponce inequality*, Comm. Partial Differential Equations **39** (2014), no. 6, 1128–1157.

[11] L. Grafakos and R. H. Torres, *Multilinear Calderón-Zygmund theory*, Adv. Math. **165** (2002), no. 1, 124–164.

[12] A. Gulisashvili and M. A. Kon, *Exact smoothing properties of Schrödinger semigroups*, Amer. J. Math. **118** (1996), no. 6, 1215–1248.

[13] J. Hart, R. H. Torres, and X. Wu, *Smoothing properties of bilinear operators and Leibniz-type rules in Lebesgue and mixed Lebesgue spaces*, Trans. Amer. Math. Soc. **370** (2018), no. 12, 8581–8612.

[14] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math. **41** (1988), no. 7, 891–907.

[15] C. E. Kenig, *On the local and global well-posedness theory for the KP-I equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **21** (2004), no. 6, 827–838.

[16] C. E. Kenig, G. Ponce, and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), no. 4, 527–620.

[17] A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres, and R. Trujillo-González, *New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory*, Adv. Math. **220** (2009), no. 4, 1222–1264.

[18] C. Muscalu, J. Pipher, T. Tao, and C. Thiele, *Bi-parameter paraproducts*, Acta Math. **193** (2004), no. 2, 269–296.

[19] C. Muscalu and W. Schlag, Classical and Multilinear Harmonic Analysis, Vol. II, Vol. 138 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge (2013), `ISBN 978-1-107-03182-1`.

[20] T. Tao, Nonlinear Dispersive Equations, Vol. 106 of CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI (2006), `ISBN 0-8218-4143-2`. Local and global analysis.

[21] R. H. Torres and E. L. Ward, *Leibniz's rule, sampling and wavelets on mixed Lebesgue spaces*, J. Fourier Anal. Appl. **21** (2015), no. 5, 1053–1076.

Department of Mathematics
Western New England University
Springfield, MA-01119, USA
E-mail address: seungly.oh@wne.edu

Department of Mathematics
China University of Mining & Technology (Beijing)
Beijing, 100083, China
E-mail address: wuxf@cumtb.edu.cn

Received October 22, 2018
Accepted March 22, 2019