Math. Res. Lett.
Volume 27, Number 4, 1195–1236, 2020

Sphere theorems for submanifolds in Kähler manifold

Jun Sun and Linlin Sun*

In this paper, we prove some differentiable sphere theorems and topological sphere theorems for submanifolds in Kähler manifold, especially in complex space forms.

1. Introduction

The study of the relation between curvature and topology is a fundamental problem in differential geometry. Sphere theorems play an important role in such a study. There are two types of differentiable sphere theorems: one is for the Riemannian manifold itself (i.e., intrinsic version), the other is for submanifolds in a Riemannian manifold (i.e., extrinsic version). The typical example of the former one is the classical 1/4-pinched differentiable sphere theorem, which states that a compact Riemannian manifold M of dimension $n \geq 4$ with pointwise 1/4-pinched sectional curvature is diffeomorphic to a

*L. Sun is the corresponding author.

spherical space form. This theorem was finally proved by Brendle-Schoen ([3], [4]).

The study of sphere theorems for submanifolds in a Riemannian manifolds also has a long history. At first, people concerned on the rigidity sphere theorems for minimal submanifolds in unit sphere. For example, Simons ([18]) and Chern-do Carmo-Kobayashi ([5]) showed that for a compact minimal submanifold M^n in unit sphere $\mathbb{S}^{n+p}$, if $|\mathbf{B}|^2 \leq \frac{n}{2-\frac{1}{p}}$, then M is either totally geodesic, or a Clifford hypersurface, or a Veronese surface in $\mathbb{S}^4$. Later on, Li-Li ([12]) proved that M is either totally geodesic or a Veronese surface in $\mathbb{S}^4$ if $|\mathbf{B}|^2 \leq \frac{2}{3}n$. Topological sphere theorems for submanifolds have also been considered. Lawson-Simons ([10]) considered the vanishing theorem of integral current in an n-dimensional submanifold in unit sphere (the case of submanifold in Euclidean space was considered by Xin ([20]) and showed that an n-dimensional submanifold in unit sphere with $|\mathbf{B}|^2 < \min\{n-1, 2\sqrt{n-1}\}$ is a homotopy sphere. Leung ([11]) proved that an n-dimensional minimal submanifold in unit sphere with $|\mathbf{B}|^2 < n$ and $n > 3$ is homeomorphic to a sphere. Later on, Shiohama and Xu ([17]) improved Lawson-Simons' result to complete submanifold in space forms with nonnegative sectional curvature. By using mean curvature flow, Andrews-Baker ([1]) proved a differentiable sphere theorem for submanifolds in $\mathbb{R}^{n+p}$ under the pinching assumption relating $|\mathbf{B}|^2$ and $|\mathbf{H}|^2$. Recently, Cui-Sun ([6]) and Gu-Xu ([7], [21], [22], etc.) also proved some topological and differentiable sphere theorems for submanifolds in general Riemannian manifold. Furthermore, Li-Wang ([13]) proved some differentiable sphere theorems for Lagrangian submanifolds in complex space form. In general, the conditions of sphere theorems for submanifolds in a Riemannian manifold are expressed in terms of the scalar curvature, Ricci curvature or the sectional curvature and the mean curvature of the submanifold, as well as the sectional curvature of the ambient manifold.

In this paper, we will consider sphere theorems for submanifolds in Kähler manifold, which are special cases comparing with the above mentioned results for general Riemannian manifold. Contrary to the above mentioned sphere theorems, we will express the condition in terms of the holomorphic sectional curvature of the ambient manifold instead of its sectional curvature.

Let M be a smooth n-dimensional submanifold of a Kähler manifold N^{2m}. We will denote the curvature tensors on M and N by R and K, respectively. Recall that the sectional curvature is given by

$$K(X,Y) := K(X,Y,X,Y)$$

and the holomorphic sectional curvature is given by

$$K(X) := K(X, JX) := K(X, JX, X, JX),$$

where X and Y are tangent vector fields on M. Denote the minimal and maximal holomorphic sectional curvatures by

$$\tilde{K}_{\min} := \min_{|X|=1} K(X), \quad \tilde{K}_{\max} := \max_{|X|=1} K(X). \tag{1.1}$$

Our first theorem is as follows:

Theorem A. *Let M be a smooth $n(\geq 2)$-dimensional closed simply connected submanifold of a Kähler manifold N^{2m}. If the scalar curvature of M satisfies the following condition:*

$$R_M \geq \begin{cases} \frac{3n^2+8}{4}\tilde{K}_{\max} - \frac{n^2-n+4}{2}\tilde{K}_{\min} + \frac{n-2}{n-1}|\mathbf{H}|^2, & if\ \tilde{K}_{\min} \geq 0; \\ \frac{3n^2+8}{4}\tilde{K}_{\max} - \frac{n^2-n+8}{2}\tilde{K}_{\min} + \frac{n-2}{n-1}|\mathbf{H}|^2, & if\ \tilde{K}_{\min} \leq 0 \leq \tilde{K}_{\max}; \\ \frac{3(n^2-n+2)}{4}\tilde{K}_{\max} - \frac{n^2-n+8}{2}\tilde{K}_{\min} + \frac{n-2}{n-1}|\mathbf{H}|^2, & if\ \tilde{K}_{\max} \leq 0, \end{cases} \tag{1.2}$$

and we further assume that the strict inequality holds for some point $x_0 \in M$ if $\tilde{K}_{\max} = \tilde{K}_{\min}$. Then M is diffeomorphic to $\mathbb{S}^n$.

Recall that a submanifold M in a Kähler manifold N is said to be *totally real* in N if $JT_x(M) \subset N_x(M)$ for each $x \in M$, where J is the complex structure on N and $N_x(M)$ is the normal space of M in N at x. When the submanifold is totally real, Theorem A can be improved to be the following:

Corollary 1.1. *Let M be a smooth $n(\geq 2)$-dimensional closed simply connected totally real submanifold of a Kähler manifold N^{2m}. If M satisfies the following condition:*

$$R_M \geq \frac{3(n^2-n+2)}{4}\tilde{K}_{\max} - \frac{n^2-n+4}{2}\tilde{K}_{\min} + \frac{n-2}{n-1}|\mathbf{H}|^2,$$

and we further assume that the strict inequality holds for some point $x_0 \in M$ if $\tilde{K}_{\max} = \tilde{K}_{\min} = 0$. Then M is diffeomorphic to $\mathbb{S}^n$.

In particular, when N is a complex space form with constant holomorphic sectional curvature c, we have:

Corollary 1.2. *Let M be a smooth $n(\geq 2)$-dimensional closed simply connected totally real submanifold of complex space form N^{2m} with holomorphic* *sectional curvature c. If M satisfies the following condition:*

$$R_M \geq \frac{(n-2)(n+1)}{4}c + \frac{n-2}{n-1}|\mathbf{H}|^2,$$

and we further assume that the strict inequality holds for some point $x_0 \in M$ if $c = 0$. Then M is diffeomorphic to $\mathbb{S}^n$.

Next, we plan to examine differentiable sphere theorems under Ricci curvature pinching condition.

Theorem B: *For fixed $0 < \varepsilon \leq 1$, set $\delta(\varepsilon, n) = \frac{((n-4)\varepsilon+2)^2}{4(2+(n^2-4n+2)\varepsilon)}$. Let M be a smooth $n(\geq 4)$-dimensional closed simply connected submanifold of a Kähler manifold N^{2m}. If M satisfies the following condition:*

$$Ric^{[2]}_{\min} \geq \begin{cases} \frac{3n+4\varepsilon}{2}\tilde{K}_{\max} - (n-1+2\varepsilon)\tilde{K}_{\min} + \delta(\varepsilon,n)|\mathbf{H}|^2, \\ \quad if\ \tilde{K}_{\min} \geq 0; \\ \frac{3n+4\varepsilon}{2}\tilde{K}_{\max} - (n-1+4\varepsilon)\tilde{K}_{\min} + \delta(\varepsilon,n)|\mathbf{H}|^2, \\ \quad if\ \tilde{K}_{\min} \leq 0 \leq \tilde{K}_{\max}; \\ \frac{3(n-1+\varepsilon)}{2}\tilde{K}_{\max} - (n-1+4\varepsilon)\tilde{K}_{\min} + \delta(\varepsilon,n)|\mathbf{H}|^2, \\ \quad if\ \tilde{K}_{\max} \leq 0, \end{cases} \tag{1.3}$$

and the strict inequality holds for some point $x_0 \in M$. Then M is diffeomorphic to $\mathbb{S}^n$.

Corollary 1.3. *For fixed $0 < \varepsilon \leq 1$, set $\delta(\varepsilon, n) = \frac{((n-4)\varepsilon+2)^2}{4(2+(n^2-4n+2)\varepsilon)}$. Let M be a smooth $n(\geq 4)$-dimensional closed simply connected totally real submanifold of a Kähler manifold N^{2m}. If M satisfies the following condition:*

$$Ric^{[2]}_{\min} \geq \frac{3(n-1+\varepsilon)}{2}\tilde{K}_{\max} - (n-1+2\varepsilon)\tilde{K}_{\min} + \delta(\varepsilon,n)|\mathbf{H}|^2,$$

and the strict inequality holds for some point $x_0 \in M$. Then M is diffeomorphic to $\mathbb{S}^n$.

Corollary 1.4. *For fixed $0 < \varepsilon \leq 1$, set $\delta(\varepsilon, n) = \frac{((n-4)\varepsilon+2)^2}{4(2+(n^2-4n+2)\varepsilon)}$. Let M be a smooth $n(\geq 4)$-dimensional closed simply connected totally real submanifold of complex space form N^{2m} with holomorphic sectional curvature c. If*

M satisfies the following condition:

$$Ric^{[2]}_{\min} \geq \frac{n-1-\varepsilon}{2}c + \delta(\varepsilon, n)|\mathbf{H}|^2,$$

and the strict inequality holds for some point $x_0 \in M$. Then M is diffeomorphic to $\mathbb{S}^n$.

Remark 1.5. *If $\varepsilon = 1$, then $\delta(\varepsilon, n) = \frac{1}{4}$.*

For a submanifold in a Kähler manifold, we also have the following topological sphere theorem:
Theorem C: *Let M be a smooth $n(\geq 4)$-dimensional closed simply connected submanifold of a Kähler manifold N^{2m}. If the scalar curvature of M satisfies the following condition:*
(1.4)

$$R_M \geq \begin{cases} \frac{3n^2+16}{4}\tilde{K}_{\max} - \frac{n^2-n+8}{2}\tilde{K}_{\min} + \frac{n-3}{n-2}|\mathbf{H}|^2, & if\ \tilde{K}_{\min} \geq 0; \\ \frac{3n^2+16}{4}\tilde{K}_{\max} - \frac{n^2-n+16}{2}\tilde{K}_{\min} + \frac{n-3}{n-2}|\mathbf{H}|^2, & if\ \tilde{K}_{\min} \leq 0 \leq \tilde{K}_{\max}; \\ \frac{3(n^2-n+4)}{4}\tilde{K}_{\max} - \frac{n^2-n+16}{2}\tilde{K}_{\min} + \frac{n-3}{n-2}|\mathbf{H}|^2, & if\ \tilde{K}_{\max} \leq 0, \end{cases}$$

and the strict inequality holds for some point $x_0 \in M$. Then M is homeomorphic to $\mathbb{S}^n$.

Corollary 1.6. *Let M be a smooth $n(\geq 4)$-dimensional closed simply connected totally real submanifold of a Kähler manifold N^{2m}. If M satisfies the following condition:*

$$R_M \geq \frac{3(n^2-n+4)}{4}\tilde{K}_{\max} - \frac{n^2-n+8}{2}\tilde{K}_{\min} + \frac{n-3}{n-2}|\mathbf{H}|^2,$$

and the strict inequality holds for some point $x_0 \in M$. Then M is homeomorphic to $\mathbb{S}^n$.

In particular, when N is a complex space form with constant holomorphic sectional curvature c, we have (comparing with Corollary 1.2):

Corollary 1.7. *Let M be a smooth $n(\geq 4)$-dimensional closed simply connected totally real submanifold of complex space form N^{2m} with holomorphic* *sectional curvature c. If M satisfies the following condition:*

$$R_M \geq \frac{n^2-n-4}{4}c + \frac{n-3}{n-2}|\mathbf{H}|^2,$$

and the strict inequality holds for some point $x_0 \in M$. Then M is homeomorphic to $\mathbb{S}^n$.

Theorem D: *Let M be a smooth $n(\geq 4)$-dimensional closed simply connected submanifold of a Kähler manifold N^{2m}. If M satisfies the following condition:*

(1.5)

$$Ric_{\min}^{[4]} \geq \begin{cases} (3n+4)\tilde{K}_{\max} - 2(n+1)\tilde{K}_{\min} + \frac{1}{2}|\mathbf{H}|^2, & if\ \tilde{K}_{\min} \geq 0; \\ (3n+4)\tilde{K}_{\max} - 2(n+3)\tilde{K}_{\min} + \frac{1}{2}|\mathbf{H}|^2, & if\ \tilde{K}_{\min} \leq 0 \leq \tilde{K}_{\max}; \\ 3n\tilde{K}_{\max} - 2(n+3)\tilde{K}_{\min} + \frac{1}{2}|\mathbf{H}|^2, & if\ \tilde{K}_{\max} \leq 0, \end{cases}$$

and the strict inequality holds for some point $x_0 \in M$. Then M is homeomorphic to $\mathbb{S}^n$.

Corollary 1.8. *Let M be a smooth $n(\geq 4)$-dimensional simply connected compact totally real submanifold of a Kähler manifold N^{2m}. If M satisfies the following condition:*

$$Ric_{\min}^{[4]} \geq 3n\tilde{K}_{\max} - 2(n+1)\tilde{K}_{\min} + \frac{1}{2}|\mathbf{H}|^2,$$

and the strict inequality holds for some point $x_0 \in M$. Then M is homeomorphic to $\mathbb{S}^n$.

In particular, if N is a complex space form, then we have the following topological sphere theorem for totally real submanifold (comparing with Remark 1.5):

Corollary 1.9. *Let M be a smooth $n(\geq 4)$-dimensional simply connected compact totally real submanifold of a complex space form N^{2m} with holomorphic sectional curvature c. If M satisfies the following condition:*

$$Ric_{\min}^{[4]} \geq (n-2)c + \frac{1}{2}|\mathbf{H}|^2,$$

and the strict inequality holds for some point $x_0 \in M$. Then M is homeomorphic to $\mathbb{S}^n$.

Remark 1.10. *All results mentioned above are sharp.*

- *Consider the totally embedding* $\mathbb{C}P^{n/2}(4) \subset \mathbb{C}P^m(4)$ *where* n *is an even number. Then* $Ric = (n+2)g$ *and* $R_M = n(n+2)$*. Thus Theorem A, Theorem B, Theorem C and Theorem D are sharp.*
- *Consider* $M_{p,\mu} := \mathrm{S}^{n-p}\left(\frac{\mu}{\sqrt{1+\mu^2}}\right) \times \mathrm{S}^p\left(\frac{1}{\sqrt{1+\mu^2}}\right)\left(\subset \mathrm{S}^{n+1}(1)\right) \subset \mathbb{C}P^{n+1}(4)$ *where* $0 < \mu < 1$*, then* $M_{p,\mu}$ *is a totally real submanifold of* $\mathbb{C}P^{n+1}(4)$*. Moreover,*

$$R_{M_{1,\mu}} - \frac{n-2}{n-1}|\mathbf{H}|^2 - (n-2)(n+1) = -\frac{n-2}{n-1}\mu^2 \to 0, \quad \textit{as } \mu \to 0,$$
$$R_{M_{2,\mu}} - \frac{n-3}{n-2}|\mathbf{H}|^2 - (n^2-n-4) = -\frac{2(n-4)}{n-2}\mu^2 \to 0, \quad \textit{as } \mu \to 0.$$

  Therefore, Corollary 1.1, Corollary 1.2, Corollary 1.6 and Corollary 1.7 are optimal.
- *For* $\varepsilon = 1$*, Corollary 1.3, Corollary 1.4 are optimal for* $n = 4$*. Corollary 1.8 and Corollary 1.9 are optimal for* $n = 4$*. We refer the reader to [22].*

In another paper, we will consider differentiable sphere theorems and topological sphere theorems for Lagrangian submanifods in Kähler manifold ([19]). Similar argument can also prove some sphere theorems for submanifolds in Sasaki space forms.

2. Preliinaries

In this section, we will provide some basic materials about Kähler manifold that will be used in the proof of the main theorems. First recall the following expression of the sectional curvature and curvature tensor in terms of holomorphic sectional curvature:

Lemma 2.1 (cf. [9]). *Let* N *be a Riemannian manifold and* X, Y, Z, W *be vector fields on* N*. Then we have*

$$\begin{aligned} 24K(X,Y,Z,W) = {} & K(X+Z,Y+W) + K(X-Z,Y-W) \\ & + K(X+W,Y-Z) + K(X-W,Y+Z) \\ & - K(X+Z,Y-W) - K(X-Z,Y+W) \\ & - K(X+W,Y+Z) - K(X-W,Y-Z). \end{aligned} \tag{2.1}$$

Lemma 2.2 (cf. [23]). *Let N be a Kähler manifold and X, Y be vector fields on N. Then we have*

$$32K(X,Y) = 3K(X+JY) + 3K(X-JY) - K(X+Y) - K(X-Y) - 4K(X) - 4K(Y). \tag{2.2}$$

Putting (2.2) into (2.1), we get that

Corollary 2.3. *Let N be a Kähler manifold and X, Y, Z, W be vector fields on N. Then we have*

$$\begin{aligned} 256K(X,Y,Z,W) = {} & K(X+Z+JY+JW) + K(X+Z-JY-JW) \\ & - K(X+Z+JY-JW) - K(X+Z-JY+JW) \\ & + K(X-Z+JY-JW) + K(X-Z-JY+JW) \\ & - K(X-Z+JY+JW) - K(X-Z-JY-JW) \\ & + K(X+W+JY-JZ) + K(X+W-JY+JZ) \\ & - K(X+W+JY+JZ) - K(X+W-JY-JZ) \\ & + K(X-W+JY+JZ) + K(X-W-JY-JZ) \\ & - K(X-W+JY-JZ) - K(X-W-JY+JZ). \end{aligned} \tag{2.3}$$

Let M^n be an n-dimensional submanifold in Riemannian manifold N^d. Choose local orthonormal frame $\{e_1, \ldots, e_d\}$ on N so that $\{e_1, \ldots, e_n\}$ are tangent to M and $\{e_{n+1}, \ldots, e_d\}$ are normal to M. Denote R and K the curvature tensors on M and N, respectively, and $h^\alpha_{ij} = \langle \mathbf{B}(e_i, e_j), e_\alpha\rangle$ the component of the second fundamental form of M in N. The mean curvature vector is given by $\mathbf{H} = \sum_{\alpha=n+1}^d H^\alpha e_\alpha$, where $H^\alpha = \sum_{i=1}^n h^\alpha_{ii}$. Then the Gauss equation can be written as

$$R_{ijkl} = K_{ijkl} + \sum_{\alpha=n+1}^{d} (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}). \tag{2.4}$$

In particular, the Ricci curvature and the scalar curvature satisfies

$$\begin{aligned} Ric(e_i) = R_{ii} &= \sum_{j=1}^n K_{ijij} + \sum_{\alpha=n+1}^d \sum_{j=1}^n [h^\alpha_{ii} h^\alpha_{jj} - (h^\alpha_{ij})^2], \\ R_M &= \sum_{i,j=1}^n K_{ijij} + |\mathbf{H}|^2 - |\mathbf{B}|^2. \end{aligned} \tag{2.5}$$

Fix $p \in M$, $X, Y \in T_pM$ and an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM, the following notations will be used in this paper:

$$\begin{aligned} Ric(X,Y) &= \sum_{i=1}^{n} R(X, e_i, Y, e_i), \quad Ric_{jj} = Ric(e_j, e_j), \\ [e_{i_1}, \ldots, e_{i_k}] &= span\{e_{i_1}, \ldots, e_{i_k}\}, \quad \forall 1 \le i_1 < i_2 < \cdots < i_k \le n, \\ Ric^{[k]}[e_{i_1}, \ldots, e_{i_k}] &= \sum_{j=1}^{k} Ric_{i_j i_j}, \\ Ric^{[k]}_{\min}(p) &= \min_{[e_{i_1},\ldots,e_{i_k}] \subset T_pM} Ric^{[k]}[e_{i_1}, \ldots, e_{i_k}], \end{aligned}$$

where $Ric^{[k]}[e_{i_1}, \ldots, e_{i_k}]$ is called the *k-th weak Ricci curvature* of $[e_{i_1}, \ldots, e_{i_k}]$, which was first introduced by Gu-Xu in [7].

At the end of this section, we will state some lemmas which will be crucial in the proof of our main theorems. The first result is due to Aubin:

Lemma 2.4 ([2]). *Let M be a compact n-dimensional Riemannian manifold. If M has nonnegative Ricci curvature everywhere and has positive Ricci curvature at some point, then M admits a metric with positive Ricci curvature everywhere.*

A Riemannian manifold M is said to have *nonnegative (positive, respectively) isotropic curvature*, if

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \ge 0 (> 0, respectively)$$

for all orthonormal four-frames $\{e_1, e_2.e_3.e_4\}$. This conception was introduced by Micallef-Moore and they proved the following topological sphere theorem:

Lemma 2.5 ([14]). *Let M be a compact simply connected $n(\ge 4)$-dimensional Riemannian manifold which has positive isotropic curvature, then M is homeomorphic to a sphere.*

In addition, Micallef-Wang proved the following topological result for manifold with positive isotropic curvature:

Lemma 2.6 ([15]). *Let M be a closed even-dimensional Riemannian manifold which has positive isotropic curvature, then $b_2(M) = 0$.*

Furthermore, Seshadri proved the following result for manifold with nonnegative isotropic curvature:

Lemma 2.7 ([16]). *Let M be a compact n-dimensional Riemannian manifold. If M has nonnegative isotropic curvature everywhere and has positive isotropic curvature at some point, then M admits a metric with positive isotropic curvature.*

The 1/4-differentiable sphere theorem was finally proved by Brendle-Schoen ([3], [4]) using the Ricci flow method. They proved that:

Theorem 2.8 ([3]). *Let (M, g_0) be a compact, locally irreducible Riemannian manifold of dimension $n(\geq 4)$ with curvature tensor R. Assume that $M \times \mathbb{R}^2$ has nonnegative isotropic curvature, i.e.,*

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234} \geq 0 \tag{2.6}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda, \mu \in [-1, 1]$. Then one of the following statements holds:

(i) M is diffeomorphic to a spherical space form;

(ii) $n = 2m$ and the universal covering of M is a Kähler manifold biholomorphic to $\mathbb{CP}^m$;

(iii) The universal covering of M is isometric to a compact symmetric space.

3. Some algebraic estimates

In this section, we will prove some algebraic estimates that are used in the proof of the main theorems.

In this section, we always assume $n \geq 4$. We say that R is an algebraic curvature on $\mathbb{R}^n$ if R is a fourth tensor such that for every $x, y, z, w \in \mathbb{R}^n$,

$$\begin{cases} R(x, y, z, w) = -R(y, x, z, w) = -R(x, y, w, z) = R(z, w, x, y), \\ R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0. \end{cases}$$

Let $\{e_i\}_{i=1}^n$ be an orthonormal frame of $\mathbb{R}^n$.

Example 3.1. *If* $B = (h_{ij}^{\alpha}) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^p$ *is a bilinear operator, we obtain an algebraic curvature tensor* $\tilde{R}$ *defined by:*

$$\tilde{R}_{ijkl} := \sum_{\alpha=1}^{p} h_{ik}^{\alpha} h_{jl}^{\alpha} - \sum_{\alpha=1}^{p} h_{il}^{\alpha} h_{jk}^{\alpha}, \quad \forall 1 \leq i, j, k, l \leq n.$$

Lemma 3.2. *Let* R *be an algebraic curvature tensor* R*. Suppose there is a constant* c *such that for every orthonormal four-frames* $\{e_1, e_2, e_3, e_4\}$,

$$R_{1212} + R_{1234} \geq c,$$

then for every $\lambda, \mu \in [-1, 1]$ *and every orthonormal four-frames* $\{e_1, e_2, e_3, e_4\}$

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda\mu R_{1234} \geq \left(1 + \lambda^2\right) \left(1 + \mu^2\right) c.$$

Proof. The assumption implies that for every orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$,

$$R_{1212} - |R_{1234}| \geq c.$$

The Bianchi identity yields that

$$R_{1234} = R_{1324} + R_{1432}.$$

Therefore,

$$\begin{aligned}
& R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda\mu R_{1234} \\
= & R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda\mu (R_{1324} + R_{1432}) \\
\geq & R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} \\
& - \left(1 + \lambda^2 \mu^2\right) |R_{1324}| - \left(\lambda^2 + \mu^2\right) |R_{1432}| \\
= & \left(R_{1313} - |R_{1324}|\right) + \lambda^2 \left(R_{1414} - |R_{1432}|\right) \\
& + \mu^2 \left(R_{2323} - |R_{2314}|\right) + \lambda^2 \mu^2 \left(R_{2424} - |R_{2413}|\right) \\
\geq & \left(1 + \lambda^2 + \mu^2 + \lambda^2 \mu^2\right) c \\
= & \left(1 + \lambda^2\right) \left(1 + \mu^2\right) c.
\end{aligned}$$

□

Lemma 3.3. *Let R be an algebraic curvature tensor R. Suppose there is a constant c such that for every orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$,*

$$R_{1313} + R_{2323} + R_{1234} \geq c,$$

then for every $\lambda \in [-1, 1]$ and every orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \geq \left(1 + \lambda^2\right) c.$$

Proof. A straightforward verification. □

Lemma 3.4. *Let $B = (h_{ij}^\alpha) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^p$ is a bilinear operator. Define $H^\alpha := \sum_{i=1}^n h_{ii}^\alpha$ and*

$$\tilde{R}_{ijkl} := \sum_{\alpha=1}^{p} h_{ik}^\alpha h_{jl}^\alpha - \sum_{\alpha=1}^{p} h_{il}^\alpha h_{jk}^\alpha, \quad \textit{for all } 1 \leq i, j, k, l \leq n.$$

Then for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$, we have

$$\tilde{R}_{1212} + \tilde{R}_{1234} \geq \frac{1}{2} \left[\frac{\sum_{\alpha=1}^p (H^\alpha)^2}{n-1} - \sum_{i,j=1}^{n} \sum_{\alpha=1}^{p} \left(h_{ij}^\alpha\right)^2 \right], \tag{3.1}$$

with equality holds if and only if $h_{ii}^\alpha = h_{11}^\alpha + h_{22}^\alpha$ for all $i \neq 1, 2$ and $h_{ij}^\alpha = 0$ for all distinct i, j with $\{i, j\} \neq \{1, 2\}$. We also have

$$\sum_{i=1}^{2} \sum_{j=3}^{4} \tilde{R}_{ijij} - 2\tilde{R}_{1234} \geq \frac{\sum_{\alpha=1}^p (H^\alpha)^2}{n-2} - \sum_{i,j=1}^{n} \sum_{\alpha=1}^{p} \left(h_{ij}^\alpha\right)^2. \tag{3.2}$$

Proof. For the proof of this Lemma, we refer the reader to Gu-Xu's paper [7]. We only need to notice that (3.1) follows from the inequality

$$2h_{mm}^\alpha h_{ll}^\alpha \geq \sum_{i \neq j} (h_{ij}^\alpha)^2 + \frac{(H^\alpha)^2}{n-1} - \sum_{i,j=1}^{n} (h_{ij}^\alpha)^2, \tag{3.3}$$

for all distinct m, l, and the equality holds if and only if

$$h_{ii}^\alpha = h_{mm}^\alpha + h_{ll}^\alpha, \quad \textit{for all } i \neq m, l. \tag{3.4}$$

Furthermore, (3.2) follows from the inequality

$$2h_{pp}^\alpha h_{qq}^\alpha + 2h_{mm}^\alpha h_{ll}^\alpha \geq \sum_{i \neq j} (h_{ij}^\alpha)^2 + \frac{(H^\alpha)^2}{n-2} - \sum_{i,j=1}^{n} (h_{ij}^\alpha)^2, \tag{3.5}$$

for all distinct p, q, m, l, and the equality holds if and only if

$$h_{ii}^{\alpha} = h_{pp}^{\alpha} + h_{qq}^{\alpha} = h_{mm}^{\alpha} + h_{ll}^{\alpha}, \quad \text{for all } i \neq p, q, m, l. \tag{3.6}$$ □

Lemma 3.5. *Let B and $\tilde{R}$ be as in Lemma 3.4. Assume*

$$\sum_{k=1}^{n} \tilde{R}_{ikik} + \sum_{k=1}^{n} \tilde{R}_{jkjk} \geq 2D, \quad \text{for all } 1 \leq i < j \leq n,$$

then for every $0 < \varepsilon \leq 1$ and all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$

$$\tilde{R}_{1212} + \tilde{R}_{1234} \geq \frac{1}{\varepsilon}\left[D - \frac{((n-4)\varepsilon + 2)^2}{8\,(2 + (n^2 - 4n + 2)\,\varepsilon)} \sum_{\alpha=1} (H^{\alpha})^2\right].$$

Proof. The proof can be found in [6]. For reader's convenience, we give another but direct proof. Set

$$h_{ij}^{\alpha} := \mathring{h}_{ij}^{\alpha} + \frac{1}{n} H^{\alpha} \delta_{ij}, \quad T^{\alpha} := \frac{1}{n} H^{\alpha}.$$

One can check that

$$\sum_{i,j=1}^{n} \left(h_{ij}^{\alpha}\right)^2 = \sum_{i,j=1}^{n} \left(\mathring{h}_{ij}^{\alpha}\right)^2 + n\,(T^{\alpha})^2, \quad \text{for all } 1 \leq \alpha \leq p.$$

Denoted by $\tilde{R}_{ii} := \sum_{j=1}^{n} \tilde{R}_{ijij}$, we get

$$\begin{aligned}
\tilde{R}_{ii} &= H^{\alpha} h_{ii}^{\alpha} - \sum_{j=1}^{n} \sum_{\alpha=1}^{p} h_{ij}^{\alpha} h_{ij}^{\alpha} \\
&= (n-1) \sum_{\alpha=1}^{p} (T^{\alpha})^2 + \sum_{\alpha=1}^{p} \left[(n-2) T^{\alpha} \mathring{h}_{ii}^{\alpha} - \sum_{j=1}^{n} \left(\mathring{h}_{ij}^{\alpha}\right)^2\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{2} \sum_{i=1}^{2} \tilde{R}_{ii} = (n-1) \sum_{\alpha=1}^{p} (T^{\alpha})^2 \\
+ \frac{1}{2} \sum_{\alpha=1}^{p} \left[(n-2) T^{\alpha} \sum_{i=1}^{2} \mathring{h}_{ii}^{\alpha} - \sum_{j=1}^{n} \sum_{i=1}^{2} \left(\mathring{h}_{ij}^{\alpha}\right)^2\right],
\end{aligned}$$

$$\frac{1}{n-2}\sum_{i=3}^{n}\tilde{R}_{ii} = (n-1)\sum_{\alpha=1}^{p}(T^{\alpha})^2 + \frac{1}{n-2}\sum_{\alpha=1}^{p}\left[(n-2)T^{\alpha}\sum_{i=3}^{n}\mathring{h}_{ii}^{\alpha} - \sum_{j=1}^{n}\sum_{i=3}^{n}\left(\mathring{h}_{ij}^{\alpha}\right)^2\right].$$

By assumption,

$$\tilde{R}_{ii} + \tilde{R}_{jj} \geq 2D, \quad \forall 1 \leq i < j \leq n,$$

then

$$\frac{1}{2}\sum_{i=1}^{2}\tilde{R}_{ii} \geq D, \quad \frac{1}{n-2}\sum_{i=3}^{n}\tilde{R}_{ii} \geq D.$$

Now for every $\varepsilon \in (0,1]$, we get

$$\begin{aligned}
D &\leq \frac{\varepsilon}{2}\sum_{i=1}^{2}\tilde{R}_{ii} + \frac{1-\varepsilon}{n-2}\sum_{i=3}^{n}\tilde{R}_{ii} \\
&= (n-1)\sum_{\alpha=1}^{p}(T^{\alpha})^2 + \frac{n\varepsilon-2}{2}\sum_{\alpha=1}^{p}T^{\alpha}\sum_{i=1}^{2}\mathring{h}_{ii}^{\alpha} - \frac{\varepsilon}{2}\sum_{\alpha=1}^{p}\sum_{j=1}^{n}\sum_{i=1}^{2}\left(\mathring{h}_{ij}^{\alpha}\right)^2 \\
&\quad - \frac{1-\varepsilon}{n-2}\sum_{\alpha=1}^{p}\sum_{j=1}^{n}\sum_{i=3}^{n}\left(\mathring{h}_{ij}^{\alpha}\right)^2 \\
&= \varepsilon\left(\tilde{R}_{1212} + \tilde{R}_{1234}\right) + (n-1-\varepsilon)\sum_{\alpha=1}^{p}(T^{\alpha})^2 + \frac{(n-2)\varepsilon-2}{2}\sum_{\alpha=1}^{p}T^{\alpha}\sum_{i=1}^{2}\mathring{h}_{ii}^{\alpha} \\
&\quad - \frac{\varepsilon}{2}\sum_{\alpha=1}^{p}\left(\sum_{i=1}^{2}\mathring{h}_{ii}^{\alpha}\right)^2 - \frac{\varepsilon}{2}\sum_{\alpha=1}^{p}\sum_{j=3}^{n}\sum_{i=1}^{2}\left(\mathring{h}_{ij}^{\alpha}\right)^2 - \frac{1-\varepsilon}{n-2}\sum_{\alpha=1}^{p}\sum_{j=1}^{n}\sum_{i=3}^{n}\left(\mathring{h}_{ij}^{\alpha}\right)^2 \\
&\quad - \varepsilon\sum_{\alpha=1}^{p}\left(\mathring{h}_{13}^{\alpha}\mathring{h}_{24}^{\alpha} - \mathring{h}_{14}^{\alpha}\mathring{h}_{23}^{\alpha}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \varepsilon\left(\tilde{R}_{1212}+\tilde{R}_{1234}\right)+(n-1-\varepsilon)\sum_{\alpha=1}^{p}(T^{\alpha})^2+\frac{(n-2)\varepsilon-2}{2}\sum_{\alpha=1}^{p}T^{\alpha}\sum_{i=1}^{2}\mathring{h}_{ii}^{\alpha}\\
&\quad-\left[\frac{\varepsilon}{2}+\frac{1-\varepsilon}{(n-2)^2}\right]\sum_{\alpha=1}^{p}\left(\sum_{i=1}^{2}\mathring{h}_{ii}^{\alpha}\right)^2\\
&\leq \varepsilon\left(\tilde{R}_{1212}+\tilde{R}_{1234}\right)+\frac{\left((n-4)\varepsilon+2\right)^2n^2}{8\left(2+(n^2-4n+2)\varepsilon\right)}\sum_{\alpha=1}^{p}(T^{\alpha})^2.
\end{aligned}
$$

□

Lemma 3.6. *Let B and $\tilde{R}$ be as in Lemma 3.4. Assume that for every orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$,*

$$
\sum_{i=1}^{4}\sum_{j=1}^{n}\tilde{R}_{ijij}\geq 4D,
$$

then for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$,

$$
\sum_{i=1}^{2}\sum_{j=3}^{4}\tilde{R}_{ijij}-2\tilde{R}_{1234}\geq 4D-\frac{1}{2}\sum_{\alpha=1}^{p}(H^{\alpha})^2.
$$

Proof. As notations in the proof of Lemma 3.5, we get

$$
\begin{aligned}
&\frac{1}{4}\sum_{i=1}^{4}\tilde{R}_{ii}=(n-1)\sum_{\alpha=1}^{p}(T^{\alpha})^2+\frac{1}{4}\sum_{\alpha=1}^{p}\left[(n-2)T^{\alpha}\sum_{i=1}^{4}\mathring{h}_{ii}^{\alpha}-\sum_{j=1}^{n}\sum_{i=1}^{4}\left(\mathring{h}_{ij}^{\alpha}\right)^2\right]\\
&=(n-1)\sum_{\alpha=1}^{p}(T^{\alpha})^2+\frac{n-2}{4}\sum_{\alpha=1}^{p}T^{\alpha}\sum_{i=1}^{4}\mathring{h}_{ii}^{\alpha}-\frac{1}{4}\sum_{\alpha=1}^{p}\sum_{i,j=1}^{2}\left(\mathring{h}_{ij}^{\alpha}\right)^2\\
&\quad-\frac{1}{4}\sum_{\alpha=1}^{p}\sum_{i,j=3}^{4}\left(\mathring{h}_{ij}^{\alpha}\right)^2-\frac{1}{2}\sum_{\alpha=1}^{p}\sum_{i=1}^{2}\sum_{j=3}^{4}\left(\mathring{h}_{ij}^{\alpha}\right)^2-\frac{1}{4}\sum_{\alpha=1}^{p}\sum_{j=5}^{n}\sum_{i=1}^{4}\left(\mathring{h}_{ij}^{\alpha}\right)^2\\
&\leq(n-1)\sum_{\alpha=1}^{p}(T^{\alpha})^2+\frac{n-2}{4}\sum_{\alpha=1}^{p}T^{\alpha}\sum_{i=1}^{4}\mathring{h}_{ii}^{\alpha}-\frac{1}{8}\sum_{\alpha=1}^{p}\left(\sum_{i=1}^{2}\mathring{h}_{ii}^{\alpha}\right)^2\\
&\quad-\frac{1}{8}\sum_{\alpha=1}^{p}\left(\sum_{i=3}^{4}\mathring{h}_{ii}^{\alpha}\right)^2-\frac{1}{4}\sum_{\alpha=1}^{p}\left(\mathring{h}_{13}^{\alpha}+\mathring{h}_{24}^{\alpha}\right)^2-\frac{1}{4}\sum_{\alpha=1}^{p}\left(\mathring{h}_{14}^{\alpha}-\mathring{h}_{23}^{\alpha}\right)^2.
\end{aligned}
$$

Notice that

$$\begin{aligned}\sum_{i=1}^{2}\sum_{j=3}^{4}\tilde{R}_{ijij}-2\tilde{R}_{1234}=&\,4\sum_{\alpha=1}^{p}(T^\alpha)^2+2\sum_{\alpha=1}^{p}T^\alpha\sum_{i=1}^{4}\mathring{h}_{ii}^\alpha\\&+\sum_{\alpha=1}^{p}\left(\sum_{i=1}^{2}\mathring{h}_{ii}^\alpha\right)\left(\sum_{j=3}^{4}\mathring{h}_{jj}^\alpha\right)\\&-\sum_{\alpha=1}^{p}\left(\mathring{h}_{13}^\alpha+\mathring{h}_{24}^\alpha\right)^2-\sum_{\alpha=1}^{p}\left(\mathring{h}_{14}^\alpha-\mathring{h}_{23}^\alpha\right)^2.\end{aligned}$$

We obtain

$$\begin{aligned}D\leq&\,\frac{1}{4}\left[\sum_{i=1}^{2}\sum_{j=3}^{4}\tilde{R}_{ijij}-2\tilde{R}_{1234}\right]+(n-1)\sum_{\alpha=1}^{p}(T^\alpha)^2+\frac{n-2}{4}\sum_{\alpha=1}^{p}T^\alpha\sum_{i=1}^{4}\mathring{h}_{ii}^\alpha\\&-\frac{1}{8}\sum_{\alpha=1}^{p}\left(\sum_{i=1}^{4}\mathring{h}_{ii}^\alpha\right)^2-\sum_{\alpha=1}^{p}(T^\alpha)^2-\frac{1}{2}\sum_{\alpha=1}^{p}T^\alpha\sum_{i=1}^{4}\mathring{h}_{ii}^\alpha\\=&\,\frac{1}{4}\left[\sum_{i=1}^{2}\sum_{j=3}^{4}\tilde{R}_{ijij}-2\tilde{R}_{1234}\right]+(n-2)\sum_{\alpha=1}^{p}(T^\alpha)^2\\&+\frac{n-4}{4}\sum_{\alpha=1}^{p}T^\alpha\sum_{i=1}^{4}\mathring{h}_{ii}^\alpha-\frac{1}{8}\sum_{\alpha=1}^{p}\left(\sum_{i=1}^{4}\mathring{h}_{ii}^\alpha\right)^2\\\leq&\,\frac{1}{4}\left[\sum_{i=1}^{2}\sum_{j=3}^{4}\tilde{R}_{ijij}-2\tilde{R}_{1234}\right]+\frac{n^2}{8}\sum_{\alpha=1}^{p}(T^\alpha)^2.\end{aligned}$$

□

4. Proof of Theorem A

In this section, we will prove the differentiable sphere theorems for submanifolds in Kähler manifold.

Proof of Theorem A. By (1.1), we have for any vector field X on N that

$$\tilde{K}_{\min}|X|^4\leq K(X)\leq\tilde{K}_{\max}|X|^4. \tag{4.1}$$

By (2.2) and (4.1), we have for orthonormal pair (X,Y) on N

$$\begin{aligned}32K(X,Y) &\le 3\tilde{K}_{\max}\left(|X+JY|^4+|X-JY|^4\right)\\ &\quad -\tilde{K}_{\min}\left(|X+Y|^4+|X-Y|^4+4|X|^4+4|Y|^4)\right)\\ &= 24(1+\langle X,JY\rangle^2)\tilde{K}_{\max}-16\tilde{K}_{\min}.\end{aligned}$$

Similarly we have

$$32K(X,Y) \ge 24(1+\langle X,JY\rangle^2)\tilde{K}_{\min}-16\tilde{K}_{\max}.$$

Therefore, we have

$$\begin{aligned}&\frac{3}{4}(1+\langle X,JY\rangle^2)\tilde{K}_{\min}-\frac{1}{2}\tilde{K}_{\max}\\ &\le K(X,Y)\le \frac{3}{4}(1+\langle X,JY\rangle^2)\tilde{K}_{\max}-\frac{1}{2}\tilde{K}_{\min}.\end{aligned} \tag{4.2}$$

By (2.3) and (4.1), we have for any orthonormal four-frames $\{X,Y,Z,W\}$ on N

$$\begin{aligned}&256K(X,Y,Z,W)\\ &\le \tilde{K}_{\max}\big(|X+Z+JY+JW|^4+|X+Z-JY-JW|^4\\ &\qquad +|X-Z+JY-JW|^4+|X-Z-JY+JW|^4\\ &\qquad +|X+W+JY-JZ|^4+|X+W-JY+JZ|^4\\ &\qquad +|X-W+JY+JZ|^4+|X-W-JY-JZ|^4\big)\\ &\quad -\tilde{K}_{\min}\big(|X+Z+JY-JW|^4+|X+Z-JY+JW|^4\\ &\qquad +|X-Z+JY+JW|^4+|X-Z-JY-JW|^4\\ &\qquad +|X+W+JY+JZ|^4+|X+W-JY-JZ|^4\\ &\qquad +|X-W+JY-JZ|^4+|X-W-JY+JZ|^4\big)\\ &= \tilde{K}_{\max}\big[128+8(\langle X+Z,JY+JW\rangle^2+\langle X-Z,JY-JW\rangle^2\\ &\qquad +\langle X+W,JY-JZ\rangle^2+\langle X-W,JY+JZ\rangle^2)\big]\\ &\quad -\tilde{K}_{\min}\big[128+8(\langle X+Z,JY-JW\rangle^2+\langle X-Z,JY+JW\rangle^2\\ &\qquad +\langle X+W,JY+JZ\rangle^2+\langle X-W,JY-JZ\rangle^2)\big].\end{aligned} \tag{4.3}$$

Similarly, we have

$$
\begin{aligned}
(4.4)\qquad & 256K(X,Y,Z,W)\\
&\geq \tilde{K}_{\min}\left[128+8(\langle X+Z,JY+JW\rangle^2+\langle X-Z,JY-JW\rangle^2\right.\\
&\qquad\left.+\langle X+W,JY-JZ\rangle^2+\langle X-W,JY+JZ\rangle^2)\right]\\
&\quad-\tilde{K}_{\max}\left[128+8(\langle X+Z,JY-JW\rangle^2+\langle X-Z,JY+JW\rangle^2\right.\\
&\qquad\left.+\langle X+W,JY+JZ\rangle^2+\langle X-W,JY-JZ\rangle^2)\right].
\end{aligned}
$$

Next we will show that under our assumption, $M\times\mathbb{R}^2$ has nonnegative isotropic curvature, i.e., (2.6) holds for all orthonormal four-frames $\{e_1,e_2,e_3,e_4\}$ and all $\lambda,\mu\in[-1,1]$. For that purpose, we first extend the four-frame $\{e_1,e_2,e_3,e_4\}$ to be an orthonormal frame $\{e_1,\ldots,e_{2m}\}$ of N such that $\{e_1,\ldots,e_n\}$ are tangent to M and $\{e_{n+1},\ldots,e_{2m}\}$ are normal to M. The Gauss equation (2.4) implies that

$$
(4.5)\qquad \tilde{R}(X,Y,Z,W):=R(X,Y,Z,W)-K(X,Y,Z,W)
$$

is an algebraic curvature. Lemma 3.4 implies that

$$
\tilde{R}_{1212}+\tilde{R}_{1234}\geq\frac{1}{2}\left[\frac{\sum_{\alpha=1}^{p}(H^\alpha)^2}{n-1}-\sum_{i,j=1}^{n}\sum_{\alpha=1}^{p}\left(h_{ij}^{\alpha}\right)^2\right]=\frac{1}{2}\left(\frac{|\mathbf{H}|^2}{n-1}-|\mathbf{B}|^2\right).
$$

Lemma 3.2 implies that for every orthonormal four-frames $\{e_1,e_2,e_3,e_4\}$ and every $\lambda,\mu\in[-1,1]$

$$
\begin{aligned}
&\tilde{R}_{1313}+\lambda^2\tilde{R}_{1414}+\mu^2\tilde{R}_{2323}+\lambda^2\mu^2\tilde{R}_{2424}-2\lambda\mu\tilde{R}_{1234}\\
&\geq\frac{(1+\lambda^2)(1+\mu^2)}{2}\left(\frac{|\mathbf{H}|^2}{n-1}-|\mathbf{B}|^2\right),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
(4.6)\qquad & R_{1313}+\lambda^2R_{1414}+\mu^2R_{2323}+\lambda^2\mu^2R_{2424}-2\lambda\mu R_{1234}\\
&\geq K_{1313}+\lambda^2K_{1414}+\mu^2K_{2323}+\lambda^2\mu^2K_{2424}-2\lambda\mu K_{1234}\\
&\quad+\frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2}\left(\frac{|\mathbf{H}|^2}{n-1}-|\mathbf{B}|^2\right).
\end{aligned}
$$

By (2.5), we have

$$
(4.7)\qquad |\mathbf{B}|^2-\frac{1}{n-1}|\mathbf{H}|^2=\sum_{i,j=1}^{n}K_{ijij}+\frac{n-2}{n-1}|\mathbf{H}|^2-R_M.
$$

Putting (4.7) into (4.6) yields

$$
\begin{aligned}
(4.8) \qquad & R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234} \\
&\geq K_{1313} + \lambda^2 K_{1414} + \mu^2 K_{2323} + \lambda^2\mu^2 K_{2424} - 2\lambda\mu K_{1234} \\
&\quad + \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2}\left(R_M - \frac{n-2}{n-1}|\mathbf{H}|^2 - \sum_{i,j=1}^{n} K_{ijij}\right).
\end{aligned}
$$

Therefore, it suffices to estimate the terms involving the curvature tensor K on N. By (4.2), for every $i \neq j$, we have

$$
(4.9) \qquad K_{ijij} \geq \frac{3}{4}(1 + \langle e_i, Je_j\rangle^2)\tilde{K}_{\min} - \frac{1}{2}\tilde{K}_{\max},
$$

and

$$
(4.10) \qquad K_{ijij} \leq \frac{3}{4}(1 + \langle e_i, Je_j\rangle^2)\tilde{K}_{\max} - \frac{1}{2}\tilde{K}_{\min}.
$$

Therefore,

(4.11)

$$
\begin{aligned}
& K_{1313} + \lambda^2 K_{1414} + \mu^2 K_{2323} + \lambda^2\mu^2 K_{2424} \\
&\quad - \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2}\sum_{i,j=1}^{n} K_{ijij} \\
&\geq (1+\lambda^2+\mu^2+\lambda^2\mu^2)\left(\frac{3}{4}\tilde{K}_{\min} - \frac{1}{2}\tilde{K}_{\max}\right) \\
&\quad + \frac{3}{4}\left(\langle e_1, Je_3\rangle^2 + \lambda^2\langle e_1, Je_4\rangle^2 + \mu^2\langle e_2, Je_3\rangle^2 + \lambda^2\mu^2\langle e_2, Je_4\rangle^2\right)\tilde{K}_{\min} \\
&\quad - \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2} \\
&\qquad \times \left[n(n-1)\left(\frac{3}{4}\tilde{K}_{\max} - \frac{1}{2}\tilde{K}_{\min}\right) + \frac{3}{4}\sum_{i,j=1}^{n}\langle e_i, Je_j\rangle^2\tilde{K}_{\max}\right] \\
&= (1+\lambda^2+\mu^2+\lambda^2\mu^2)\left(\frac{n^2-n+3}{4}\tilde{K}_{\min} - \frac{3n^2-3n+4}{8}\tilde{K}_{\max}\right) \\
&\quad + \frac{3}{4}\left(\langle e_1, Je_3\rangle^2 + \lambda^2\langle e_1, Je_4\rangle^2 + \mu^2\langle e_2, Je_3\rangle^2 + \lambda^2\mu^2\langle e_2, Je_4\rangle^2\right)\tilde{K}_{\min} \\
&\quad - \frac{3(1+\lambda^2+\mu^2+\lambda^2\mu^2)}{8}\sum_{i,j=1}^{n}\langle e_i, Je_j\rangle^2\tilde{K}_{\max}.
\end{aligned}
$$

We will consider three cases:

Case 1: $\tilde{K}_{\min} \geq 0$. In this case, we have from (4.11) that

$$\begin{aligned}
&K_{1313} + \lambda^2 K_{1414} + \mu^2 K_{2323} + \lambda^2\mu^2 K_{2424} - \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2}\sum_{i,j=1}^{n} K_{ijij} \\
&\geq (1+\lambda^2+\mu^2+\lambda^2\mu^2)\left(\frac{n^2-n+3}{4}\tilde{K}_{\min} - \frac{3n^2-3n+4}{8}\tilde{K}_{\max}\right) \\
&\quad - \frac{3n(1+\lambda^2+\mu^2+\lambda^2\mu^2)}{8}\tilde{K}_{\max} \\
&= (1+\lambda^2+\mu^2+\lambda^2\mu^2)\left(\frac{n^2-n+3}{4}\tilde{K}_{\min} - \frac{3n^2+4}{8}\tilde{K}_{\max}\right).
\end{aligned}$$

By (4.3) and (4.4), we have

$$\frac{1}{2}\tilde{K}_{\min} - \tilde{K}_{\max} \leq K_{1234} \leq \tilde{K}_{\max} - \frac{1}{2}\tilde{K}_{\min}. \tag{4.12}$$

Therefore, we have

$$\begin{aligned}
&K_{1313} + \lambda^2 K_{1414} + \mu^2 K_{2323} + \lambda^2\mu^2 K_{2424} - 2\lambda\mu K_{1234} \\
&\quad - \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2}\sum_{i,j=1}^{n} K_{ijij} \\
&\geq (1+\lambda^2+\mu^2+\lambda^2\mu^2)\left(\frac{n^2-n+3}{4}\tilde{K}_{\min} - \frac{3n^2+4}{8}\tilde{K}_{\max}\right) \\
&\quad - \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2}\left(\tilde{K}_{\max} - \frac{1}{2}\tilde{K}_{\min}\right) \\
&= \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2}\left(\frac{n^2-n+4}{2}\tilde{K}_{\min} - \frac{3n^2+8}{4}\tilde{K}_{\max}\right).
\end{aligned} \tag{4.13}$$

Putting (4.13) into (4.8) yields

$$\begin{aligned}
&R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234} \\
&\geq \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2} \\
&\quad \times \left(R_M - \frac{3n^2+8}{4}\tilde{K}_{\max} + \frac{n^2-n+4}{2}\tilde{K}_{\min} - \frac{n-2}{n-1}|\mathbf{H}|^2\right).
\end{aligned} \tag{4.14}$$

Case 2: $\tilde{K}_{\min} \le 0 \le \tilde{K}_{\max}$. In this case, we have from (4.11) that

$$\begin{aligned}
&K_{1313} + \lambda^2 K_{1414} + \mu^2 K_{2323} + \lambda^2\mu^2 K_{2424} \\
&\quad - \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2} \sum_{i,j=1}^{n} K_{ijij} \\
&\ge (1+\lambda^2+\mu^2+\lambda^2\mu^2)\left(\frac{n^2-n+3}{4}\tilde{K}_{\min} - \frac{3n^2-3n+4}{8}\tilde{K}_{\max}\right) \\
&\quad + \frac{3(1+\lambda^2+\mu^2+\lambda^2\mu^2)}{4}\tilde{K}_{\min} - \frac{3n(1+\lambda^2+\mu^2+\lambda^2\mu^2)}{8}\tilde{K}_{\max} \\
&= (1+\lambda^2+\mu^2+\lambda^2\mu^2)\left(\frac{n^2-n+6}{4}\tilde{K}_{\min} - \frac{3n^2+4}{8}\tilde{K}_{\max}\right).
\end{aligned}$$

By (4.3) and (4.4), we have

$$\tilde{K}_{\min} - \tilde{K}_{\max} \le K_{1234} \le \tilde{K}_{\max} - \tilde{K}_{\min}. \tag{4.15}$$

Therefore, we have

$$\begin{aligned}
&K_{1313} + \lambda^2 K_{1414} + \mu^2 K_{2323} + \lambda^2\mu^2 K_{2424} - 2\lambda\mu K_{1234} \\
&\quad - \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2} \sum_{i,j=1}^{n} K_{ijij} \\
&\ge (1+\lambda^2+\mu^2+\lambda^2\mu^2)\left(\frac{n^2-n+6}{4}\tilde{K}_{\min} - \frac{3n^2+4}{8}\tilde{K}_{\max}\right) \\
&\quad - \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2}\left(\tilde{K}_{\max} - \tilde{K}_{\min}\right) \\
&= \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2}\left(\frac{n^2-n+8}{2}\tilde{K}_{\min} - \frac{3n^2+8}{4}\tilde{K}_{\max}\right).
\end{aligned} \tag{4.16}$$

Putting (4.16) into (4.8) yields

$$\begin{aligned}
&R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234} \\
&\ge \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2} \\
&\quad \times \left(R_M - \frac{3n^2+8}{4}\tilde{K}_{\max} + \frac{n^2-n+8}{2}\tilde{K}_{\min} - \frac{n-2}{n-1}|\mathbf{H}|^2\right).
\end{aligned} \tag{4.17}$$

Case 3: $\tilde{K}_{\max} \leq 0$. In this case, we have from (4.11) that

$$\begin{aligned}
&K_{1313} + \lambda^2 K_{1414} + \mu^2 K_{2323} + \lambda^2\mu^2 K_{2424} \\
&\quad - \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2} \sum_{i,j=1}^{n} K_{ijij} \\
&\geq (1+\lambda^2+\mu^2+\lambda^2\mu^2)\left(\frac{n^2-n+3}{4}\tilde{K}_{\min} - \frac{3n^2-3n+4}{8}\tilde{K}_{\max}\right) \\
&\quad + \frac{3(1+\lambda^2+\mu^2+\lambda^2\mu^2)}{4}\tilde{K}_{\min} \\
&= (1+\lambda^2+\mu^2+\lambda^2\mu^2)\left(\frac{n^2-n+6}{4}\tilde{K}_{\min} - \frac{3n^2-3n+4}{8}\tilde{K}_{\max}\right).
\end{aligned}$$

By (4.3) and (4.4), we have

$$\tilde{K}_{\min} - \frac{1}{2}\tilde{K}_{\max} \leq K_{1234} \leq \frac{1}{2}\tilde{K}_{\max} - \tilde{K}_{\min}. \tag{4.18}$$

Therefore, we have

$$\begin{aligned}
&K_{1313} + \lambda^2 K_{1414} + \mu^2 K_{2323} + \lambda^2\mu^2 K_{2424} - 2\lambda\mu K_{1234} \\
&\geq (1+\lambda^2+\mu^2+\lambda^2\mu^2)\left(\frac{n^2-n+6}{4}\tilde{K}_{\min} - \frac{3n^2-3n+4}{8}\tilde{K}_{\max}\right) \\
&\quad - \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2}\left(\frac{1}{2}\tilde{K}_{\max} - \tilde{K}_{\min}\right) \\
&= \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2}\left(\frac{n^2-n+8}{2}\tilde{K}_{\min} - \frac{3n^2-3n+6}{4}\tilde{K}_{\max}\right).
\end{aligned} \tag{4.19}$$

Putting (4.19) into (4.8) yields

$$\begin{aligned}
&R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234} \\
&\geq \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2} \\
&\quad \times \left(R_M - \frac{3(n^2-n+2)}{4}\tilde{K}_{\max} + \frac{n^2-n+8}{2}\tilde{K}_{\min} - \frac{n-2}{n-1}|\mathbf{H}|^2\right).
\end{aligned} \tag{4.20}$$

From (4.14), (4.17) and (4.20), we see that in any case, under our assumption (1.2), $M \times \mathbb{R}^2$ always has nonnegative isotropic curvature, i.e.,

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234} \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda, \mu \in [-1, 1]$.

Next, we will estimate the Ricci curvature on M. We will assume that $n \geq 3$. By the Gauss equation (2.4), (3.3), (3.4), (4.7), (4.9) and (4.10), we have for $i \neq j$

(4.21)

$$\begin{aligned} R_{ijij} &= K_{ijij} + \sum_{\alpha=n+1}^{2m} [h_{ii}^{\alpha} h_{jj}^{\alpha} - (h_{ij}^{\alpha})^2] \\ &\geq \frac{1}{2}\left(\frac{3}{2}(1+\langle e_i, Je_j\rangle^2)\tilde{K}_{\min} - \tilde{K}_{\max} + R_M - \sum_{i,j=1}^{n} K_{ijij} - \frac{n-2}{n-1}|\mathbf{H}|^2\right) \\ &\geq \frac{1}{2}\left(R_M - \frac{3n^2-3n+4}{4}\tilde{K}_{\max} + \frac{n^2-n+3}{2}\tilde{K}_{\min} - \frac{n-2}{n-1}|\mathbf{H}|^2\right) \\ &\quad + \frac{1}{2}\left(\frac{3}{2}\langle e_i, Je_j\rangle^2 \tilde{K}_{\min} - \frac{3}{4}\sum_{i,j=1}^{n}\langle e_i, Je_j\rangle^2 \tilde{K}_{\max}\right), \end{aligned}$$

with the first equality holds only if

$$h_{kl}^{\alpha} = 0, \quad \textit{for all } k \neq l, \{k,l\} \neq \{i,j\} \textit{ and any } \alpha \tag{4.22}$$

and

$$h_{kk}^{\alpha} = h_{ii}^{\alpha} + h_{jj}^{\alpha}, \quad \textit{for all } k \neq i, j, \textit{ and any } \alpha. \tag{4.23}$$

We will also consider three cases:

Case 1: $\tilde{K}_{\min} \geq 0$. In this case, we have from (4.21) and the assumption (1.2) that

$$R_{ijij} \geq \frac{1}{2}\left(R_M - \frac{3n^2+4}{4}\tilde{K}_{\max} + \frac{n^2-n+3}{2}\tilde{K}_{\min} - \frac{n-2}{n-1}|\mathbf{H}|^2\right)$$

with equality holds only if (4.22) and (4.23) hold. In particular, we see that for any $1 \leq i \leq n$,

$$Ric_{ii} \geq \frac{n-1}{2}\left(R_M - \frac{3n^2+4}{4}\tilde{K}_{\max} + \frac{n^2-n+3}{2}\tilde{K}_{\min} - \frac{n-2}{n-1}|\mathbf{H}|^2\right),$$

with equality holds only if

$$h_{ii}^{\alpha} = 0, \; h_{kl}^{\alpha} = 0, \quad \textit{for all } k \neq l, \textit{ and any } \alpha$$

and

$$h^\alpha_{kk} = h^\alpha_{ll}, \quad for\ all\ k, l \neq i,\ and\ any\ \alpha,$$

which implies

$$|\mathbf{B}|^2 = \frac{|\mathbf{H}|^2}{n-1}. \tag{4.24}$$

By assumption (1.2), we have

$$Ric_{ii} \geq \frac{n-1}{2}\left(\tilde{K}_{\max} - \frac{1}{2}\tilde{K}_{\min}\right).$$

Case 2: $\tilde{K}_{\min} \leq 0 \leq \tilde{K}_{\max}$. In this case, similar arguments as above shows that

$$Ric_{ii} \geq \frac{n-1}{2}\left(\tilde{K}_{\max} - \tilde{K}_{\min}\right),$$

with equality holds only if (4.24) holds.

Case 3: $\tilde{K}_{\max} \leq 0$. In this case, we have from (4.21) and the assumption (1.2) that

$$Ric_{ii} \geq \frac{n-1}{2}\left(\frac{1}{2}\tilde{K}_{\max} - \tilde{K}_{\min}\right),$$

with equality holds only if (4.24) holds.

If $\tilde{K}_{\max}$ and $\tilde{K}_{\min}$ are not both zero, then we can easily see from above that Ric_M is positive everywhere on M.

If $\tilde{K}_{\max} = \tilde{K}_{\min} = 0$, then by assumption, M has nonnegative Ricci curvature everywhere and has positive Ricci curvature at least at some point. By Aubin's theorem (Lemma 2.4), M admits a metric with positive Ricci curvature. Now we can finish the proof of the theorem:

If $n = 2$, then by our assumption (1.2), we see that M has nonnegative Gauss curvature and has positive Gauss curvature at least at some point. Hence M is diffeomorphic to $\mathbb{S}^2$ or $\mathbb{R}P^2$. In particular, since M is simply connected, M is diffeomorphic to $\mathbb{S}^2$.

If $n = 3$, then from the above argument, M admits a metric with positive Ricci curvature. Therefore, M admits a metric with constant positive sectional curvature by Hamilton's theorem ([8]). Hence, M is diffeomorphic to a spherical space form. Since M is simply connected, M is diffeomorphic to $\mathbb{S}^3$.

If $n \geq 4$, then $M \times \mathbb{R}^2$ has nonnegative isotropic curvature. On the other hand, putting $\lambda = \mu = 1$ in (4.6) and from the above arguments (by considering three cases), we see that under our assumption (1.2), we have

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$.

Claim. *M has nonnegative isotropic curvature and has positive isotropic curvature at some point x_0 on M.*

Proof of the claim. We will also consider three cases according to the sign of the holomorphic sectional curvature as above.

If $\tilde{K}_{\min} \geq 0$, then we have from (4.14), (3.3) and (3.4) that

$$\begin{aligned}
&R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\
&\geq 2\left(R_M - \frac{3n^2+8}{4}\tilde{K}_{\max} + \frac{n^2-n+4}{2}\tilde{K}_{\min} - \frac{n-2}{n-1}|\mathbf{H}|^2\right) \\
&\geq 0,
\end{aligned}$$

with the first equality holds only if $h^\alpha_{ij} = 0$ for all $1 \leq i, j \leq n$. We will show that if $\tilde{K}_{\max} \neq \tilde{K}_{\min}$ at some point $p \in M$, then the first equality cannot achieve at p. Actually, if the first equality holds at p, then we have at p that $R_M = \sum_{i,j=1}^n K_{ijij}$ by (2.5), since p is a totally geodesic point. Now our assumption reduces to

$$\sum_{i,j=1}^n K_{ijij} \geq \frac{3n^2+8}{4}\tilde{K}_{\max} - \frac{n^2-n+4}{2}\tilde{K}_{\min}.$$

Using (4.10), we compute

$$\frac{3n^2+8}{4}\tilde{K}_{\max} - \frac{n^2-n+4}{2}\tilde{K}_{\min} \leq \sum_{i,j=1}^n K_{ijij} \leq \frac{3n^2}{4}\tilde{K}_{\max} - \frac{n^2-n}{2}\tilde{K}_{\min},$$

which implies that $\tilde{K}_{\max} = \tilde{K}_{\min}$, contradicting to our assumption. Therefore, if $\tilde{K}_{\max} \neq \tilde{K}_{\min}$ at p, them M has positive isotropic curvature at p. If $\tilde{K}_{\max} = \tilde{K}_{\min}$ at p, then M has also positive isotropic curvature at p by assumption.

The proof of the other two cases are similar and we omit the details here. This completes the proof of the claim. □

By Lemma 2.7 and the above claim, M admits a metric with positive isotropic curvature, and hence M is homeomorphic to a sphere by Micallef-Moore's theorem (Lemma 2.5). In particular, M is locally irreducible. Now Brendle-Schoen's theorem (Theorem 2.8) applying to M gives us that M is either diffeomorphic to a round sphere $\mathbb{S}^n$, or is a Kähler manifold biholomorphic to complex projective space, or is isometric to a compact symmetric space. Since, M admits a metric with positive isotropic curvature, Lemma 2.6 shows that $b_2(M) = 0$ if M has even dimension, and hence M cannot be a Kähler manifold. Furthermore, Seshadri ([16]) proved that any locally symmetric metric on M must be of constant sectional curvature. Thus, we have shown that M must be diffeomorphic to a round sphere $\mathbb{S}^n$. This finishes the proof of the theorem. □

From the proof of Theorem A, we can easily see that the assumption of Theorem A can be weaken if the submanifold is totally real, which is given by Corollary 1.1.

Proof of Corollary 1.1. We choose any orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ on M. Since M is totally real in N, we see that Je_i is normal to TM for any $1 \le i \le 4$. Therefore, we have by (4.9) and (4.10) that for $1 \le i, j \le 4$

$$K_{ijij} \ge \frac{3}{4}\tilde{K}_{\min} - \frac{1}{2}\tilde{K}_{\max}, \tag{4.25}$$

and

$$K_{ijij} \le \frac{3}{4}\tilde{K}_{\max} - \frac{1}{2}\tilde{K}_{\min},$$

Also by (4.3) and (4.4) we have that

$$\frac{1}{2}(\tilde{K}_{\min} - \tilde{K}_{\max}) \le K_{1234} \le \frac{1}{2}(\tilde{K}_{\max} - \tilde{K}_{\min}). \tag{4.26}$$

From (4.8), (4.25) and (4.26), we see that

$$\begin{aligned}
&R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234} \\
\ge\ &\frac{1 + \lambda^2 + \mu^2 + \lambda^2\mu^2}{2} \\
&\times \left(R_M - \frac{3(n^2 - n + 2)}{4}\tilde{K}_{\max} + \frac{n^2 - n + 4}{2}\tilde{K}_{\min} - \frac{n-2}{n-1}|\mathbf{H}|^2 \right).
\end{aligned}$$

The remaining part of the proof is similar to that of the proof of Theorem A and we omit the details. We only need to notice that in order to show that the isotropic curvature is nonnegative everywhere and positive at some point on M, we have

$$\begin{aligned}&R_{1313}+R_{1414}+R_{2323}+R_{2424}-2R_{1234}\\&\geq 2\left(R_M-\frac{3(n^2-n+2)}{4}\tilde{K}_{\max}+\frac{n^2-n+4}{2}\tilde{K}_{\min}-\frac{n-2}{n-1}|\mathbf{H}|^2\right)\\&\geq 0,\end{aligned}$$

with the first equality holds at $p\in M$ only if p is a totally geodesic point. Then at p, we have $R_M=\sum_{i,j=1}^n K_{ijij}$, and our assumption reduces to

$$\begin{aligned}&\frac{3(n^2-n+2)}{4}\tilde{K}_{\max}-\frac{n^2-n+4}{2}\tilde{K}_{\min}\\&\leq \sum_{i,j=1}^n K_{ijij}\leq n(n-1)\left(\frac{3}{4}\tilde{K}_{\max}-\frac{1}{2}\tilde{K}_{\min}\right),\end{aligned}$$

which implies that $\tilde{K}_{\max}\leq \frac{4}{3}\tilde{K}_{\min}$. But at p we also have

$$\begin{aligned}&R_{1313}+R_{1414}+R_{2323}+R_{2424}-2R_{1234}\\&=K_{1313}+K_{1414}+K_{2323}+K_{2424}-2K_{1234}\\&\geq 4\tilde{K}_{\min}-3\tilde{K}_{\max},\end{aligned}$$

which implies that $\frac{4}{3}\tilde{K}_{\min}\leq\tilde{K}_{\max}$ if $R_{1313}+R_{1414}+R_{2323}+R_{2424}-2R_{1234}=0$. Therefore, $\tilde{K}_{\max}=\tilde{K}_{\min}$ at p. This finished the proof of the corollary. □

Proof of Corollary 1.2. As in the proof of Corollary 1.1, we have

$$\begin{aligned}&R_{1313}+\lambda^2R_{1414}+\mu^2R_{2323}+\lambda^2\mu^2R_{2424}-2\lambda\mu R_{1234}\\&\geq\frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{2}\left(R_M-\frac{n^2-n-2}{4}c-\frac{n-2}{n-1}|\mathbf{H}|^2\right).\end{aligned}$$

It suffices to estimate the isotropic curvature of M. By taking $\lambda=\mu=1$, we obtain

$$\begin{aligned}&R_{1313}+R_{1414}+R_{2323}+R_{2424}-2R_{1234}\\&\geq 2\left(R_M-\frac{n^2-n-2}{4}c-\frac{n-2}{n-1}|\mathbf{H}|^2\right)\geq 0,\end{aligned}$$

with the first equality holds at $p\in M$ only if p is a totally geodesic point. Then at p, we have $R_M=\sum_{i,j=1}^n K_{ijij}=\frac{n(n-1)c}{4}$. We conclude that $c\geq 0$.

However, at p,

$$\begin{aligned}&R_{1313}+R_{1414}+R_{2323}+R_{2424}-2R_{1234}\\ &=K_{1313}+K_{1414}+K_{2323}+K_{2424}-2K_{1234}\geq c.\end{aligned}$$

Therefore, if $c\neq 0$, then the isotropic curvature of M is positive everywhere. If $c=0$, then by assumption M has nonnegative isotropic curvature and has positive isotropic curvature at some point x_0 on M. The remaining part of the proof is similar to that of Theorem A. □

5. Proof of Theorem B

In this section, we will consider differentiable sphere theorem for compact submanifolds in Kähler manifold under the Ricci curvature pinching condition.

Proof of Theorem B. We will show that under our assumption, $M\times\mathbb{R}^2$ has nonnegative isotropic curvature, i.e., (2.6) holds for all orthonormal four-frames $\{e_1,e_2,e_3,e_4\}$ and all $\lambda,\mu\in[-1,1]$. As in the proof of Theorem A, we first extend the four-frame $\{e_1,e_2,e_3,e_4\}$ to be an orthonormal frame $\{e_1,\ldots,e_{2m}\}$ of N such that $\{e_1,\ldots,e_n\}$ are tangent to M and $\{e_{n+1},\ldots,e_{2m}\}$ are normal to M. Define the operator $\tilde{R}$ by (4.5), which is an algebraic curvature. Then for any $1\leq i<j\leq n$, we have from (4.10) that

$$\begin{aligned}\sum_{k=1}^{n}\tilde{R}_{ikik}+\sum_{k=1}^{n}\tilde{R}_{jkjk}&=Ric_{ii}+Ric_{jj}-\sum_{k=1}^{n}K_{ikik}-\sum_{k=1}^{n}K_{jkjk}\\ &\geq Ric^{[2]}_{\min}-(n-1)\left(\frac{3}{2}\tilde{K}_{\max}-\tilde{K}_{\min}\right)\\ &\quad-\frac{3}{4}\sum_{k=1}^{n}\left(\langle e_i,Je_k\rangle^2+\langle e_j,Je_k\rangle^2\right)\tilde{K}_{\max}.\end{aligned}\tag{5.1}$$

Now we will consider three cases:

Case 1: $\tilde{K}_{\min}\geq 0$. In this case, we have from (5.1) that

$$\sum_{k=1}^{n}\tilde{R}_{ikik}+\sum_{k=1}^{n}\tilde{R}_{jkjk}\geq Ric^{[2]}_{\min}-\frac{3n}{2}\tilde{K}_{\max}+(n-1)\tilde{K}_{\min}.$$

By taking $2D = Ric^{[2]}_{\min} - \frac{3n}{2}\tilde{K}_{\max} + (n-1)\tilde{K}_{\min}$ in Lemma 3.5, we obtain for every $0 < \varepsilon \leq 1$ and all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$

$$\tilde{R}_{1212} + \tilde{R}_{1234} \geq \frac{1}{2\varepsilon}\left[Ric^{[2]}_{\min} - \frac{3n}{2}\tilde{K}_{\max} + (n-1)\tilde{K}_{\min} - \delta(\varepsilon, n)|\mathbf{H}|^2\right],$$

where $\delta(\varepsilon, n) = \frac{((n-4)\varepsilon+2)^2}{4(2+(n^2-4n+2)\varepsilon)}$. Lemma 3.2 implies that for every $\lambda, \mu \in [-1, 1]$ and every orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$

$$\begin{aligned}&\tilde{R}_{1313} + \lambda^2\tilde{R}_{1414} + \mu^2\tilde{R}_{2323} + \lambda^2\mu^2\tilde{R}_{2424} - 2\lambda\mu\tilde{R}_{1234}\\ \geq\ &\frac{(1+\lambda^2)(1+\mu^2)}{2\varepsilon}\left[Ric^{[2]}_{\min} - \frac{3n}{2}\tilde{K}_{\max} + (n-1)\tilde{K}_{\min} - \delta(\varepsilon, n)|\mathbf{H}|^2\right],\end{aligned}$$

i.e.,

(5.2)
$$\begin{aligned}&2\varepsilon(R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234})\\ \geq\ &2\varepsilon(K_{1313} + \lambda^2 K_{1414} + \mu^2 K_{2323} + \lambda^2\mu^2 K_{2424} - 2\lambda\mu K_{1234})\\ &+ (1+\lambda^2)(1+\mu^2)\left[Ric^{[2]}_{\min} - \frac{3n}{2}\tilde{K}_{\max} + (n-1)\tilde{K}_{\min} - \delta(\varepsilon, n)|\mathbf{H}|^2\right].\end{aligned}$$

Since $\tilde{K}_{\min} \geq 0$, we have from (4.9) and (4.12) that

$$\begin{aligned}&K_{1313} + \lambda^2 K_{1414} + \mu^2 K_{2323} + \lambda^2\mu^2 K_{2424} - 2\lambda\mu K_{1234}\\ \geq\ &(1+\lambda^2)(1+\mu^2)\left(\frac{3}{4}\tilde{K}_{\min} - \frac{1}{2}\tilde{K}_{\max}\right)\\ &- \frac{(1+\lambda^2)(1+\mu^2)}{2}\left(\tilde{K}_{\max} - \frac{1}{2}\tilde{K}_{\min}\right)\\ =\ &(1+\lambda^2)(1+\mu^2)\left(\tilde{K}_{\min} - \tilde{K}_{\max}\right).\end{aligned} \tag{5.3}$$

Inserting (5.3) into (5.2), we have

$$\begin{aligned}&2\varepsilon(R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234})\\ \geq\ &(1+\lambda^2)(1+\mu^2)\\ &\times\left[Ric^{[2]}_{\min} - \frac{3n+4\varepsilon}{2}\tilde{K}_{\max} + (n-1+2\varepsilon)\tilde{K}_{\min} - \delta(\varepsilon, n)|\mathbf{H}|^2\right]\\ \geq\ &0,\end{aligned}$$

the strict inequality holds for some point $x_0 \in M$, where the last inequality follows from our assumption (1.3). The same argument as in the proof of Theorem A implies that M is diffeomorphic to $\mathbb{S}^n$.

Case 2: $\tilde{K}_{\min} \leq 0 \leq \tilde{K}_{\max}$. In this case, following the same argument as Case 1, we also have (5.2). By (4.9) and (4.15), we have

$$\begin{aligned}(5.4)\qquad & K_{1313} + \lambda^2 K_{1414} + \mu^2 K_{2323} + \lambda^2\mu^2 K_{2424} - 2\lambda\mu K_{1234} \\ &\geq (1+\lambda^2)(1+\mu^2)\left(\frac{3}{2}\tilde{K}_{\min} - \frac{1}{2}\tilde{K}_{\max}\right) \\ &\quad - \frac{(1+\lambda^2)(1+\mu^2)}{2}\left(\tilde{K}_{\max} - \tilde{K}_{\min}\right) \\ &= (1+\lambda^2)(1+\mu^2)\left(2\tilde{K}_{\min} - \tilde{K}_{\max}\right).\end{aligned}$$

Inserting (5.4) into (5.2), we have

$$\begin{aligned}& 2\varepsilon(R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234}) \\ &\geq (1+\lambda^2)(1+\mu^2) \\ &\quad \times \left[Ric^{[2]}_{\min} - \frac{3n+4\varepsilon}{2}\tilde{K}_{\max} + (n-1+4\varepsilon)\tilde{K}_{\min} - \delta(\varepsilon,n)|\mathbf{H}|^2\right] \\ &\geq 0,\end{aligned}$$

the strict inequality holds for some point $x_0 \in M$, where the last inequality follows from our assumption (1.3). The same argument as in the proof of Theorem A implies that M is diffeomorphic to $\mathbb{S}^n$.

Case 3: $\tilde{K}_{\max} \leq 0$. In this case, we have from (5.1) that

$$\sum_{k=1}^{n} \tilde{R}_{ikik} + \sum_{k=1}^{n} \tilde{R}_{jkjk} \geq Ric^{[2]}_{\min} - \frac{3(n-1)}{2}\tilde{K}_{\max} + (n-1)\tilde{K}_{\min}.$$

By taking $2D = Ric^{[2]}_{\min} - \frac{3(n-1)}{2}\tilde{K}_{\max} + (n-1)\tilde{K}_{\min}$ in Lemma 3.5, we obtain for every $0 < \varepsilon \leq 1$ and all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$

$$\tilde{R}_{1212} + \tilde{R}_{1234} \geq \frac{1}{2\varepsilon}\left[Ric^{[2]}_{\min} - \frac{3(n-1)}{2}\tilde{K}_{\max} + (n-1)\tilde{K}_{\min} - \delta(\varepsilon,n)|\mathbf{H}|^2\right].$$

Lemma 3.2 implies that for every orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and every $\lambda, \mu \in [-1, 1]$

$$\begin{aligned}
&\tilde{R}_{1313} + \lambda^2 \tilde{R}_{1414} + \mu^2 \tilde{R}_{2323} + \lambda^2\mu^2 \tilde{R}_{2424} - 2\lambda\mu \tilde{R}_{1234} \\
\geq\ & \frac{(1+\lambda^2)(1+\mu^2)}{2\varepsilon} \\
&\times \left[Ric^{[2]}_{\min} - \frac{3(n-1)}{2}\tilde{K}_{\max} + (n-1)\tilde{K}_{\min} - \delta(\varepsilon, n)|\mathbf{H}|^2 \right],
\end{aligned}$$

i.e.,

$$\begin{aligned}
&2\varepsilon(R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234}) \\
&\geq 2\varepsilon(K_{1313} + \lambda^2 K_{1414} + \mu^2 K_{2323} + \lambda^2\mu^2 K_{2424} - 2\lambda\mu K_{1234}) \\
&\quad + (1+\lambda^2)(1+\mu^2) \\
&\quad \times \left[Ric^{[2]}_{\min} - \frac{3(n-1)}{2}\tilde{K}_{\max} + (n-1)\tilde{K}_{\min} - \delta(\varepsilon, n)|\mathbf{H}|^2 \right].
\end{aligned} \tag{5.5}$$

Since $\tilde{K}_{\max} \leq 0$, we have from (4.9) and (4.18) that

$$\begin{aligned}
&K_{1313} + \lambda^2 K_{1414} + \mu^2 K_{2323} + \lambda^2\mu^2 K_{2424} - 2\lambda\mu K_{1234} \\
&\geq (1+\lambda^2)(1+\mu^2)\left(\frac{3}{2}\tilde{K}_{\min} - \frac{1}{2}\tilde{K}_{\max}\right) \\
&\quad - \frac{(1+\lambda^2)(1+\mu^2)}{2}\left(\frac{1}{2}\tilde{K}_{\max} - \tilde{K}_{\min}\right) \\
&= (1+\lambda^2)(1+\mu^2)\left(2\tilde{K}_{\min} - \frac{3}{4}\tilde{K}_{\max}\right).
\end{aligned} \tag{5.6}$$

Inserting (5.6) into (5.5), we have

$$\begin{aligned}
&2\varepsilon(R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234}) \\
&\geq (1+\lambda^2)(1+\mu^2) \\
&\quad \times \left[Ric^{[2]}_{\min} - \frac{3(n-1+\varepsilon)}{2}\tilde{K}_{\max} + (n-1+4\varepsilon)\tilde{K}_{\min} - \delta(\varepsilon, n)|\mathbf{H}|^2 \right] \\
&\geq 0,
\end{aligned}$$

the strict inequality holds for some point $x_0 \in M$, where the last inequality follows from our assumption (1.3). The same argument as in the proof of Theorem A implies that M is diffeomorphic to $\mathbb{S}^n$. This finishes the proof of the theorem. $\square$

Proof of Corollary 1.3. Let M^n be a totally real submanifold of a Kähler manifold N^{2m}. Using the notations as in the proof of Theorem B, we have from (5.1) that

$$\sum_{k=1}^{n} \tilde{R}_{ikik} + \sum_{k=1}^{n} \tilde{R}_{jkjk} \geq Ric^{[2]}_{\min} - (n-1)\left(\frac{3}{2}\tilde{K}_{\max} - \tilde{K}_{\min}\right).$$

By taking $2D = Ric^{[2]}_{\min} - \frac{3(n-1)}{2}\tilde{K}_{\max} + (n-1)\tilde{K}_{\min}$ in Lemma 3.5, we obtain for every $0 < \varepsilon \leq 1$ and all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$

$$\tilde{R}_{1212} + \tilde{R}_{1234} \geq \frac{1}{2\varepsilon}\left[Ric^{[2]}_{\min} - \frac{3(n-1)}{2}\tilde{K}_{\max} + (n-1)\tilde{K}_{\min} - \delta(\varepsilon, n)|\mathbf{H}|^2\right].$$

Lemma 3.2 implies that for every orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and every $\lambda, \mu \in [-1, 1]$

$$\begin{aligned} &\tilde{R}_{1313} + \lambda^2\tilde{R}_{1414} + \mu^2\tilde{R}_{2323} + \lambda^2\mu^2\tilde{R}_{2424} - 2\lambda\mu\tilde{R}_{1234} \\ \geq &\frac{(1+\lambda^2)(1+\mu^2)}{2\varepsilon}\left[Ric^{[2]}_{\min} - \frac{3(n-1)}{2}\tilde{K}_{\max} + (n-1)\tilde{K}_{\min} - \delta(\varepsilon, n)|\mathbf{H}|^2\right], \end{aligned}$$

i.e.,

$$\begin{aligned} (5.7) \qquad &2\varepsilon(R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234}) \\ &\geq 2\varepsilon(K_{1313} + \lambda^2 K_{1414} + \mu^2 K_{2323} + \lambda^2\mu^2 K_{2424} - 2\lambda\mu K_{1234}) \\ &\quad + (1+\lambda^2)(1+\mu^2) \\ &\qquad \times \left[Ric^{[2]}_{\min} - \frac{3(n-1)}{2}\tilde{K}_{\max} + (n-1)\tilde{K}_{\min} - \delta(\varepsilon, n)|\mathbf{H}|^2\right]. \end{aligned}$$

By (4.25) and (4.26), we have

$$\begin{aligned} (5.8) \qquad &K_{1313} + \lambda^2 K_{1414} + \mu^2 K_{2323} + \lambda^2\mu^2 K_{2424} - 2\lambda\mu K_{1234} \\ &\geq (1+\lambda^2)(1+\mu^2)\left(\frac{3}{4}\tilde{K}_{\min} - \frac{1}{2}\tilde{K}_{\max}\right) \\ &\quad - \frac{(1+\lambda^2)(1+\mu^2)}{2}\left(\frac{1}{2}\tilde{K}_{\max} - \frac{1}{2}\tilde{K}_{\min}\right) \\ &= (1+\lambda^2)(1+\mu^2)\left(\tilde{K}_{\min} - \frac{3}{4}\tilde{K}_{\max}\right). \end{aligned}$$

Inserting (5.8) into (5.7), we have

$$\begin{aligned}
&2\varepsilon(R_{1313}+\lambda^2R_{1414}+\mu^2R_{2323}+\lambda^2\mu^2R_{2424}-2\lambda\mu R_{1234})\\
&\geq(1+\lambda^2)(1+\mu^2)\\
&\times\left[Ric^{[2]}_{\min}-\frac{3(n-1+\varepsilon)}{2}\tilde{K}_{\max}+(n-1+2\varepsilon)\tilde{K}_{\min}-\delta(\varepsilon,n)|\mathbf{H}|^2\right]\\
&\geq 0,
\end{aligned}$$

the strict inequality holds for some point $x_0\in M$, where the last inequality follows from our assumption (1.3). The same argument as in the proof of Theorem A implies that M is diffeomorphic to $\mathbb{S}^n$. This finishes the proof of the corollary. □

6. Proof of Theorem C and Theorem D

In this section, we will prove the topological sphere theorem for submanifolds in Kähler manifold.

Proof of Theorem C. As before, we will show that under our assumption, $M\times\mathbb{R}^2$ has nonnegative isotropic curvature. For any orthonormal four-frame $\{e_1,e_2,e_3,e_4\}$, we first extend it to be an orthonormal frame $\{e_1,\dots,e_{2m}\}$ of N such that $\{e_1,\dots,e_n\}$ are tangent to M and $\{e_{n+1},\dots,e_{2m}\}$ are normal to M. The tensor $\tilde{R}$ defined by (4.5) is an algebraic curvature. Then (3.2) and (3.5), (3.6) implie that

$$\sum_{i=1}^{2}\sum_{j=3}^{4}\tilde{R}_{ijij}-2\tilde{R}_{1234}\geq\frac{\sum_{\alpha=1}^{p}(H^\alpha)^2}{n-2}-\sum_{i,j=1}^{n}\sum_{\alpha=1}^{p}\left(h^\alpha_{ij}\right)^2=\frac{|\mathbf{H}|^2}{n-2}-|\mathbf{B}|^2.$$

i.e.,

$$\begin{aligned}
&R_{1313}+R_{1414}+R_{2323}+R_{2424}-2R_{1234}\\
&\geq K_{1313}+K_{1414}+K_{2323}+K_{2424}-2K_{1234}+\frac{|\mathbf{H}|^2}{n-2}-|\mathbf{B}|^2.
\end{aligned}\tag{6.1}$$

Putting (4.7) into (6.1) yields

$$\begin{aligned}
&R_{1313}+R_{1414}+R_{2323}+R_{2424}-2R_{1234}\\
&\geq K_{1313}+K_{1414}+K_{2323}+K_{2424}-2K_{1234}\\
&+R_M-\frac{n-3}{n-2}|\mathbf{H}|^2-\sum_{i,j=1}^{n}K_{ijij}.
\end{aligned}\tag{6.2}$$

Therefore, it suffices to estimate the terms involving the curvature tensor K on N. As in the proof of Theorem A, we will consider three cases:

Case 1: $\tilde{K}_{\min} \geq 0$. In this case, we have from (4.9), (4.10) and (4.12) that

$$
\begin{aligned}
&K_{1313} + K_{1414} + K_{2323} + K_{2424} - 2K_{1234} - \sum_{i,j=1}^{n} K_{ijij} \\
&\geq 4\left(\frac{3}{4}\tilde{K}_{\min} - \frac{1}{2}\tilde{K}_{\max}\right) - 2\left(\tilde{K}_{\max} - \frac{1}{2}\tilde{K}_{\min}\right) \\
&\quad - n(n-1)\left(\frac{3}{4}\tilde{K}_{\max} - \frac{1}{2}\tilde{K}_{\min}\right) - \frac{3}{4}\sum_{i,j=1}^{n}\langle e_i, Je_j\rangle^2 \tilde{K}_{\max} \\
&\geq \frac{n^2 - n + 8}{2}\tilde{K}_{\min} - \frac{3n^2 + 16}{4}\tilde{K}_{\max}.
\end{aligned} \tag{6.3}
$$

Putting (6.3) into (6.2) yields

$$
\begin{aligned}
&R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\
&\geq R_M - \frac{3n^2 + 16}{4}\tilde{K}_{\max} + \frac{n^2 - n + 8}{2}\tilde{K}_{\min} - \frac{n-3}{n-2}|\mathbf{H}|^2.
\end{aligned} \tag{6.4}
$$

Case 2: $\tilde{K}_{\min} \leq 0 \leq \tilde{K}_{\max}$. In this case, we have from (4.9), (4.10) and (4.15) that

$$
\begin{aligned}
&K_{1313} + K_{1414} + K_{2323} + K_{2424} - 2K_{1234} - \sum_{i,j=1}^{n} K_{ijij} \\
&\geq 4\left(\frac{3}{2}\tilde{K}_{\min} - \frac{1}{2}\tilde{K}_{\max}\right) - 2\left(\tilde{K}_{\max} - \tilde{K}_{\min}\right) \\
&\quad - n(n-1)\left(\frac{3}{2}\tilde{K}_{\max} - \frac{1}{2}\tilde{K}_{\min}\right) - \frac{3}{4}\sum_{i,j=1}^{n}\langle e_i, Je_j\rangle^2 \tilde{K}_{\max} \\
&= \frac{n^2 - n + 16}{2}\tilde{K}_{\min} - \frac{3n^2 + 16}{4}\tilde{K}_{\max}.
\end{aligned} \tag{6.5}
$$

Putting (6.5) into (6.2) yields

$$
\begin{aligned}
&R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\
&\geq R_M - \frac{3n^2 + 16}{4}\tilde{K}_{\max} + \frac{n^2 - n + 16}{2}\tilde{K}_{\min} - \frac{n-3}{n-2}|\mathbf{H}|^2.
\end{aligned} \tag{6.6}
$$

Case 3: $\tilde{K}_{\max} \leq 0$. In this case, we have from (4.9), (4.10) and (4.18) that

$$
\begin{aligned}
&K_{1313} + K_{1414} + K_{2323} + K_{2424} - 2K_{1234} - \sum_{i,j=1}^{n} K_{ijij} \\
&\geq 4\left(\frac{3}{2}\tilde{K}_{\min} - \frac{1}{2}\tilde{K}_{\max}\right) - 2\left(\frac{1}{2}\tilde{K}_{\max} - \tilde{K}_{\min}\right) \\
&\quad - n(n-1)\left(\frac{3}{4}\tilde{K}_{\max} - \frac{1}{2}\tilde{K}_{\min}\right) \\
&= \frac{n^2 - n + 16}{2}\tilde{K}_{\min} - \frac{3(n^2 - n + 4)}{4}\tilde{K}_{\max}.
\end{aligned} \tag{6.7}
$$

Putting (6.7) into (6.2) yields

$$
\begin{aligned}
&R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\
&\geq R_M - \frac{3(n^2 - n + 4)}{4}\tilde{K}_{\max} + \frac{n^2 - n + 16}{2}\tilde{K}_{\min} - \frac{n-3}{n-2}|\mathbf{H}|^2.
\end{aligned} \tag{6.8}
$$

From (6.4), (6.6) and (6.8), we see that in any case, under our assumption (1.4), M always has nonnegative isotropic curvature and has positive isotropic curvature at some point. By Lemma 2.7, M admits a metric with positive isotropic curvature. Since M is simply connected, M is homeomorphic to $\mathbb{S}^n$ by Lemma 2.5. □

Proof of Corollary 1.6. Let M^n be a totally real submanifold of a Kähler manifold N^{2m}. In this case, (6.2) is still true. By (4.25) and (4.26), we have

$$
\begin{aligned}
&K_{1313} + K_{1414} + K_{2323} + K_{2424} - 2K_{1234} - \sum_{i,j=1}^{n} K_{ijij} \\
&\geq 4\left(\frac{3}{4}\tilde{K}_{\min} - \frac{1}{2}\tilde{K}_{\max}\right) - 2\left(\frac{1}{2}\tilde{K}_{\max} - \frac{1}{2}\tilde{K}_{\min}\right) \\
&\quad - n(n-1)\left(\frac{3}{4}\tilde{K}_{\max} - \frac{1}{2}\tilde{K}_{\min}\right) \\
&= \frac{n^2 - n + 8}{2}\tilde{K}_{\min} - \frac{3(n^2 - n + 4)}{4}\tilde{K}_{\max}.
\end{aligned} \tag{6.9}
$$

Inserting (6.9) into (6.2), we have

$$\begin{aligned}
&R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\
&\geq R_M - \frac{3(n^2-n+4)}{4}\tilde{K}_{\max} + \frac{n^2-n+8}{2}\tilde{K}_{\min} - \frac{n-3}{n-2}|\mathbf{H}|^2 \\
&\geq 0,
\end{aligned}$$

and the strict inequality holds for some point $x_0 \in M$, where the last inequality follows from our assumption. Then the corollary follows from Lemma 2.7 and Lemma 2.5. □

Proof of Theorem D. Using the same notations as in the proof of Theorem B, we have from (4.10)

$$\begin{aligned}
\sum_{i=1}^{4}\sum_{j=1}^{n}\tilde{R}_{ijij} &= \sum_{i=1}^{4} Ric_{ii} - \sum_{i=1}^{4}\sum_{j=1}^{n} K_{ijij} \\
&\geq Ric_{\min}^{[4]} - (n-1)\left(3\tilde{K}_{\max} - 2\tilde{K}_{\min}\right) \\
&\quad - \frac{3}{4}\sum_{i=1}^{4}\sum_{j=1}^{n}\langle e_i, Je_j\rangle^2\tilde{K}_{\max}.
\end{aligned} \tag{6.10}$$

Now we will consider three cases:
Case 1: $\tilde{K}_{\min} \geq 0$. In this case, we have from (6.10) that

$$\sum_{k=1}^{n}\tilde{R}_{ikik} + \sum_{k=1}^{n}\tilde{R}_{jkjk} \geq Ric_{\min}^{[4]} - 3n\tilde{K}_{\max} + 2(n-1)\tilde{K}_{\min}.$$

By taking $4D = Ric_{\min}^{[4]} - 3n\tilde{K}_{\max} + 2(n-1)\tilde{K}_{\min}$ in Lemma 3.6, we obtain for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$,

$$\sum_{i=1}^{2}\sum_{j=3}^{4}\tilde{R}_{ijij} - 2\tilde{R}_{1234} \geq Ric_{\min}^{[4]} - 3n\tilde{K}_{\max} + 2(n-1)\tilde{K}_{\min} - \frac{1}{2}|\mathbf{H}|^2.$$

In other word,

$$\begin{aligned}
&R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\
&\geq K_{1313} + K_{1414} + K_{2323} + K_{2424} - 2K_{1234} \\
&\quad + Ric_{\min}^{[4]} - 3n\tilde{K}_{\max} + 2(n-1)\tilde{K}_{\min} - \frac{1}{2}|\mathbf{H}|^2.
\end{aligned} \tag{6.11}$$

Since $\tilde{K}_{\min} \geq 0$, we have from (4.9) and (4.12) that

$$
\begin{aligned}
&K_{1313} + K_{1414} + K_{2323} + K_{2424} - 2K_{1234} \\
&\geq 4\left(\frac{3}{4}\tilde{K}_{\min} - \frac{1}{2}\tilde{K}_{\max}\right) - 2\left(\tilde{K}_{\max} - \frac{1}{2}\tilde{K}_{\min}\right) \\
&= 4\left(\tilde{K}_{\min} - \tilde{K}_{\max}\right).
\end{aligned}
\tag{6.12}
$$

Inserting (6.12) into (6.11), we have

$$
\begin{aligned}
&R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\
&\geq Ric^{[4]}_{\min} - (3n+4)\tilde{K}_{\max} + 2(n+1)\tilde{K}_{\min} - \frac{1}{2}|\mathbf{H}|^2 \\
&\geq 0
\end{aligned}
$$

the strict inequality holds for some point $x_0 \in M$, where the last inequality follows from our assumption (1.5). By Lemma 2.7, M admits a metric with positive isotropic curvature. Since M is simply connected, M is homeomorphic to $\mathbb{S}^n$ by Lemma 2.5.

Case 2: $\tilde{K}_{\min} \leq 0 \leq \tilde{K}_{\max}$. In this case, following the same argument as Case 1, we also have (6.11). By (4.9) and (4.15), we have

$$
\begin{aligned}
&K_{1313} + K_{1414} + K_{2323} + K_{2424} - 2K_{1234} \\
&\geq 4\left(\frac{3}{2}\tilde{K}_{\min} - \frac{1}{2}\tilde{K}_{\max}\right) - 2\left(\tilde{K}_{\max} - \tilde{K}_{\min}\right) \\
&= 4\left(2\tilde{K}_{\min} - \tilde{K}_{\max}\right).
\end{aligned}
\tag{6.13}
$$

Inserting (6.13) into (6.11), we have

$$
\begin{aligned}
&R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\
&\geq Ric^{[4]}_{\min} - (3n+4)\tilde{K}_{\max} + 2(n+3)\tilde{K}_{\min} - \frac{1}{2}|\mathbf{H}|^2 \\
&\geq 0
\end{aligned}
$$

the strict inequality holds for some point $x_0 \in M$, where the last inequality follows from our assumption (1.5). Then the theorem follows from Lemma 2.7 and Lemma 2.5.

Case 3: $\tilde{K}_{\max} \leq 0$. In this case, we have from (6.10) that

$$\sum_{k=1}^{n} \tilde{R}_{ikik} + \sum_{k=1}^{n} \tilde{R}_{jkjk} \geq Ric_{\min}^{[4]} - 3(n-1)\tilde{K}_{\max} + 2(n-1)\tilde{K}_{\min}.$$

By taking $4D = Ric_{\min}^{[4]} - 3(n-1)\tilde{K}_{\max} + 2(n-1)\tilde{K}_{\min}$ in Lemma 3.6, we obtain for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$,

$$\sum_{i=1}^{2}\sum_{j=3}^{4} \tilde{R}_{ijij} - 2\tilde{R}_{1234} \geq Ric_{\min}^{[4]} - 3(n-1)\tilde{K}_{\max} + 2(n-1)\tilde{K}_{\min} - \frac{1}{2}|\mathbf{H}|^2.$$

In other word,

$$\begin{aligned} &R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\ \geq\ &K_{1313} + K_{1414} + K_{2323} + K_{2424} - 2K_{1234} \\ &+ Ric_{\min}^{[4]} - 3(n-1)\tilde{K}_{\max} + 2(n-1)\tilde{K}_{\min} - \frac{1}{2}|\mathbf{H}|^2. \end{aligned} \tag{6.14}$$

Since $\tilde{K}_{\max} \leq 0$, we have from (4.9) and (4.18) that

$$\begin{aligned} &K_{1313} + K_{1414} + K_{2323} + K_{2424} - 2K_{1234} \\ &\geq 4\left(\frac{3}{2}\tilde{K}_{\min} - \frac{1}{2}\tilde{K}_{\max}\right) - 2\left(\frac{1}{2}\tilde{K}_{\max} - \tilde{K}_{\min}\right) \\ &= 8\tilde{K}_{\min} - 3\tilde{K}_{\max}. \end{aligned} \tag{6.15}$$

Inserting (6.15) into (6.14), we have

$$\begin{aligned} &R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\ &\geq Ric_{\min}^{[4]} - 3n\tilde{K}_{\max} + 2(n+3)\tilde{K}_{\min} - \frac{1}{2}|\mathbf{H}|^2 \\ &\geq 0 \end{aligned}$$

the strict inequality holds for some point $x_0 \in M$, where the last inequality follows from our assumption (1.5). Then the theorem follows from Lemma 2.7 and Lemma 2.5. This finishes the proof of the theorem. □

Proof of Corollary 1.8. Let M^n be a totally real submanifold of a Kähler manifold N^{2m}. Using the notations as in the proof of Theorem D, we have from (6.10) that

$$\sum_{i=1}^{4}\sum_{j=1}^{n}\tilde{R}_{ijij} \geq Ric_{\min}^{[4]} - (n-1)\left(3\tilde{K}_{\max} - 2\tilde{K}_{\min}\right).$$

By taking $4D = Ric_{\min}^{[4]} - (n-1)\left(3\tilde{K}_{\max} - 2\tilde{K}_{\min}\right)$ in Lemma 3.6, we obtain for every orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$

$$\sum_{i=1}^{2}\sum_{j=3}^{4}\tilde{R}_{ijij} - 2\tilde{R}_{1234} \geq Ric_{\min}^{[4]} - 3(n-1)\tilde{K}_{\max} + 2(n-1)\tilde{K}_{\min} - \frac{1}{2}|\mathbf{H}|^2.$$

In other word, (6.14) is true. By (4.25) and (4.26), we have

$$\begin{aligned}
&K_{1313} + K_{1414} + K_{2323} + K_{2424} - 2K_{1234} \\
&\geq 4\left(\frac{3}{4}\tilde{K}_{\min} - \frac{1}{2}\tilde{K}_{\max}\right) - 2\left(\frac{1}{2}\tilde{K}_{\max} - \frac{1}{2}\tilde{K}_{\min}\right) \\
&= 4\tilde{K}_{\min} - 3\tilde{K}_{\max}.
\end{aligned} \tag{6.16}$$

Inserting (6.16) into (6.14), we have

$$\begin{aligned}
&R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\
&\geq Ric_{\min}^{[4]} - 3n\tilde{K}_{\max} + 2(n+1)\tilde{K}_{\min} - \frac{1}{2}|\mathbf{H}|^2 \\
&\geq 0
\end{aligned}$$

the strict inequality holds for some point $x_0 \in M$, where the last inequality follows from our assumption. Then the corollary follows from Lemma 2.7 and Lemma 2.5. □

Acknowledgements

The first author was supported by the National Natural Science Foundation of China (Grant Nos. 11401440, 12071352). Part of the work was finished when the first author was a visiting scholar at MIT supported by China Scholarship Council (CSC) and Wuhan University. The author would like to express his gratitude to Professor Tobias Colding for his invitation, to MIT for their hospitality, and to CSC and Wuhan University for their support. The second author was supported by the National Natural Science Foundation of China (Grant No. 11801420) and Fundamental Research Funds for the Central Universities (Grant No. 2042018kf0044).

The authors would like to thank the referees for their careful reading and useful comments.

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School of Mathematics and Statistics
Wuhan University, 430072 Wuhan, China
Hubei Key Laboratory of Computational Science
Wuhan University, 430072 Wuhan, China
E-mail address: `sunjun@whu.edu.cn`
E-mail address: `sunll@whu.edu.cn`

Received October 17, 2018
Accepted February 25, 2019