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The Eisenbud-Green-Harris conjecture for defect two quadratic ideals

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The Eisenbud-Green-Harris (EGH) conjecture states that a homogeneous ideal in a polynomial ring $K[x_1, \ldots, x_n]$ over a field K that contains a regular sequence $f_1, \ldots, f_n$ with degrees a_i, $i = 1, \ldots, n$ has the same Hilbert function as a lex-plus-powers ideal containing the powers $x_i^{a_i}$, $i = 1, \ldots, n$. In this paper, we discuss a case of the EGH conjecture for homogeneous ideals generated by $n+2$ quadrics containing a regular sequence $f_1, \ldots, f_n$ and give a complete proof for EGH when $n = 5$ and $a_1 = \cdots = a_5 = 2$.

1. Introduction

Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K with the homogeneous lexicographic order in which $x_1 > \cdots > x_n$ and with the standard grading $R = \bigoplus_{i \geq 0} R_i$. We denote the Hilbert function of a $\mathbb{Z}$-graded R-module M by $\mathrm{Hilb}_M(i) := \dim_K M_i$, where M_i is the homogeneous component of M in degree i. When I is a homogeneous ideal of R and M is R, or I, or R/I, the Hilbert function has value 0 when $i < 0$. When the Hilbert function of M is 0 in negative degree, we may discuss the Hilbert function of M by giving the sequence of its values, and we refer to this sequence of integers as the *O-sequence* of M.

In 1927, Macaulay [13] showed that the Hilbert function of any homogeneous ideal of R is attained by a lexicographic ideal in R. Later, in Kruskal-Katona's theorem [11, 12], it is shown that the polynomial ring R in Macaulay's result can be replaced with the quotient $R/(x_1^2, \ldots, x_n^2)$. After this result, Clement and Lindström, in [5], generalized the result to $R/(x_1^{a_1}, \ldots, x_n^{a_n})$ if $a_1 \leq \cdots \leq a_n < \infty$.

In [7] Eisenbud, Green and Harris conjectured a generalization of the Clement-Lindström result. Let $\underline{\mathbf{a}} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, where $2 \leq a_1 \leq \cdots \leq a_n$.

Conjecture 1.1 (Eisenbud-Green-Harris ($\mathrm{EGH}_{\underline{\mathbf{a}},n}$) Conjecture [7]). *If I is a homogeneous ideal in $R = K[x_1, \ldots, x_n]$ containing a regular sequence $f_1, f_2, \ldots, f_n$ with degrees $\deg f_i = a_i$, then there is a monomial ideal $\mathcal{L} = (x_1^{a_1}, \ldots, x_n^{a_n}) + J$, where J is a lexicographic ideal in R, such that $R/\mathcal{L}$ and R/I have the same Hilbert function.*

Although there has been some progress on the conjecture, it remains open. The conjecture is shown to be true for $n = 2$ by Richert in [14]. Francisco [8] shows part of the conjecture in the case of an almost complete intersection: see Theorem 2.3. Caviglia and Maclagan in [2] prove the result if $a_i > \sum_{j=1}^{i-1}(a_j - 1)$ for $2 \leq i \leq n$. The rapid growth required for the degrees does not yield much insight into cases like the one in which the regular sequence consists of quadratic forms. When $n = 3$, Cooper in [6] proves the EGH conjecture for the cases where $(a_1, a_2, a_3) = (2, a_2, a_3)$ and $(a_1, a_2, a_3) = (3, a_2, a_3)$ with $a_2 \leq a_3 \leq a_2 + 1$.

One of the most intriguing cases is when $a_1 = \cdots = a_n = 2$ for any $n \geq 2$, which is the case for which Eisenbud, Green and Harris originally stated their conjecture. It is known that the conjecture holds for homogeneous ideals minimally generated by generic quadrics: the case where $\operatorname{char} K = 0$ was proved by Herzog and Popescu [10] and the case of arbitrary characteristic was proved by Gasharov [9] around the same time. There have been several other results on the EGH conjecture. More recently, the case when every f_i, $i = 1, \ldots, n$, in the regular sequence is a product of linear forms is settled by Abedelfatah in [1], and results on the EGH conjecture using linkage theory are given by Chong [4].

In this paper we focus on the case when the degrees of the elements of the regular sequence are $a_1 = \cdots = a_n = 2$. In [14], Richert claimed that the conjecture for quadratic regular sequences is true for $2 \leq n \leq 5$, but this work has not been published, and other researchers have been unable to verify this for $n = 5$ thus far. Chen, in [3], has given a proof for the case where $n \leq 4$ when $a_1 = \cdots = a_n = 2$.

In §2 we recall some definitions and results from the papers of Francisco [8], Caviglia-Maclagan [2] and Chen [3]. In §3 we study homogeneous ideals I generated by $n+2$ quadratic forms in n variables containing a regular sequence of length n, and Theorem 3.17 shows that there is a monomial ideal $\mathcal{L} = (x_1^2, \ldots, x_n^2) + J$, where J is a lexicographic ideal in R, such that R/I and $R/\mathcal{L}$ have the same Hilbert function in degree 2 and 3 (i.e., $\mathrm{EGH}_{(2,\ldots,2),n}(2)$ holds: see Definition 2.5). In §4 we give a proof to the claim of Richert for the quadratic regular sequence case when $n = 5$.

2. Background and preliminaries

In this section we recall some definitions and state some known results that are used throughout the paper.

Definition 2.1. Let $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$ be monomials in R of the same degree. We say that u is greater than v with respect to the *lexicographic* (or *lex*) order if there exists an i such that $a_i > b_i$ and $a_j = b_j$ for all $j < i$.

A monomial ideal $J \subseteq R$ is called a *lexicographic ideal* (or *lex ideal*) if, for all degrees d, the d-th degree component of J, denoted by J_d, is spanned over the base field K by an initial segment of the degree d monomials in the lexicographic order.

Definition 2.2. Given $2 \le a_1 \le \cdots \le a_n$, a *lex-plus-powers ideal* (LPP ideal) $\mathcal{L}$ is a monomial ideal in R that can be written as $\mathcal{L} = (x_1^{a_1}, \ldots, x_n^{a_n}) + J$ where J is a lex ideal in R.

This definition agrees with the one in [2]. Some authors require that the $x_i^{a_i}$ be minimal generators of $\mathcal{L}$, which we do not. However, since we consider only nondegenerate homogeneous ideals in this paper, i.e., ideals contained in $(x_1, \ldots, x_n)^2$, in the case where $a_1 = \cdots = a_n = 2$ it is automatic that the x_i^2 are minimal generators of the ideal under consideration.

In [8] Francisco showed the following for almost complete intersections.

Theorem 2.3 (Francisco [8]). *Let integers $2 \le a_1 \le a_2 \le \cdots \le a_n$ and $d \ge a_1$ be given. Let the ideal I have minimal generators $f_1, \ldots, f_n, g$ where $f_1, \ldots, f_n$ form a regular sequence with $\deg f_i = a_i$ and g has degree d. Let $\mathcal{L} = (x_1^{a_1}, \ldots, x_n^{a_n}, m)$ be the lex-plus-powers ideal where m is the greatest monomial in lex order in degree d that is not in $(x_1^{a_1}, \ldots, x_n^{a_n})$. Then $\mathrm{Hilb}_{R/I}(d+1) \le \mathrm{Hilb}_{R/\mathcal{L}}(d+1)$.*

Note that, necessarily, $d \le \sum_{i=1}^{n}(a_i - 1)$, since $(f_1, \ldots, f_n)$ contains all forms of degree larger than that. If $a_1 = \cdots = a_n = 2$, then $d \le n$.

The following corollary is an immediate consequence of Theorem 2.3 above. If $g \in R$ is a nonzero form of degree i we write gR_j for the vector space $\{gh : h \in R_j\} \subseteq R_{i+j}$.

Corollary 2.4. *Let $I = (f_1, \ldots, f_n, g)$ be an almost complete intersection as in Theorem 2.3 above such that $a_1 = \cdots = a_n = 2$. Then*

$$\dim_K \big((f_1, \ldots, f_n)_{d+1} \cap gR_1\big) \le d.$$

Proof. We can write

$$\begin{aligned}\dim_K I_{d+1} = \dim_K(f_1, \ldots, f_n)_{d+1} + \dim_K gR_1 \\ - \dim_K \big((f_1, \ldots, f_n)_{d+1} \cap gR_1\big),\end{aligned}$$

where $\dim_K gR_1 = n$. Then by Theorem 2.3, we have

$$\begin{aligned}\dim_K I_{d+1} &\geq \dim_K(x_1^2, \ldots, x_n^2, x_1 \cdots x_d)_{d+1} \\ &= \dim_K(x_1^2, \ldots, x_n^2)_{d+1} + n - d\end{aligned}$$

Since $\mathrm{Hilb}_{R/(f_1, \ldots, f_n)}(i) = \mathrm{Hilb}_{R/(x_1^2, \ldots, x_n^2)}(i)$ for all $i \geq 0$, we can conclude that

$$\dim_K \big((f_1, \ldots, f_n)_{d+1} \cap gR_1\big) \leq d.$$

□

The next statement is a weaker version of the $\mathrm{EGH}_{\underline{\mathbf{a}},n}$ conjecture. It focuses on the Hilbert function of the given homogeneous ideal only at the two consecutive degrees d and $d+1$ for some non-negative integer d.

Definition 2.5 ($\mathrm{EGH}_{\underline{\mathbf{a}},\mathbf{n}}(\mathbf{d})$)**.** Following Caviglia-Maclagan in their paper [2], we say that "$\mathrm{EGH}_{\underline{\mathbf{a}},n}(d)$ holds" if for any homogeneous ideal $I \in K[x_1, \ldots, x_n]$ containing a regular sequence of degrees $\underline{\mathbf{a}} = (a_1, \ldots, a_n)$, where $2 \leq a_1 \leq \cdots \leq a_n$, there exists a lex-plus-powers ideal $\mathcal{L}$ containing $\{x_i^{a_i} : 1 \leq i \leq n\}$ such that

$$\dim_K I_d = \dim_K \mathcal{L}_d \quad \text{and} \quad \dim_K I_{d+1} = \dim_K \mathcal{L}_{d+1}\,.$$

Lemma 2.6. *The condition* $\mathrm{EGH}_{(d, \ldots, d),n}(d)$ *on a polynomial ring* $K[x_1, \ldots, x_n]$ *is equivalent to the statement that for the ideal* I *generated by* $n+\delta$ K*-linearly independent forms of degree* d *containing a regular sequence, one has that* $\dim_K I_{d+1} \geq \dim_K \mathcal{L}_{d+1}$*, where* $\mathcal{L} = (x_1^d, \ldots, x_n^d) + J'$ *and* J' *is minimally generated by the greatest in lex order* δ *forms of degree* d *not already in* $(x_1^d, \ldots, x_n^d)$*.*

Proof. If there is an LPP ideal $(x_1^d, \ldots, x_n^d) + J$, where J is a lex ideal, with the same Hilbert function as I in degrees d and $d+1$, it is clear that J_d must be spanned over K by the specified generators of J', so that $(x_1^d, \ldots, x_n^d) + J' \subseteq (x_1^d, \ldots, x_n^d) + J$, which implies the specified inequality on the Hilbert functions. Moreover, when that inequality holds we may increase $\mathcal{L} := (x_1^d, \ldots, x_n^d) + J'$ to an LPP ideal with the same Hilbert function as I in degrees d and $d+1$: if $\Delta = \mathrm{Hilb}_I(d+1) - \mathrm{Hilb}_{\mathcal{L}}(d+1)$, we may simply include the greatest (in lex order) Δ forms of degree $d+1$ not already in $\mathcal{L}$. $\square$

Remark 2.7. We shall eventually be focused on $\mathrm{EGH}_{\underline{\mathbf{a}},n}(d)$ in the case where $a_1 = \cdots = a_n = d = 2$, simply referred as $\mathrm{EGH}_{(2,\ldots,2),n}(2)$ or $\mathrm{EGH}_{\underline{\mathbf{2}},n}(2)$. We shall routinely make use of this lemma in this case of quadratic regular sequence and $d = 2$.

Lemma 2.8 (Caviglia-Maclagan [2]). *Fix* $\underline{\mathbf{a}} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ *where* $2 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ *and set* $s = \sum_{i=1}^{n}(a_i - 1)$. *Then for any* $0 \leq d \leq s-1$, $\mathrm{EGH}_{\underline{\mathbf{a}},n}(d)$ *holds if and only if* $\mathrm{EGH}_{\underline{\mathbf{a}},n}$*(s-1-d) holds.*
Furthermore, the $\mathrm{EGH}_{\underline{\mathbf{a}},n}$ *conjecture holds if and only if* $\mathrm{EGH}_{\underline{\mathbf{a}},n}(d)$ *holds for all degrees* $d \geq 0$.

From now on, we always assume $\underline{\mathbf{a}} = \underline{\mathbf{2}} = (2, \ldots, 2)$ for $n \geq 2$, unless it is stated otherwise.

Remark 2.9. For any $n \geq 2$, $\mathrm{EGH}_{\underline{\mathbf{2}},n}(0)$ holds trivially. In [3, Proposition 2.1], Chen showed that $\mathrm{EGH}_{\underline{\mathbf{2}},n}(1)$ is true for any $n \geq 2$.

Chen proved the following.

Theorem 2.10 (Chen [3]). *The* $\mathrm{EGH}_{\underline{\mathbf{2}},n}$ *conjecture holds when* $2 \leq n \leq 4$.

Chen's proof of this uses Lemma 2.8 above, and the observation that, when $n = 4$, to demonstrate that the $\mathrm{EGH}_{\underline{\mathbf{2}},4}$ conjecture is true, it suffices to show that $\mathrm{EGH}_{\underline{\mathbf{2}},4}(0)$ and $\mathrm{EGH}_{\underline{\mathbf{2}},4}(1)$ are true.

3. $\mathrm{EGH}_{\underline{\mathbf{2}},n}(2)$ for defect two ideals

In this section, we focus on the homogeneous ideals in $K[x_1, \ldots, x_n]$ for $n \geq 5$ that are generated by $n+2$ quadratic forms containing a regular sequence. In particular, we study their Hilbert functions in degree 3.

Definition 3.1. If I is a homogeneous ideal minimally generated by $n + \delta$ forms that contain a regular sequence of length n, then I is said to be a *defect* δ ideal.

Clearly, when $\delta = 0$ then I is generated by a regular sequence, it is a complete intersection, and we understand the Hilbert function completely. If $\delta = 1$, then I is an almost complete intersection.

Definition 3.2. We call a homogeneous ideal a *quadratic ideal* if it is generated by quadratic forms.

Let $I = (f_1, \ldots, f_n, g, h)$ be a homogeneous ideal minimally generated by $n+2$ quadrics where $f_1, \ldots, f_n$ form a regular sequence. We call such an ideal a *defect two ideal generated by quadrics* or simply a *defect two quadratic ideal.* More generally, if a quadratic ideal is a defect δ ideal, then we call it *defect* δ quadratic ideal.

Example 3.3. The lex-plus-powers ideal $\mathcal{L} = (x_1^2, \ldots, x_n^2, x_1x_2, x_1x_3)$ in R is also a defect two quadratic ideal.

Further, for any homogeneous defect two quadratic ideal I, we have the equality

$$\dim_K I_2 = n + 2 = \dim_K \mathcal{L}_2 \,.$$

Main Question 3.4 ($\mathrm{EGH}_{\underline{2},n}(2)$ **for defect two quadratic ideals).** For any $n \geq 5$, is it true that

$$\dim_K I_3 \geq n^2 + 2n - 5 = \dim_K \mathcal{L}_3?$$

An affirmative answer for this question is proved completely in Theorem 3.17 below.

Notation 3.5. Throughout the rest of the paper we write $\mathfrak{f}$ for the ideal $(f_1, \ldots, f_n)R$ when $f_1, \ldots, f_n$ is a regular sequence of quadratic forms, and in the defect δ quadratic ideal case we write $\mathfrak{g}$ for the additional generators $g_1, \ldots, g_\delta$ of the quadratic ideal. Here, $f_1, \ldots, f_n, g_1, \ldots, g_\delta$ are assumed to be linearly independent over K. Moreover, henceforth, we write J for the ideal $\mathfrak{f} + (g_1, \ldots, g_{\delta-1})$. However, when $\delta = 1$ or 2 we may write g, h for g_1, g_2, so that whenever $\delta = 2$ we henceforth write J for the ideal $\mathfrak{f} + (g_1) = \mathfrak{f} + (g)$. We denote the graded Gorenstein Artin K-algebra $R/\mathfrak{f}$ by A.

We know that, if $a_1 = \cdots = a_n = \deg g = 2$, Theorem 2.3 shows that

$$\dim_K J_3 \geq n^2 + n - 2$$

and then Corollary 2.4 gives $\dim_K \big(\mathfrak{f}_3 \cap gR_1\big) \leq 2$.

Remark 3.6. In [3, Proposition 3.7] Chen gave a positive answer to the Question 3.4 for defect two quadratic ideals $I = \mathfrak{f} + (g, h)$ if $\dim_K \big(\mathfrak{f}_3 \cap gR_1\big) = 2$. We shall make repeated use of this fact in the sequel.

In this section we show $\mathrm{EGH}_{\underline{\mathbf{2}},n}(2)$ for a defect two quadratic ideal $I = \mathfrak{f} + (g, h)$ under the condition that $\dim_K \big(\mathfrak{f}_3 \cap g' R_1\big) \leq 1$ for all $g' \in Kg + Kh - \{0\}$: this covers all the cases for which Chen's result in Proposition 3.6 is not applicable.

Lemma 3.7. *As in Notation 3.5, J is the defect 1 quadratic ideal $\mathfrak{f} + gR$. Then:*

$$\dim_K I_3 = n^2 + 2n - \dim_K \big(\mathfrak{f}_3 \cap gR_1\big) - \dim_K \big(J_3 \cap hR_1\big).$$

Consequently, for the cases that are not covered by the Proposition 3.6 we have:

- **(i)** *If* $\dim_K \big(\mathfrak{f}_3 \cap gR_1\big) = 1$ *then* $\dim_K I_3 = n^2 + 2n - 1 - \dim_K \big(J_3 \cap hR_1\big)$, *and* $\mathrm{EGH}_{\underline{\mathbf{2}},n}(2)$ *holds for a defect two quadratic ideal I if and only if* $\dim_K \big(J_3 \cap hR_1\big) \leq 4$.
- **(ii)** *If* $\dim_K \big(\mathfrak{f}_3 \cap gR_1\big) = 0$ *then* $\dim_K I_3 = n^2 + 2n - \dim_K \big(J_3 \cap hR_1\big)$, *and* $\mathrm{EGH}_{\underline{\mathbf{2}},n}(2)$ *holds for I if and only if* $\dim_K \big(J_3 \cap hR_1\big) \leq 5$.

Proof. We have:

$$\begin{aligned} \dim_K I_3 &= \dim_K J_3 + \dim_K(hR_1) - \dim_K \big(J_3 \cap hR_1\big) \\ &= \big(\dim_K \mathfrak{f}_3 + \dim_K(gR_1) - \dim_K \big(\mathfrak{f}_3 \cap gR_1\big)\big) \\ &\quad + n - \dim_K \big(J_3 \cap hR_1\big) \\ &= n^2 + 2n - \dim_K \big(\mathfrak{f}_3 \cap gR_1\big) - \dim_K \big(J_3 \cap hR_1\big), \end{aligned}$$

and then (i) and (ii) are immediate. □

Remark 3.8. Let $n=5$, so that $\mathfrak{f}=(f_1, \ldots, f_5)$. For a defect two quadratic ideal $I = (\mathfrak{f}, g, h) \subseteq K[x_1, \ldots, x_5]$, if $\dim_K \big(\mathfrak{f}_3 \cap gR_1\big) = 0$ then clearly $\dim_K \big((\mathfrak{f}, g)_3 \cap hR_1\big) \leq \dim_K(hR_1) \leq 5$, therefore $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ holds for such an ideal I. However, we must give an argument to cover all possible cases, that is, when $\dim_K \big(\mathfrak{f}_3 \cap gR_1\big) = 1$, to be able to confirm $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ for every defect two quadratic ideal. In the last section, we discuss the EGH conjecture for $n = 5$ and $a_1 = \cdots = a_5 = 2$ in detail.

Next, we proceed with two useful lemmas.

Lemma 3.9. *Let A be the graded Gorenstein Artin K-algebra $R/\mathfrak{f}$ with $\dim_K A_1 = n$. Let g, h be two quadratic forms such that $gA_1 = hA_1$. Then $\mathrm{Ann}_{A_1}\, g = \mathrm{Ann}_{A_1}\, h$.*

Moreover, $\operatorname{Ann}_{A_i}(g) = \operatorname{Ann}_{A_i}(h)$ *if* $i \neq n-2$.

Proof. Suppose that the linear annihilator space of g, $\operatorname{Ann}_{A_1} g$, has dimension a and $gA_1 = hA_1$. Thus gA_1 has dimension $n-a$ and clearly hA_1 and $\operatorname{Ann}_{A_1} h$ have dimensions $n-a$ and a, respectively.

Notice that $gA(-2) \cong A/\operatorname{Ann}_A(g)$, hence it is Gorenstein and it has a symmetric O-sequence

$$(0, 0, 1, n-a, e_4, e_5, \ldots, e_5, e_4, n-a, 1),$$

where e_i denotes the dimension of $[gA]_i$ and $e_i = e_{n-i+2}$ for $2 \leq i \leq n$. Then the Hilbert function of A/gA is

$$\left(1, n, \binom{n}{2} - 1, \binom{n}{3} - n + a, \binom{n}{4} - e_4, \ldots, \binom{n}{3} - e_5, \binom{n}{2} - e_4, a, 0\right).$$

Since $\operatorname{Ann}_A(g) \cong \operatorname{Hom}_K(A/gA, A) \cong (A/gA)^\vee$, the Hilbert function of $\operatorname{Ann}_A(g)$ is

$$\left(0, a, \binom{n}{2} - e_4, \ldots, \binom{n}{4} - e_4, \binom{n}{3} - n + a, \binom{n}{2} - 1, n, 1\right).$$

Recall that $gA_1 = hA_1$, $gA_i = hA_i$ for all $i \geq 2$, so $(g,h)A$ has the Hilbert function

$$(0, 0, 2, \underbrace{n-a, e_4, \ldots, e_4, n-a, 1}_{\text{the same as for } gA}).$$

Then the O-sequence of $A/(g,h)$ becomes

$$\left(1, n, \binom{n}{2} - 2, \binom{n}{3} - n + a, \binom{n}{4} - e_4, \ldots, \binom{n}{3} - e_5, \binom{n}{2} - e_4, a, 0\right),$$

and it follows that $\operatorname{Ann}_A(g,h)$ has the Hilbert function

$$\left(0, a, \binom{n}{2} - e_4, \ldots, \binom{n}{4} - e_4, \binom{n}{3} - n + a, \binom{n}{2} - 2, n, 1\right).$$

We know that $\operatorname{Ann}_A(g,h) = \operatorname{Ann}_A(g) \cap \operatorname{Ann}_A(h)$, and in degree 1, $\operatorname{Ann}_A(g,h)$ has dimension a, so $\operatorname{Ann}_A(g,h) = \operatorname{Ann}_{A_1}(g) = \operatorname{Ann}_{A_1}(h)$. Further, $\operatorname{Ann}_A(g)$ and $\operatorname{Ann}_A(h)$ are the same in every degrees except in degree $n-2$. $\square$

Lemma 3.10. *Let g, h be two quadratic forms in a graded Gorenstein Artin K-algebra A such that $gA_i = hA_i$ and g, h have the same annihilator space V in A_i for some $i \geq 1$. Then there exists $g' \in Kg + Kh - \{0\}$ such that*

$$\dim_K \operatorname{Ann}_{A_i}(g') \geq \dim_K V + 1.$$

Proof. Consider the multiplication maps by g and h,

$$\phi_g : A_i/V \to gA_i \quad \text{and} \quad \phi_h : A_i/V \to hA_i$$

whose images gA_i, hA_i are subspaces in A_{i+2} and $gA_i = hA_i$ by assumption. Then there is a automorphism

$$T : A_i/V \to A_i/V$$

such that $g\ell = hT(\ell)$ for any $\ell \in A_i/V$. However, T has at least one nonzero eigenvector u with $T(u) = cu$ for some $c \in K$. Say ℓ_u be a form in degree i represented by this eigenvector u in A_i and not in the annihilator space V, thus $g\ell_u = hc\ell_u$. Then there is a quadratic form $g' := g - ch \in Kg + Kh - \{0\}$ such that g' is annihilated by the space V and also by $\ell_u \in A_i \setminus V$. Hence $\dim_K \operatorname{Ann}_{A_i}(g') \geq \dim_K V + 1$. □

From now on, $I = (f_1, \ldots, f_n, g, h) = \mathfrak{f} + (g, h)$ is a homogeneous ideal where $\dim_K \left(\mathfrak{f}_3 \cap g'R_1\right) \neq 2$ for a quadratic form $g' \in Kg + Kh - \{0\}$, which means that $\dim_K g'A_1 \neq n - 2$. Therefore $\dim_K g'A_1$ is either n or $n - 1$.

Proposition 3.11. *For the graded Gorenstein Artin K-algebra A, if $gA_1 = hA_1$ with $\dim_K gA_1 = n - 1 = \dim_K hA_1$, that is*

$$\dim_K(\mathfrak{f}_3 \cap gR_1) = \dim_K(\mathfrak{f}_3 \cap hR_1) = 1,$$

then $\mathrm{EGH}_{\underline{\mathbf{2}},n}(2)$ holds for the homogeneous defect two quadratic ideal $I = \mathfrak{f} + (g, h)$.

Proof. Since $\dim_K \operatorname{Ann}_{A_1}(g) = \dim_K \operatorname{Ann}_{A_1}(h) = 1$ there is some $g' \in Kg + Kh - \{0\}$ with $\dim_K \operatorname{Ann}_{A_i}(g') = 2$ by Lemma 3.10. In consequence, $\dim_K \left(\mathfrak{f}_3 \cap g'R_1\right) = 2$, and so we are done by Proposition 3.6. □

Proposition 3.12. *For the graded Gorenstein Artin K-algebra A, if $\dim_K gA_1 = \dim_K hA_1 = n$, then there exists a quadratic form g' in $Kg + Kh$ with a nonzero linear annihilator in A.*

Proof. By assumption $\dim_K A_1 = \dim_K gA_1 = \dim_K hA_1 = n$, and so we may consider again the multiplication maps $\phi_g : A_1 \to gA_1$ and $\phi_h : A_1 \to hA_1$. Then we obtain a automorphism $T : A_1 \to A_1$ and there exists an nonzero linear form $\ell \in A_1$ such that $T(\ell) = c\ell$ for some $c \in K$, that is $g\ell = ch\ell$. Consider $g' = g - ch \in Kg + Kh$. Clearly, $\ell \in \operatorname{Ann}_{A_1}(g')$. □

Next we assume that there is a linear annihilator $L \in A_1$ of g where $Lh \neq 0$ over the Gorenstein ring $A = R/\mathfrak{f}$. This case may come up either when $\dim_K gA_1 = \dim_K hA_1 = n - 1$ and the linear annihilator spaces $\operatorname{Ann}_{A_1}(g)$ and $\operatorname{Ann}_{A_1}(h)$ are distinct, or when $\dim_K gA_1 = n - 1$ and $\dim_K hA_1 = n$.

We shall make repeated use of the following result, which is Lemma 3.3 of Chen's paper [3].

Lemma 3.13 (Chen [3]). *If $f_1, \ldots, f_n$ is a regular sequence of 2-forms in R and we have a relation $u_1 f_1 + u_2 f_2 + \cdots + u_n f_n = 0$ for some t-forms $u_1, \ldots, u_n$, then $u_1, \ldots, u_n \in (f_1, \ldots, f_n)_t$. More precisely, we have that $t \geq 2$ and there exists a skew-symmetric $n \times n$ matrix B of $(t-2)$-forms such that $(u_1\, u_2 \cdots u_n) = (f_1\, f_2 \cdots f_n)B$.*

Proposition 3.14. *Let $I = \mathfrak{f} + \mathfrak{g}$ be a defect δ, where $2 \leq \delta \leq n - 1$, quadratic ideal of R as in Notation 3.5. If there is a linear form L in $\operatorname{Ann}_A(g_1, \ldots, g_{\delta-1})$ such that $Lg_\delta \neq 0$ in A, then*

$$\dim_K \big((f_1, \ldots, f_n, g_1, \ldots, g_{\delta-1})_3 \cap g_\delta R_1\big) \leq 3$$

Chen [3] used an argument involving the Koszul relations on $(x_1, \ldots, x_r)$ for $r \leq n$ while introducing another proof for Theorem 2.3. In the proof of this proposition we use a very similar argument.

Proof. As in Notation 3.5, let $J = \mathfrak{f} + (g_1, \ldots, g_{\delta-1})$, and denote the row vector of the regular sequence $f_1, \ldots, f_n$ by $\vec{\mathbf{f}}$ and the row vector of quadratic forms $g_1, \ldots, g_{\delta-1}$ by $\vec{\mathbf{g}}$.

Suppose $\dim_K(J_3 \cap g_\delta R_1) \geq 4$, and without loss of generality we may assume that

$$\begin{aligned}
x_1 g_\delta &= \vec{\mathbf{g}} \cdot \vec{\ell_1} + \vec{\mathbf{f}} \cdot \vec{p_1} \\
x_2 g_\delta &= \vec{\mathbf{g}} \cdot \vec{\ell_2} + \vec{\mathbf{f}} \cdot \vec{p_2} \\
x_3 g_\delta &= \vec{\mathbf{g}} \cdot \vec{\ell_3} + \vec{\mathbf{f}} \cdot \vec{p_3} \\
x_4 g_\delta &= \vec{\mathbf{g}} \cdot \vec{\ell_4} + \vec{\mathbf{f}} \cdot \vec{p_4}
\end{aligned}$$

where $\vec{\ell_i}$ and $\vec{p_i}$ are column vectors of linear forms of lengths $\delta - 1$ and n, respectively.

We assume that there is a linear form L such that $Lg_i = 0$ for each $i = 1, \ldots, \delta - 1$ but $Lg_\delta \neq 0$ in A. Then we get an $n \times (\delta - 1)$ matrix $(q_{i,j}) = \begin{pmatrix} \vec{q_1} & \vec{q_2} & \cdots & \vec{q}_{\delta-1} \end{pmatrix}$ of linear forms such that

$$L\vec{\mathbf{g}} = \vec{\mathbf{f}} \cdot (q_{i,j}).$$

We observe that each $x_i Lg_\delta$ is in $\mathfrak{f}$, and write $x_i Lg_\delta = \vec{\mathbf{f}} \cdot \vec{Q_i}$ where $\vec{Q_i}$ is a column of quadratic forms for $i = 1, 2, 3, 4$. Therefore:

$$Lg_\delta \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} = \vec{\mathbf{f}} \cdot \begin{pmatrix} \vec{Q_1} & \vec{Q_2} & \vec{Q_3} & \vec{Q_4} \end{pmatrix}. \tag{1}$$

Let

$$M_1 = \begin{pmatrix} x_2 & x_3 & x_4 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & x_3 & x_4 & 0 \\ 0 & -x_1 & 0 & -x_2 & 0 & x_4 \\ 0 & 0 & -x_1 & 0 & -x_2 & -x_3 \end{pmatrix}.$$

Note that $\begin{pmatrix} x_1 & x_2 & \cdots & x_4 \end{pmatrix} \cdot M_1 = 0$. Multiplying the equation (1) by M_1 from right gives that $\vec{\mathbf{f}} \cdot (\vec{Q_1}\ \vec{Q_2}\ \vec{Q_3}\ \vec{Q_4}) \cdot M_1 = 0$, and so all entries are 0 in

$$\vec{\mathbf{f}}\begin{pmatrix} x_2\vec{Q_1} - x_1\vec{Q_2} & x_3\vec{Q_1} - x_1\vec{Q_3} & x_4\vec{Q_1} - x_1\vec{Q_4} & x_3\vec{Q_2} - x_2\vec{Q_3} & x_4\vec{Q_2} - x_2\vec{Q_4} & x_4\vec{Q_3} - x_3\vec{Q_4} \end{pmatrix}$$

By Lemma 3.13, there are alternating $n \times n$ matrices B_{12}, B_{13}, B_{14}, B_{23}, B_{24}, B_{34} of <ins>linear</ins> forms such that

$$\begin{pmatrix} \underbrace{x_2\vec{Q_1} - x_1\vec{Q_2}}_{\substack{\text{a column vector}\\ \text{of cubic forms}}} & \cdots & x_4\vec{Q_3} - x_3\vec{Q_4} \end{pmatrix} = \begin{pmatrix} B_{12}\vec{\mathbf{f}}^T & \cdots & B_{34}\vec{\mathbf{f}}^T \end{pmatrix} \tag{2}$$

Similarly, consider the matrix $M_2 = \begin{pmatrix} x_3 & x_4 & 0 & 0 \\ -x_2 & 0 & x_4 & 0 \\ 0 & -x_2 & -x_3 & 0 \\ x_1 & 0 & 0 & x_4 \\ 0 & x_1 & 0 & -x_3 \\ 0 & 0 & x_1 & x_2 \end{pmatrix}$ such that $M_1 \cdot M_2 = \mathbf{0}$ and multiply equation (2) by M_2 from right to obtain:

$$\begin{pmatrix} \underbrace{(x_3B_{12} - x_2B_{13} + x_1B_{23})}_{\substack{n \times n \text{ matrix of}\\ \text{quadratic forms}}}\vec{\mathbf{f}}^T & \cdots & (x_4B_{23} - x_3B_{24} + x_2B_{34})\vec{\mathbf{f}}^T \end{pmatrix} = \mathbf{0}.$$

Then again by Lemma 3.13, there are alternating $n \times n$ matrices

$$C_1^{123}, \ldots, C_n^{123}, C_1^{124}, \ldots, C_n^{124}, \ldots, C_1^{234}, \ldots, C_n^{234}$$

of <u>scalars</u> such that

$$\begin{aligned}
x_3B_{12} - x_2B_{13} + x_1B_{23} &= \begin{pmatrix} \vec{\mathbf{f}}C_1^{123} \\ \vdots \\ \vec{\mathbf{f}}C_n^{123} \end{pmatrix} \\
x_4B_{12} - x_2B_{14} + x_1B_{24} &= \begin{pmatrix} \vec{\mathbf{f}}C_1^{124} \\ \vdots \\ \vec{\mathbf{f}}C_n^{124} \end{pmatrix} \\
x_4B_{13} - x_3B_{14} + x_1B_{34} &= \begin{pmatrix} \vec{\mathbf{f}}C_1^{134} \\ \vdots \\ \vec{\mathbf{f}}C_n^{134} \end{pmatrix} \\
x_4B_{23} - x_3B_{24} + x_2B_{34} &= \begin{pmatrix} \vec{\mathbf{f}}C_1^{234} \\ \vdots \\ \vec{\mathbf{f}}C_n^{234} \end{pmatrix}
\end{aligned} \tag{3}$$

Repeating the previous steps with $M_3 = \begin{pmatrix} x_4 \\ -x_3 \\ x_2 \\ -x_1 \end{pmatrix}$, so that $M_2 \cdot M_3 = \mathbf{0}$, we get

$$\begin{aligned}
\mathbf{0} &= (B_{12} \ \ B_{13} \ \ B_{14} \ \ B_{23} \ \ B_{24} \ \ B_{34})M_2M_3 \\
&= \begin{pmatrix} \vec{\mathbf{f}}C_1^{123} & \vec{\mathbf{f}}C_1^{124} & \vec{\mathbf{f}}C_1^{134} & \vec{\mathbf{f}}C_1^{234} \\ \vdots & \vdots & \vdots & \vdots \\ \vec{\mathbf{f}}C_n^{123} & \vec{\mathbf{f}}C_n^{124} & \vec{\mathbf{f}}C_n^{134} & \vec{\mathbf{f}}C_n^{234} \end{pmatrix} M_3
\end{aligned}$$

and then for all $i = 1, 2, \ldots, n$ we obtain

$$\vec{\mathbf{f}}(x_4C_i^{123} - x_3C_i^{124} + x_2C_i^{134} - x_1C_i^{234}) = 0.$$

Then, finally, $x_4C_i^{123} - x_3C_i^{124} + x_2C_i^{134} - x_1C_i^{234} = 0$ for all $i = 1, 2, \ldots, n$. Hence,

$$C_i^{123} = C_i^{124} = C_i^{134} = C_i^{234} = 0 \ \text{ for all } \ i = 1, 2, \ldots, n.$$

Thus, in (3) we get $x_3B_{12} - x_2B_{13} + x_1B_{23} = 0$. This shows that x_3 divides every entry in $x_2B_{13} - x_1B_{23}$. Therefore we may rewrite $B_{13} = x_3\widetilde{B_{13}} + D_{13}$ and $B_{23} = x_3\widetilde{B_{23}} + D_{23}$, where $\widetilde{B_{13}}$ and $\widetilde{B_{23}}$ are alternating matrices of scalars, D_{13} and D_{23} are alternating matrices of linear forms that do not contain x_3, and $x_2D_{13} - x_1D_{23} = 0$. We obtain the following

$$B_{12} = \frac{1}{x_3}(x_2B_{13} - x_1B_{23}) = x_2\widetilde{B_{13}} - x_1\widetilde{B_{23}}$$

Returning to equation (2), we obtain

$$x_2\vec{Q_1} - x_1\vec{Q_2} = B_{12}\vec{\mathbf{f}}^T = (x_2\widetilde{B_{13}} - x_1\widetilde{B_{23}})\vec{\mathbf{f}}^T.$$

Consequently,

$$x_1\big(\vec{Q_2} - \widetilde{B_{23}}\vec{\mathbf{f}}^T\big) = x_2\big(\vec{Q_1} - \widetilde{B_{13}}\vec{\mathbf{f}}^T\big)$$

which tells us that x_1 divides every entry of $\vec{Q_1} - \widetilde{B_{13}}\vec{\mathbf{f}}^T$. It follows that

$$\begin{aligned}\vec{\mathbf{f}}\big(\vec{Q_1} - \widetilde{B_{13}}\vec{\mathbf{f}}^T\big) &= \vec{\mathbf{f}}\vec{Q_1} && \text{as } \widetilde{B_{13}} \text{ is alternating and } \vec{\mathbf{f}}\widetilde{B_{13}}\vec{\mathbf{f}}^T = 0\\ &= x_1Lg_\delta && \text{by equation (1).}\end{aligned}$$

This shows that $Lg_\delta = \vec{\mathbf{f}}\frac{1}{x_1}\left(\vec{Q_1} - \widetilde{B_{13}}\vec{\mathbf{f}}^T\right) \in (f_1, \ldots, f_n)_3$, which contradicts our assumption $L \notin \operatorname{Ann}_A(g_\delta)$. $\square$

Corollary 3.15. *Let $I = \mathfrak{f} + \mathfrak{g} \subseteq R$ be a defect δ quadratic ideal with $2 \leq \delta \leq n-1$. Suppose that*

$$(\dagger) \qquad \operatorname{Ann}_{A_1}(g_1, \ldots, g_{\delta-1}) \setminus \operatorname{Ann}_{A_1}(g_\delta) \neq \emptyset.$$

Then

$$\dim_K I_3 \geq \dim_K \mathcal{L}_3$$

where $\mathcal{L} = (x_1^2, \ldots, x_n^2) + (x_1x_2, x_1x_3, \ldots, x_1x_{\delta+1})$ is the defect δ lex-plus-powers ideal of R. That is, $\mathrm{EGH}_{\mathbf{2},n}(2)$ holds for any defect δ quadratic ideal with property $(\dagger)$.

Proof. Notice that $\dim_K \mathcal{L}_3 = n^2 + n\delta - \frac{\delta(\delta+3)}{2}$. We use induction on δ. Let $J = \mathfrak{f} + (g_1, \ldots, g_{\delta-1})$ be the defect $\delta - 1$ quadratic ideal.

$$\begin{aligned}\dim_K I_3 &= \dim_K J_3 + n - \dim_K \left(J_3 \cap g_\delta R_1\right)\\ &\geq \left(n^2 + (\delta-1)n - \frac{(\delta-1)(\delta+2)}{2}\right) + n - 3\\ &= n^2 + n\delta - \frac{\delta(\delta+3)}{2} + \delta - 2\\ &\geq n^2 + n\delta - \frac{(\delta)(\delta+3)}{2}.\end{aligned}$$

□

We notice that a special case of Corollary 3.15 when $\delta = 2$ shows that the inequality is strict.

Corollary 3.16. *Let $I = \mathfrak{f} + (g, h)$ be a defect two ideal generated by quadrics in R. If $\mathrm{Ann}_{A_1}(g) = \mathrm{Span}\{L\}$ for some $L \in R_1$ and L does not annihilate h in $A = R/\mathfrak{f}$, then*

$$\dim_K I_3 \geq n^2 + 2n - 4 > \dim_K(x_1^2, \ldots, x_n^2, x_1x_2, x_1x_3)_3 = n^2 + 2n - 5$$

Proof. The result follows from Proposition 3.14 as

$$\dim_K I_3 = n^2 + 2n - \underbrace{\dim_K(\mathfrak{f}_3 \cap gR_1)}_{=\dim_K \mathrm{Ann}_{A_1}(g)=1} - \underbrace{\dim_K(J_3 \cap hR_1)}_{\leq 3}$$

which is $\geq n^2 + 2n - 4$. □

Finally, we give an affirmative answer to the Main Question 3.4.

Theorem 3.17. *Let $I = \mathfrak{f} + (g, h) \subseteq R = K[x_1, \ldots, x_n]$ for $n \geq 5$ be a defect two ideal quadratic ideal. Then*

$$\dim_K I_3 \geq n^2 + 2n - 5.$$

More precisely, $\mathrm{EGH}_{\underline{\mathbf{2}},n}(2)$ *holds for homogeneous defect two quadratic ideals in R for any $n \geq 5$.*

Proof. If the given defect two ideal satisfies Proposition 3.6 , then, by Chen's result, the theorem is proved.

Assume that $\dim_K \left(\mathfrak{f}_3 \cap g'R_1\right) \neq 2$ for any $g' \in Kg + Kh \setminus \{0\}$. If $\dim_K \left(\mathfrak{f}_3 \cap gR_1\right) = \dim_K \left(\mathfrak{f}_3 \cap hR_1\right) = 0$, by Proposition 3.12, we can always find another quadratic form $g' \in Kg + Kh \setminus \{0\}$ so that g' has a linear annihilator in A. Then we can apply Corollary 3.16. If $\dim_K \left(\mathfrak{f}_3 \cap gR_1\right) = \dim_K \left(\mathfrak{f}_3 \cap hR_1\right) = 1$ and the same linear form annihilates both g and h in A, by Proposition 3.11. we have a situation that contradicts our assumption. □

Corollary 3.18. $\mathrm{EGH}_{\underline{\mathbf{2}},n}(2)$ *holds for every defect two ideal containing a regular sequence of quadratic forms.*

Proof. This result follows from Lemma 2.6 and Theorem 3.17. □

4. The EGH conjecture when $n = 5$ and $a_1 = \cdots = a_5 = 2$

In this section $R = K[x_1, \ldots, x_5]$ and $I = (f_1, \ldots, f_5) + (g_1, \ldots, g_\delta) = \mathfrak{f} + \mathfrak{g}$ is a homogeneous defect δ ideal in R, where $f_1, \ldots, f_5$ is a regular sequence of quadrics and $\deg g_j \geq 2$ for $j = 1, \ldots, \delta$. Throughout, we shall write $A := R/\mathfrak{f}$, which is a graded Gorenstein local Artin ring. We will show the existence of a lex-plus-powers ideal $\mathcal{L} \subseteq R$ containing x_i^2 for $i = 1, \ldots, 5$ with the same Hilbert function as I by proving the following main theorem.

Theorem 4.1. *The EGH conjecture holds for all homogeneous ideals containing a regular sequence of quadrics in* $K[x_1, \ldots, x_5]$.

Lemma 2.8 of Caviglia-Maclagan tells us that $\mathrm{EGH}_{\underline{\mathbf{2}},5}(d)$ holds if and only if $\mathrm{EGH}_{\underline{\mathbf{2}},5}(5-d-1)$ holds. Thus it will be enough to show $\mathrm{EGH}_{\underline{\mathbf{2}},5}(d)$ when $d = 0, 1, 2$. By Remark 2.9 we know that $\mathrm{EGH}_{\underline{\mathbf{2}},5}(d)$ is true when $d = 0, 1$, therefore $\mathrm{EGH}_{\underline{\mathbf{2}},5}(3)$ and $\mathrm{EGH}_{\underline{\mathbf{2}},5}(4)$ both hold as well.

Our goal in this section is to prove $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ for any homogeneous ideal containing a regular sequence of quadrics: this will complete the proof of $\mathrm{EGH}_{\underline{\mathbf{2}},5}$. To achieve this, it suffices to understand $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ for quadratic ideals with arbitrary defect δ (but, of course, $\delta \leq 10$, since $\dim_K R_2 = 15$), by Lemma 2.6.

Remark 4.2. As a result of Corollary 3.18, we see that $\mathrm{EGH}_{\underline{\mathbf{2}},n}$ holds for any defect $\delta = 2$ quadratic ideal in $K[x_1, \ldots, x_n]$ for $n = 5$.

To accomplish our goal we will prove $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ for defect $\delta \geq 3$ quadratic ideals. In the next subsection, we prove that if one knows the case where $\delta = 3$, one obtains all the cases for $\delta \geq 4$. In the final subsection we finish the proof by establishing $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ for $\delta = 3$.

Quadratic ideals with defect $\delta \geq 4$

Lemma 4.3. *If* $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ *holds for all defect three quadratic ideals, then it holds for all quadratic ideals with defect* $\delta \geq 4$.

Proof. Let $I = (f_1, \ldots, f_5, g_1, g_2, g_3, g_4) = \mathfrak{f} + \mathfrak{g} \subseteq R$ be a defect 4 homogeneous ideal generated by quadrics, where $f_1, \ldots, f_5$ form a regular sequence. By assumption the defect three quadratic ideal $J = \mathfrak{f} + (g_1, g_2, g_3) \subseteq I$ satisfies $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$, that is, $\dim_K J_3 \geq 31$.

Let $\mathcal{L} = (x_1^2, \ldots, x_5^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5)$ be the LPP ideal with $\dim_K \mathcal{L}_2 = \dim_K I_2 = 9$. Then we get $\dim_K I_3 \geq \dim_K J_3 \geq 31 = \dim_K \mathcal{L}_3$, as we need for the case of defect $\delta = 4$.

Now assume $5 \leq \delta \leq 10$. Let ${}^{\delta}I$ denote an arbitrary defect δ quadratic ideal, and let ${}^{\delta}\mathcal{L}$ denote the lex-plus-power ideal with defect $\delta \geq 5$. More precisely, ${}^{\delta}\mathcal{L} := (x_1^2, \ldots, x_5^2) + (m_1, \ldots, m_\delta)$ where m_i are the next greatest quadratic square-free monomials with respect to lexicographic order. We need to show that $\mathrm{Hilb}_{R/{}^{\delta}I}(3) \leq \mathrm{Hilb}_{R/{}^{\delta}\mathcal{L}}(3)$.

We assume that $\mathrm{Hilb}_{R/{}^{\delta}I}(3) \geq \mathrm{Hilb}_{R/{}^{\delta}\mathcal{L}}(3) + 1$, and we shall obtain a contradiction.

Using duality for Gorenstein rings, we know that for $0 \leq d \leq 5$ we have that

$$\mathrm{Hilb}_{R/{}^{\delta}I}(d) = \mathrm{Hilb}_{R/\mathfrak{f}}(d) - \mathrm{Hilb}_{R/(\mathfrak{f}:{}^{\delta}I)}(5-d).$$

Then, for $d = 3$, using the assumption we get

$$\begin{aligned}\mathrm{Hilb}_{R/(\mathfrak{f}:{}^{\delta}I)}(2) &= \mathrm{Hilb}_{R/\mathfrak{f}}(3) - \mathrm{Hilb}_{R/{}^{\delta}I}(3) \;\leq\; 10 - (\mathrm{Hilb}_{R/{}^{\delta}\mathcal{L}}(3) + 1)\\ &\leq 9 - \mathrm{Hilb}_{R/{}^{\delta}\mathcal{L}}(3) = \begin{cases} 7 & \text{if } \delta = 5, \\ 8 & \text{if } \delta = 6, 7, \\ 9 & \text{if } \delta = 8, 9, 10. \end{cases}\end{aligned}$$

We next show that $\dim_K(\mathfrak{f} : {}^{\delta}I)_1 = 0$. If there is a nonzero linear form $\ell \in \mathfrak{f} : {}^{\delta}I$ then $\dim_K \mathrm{Ann}_{A_2}\, \ell A \geq \delta \geq 5$, so we get that $\dim_K A_3/\ell A_2 \geq 5$. On the other hand, we see that $A_3/\ell A_2 \cong [R/(\bar{f}_1, \ldots, \bar{f}_4, \bar{f}_5, l)]_3$ where the $\bar{f}_i$ are the images of the f_i, and the dimension of $[R/(\bar{f}_1, \ldots, \bar{f}_4, \bar{f}_5, l)]_3$ as a K-vector space is at most 4.

Then we can find a defect γ quadratic ideal ${}^{\gamma}J \subseteq \mathfrak{f} : {}^{\delta}I$ for $\gamma = 3, 2, 1$ if the defect of ${}^{\delta}I$ is $\delta = 5$ or $\delta = 6, 7$ or $\delta = 8, 9, 10$, respectively. We then have the inequalities shown below, where the first is obvious as ${}^{\gamma}J$ is contained in $\mathfrak{f} : {}^{\delta}I$ and the second follows by comparison with Hilbert functions of quotients by

LPP ideals in degree 3 and the fact that, by assumption, $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ holds for quadratic ideals with defect less than or equal to three.

$$\begin{aligned}
&\mathrm{Hilb}_{R/(\mathfrak{f}:{}^{\delta}I)}(3) \leq \mathrm{Hilb}_{R/{}^{\gamma}J}(3) \\
&\leq \begin{cases} 4 \text{ if } {}^{\gamma}J \text{ is a defect } \gamma = 3 \text{ quadratic ideal when } \delta = 5, \\ 5 \text{ if } {}^{\gamma}J \text{ is a defect } \gamma = 2 \text{ quadratic ideal when } 6 \leq \delta \leq 7, \\ 7 \text{ if } {}^{\gamma}J \text{ is a defect } \gamma = 1 \text{ quadratic ideal when } 8 \leq \delta \leq 10. \end{cases}
\end{aligned}$$

However, each of the cases above contradicts the following equality:

$$\mathrm{Hilb}_{R/(\mathfrak{f}:{}^{\delta}I)}(3) = \mathrm{Hilb}_{R/\mathfrak{f}}(2) - \mathrm{Hilb}_{R/{}^{\delta}I}(2) = \delta.$$

Thus, we get $\mathrm{Hilb}_{R/{}^{\delta}I}(3) \leq \mathrm{Hilb}_{R/{}^{\delta}\mathcal{L}}(3)$ for any defect $\delta \geq 5$ quadratic ideal ${}^{\delta}I$ in R. □

Defect three quadratic ideals

Lemma 4.4. *Let $I = \mathfrak{f} + (g_1, g_2, g_3)$ be a defect three quadratic ideal in the polynomial ring R. Then, for any $1 \leq i_1 < i_2 \leq 3$,*

$$\dim_K(\mathfrak{f} : (g_{i_1}, g_{i_2}))_1 \leq 1,$$

and, furthermore, $\dim_K(\mathfrak{f} : (g_1, g_2, g_3))_1 \leq 1$.

Proof. Suppose that $\dim_K(\mathfrak{f} : (g_1, g_2))_1 \geq 2$, and assume there are $\ell_1, \ell_2 \in R_1$ such that $\ell_i g_1, \ell_i g_2 \in \mathfrak{f}$ for both $i = 1, 2$. Without loss of generality we assume that $\ell_1 = x_1$ and $\ell_2 = x_2$.

Therefore, we can write $(x_1, x_2, f_1, \ldots, f_5) \subseteq \mathfrak{f} : (f_1, \ldots, f_5, g_1, g_2)$. Then

$$\begin{aligned}
2 &= \mathrm{Hilb}_{(f_1, \ldots, f_5, g_1, g_2)/\mathfrak{f}}(2) \\
&= \mathrm{Hilb}_{R/\left(\mathfrak{f}:(f_1, \ldots, f_5, g_1, g_2)\right)}(5-2), \quad \text{(by duality)} \\
&\leq \mathrm{Hilb}_{R/(x_1, x_2, f_1, \ldots, f_5)}(3) \\
&= \mathrm{Hilb}_{K[x_3, x_4, x_5]/(\bar{f}_1, \ldots, \bar{f}_5)}(3), \quad \text{(where } \bar{f}_i \text{ is the image of } f_i \text{ in } K[x_3, x_4, x_5],) \\
&\leq \binom{5-2}{3} = 1,
\end{aligned}$$

which is a contradiction. □

Hence, working in the graded Gorenstein Artin K-algebra $A = R/\mathfrak{f}$, we have from the lemma just above that $\mathrm{Ann}_{A_1}(g_1, g_2)$ is a K-vector space of dimension at most one, and, therefore

$$\dim_K \operatorname{Ann}_{A_1}(g_1, g_2, g_3) \leq 1$$

since $\operatorname{Ann}_{A_1}(g_1, g_2, g_3) \subseteq \operatorname{Ann}_{A_1}(g_1, g_2)$.

Remark 4.5. By Remark 4.2 we know that for any defect two quadratic ideal J in R, $\dim_K J_3$ is at least 30. Then $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ holds for the defect three quadratic ideals I containing a defect two quadratic ideal J with $\dim_K J_3 \geq 31$, as $\mathrm{Hilb}_{R/I}(3) \leq \mathrm{Hilb}_{R/J}(3) \leq 4$.

We henceforth focus on defect three quadratic ideals $I = \mathfrak{f} + (g_1, g_2, g_3)$ in R such that every defect two quadratic ideal $J \subseteq I$ containing $\mathfrak{f}$ has $\dim_K J_3 = 30$.

For such defect three quadratic ideals, we observe the following.

Lemma 4.6. *Consider the ideal $\mathcal{I} = (g_1, g_2, g_3)A$ in the Gorenstein ring A such that any ideal $(g_{i_1}, g_{i_2})A$ contained in $\mathcal{I}$ has degree three component of dimension $\dim_K(g_{i_1}, g_{i_2})A_1 = 5$. Assuming that $\dim_K \operatorname{Ann}_{A_1}(g_1) = 1$, we have that*

$$\operatorname{Ann}_{A_1}(g_1, g_2, g_3) = \operatorname{Ann}_{A_1}(g_1).$$

Furthermore, if g_1A_1 is 5-dimensional, that is, there is no linear form that annihilates g_1 in A, then for any quadric g in $Kg_1 + Kg_2 + Kg_3$ the vector space $gA_1 \subseteq A_3$ is either 3 or 5 dimensional.

Proof. Let $\dim_K \operatorname{Ann}_{A_1}(g_1) = 1$, and let the linear form L annihilate g_1 but not some form $g' \in Kg_2 + Kg_3$ in A. We define a defect two quadratic ideal

$$J = (f_1, \ldots, f_5, g_1, g') \subseteq \mathfrak{f} + (g_1, g_2, g_3)$$

in R. Hence, by Corollary 3.16, we know already that $\dim_K J_3 \geq 31$, which means that $\dim_K(g_1, g')A_1 = 6$. This contradicts our assumption. Thus, L must be in $\operatorname{Ann}_{A_1}(g_1, g_2, g_3)$. □

Recall that the following holds, by Proposition 3.14, when $\delta = 3$.

Proposition 4.7. *Let $I = \mathfrak{f} + (g_1, g_2, g_3) \subseteq K[x_1, \ldots, x_5]$ be a defect 3 quadratic ideal. As usual, let $A = R/\mathfrak{f}$. If there is a linear form $L \in \operatorname{Ann}_A(g_1, g_2)$ such that $L \notin \operatorname{Ann}_A(g_3)$, then*

$$\dim_K \left((\mathfrak{f} + (g_1, g_2))_3 \cap g_3R_1\right) \leq 3.$$

When a defect three quadratic ideal I satisfies the condition of the above proposition, we notice a sharp bound for $\mathrm{Hilb}_{R/I}(3)$.

Corollary 4.8. *Given a defect three quadratic ideal* $I = \mathfrak{f} + (g_1, g_2, g_3)$ *in* $R = K[x_1, \ldots, x_5]$, *and, as usual, let* $A = R/\mathfrak{f}$, *which is a graded Gorenstein Artin ring. If* $\dim_K \mathrm{Ann}_{A_1}(g_1, g_2) = 1$ *and* $\mathrm{Ann}_{A_1}(g_1, g_2, g_3) = 0$ *then*

$$\dim_K I_3 \geq 32 > \dim_K \mathcal{L}_3,$$

where $\mathcal{L} = (x_1^2, \ldots, x_5^2, x_1x_2, x_1x_3, x_1x_4)$ *and* $\dim_K \mathcal{L}_3 = 31$.

Proof. By assumption there is a linear form in $\mathrm{Ann}_A(g_1, g_2)$, say L, such that L does not annihilate g_3. Hence, Proposition 4.7 gives us $\dim_K \big((\mathfrak{f} + (g_1, g_2))_3 \cap g_3R_1\big) \leq 3$. Then we get

$$\begin{aligned}\dim_K(\mathfrak{f} + (g_1, g_2, g_3))_3 &= \dim_K(\mathfrak{f} + (g_1, g_2))_3 + \dim_K g_3R_1 \\ &\quad - \dim_K \big((\mathfrak{f} + (g_1, g_2))_3 \cap g_3R_1\big) \\ &\geq 30 + 5 - 3 = 32 > 31 = \dim_K \mathcal{L}_3 .\end{aligned}$$ □

Proposition 4.9. *Suppose that for all quadratic forms* g *in* $Kg_1 + Kg_2$, *the subspace* gA_1 *of* A_3 *is a* 3*-dimensional. If* $\dim_K(g_1, g_2)A_1 = 5$, *then* $\dim_K \mathrm{Ann}_{A_1}(g_1, g_2) = 1$.

We first state the following observation in a linear algebra setting, which will be useful for the proof Proposition 4.9.

Lemma 4.10. *Let* S, T *be linear transformations from* V *to* W, *both* n*-dimensional vector spaces over* K, *such that* $\mathrm{rank}(S) = \mathrm{rank}(T) = \mathrm{rank}(S - T) = r$, *and the kernels of* S, T *are disjoint. Then the images of* S *and* T *are contained in the same* $(3r - n)$*-dimensional subspace of* W.

Proof. $V_0 = \ker(S - T)$ is $(n - r)$-dimensional. S and T are injective on V_0, since for $v \in V_0$, $S(v) = 0$ iff $T(v) = 0$, and $\mathrm{Ker}(S) \cap \mathrm{Ker}(T) = 0$. Thus, $S(V_0) = T(V_0)$ is an $(n - r)$-dimensional space in $S(V) \cap T(V)$. Since $S(V), T(V)$ are r-dimensional and overlap in a space of dimension at least $n - r$, $S(V) + T(V)$ has dimension at most $r + r - (n - r) = 3r - n$. □

Proof of Proposition 4.9. Assume that $\dim_K \mathrm{Ann}_{A_1}(g_1, g_2) = 0$. Since all quadratic forms g in $Kg_1 + Kg_2$ are such that $gA_1 \subseteq A_3$ has vector space dimension 3, we have from Lemma 4.10 with $n = 5, r = 3$, that $(Kg_1 + Kg_2)A_1 \subseteq A_3$ is at most 4-dimensional. Consequently,

$$\dim_K[A/(g_1, g_2)A]_3 = \dim_K[R/\mathfrak{f} + (g_1, g_2)]_3 \geq 6,$$

contradicting $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ for defect 2 quadratic ideals. Hence,

$$\dim_K \mathrm{Ann}_{A_1}(g_1, g_2) = 1. \qquad \square$$

Proposition 4.11. *Let* $I = \mathfrak{f} + (g_1, g_2, g_3)$ *be a defect three quadratic ideal in* $R = K[x_1, \ldots, x_5]$. *If* $\dim_K \mathrm{Ann}_{A_1}(g_1, g_2, g_3) = 0$ *then* $\mathrm{Hilb}_{R/I}(3) \leq 4$.

Proof. First, by Remark 4.5 we note that it suffices to consider any defect two quadratic ideal $J \subseteq I$ with $\mathrm{Hilb}_{R/J}(3) = 5$.

Suppose that $\dim_K \mathrm{Ann}_{A_1}(g_1, g_2, g_3) = 0$. Then, clearly, no g_i, for $i = 1, 2, 3$ has a 1-dimensional linear annihilator space in A, since, otherwise, by Lemma 4.6, we obtain that $\dim_K \mathrm{Ann}_{A_1}(g_1, g_2, g_3) = 1$, which contradicts our assumption. Thus, for the rest of the proof we may assume that each g_iA_1, $i = 1, 2, 3$, is either 3 or 5 dimensional.

If all forms g in $Kg_1 + Kg_2 + Kg_3$ are such that $\dim_K gA_1 = 3$ then we can find two independent quadratic forms whose linear annihilator spaces intersect in 1-dimensional space, and the result follows from Corollary 4.8.

Let g_1A_1 be a 5-dimensional subspace of A_3 and suppose for every $g \in Kg_2 + Kg_3$, gA_1 has dimension either 3 or 5.

We complete the proof by obtaining a contradiction. We assume that $\mathrm{Hilb}_{R/I}(3) = 5$. In other words, the space $W = (Kg_1 + Kg_2 + Kg_3)A_1 \subseteq A_3$ is 5-dimensional. Then we get $W = g_1A_1 = (Kg_2 + Kg_3)A_1$.

Consider the multiplication maps by g_1, g_2 and g_3 from A_1 to the subspace W of A_3. By adjusting the bases of A_1 and W we can assume the matrix of g_1 is the identity matrix $\mathbb{I}_5$ of size 5. Denote the matrices of g_2 and g_3 by α and β, respectively. We can assume that α and β are both singular, and so have rank 3, by subtracting the suitable multiples of $\mathbb{I}_5$ from them if they are not singular.

We see that all matrices $z\mathbb{I}_5 + x\alpha + y\beta$ must have at most two eigenvalues, otherwise we can form a linear combination whose kernel is 1-dimensional, which corresponds to a quadratic form with 1-dimensional linear annihilator space. Then there are two main cases: one is that every matrix in the space spanned by $\mathbb{I}_5, \alpha$ and β has one eigenvalue. The other is that almost all matrices in the form $z\mathbb{I}_5 + x\alpha + y\beta$ have two eigenvalues, since the subset with at most one eigenvalue is Zariski closed.

Define $D(x, y, z) = \det(z\mathbb{I}_5 - x\alpha - y\beta)$, a homogeneous polynomial in x, y, z of degree 5 that is monic in z. Note that D is also the characteristic polynomial, in z, of $x\alpha + y\beta$. Notice that the singular matrices in the subspace of 5×5 matrices spanned by $\mathbb{I}, \alpha$ and β are defined by the vanishing of D.

If the determinant D is square-free (as the characteristic polynomial in z), then the ideal (D) is a radical ideal and it cannot contain a nonzero polynomial of degree less than 5, which contradicts the fact that all size 4 minors of a singular matrix must vanish, since in our situation these singular matrices have rank 3. Therefore the size 4 minors, whose degrees are at most 4, are in the radical (D).

If the determinant D is not square-free, then its squared factor must be linear or quadratic: in the latter case the other factor is linear, so that in either case D has a linear factor, say $z - ax - by$.

Consider the independent matrices $\alpha' = a\mathbb{I}_5 - \alpha$, $\beta' = b\mathbb{I}_5 - \beta$. Then we think of any linear combination of them, say $r\alpha' + s\beta' = r(a\mathbb{I}_5 - \alpha) + s(b\mathbb{I}_5 - \beta) = (ar + bs)\mathbb{I}_5 - r\alpha - s\beta$. As $z - ax - by$ is a factor of $D(x, y, z)$, and hence, D vanishes for $x = r, y = s, z = ar + bs$. This means that every linear combination of α' and β' is singular. Therefore, we can replace α, β by α' and β' and so we can assume that we are in the case where every linear combination of the two non-identity matrices is singular, and, if not 0, of rank 3. By Lemma 4.10, this implies that the kernels of α' and β' cannot be disjoint, so we are done by Proposition 4.9 and Corollary 4.8. □

Finally, we complete the proof of Theorem 4.1 by showing $\mathrm{EGH}_{\underline{\mathbf{2}},5}(2)$ for every defect three quadratic ideal $I = \mathfrak{f} + (g_1, g_2, g_3)$ in $R = K[x_1, \ldots, x_5]$ when there is a nonzero linear form $L \in \mathrm{Ann}_A(g_1, g_2, g_3)$ in the following proposition.

Proposition 4.12. *Let $I = \mathfrak{f} + (g_1, g_2, g_3)$ be a defect three quadratic ideal in R. If $\mathrm{Ann}_{A_1}(g_1, g_2, g_3)$ is a 1-dimensional K-subspace of A_1, say KL, then*

$$\mathrm{Hilb}_{R/I}(3) = 4.$$

Proof. The proof of this proposition will be completed as soon as we prove the following lemmas 4.13 and 4.15 along with propositions 4.14 and 4.16 below.

Lemma 4.13. *Let L be a nonzero linear form in $\mathrm{Ann}_A(g_1, g_2, g_3)$. Then one of the quadratic forms f_i in the regular sequence has the linear factor L.*

Proof of lemma. As $g_1, g_2, g_3 \in \mathrm{Ann}_{A_2}(L) \subseteq A_2$ for $L \in \mathrm{Ann}_{A_1}(g_1, g_2, g_3)$ we know that

$$\dim_K \mathrm{Ann}_{A_2}(L) \geq 3.$$

This tells us that $\dim_K LA_2 \leq 7$, which implies

$$\dim_K(A_3/LA_2) = \dim_K[A/LA]_3 \geq 3 \tag{4}$$

as $\dim_K A_3 = 10$.

Assume that $L = x_5$ and let $\bar{f}_i$ be the image of f_i modulo x_5 .

Suppose that $\bar{\mathfrak{f}} = (\bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4, \bar{f}_5)$ is an almost complete intersection in the polynomial ring $K[x_1, x_2, x_3, x_4]$. Thus,

$$A/LA \cong \frac{K[x_1, \ldots, x_5]}{\mathfrak{f} + (x_5)} \cong \frac{K[x_1, x_2, x_3, x_4]}{\bar{\mathfrak{f}}}.$$

However, using the Francisco's result for almost complete intersections [8], we know that

$$\dim_K \left[\frac{K[x_1, x_2, x_3, x_4]}{\bar{\mathfrak{f}}}\right]_3 \leq 2 = \dim_K \left[\frac{K[x_1, x_2, x_3, x_4]}{(x_1^2,\, x_2^2,\, x_3^2,\, x_4^2,\, x_1x_2)}\right]_3.$$

This contradicts (4).

Hence the images of f_i modulo L form a regular sequence in $K[x_1, \ldots, x_4]$, that is, one of them has a linear factor x_5. □

As a result of the claim, after a suitable change of variables, we may assume that the linear annihilator is $L = x_5$ and may consider I in two possible forms: either I is in the form of (5) in *Case 1* below, where $f_1,\, f_2,\, f_3,\, f_4,\, x_1x_5$ is the regular sequence, or I is as in (6) in *Case 2* below, where $f_1,\, f_2,\, f_3,$ $f_4,\, x_5^2$ form a quadratic regular sequence in I.

Case 1. Suppose that $f_5 = x_1x_5$. Then we can assume that $g_1 = x_1x_2,\, g_2 = x_1x_3,\, g_3 = x_1x_4$. Furthermore, after we alter the f_i by getting rid of all the terms containing x_1 except x_1^2, we may assume that the defect three quadratic ideal I looks like

$$I = (f_1,\, f_2,\, f_3,\, f_4 + cx_1^2,\, x_1x_5,\, x_1x_2,\, x_1x_3,\, x_1x_4), \tag{5}$$

where f_1, f_2, f_3, f_4 form a regular sequence in $K[x_2, x_3, x_4, x_5]$ and $c \in K$, $c \neq 0$.

Proposition 4.14. *Let* $I=(f_1,\ f_2,\ f_3,\ f_4+cx_1^2,\ x_1x_5,\ x_1x_2,\ x_1x_3,\ x_1x_4)$ *be a defect three quadratic ideal in* R *where* f_1, f_2, f_3, f_4 *is an* $K[x_2, x_3, x_4, x_5]$*-sequence. Then*

$$\mathrm{Hilb}_{R/I}(3) = 4 = \mathrm{Hilb}_{R/\mathcal{L}}(3)$$

where $\mathcal{L} = (x_1^2,\ \ldots,\ x_5^2,\ x_1x_2,\ x_1x_3,\ x_1x_4)$*.*

Proof. One can easily see that I contains all cubic monomials divisible by x_1 since $x_1x_i\in I$ for all $i=2,3,4,5$ and f_4 is a quadratic form in $K[x_2, x_3, x_4, x_5]$, therefore $x_1f_4 \in I$ and so is x_1^3. Thus, the Hilbert functions of R/I and $k[x_2, x_3, x_4, x_5]/I \cap K[x_2, x_3, x_4, x_5]$ agrees in degree 3. So

$$\begin{aligned}\mathrm{Hilb}_{R/I}(3) &= \mathrm{Hilb}_{K[x_2,x_3,x_4,x_5]/I\cap K[x_2,x_3,x_4,x_5]}(3)\\ &= \mathrm{Hilb}_{K[x_2,x_3,x_4,x_5]/(f_1,\,f_2,\,f_3,\,f_4)}(3) = 4.\end{aligned}$$

□

Case 2. Suppose that $f_5 = x_5^2$ by altering the variables and generators, and then we can assume that $g_1 = x_1x_5$, $g_2 = x_2x_5$, $g_3 = x_3x_5$. As we did in the case above, we get rid of all the terms containing x_5 except x_4x_5 in the f_i, and so the defect three quadratic ideal can be written as follows:

$$I = (f_1,\ f_2,\ f_3,\ f_4 + cx_4x_5,\ x_5^2,\ x_1x_5,\ x_2x_5,\ x_3x_5), \tag{6}$$

where f_1, f_2, f_3, f_4 form a regular sequence in $K[x_1, x_2, x_3, x_4]$ and $c \in K$.

Lemma 4.15. *Let* $\mathfrak{a} = (f_1,\ f_2,\ f_3,\ f_4 + cx_4x_5,\ x_5^2) : (x_1x_5,\ x_2x_5,\ x_3x_5)$ *be the colon ideal in* R*. Then we have* $\mathrm{Hilb}_{R/\mathfrak{a}}(2) = 6$*.*

Proof. It suffices to show $\dim_K \mathfrak{a}_2 = 9$.

We know that x_1x_5, x_2x_5, x_3x_5, x_4x_5, x_5^2 are all in $\mathfrak{a}_2$, and f_1, f_2, f_3, $f_4 \in \mathfrak{a}_2$ as well. Thus we see that $\dim_K \mathfrak{a}_2 \geq 9$.

If there is another independent quadratic form in $\mathfrak{a}$, it must be in $K[x_1, x_2, x_3, x_4]$, as we have all quadratic monomials containing x_5, so call it Q in $K[x_1, x_2, x_3, x_4]$. Then we consider the cubic form $H = x_5Q$. Clearly H is not in the R_1-span of f_1, f_2, f_3, f_4, x_5^2, therefore we can define the ideal $J = (f_1,\ f_2,\ f_3,\ f_4,\ x_5^2,\ H)$, which is an almost complete intersection in R. Then we get $\dim_K \big((f_1,\ f_2,\ f_3,\ f_4, x_5^2)_4 \cap HR_1\big) \geq 4$ as x_1H, x_2H, x_3H and x_5H are in $(f_1,\ f_2,\ f_3,\ f_4, x_5^2)_4$, but by Corollary 2.4 this dimension must be at most 3. This proves that there cannot be such a quadratic form Q in $\mathfrak{a}$. □

Proposition 4.16. *Let* $I = (f_1,\, f_2,\, f_3,\, f_4 + cx_4x_5,\, x_5^2,\, x_1x_5,\, x_2x_5,\, x_3x_5)$ *be a defect three quadratic ideal in* R *where* f_1, f_2, f_3, f_4 *is an* $K[x_1, x_2, x_3, x_4]$*-sequence. Then*

$$\mathrm{Hilb}_{R/I}(3) = 4 = \mathrm{Hilb}_{R/\mathcal{L}}(3)$$

where $\mathcal{L} = (x_1^2,\, \ldots,\, x_5^2,\, x_1x_2,\, x_1x_3,\, x_1x_4)$.

Proof. Using the duality of Gorenstein algebras, again we can obtain

$$\mathrm{Hilb}_{R/I}(3) = \mathrm{Hilb}_{R/(f_1,\, f_2,\, f_3,\, f_4 + cx_4x_5,\, x_5^2)}(3) - \mathrm{Hilb}_{R/\mathfrak{a}}(5-3),$$

where $\mathfrak{a}$ is the colon ideal $(f_1,\, f_2,\, f_3,\, f_4 + cx_4x_5,\, x_5^2) : I$.

Then proof is done, since $\mathrm{Hilb}_{R/(f_1,\, f_2,\, f_3,\, f_4+cx_4x_5,\, x_5^2)}(3) = 10$ and $\mathrm{Hilb}_{R/\mathfrak{a}}(2) = 6$ by the above lemma. □

This finishes the proof of Proposition 4.12 and hence the proof of Theorem 4.1. □

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