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Semiorthogonal decompositions of equivariant derived categories of invariant divisors

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Given a smooth variety X with an action of a finite group G, and a semiorthogonal decomposition of the derived category, $\mathcal{D}([X/G])$, of G-equivariant coherent sheaves on X into subcategories equivalent to derived categories of smooth varieties, we construct a similar semiorthogonal decomposition for a smooth G-invariant divisor in X (under certain technical assumptions). Combining this procedure with the semiorthogonal decompositions constructed in [18], we construct semiorthogonal decompositions of some equivariant derived categories of smooth projective hypersurfaces.

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1. Introduction

1.1. Semiorthogonal decompositions for $\mathcal{D}([X/G])$

Let X be a smooth quasi-projective variety over an algebraically closed field k of characteristic zero. Suppose G is a finite group acting on X by automorphisms. Then there is a decomposition of the Hochschild homology of the quotient stack $[X/G]$,

$$HH_*([X/G]) \cong \bigoplus_{\lambda \in G/\sim} HH_*(X_\lambda)^{C(\lambda)}, \tag{1.1}$$

where $G/\sim$ is the set of conjugacy classes of G, $C(\lambda)$ is the centralizer of λ, $X_\lambda \subset X$ is the invariant subvariety of λ, see [18, Lemma 2.1.1]. In [19, Theorem 1.1], the authors show that the decomposition (1.1) has a motivic origin in an appropriate sense, and that a similar decomposition exists for any additive invariant of dg-categories. In [3] a related decomposition of the equivariant zeta function is given.

In the case when the geometric quotient $X_\lambda/C(\lambda)$ is smooth one can identify $HH_*(X_\lambda)^{C(\lambda)}$ with $HH_*(X_\lambda/C(\lambda))$ (see [18, Proposition 2.1.2]). Thus, it is natural to ask whether in some cases the above decomposition can be realized at the level of derived categories of coherent sheaves.

Conjecture A ([18, Conjecture A]). *Assume a finite group G acts effectively on a smooth variety X, and all the geometric quotients $X_\lambda/C(\lambda)$ are smooth for $\lambda \in G/\sim$. Then there is a semiorthogonal decomposition of the derived category $\mathcal{D}([X/G])$ such that the components $\mathcal{C}_{[\lambda]}$ of this decomposition are in bijection with conjugacy classes in G and $\mathcal{C}_{[\lambda]} \cong \mathcal{D}(X_\lambda/C(\lambda))$.*

This conjecture was verified in [18] in the case where G is a complex reflection group of types A, B, G_2, F_4, and $G(m,1,n)$ acting on a vector space V, as well as for some actions on C^n, where C is a smooth curve. Other global results exist for cyclic quotients, see [13, Theorem 4.1] and [14, Theorem 3.3.2], and for quotients of curves, see [17, Theorem 1.2]. It is shown in [3, Theorem D] that the above conjecture fails without the assumption that G acts effectively. Note that we do not expect a natural bijection in Conjecture A as one can see in simple examples with the action of a cyclic group (see Example 4.3.3 below).

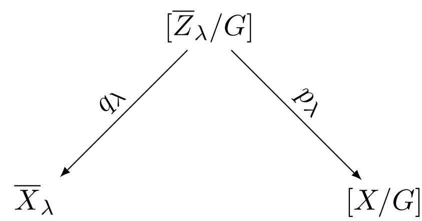
Because of the results mentioned above on the analogs of the decomposition (1.1), we refer to a semiorthogonal decomposition as in Conjecture A, as *motivic semiorthogonal decomposition.*

In all known cases of Conjecture A, the semiorthogonal decompositions are linear over $\mathcal{D}(X/G)$, where X/G is the geometric quotient, i.e., the Fourier-Mukai kernels giving the components of the semiorthogonal decomposition live on the fibered products over X/G. We describe this situation in Definition 1.1.1 below.

Let us set for brevity $\overline{X} = X/G$ (we assume that X and $\overline{X}$ are smooth). For each conjugacy class λ in G we pick a representative and denote by $X_\lambda \subset X$ the corresponding invariant locus. We set $\overline{X}_\lambda = X_\lambda/C(\lambda)$ (the geometric quotient),

$$\overline{Z}_\lambda = \overline{X}_\lambda \times_{\overline{X}} X. \tag{1.2}$$

Note that $\overline{Z}_\lambda$ is equipped with a natural G-action induced by the G-action on X, so we have a diagram



in which q_λ is finite flat (since so is the map $X \to X/G$), while p_λ is finite.

For example, for $\lambda = 1$ we have $X_1 = X$, $C(1) = G$, $\overline{X}_1 = X/G = \overline{X}$, $\overline{Z}_1 = X$, p_1 is the identity map, $q_1 : [X/G] \to X/G$ is the natural projection. For $\lambda \neq 1$ the scheme $\overline{Z}_\lambda$ is typically nonreduced (see e.g., Example 3.3.1).

Definition 1.1.1. Let us say that the action of a finite group G on a smooth quasiprojective variety X satisfies condition (MSOD)[1] if

- all the quotients $\overline{X}_\lambda = X_\lambda/C(\lambda)$ are smooth;
- there exists a collection of objects K_λ in $\mathcal{D}([\overline{Z}_\lambda/G])$, such that the corresponding Fourier-Mukai functors

  $$\Phi_{K_\lambda} : \mathcal{D}(\overline{X}_\lambda) \to \mathcal{D}([X/G]) : F \mapsto p_{\lambda *}(K_\lambda \otimes q_\lambda^* F)$$

  are fully faithful (here the functors $p_{\lambda *}$, $\otimes$ and q_λ^* are derived);
- the corresponding subcategories give a semiorthogonal decomposition

  $$\mathcal{D}([X/G]) = \langle \mathcal{D}(\overline{X}_{\lambda_1}), \dots, \mathcal{D}(\overline{X}_{\lambda_r}) \rangle$$

  with respect to some total ordering $\lambda_1, \dots, \lambda_r$ on $G/\sim$.

[1]MSOD stands for "motivic semiorthogonal decomposition"

1.2. Restricting (MSOD) to G-invariant divisors and application to S_n-invariant hypersurfaces

The main observation we make in this paper is that condition (MSOD) is preserved when passing to sufficiently generic G-invariant divisors. Namely, we assume that the action of G on X is effective and denote by $X^{fr} \subset X$ the open subset on which G acts freely. Similarly, for each λ and every connected component $Y \subset X_\lambda$, let us denote by $W(Y)$ the quotient of $C(\lambda)$ that acts effectively on Y, and let $Y^{fr} \subset X_\lambda$ denote the open subset on which $W(Y)$ acts freely. In the case when X_λ is connected we will write $W_\lambda := W(X_\lambda)$.

We will impose the following assumption on a divisor H in X:

> ($*$) for every λ and every connected component $Y \subset X_\lambda$, H does not contain Y and $H \cap Y^{fr}$ is dense in $H \cap Y$ (in particular, $H \cap X^{fr}$ is dense in H).

Theorem 1.2.1. *Assume that the pair (X, G) satisfies (MSOD), and let $H \subset X$ be a smooth G-invariant divisor satisfying ($*$). Then the pair (H, G) satisfies (MSOD).*

We will deduce this result from Kuznetsov's base change for semiorthogonal decompositions [12].

To get applications of this theorem, one should start with some pairs (X, G) for which condition (MSOD) is already known. We mostly focus on the case of the S_n action on $\mathbb{A}^n$ (in which case the semiorthogonal decomposition of the required type was constructed in [18]), and also consider pairs of the form $(C_1 \times \cdots \times C_n, G_1 \times \cdots \times G_n)$, where for each i, G_i is a finite group acting effectively on a smooth curve C_i.

We combine Theorem 1.2.1 with two simpler procedures: replacing X by a G-invariant open subset and passing to the quotient by a free action of $\mathbb{G}_m$. This leads us in the case of $[V/S_n]$, where $V = \mathbb{A}^n$, to the following semiorthogonal decomposition for the projective hypersurface given by an S_n-invariant homogeneous polynomial f.

Note that in this case the conjugacy classes in S_n are numbered by partitions λ of n. For each λ, we have the corresponding linear subspace V_λ of invariants and we denote by W_λ the quotient of $C(\lambda)$ acting effectively on V_λ (see Sec. 4.1 for details). There is an induced $\mathbb{G}_m$-action on $\overline{V}_\lambda := V_\lambda/W_\lambda$, and $f_\lambda := f|_{V_\lambda}$ descends to a quasihomogeneous polynomial $\overline{f}_\lambda$ on $\overline{V}_\lambda$. We denote by $X_{\overline{f}_\lambda} \subset \mathbb{P}\overline{V}_\lambda$ the corresponding weighted projective hypersurface stack defined as the quotient of the affine hypersurface $\overline{f}_\lambda = 0$ with the origin removed, by the action of $\mathbb{G}_m$ (here $\mathbb{P}\overline{V}_\lambda$ is the weighted projective space stack $[\overline{V}_\lambda \setminus \{0\}/\mathbb{G}_m]$).

Theorem 1.2.2. *Let f be an S_n-invariant homogeneous polynomial on $V = \mathbb{A}^n$, such that the corresponding projective hypersurface $X_f = \mathbb{P}H(f) \subset \mathbb{P}(V)$ is smooth. Then there exists a semiorthogonal decomposition*

$$\mathcal{D}([X_f/S_n]) = \langle \mathcal{D}(X_{\overline{f}_{\lambda_1}}), \dots, \mathcal{D}(X_{\overline{f}_{\lambda_r}}) \rangle,$$

where $\lambda_1 < \cdots < \lambda_r$ is a total order on partitions of n refining the dominance order.

Note that the decomposition of Theorem 1.2.2 no longer follows the pattern of Conjecture A since some components of the decompositions are themselves derived categories of stacks. The only similarity is that in both cases there is a birational morphism of stacks inducing a fully faithful embedding of derived categories via the pull-back (namely, $[X/G] \to X/G$ in Conjecture A and $[X_f/S_n] \to X_{\overline{f}_{1^n}}$ in Theorem 1.2.2), which is then extended to a semiorthogonal decomposition of the derived category of the source stack.

1.3. Outline of paper

In Section 2, after some preliminary material, we review Kuznetsov's theory of base change for semiorthogonal decompositions. In Section 3, we prove Theorem 1.2.1 and discuss the procedure of inducing the semiorthogonal decomposition on the quotient by an action of a reductive algebraic group. In Section 4, we consider applications of Theorem 1.2.1. In particular, in Section 4.2 we prove Theorem 1.2.2. In Section 4.3 we consider applications related to the stacks $[C_1 \times \cdots \times C_n/(G_1 \times \cdots \times G_n)]$, where G_i is a finite group acting on a smooth curve C_i.

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1.5. Conventions

We work over $\mathbb{C}$. All varieties are assumed to be quasiprojective (in particular, when a finite group acts on such a variety, the geometric quotient exists). All stacks are assumed to be quasiprojective DM-stacks in the sense of [10, Definition 5.5]. All functors are assumed to be derived. We denote by $\mathcal{D}(X)$ (resp., $\mathrm{Perf}(X)$), for X a variety or a stack, the bounded derived category of coherent sheaves on X (resp., the subcategory of perfect complexes). When G is an algebraic group acting on a variety X, we denote by $[X/G]$ the corresponding quotient stack, whereas X/G denotes the geometric quotient (when it exists). We always denote by $\mathbb{P}(a_1, \ldots, a_n)$ the weighted projective space *stack* obtained as the quotient stack $[\mathbb{A}^n \setminus \{0\}/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts with the weights $(a_1, \ldots, a_n)$.

2. Semiorthogonal decompositions and base change for stacks

In this section, we will prove a version of Kuznetsov's base change for semiorthogonal decompositions of derived categories of stacks.

2.1. Semiorthogonal decompositions

Recall that a *semiorthogonal decomposition* of a triangulated category $\mathcal{T}$ is a pair $\mathcal{A}, \mathcal{B}$ of full triangulated subcategories of $\mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(\mathcal{B}, \mathcal{A}) = 0$, and every object $t \in \mathcal{T}$ fits into an exact triangle

$$b \to t \to a \to b[1]$$

where $a \in \mathcal{A}, b \in \mathcal{B}$. In this case, we write $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$. We can iterate this definition to get semiorthogonal decompositions with any finite number of components $\mathcal{A}_1, \ldots, \mathcal{A}_n$ and we write

$$\mathcal{T} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle.$$

For an overview of semiorthogonal decompositions in algebraic geometry, see [5, 6].

2.2. Fourier-Mukai functors

Recall that following [10], we call a DM-stack $\mathcal{X}$ *quasiprojective* if it has quasiprojective coarse moduli space and is a global quotient of a quasiprojective scheme by a reductive algebraic group. For example, the quotient stack $[X/G]$, where G is a finite group acting on a quasiprojective scheme X, satisfies these conditions and its coarse moduli space is X/G. By [10, Prop. 5.1], such a stack has the resolution property, i.e., every coherent sheaf on it admits a surjective morphism from a vector bundle. Also, such a stack has an affine diagonal. We denote by $\mathcal{D}(\mathcal{X})$ the bounded derived category of coherent sheaves on $\mathcal{X}$ and by $\mathrm{Perf}(\mathcal{X}) \subset \mathcal{D}(\mathcal{X})$ the perfect derived category.

An object $\mathcal{K} \in \mathcal{D}(\mathcal{X} \times \mathcal{Y})$, whose support is proper over $\mathcal{Y}$, gives rise to an exact functor $\Phi_{\mathcal{K}}\colon \mathrm{Perf}(\mathcal{X}) \to \mathcal{D}(\mathcal{Y})$ defined by

$$\Phi_{\mathcal{K}}(F) = \pi_{\mathcal{Y}*}(\pi_{\mathcal{X}}^{*}F \otimes \mathcal{K}),$$

where $\pi_{\mathcal{X}} : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ and $\pi_{\mathcal{Y}} : \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ are the projections. We will refer to $\mathcal{K}$ as a *Fourier-Mukai kernel* and $\Phi_{\mathcal{K}}$ a *Fourier-Mukai functor*. Note that this functor also has a natural extension $\mathcal{D}_{qc}(\mathcal{X}) \to \mathcal{D}_{qc}(\mathcal{Y})$ to unbounded derived categories of quasicoherent sheaves, which has the right adjoint

$$\Phi^{!}_{\mathcal{K}}(G) = \pi_{\mathcal{X}*}\underline{\mathrm{Hom}}(\mathcal{K}, \pi^{!}_{\mathcal{Y}}G), \tag{2.1}$$

where $\pi^{!}_{\mathcal{Y}}$ is the right adjoint to $\pi_{\mathcal{Y}*}$.

The formalism of Fourier-Mukai functors, e.g., as in [9], extends routinely to the case of smooth DM-stacks (see [2] where Fourier-Mukai functors are considered in a much more general context).

Note that in the case when $\mathcal{X}$ and $\mathcal{Y}$ have maps to some DM-stack $\mathcal{S}$, then it is natural to consider relative Fourier-Mukai functors $\Phi_{\mathcal{K}}$ associated with kernels $\mathcal{K}$ on $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$, defined in the same way as above. (One gets the same functor by considering the usual Fourier-Mukai functor associated with the push-forward of $\mathcal{K}$ with respect to the morphism $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$.) We refer to such Fourier-Mukai functors as $\mathcal{S}$*-linear* since they commute with tensoring by the pull-backs of objects in $\mathrm{Perf}(\mathcal{S})$ (as one can easily see from the projection formula). Note that the right adjoint functor $\Phi^{!}_{\mathcal{K}}$ is also $\mathcal{S}$-linear (see [11, Lemma 2.34]). Also, under appropriate assumptions, such relative Fourier-Mukai functors are compatible with pull-backs under a base change (see Proposition 2.3.3 below).

2.3. Base change for semiorthogonal decompositions

Here we will recall the result of [12], on the base change for semiorthogonal decompositions. For our purposes, we need a slight generalization to Deligne-Mumford stacks. Throughout, $\mathcal{X}, \mathcal{X}_i, \mathcal{S}, \mathcal{T}$ will be quasiprojective DM stacks in the sense of [10].

The following technical definition plays an important role in the base change.

Definition 2.3.1. Suppose we have morphisms of quasiprojective DM stacks $f\colon \mathcal{X} \to \mathcal{S}$ and $\varphi\colon \mathcal{T} \to \mathcal{S}$. Then the cartesian diagram

$$\begin{array}{ccc} \mathcal{X}_{\mathcal{T}} & \xrightarrow{\varphi_{\mathcal{X}}} & \mathcal{X} \\ \downarrow{\scriptstyle f_{\mathcal{T}}} & & \downarrow{\scriptstyle f} \\ \mathcal{T} & \xrightarrow{\varphi} & \mathcal{S} \end{array} \tag{2.2}$$

is called **exact** if the natural map $\varphi^* f_* \to (f_{\mathcal{T}})_* \varphi_{\mathcal{X}}^*$ is an isomorphism. In this case we say that the base change $\varphi\colon \mathcal{T} \to \mathcal{S}$ is **faithful** for the map f.

For example, the cartesian diagram is exact if either f or φ is flat (this is proved similarly to [11, Corollary 2.23]).

Lemma 2.3.2. *Assume the square* (2.2) *is exact cartesian. Then the perfect derived category* $\mathrm{Perf}(\mathcal{X}_{\mathcal{T}})$ *is classically generated by objects of the form* $\varphi_{\mathcal{X}}^* F \otimes f_{\mathcal{T}}^* G$ *with* $F \in \mathrm{Perf}(\mathcal{X})$ *and* $G \in \mathrm{Perf}(\mathcal{T})$.

1st proof. Arguing as in the case of schemes (see [12, Lemma 5.2]), we see that it is enough to check that for every coherent sheaf F on $\mathcal{X}_{\mathcal{T}}$ there exists a surjection $\varphi_{\mathcal{X}}^* P_{\mathcal{X}} \otimes f_{\mathcal{T}}^* P_{\mathcal{T}} \to F$, where $P_{\mathcal{X}}$ (resp., $P_{\mathcal{T}}$) is a vector bundle on $\mathcal{X}$ (resp., $\mathcal{T}$). Let us consider the natural map

$$\alpha : \mathcal{X}_{\mathcal{T}} = \mathcal{T} \times_{\mathcal{S}} \mathcal{X} \to \mathcal{T} \times \mathcal{X}.$$

Since it is obtained by the base change from the diagonal map of $\mathcal{S}$, α is affine. Hence, the adjunction map $\alpha^* \alpha_* F \to F$ is surjective (since the image of this map under α_* is surjective). Since $\alpha_* F$ is the union of its coherent subsheaves, we can find a coherent sheaf G on $\mathcal{T} \times \mathcal{X}$ with a surjective map $\alpha^* G \to F$. Now using ample line bundles on the coarse moduli spaces of $\mathcal{T}$ and $\mathcal{X}$, as well as vector bundles on $\mathcal{T}$ and $\mathcal{X}$ that have faithful action of the stabilizer subgroups at all geometric points, we can find vector bundles

$P_{\mathcal{X}}$ and $P_{\mathcal{T}}$ on $\mathcal{X}$ and $\mathcal{T}$, respectively, and a surjection $P_{\mathcal{X}} \otimes P_{\mathcal{T}} \to G$ (see [10, Sec. 5.2]). Thus, we get the composed surjection

$$\varphi_{\mathcal{X}}^* P_{\mathcal{X}} \otimes f_{\mathcal{T}}^* P_{\mathcal{T}} \simeq \alpha^*(P_{\mathcal{X}} \otimes P_{\mathcal{T}}) \to \alpha^* G \to F.$$

2nd proof (sketch). Here we use a result of Ben-Zvi, Nadler and Preygel [2] in the context of derived algebraic geometry. Using [11, Prop. 2.19] it is easy to see that exactness of a cartesian square is equivalent to its tor-independence. Hence, the derived fiber product $\mathcal{X}_{\mathcal{T}}^h = \mathcal{X} \times_{\mathcal{S}}^L \mathcal{T}$ is equivalent to the fiber product $\mathcal{X}_{\mathcal{T}}$. Indeed, let $(\mathcal{X}_{\mathcal{T}}^h, \mathcal{A}^{\cdot})$ be the derived fiber product. Then the cohomology sheaves $\mathcal{H}^{-*}(\mathcal{A}^{\cdot})$ are given by $\mathrm{Tor}_{\mathcal{O}_{\mathcal{S}}}^{-*}(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{T}})$ which vanishes for $* \neq 0$.

Now the assertion follows from the equivalence [2, Theorem 1.2]:

$$\mathrm{Perf}(\mathcal{X}) \otimes_{\mathrm{Perf}(\mathcal{S})} \mathrm{Perf}(\mathcal{T}) \xrightarrow{\sim} \mathrm{Perf}(\mathcal{X}_{\mathcal{T}}^h) \xrightarrow{\sim} \mathrm{Perf}(\mathcal{X}_{\mathcal{T}}). \qquad \square$$

Given an $\mathcal{S}$-linear Fourier-Mukai functor $\Phi_{\mathcal{K}} : \mathrm{Perf}(\mathcal{X}) \to \mathcal{D}(\mathcal{Y})$ with the kernel $\mathcal{K}$ on $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ (with proper support over $\mathcal{Y}$), one can consider a base change $\varphi : \mathcal{T} \to \mathcal{S}$, and the corresponding $\mathcal{T}$-linear Fourier-Mukai functor

$$\Phi_{\mathcal{K}_{\mathcal{T}}} : \mathrm{Perf}(\mathcal{X}_{\mathcal{T}}) \to \mathcal{D}(\mathcal{Y}_{\mathcal{T}})$$

given by the kernel $\mathcal{K}_{\mathcal{T}}$ obtained as the pull-back of $\mathcal{K}$ with respect to the natural morphism

$$\mathcal{X}_{\mathcal{T}} \times_{\mathcal{T}} \mathcal{Y}_{\mathcal{T}} \to \mathcal{X} \times_{\mathcal{S}} \mathcal{Y}.$$

The natural question is whether the functors $\Phi_{\mathcal{K}_{\mathcal{T}}}$ and $\Phi_{\mathcal{K}}$ are compatible with the pull-back functors induced by φ. For our purposes the following criterion will suffice (see [11, Lemma 2.42]).

Proposition 2.3.3. *In the above situation assume that the map $\mathcal{Y} \to \mathcal{S}$ is flat and the base change $\varphi : \mathcal{T} \to \mathcal{S}$ is faithful for $\mathcal{X} \to \mathcal{S}$. In addition, assume that $\mathcal{X} \to \mathcal{S}$ is proper and φ has finite Tor-dimension. Then for $F \in \mathcal{D}^-(\mathcal{X})$ and $G \in \mathcal{D}^+(\mathcal{Y})$ (where $\mathcal{D}^-, \mathcal{D}^+ \subset \mathcal{D}$ denote bounded above and bounded below derived categories), one has*

$$\Phi_{\mathcal{K}_{\mathcal{T}}} \varphi_{\mathcal{X}}^*(F) \simeq \varphi_{\mathcal{Y}}^* \Phi_{\mathcal{K}}(F),$$
$$\Phi^!_{\mathcal{K}_{\mathcal{T}}} \varphi_{\mathcal{Y}}^*(G) \simeq \varphi_{\mathcal{X}}^* \Phi^!_{\mathcal{K}}(G).$$

Here $\varphi_{\mathcal{X}} : \mathcal{X}_{\mathcal{T}} \to \mathcal{X}$ and $\varphi_{\mathcal{Y}} : \mathcal{Y}_{\mathcal{T}} \to \mathcal{Y}$ are the natural projections.

Proof. This is proved in the same way as [11, Lemma 2.42], by a calculation on the commutative cube obtained as the product over $\mathcal{S}$ of the cartesian square

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{S} \end{array}$$

with the arrow $\varphi : \mathcal{T} \to \mathcal{S}$. One has to observe that all the faces of this cube are exact cartesian and use the base change. The assumption that φ is of finite Tor-dimension is used to check that φ^* commutes with $\underline{\mathrm{Hom}}$. □

Remark 2.3.4. The condition that $\mathcal{Y} \to \mathcal{S}$ is flat in Proposition 2.3.3 can be replaced by a weaker condition that $\varphi : \mathcal{T} \to \mathcal{S}$ is faithful for $\mathcal{Y} \to \mathcal{S}$ and for $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{S}$, as is done in [11, Lemma 2.42]. Note that there is a slight mistake in the proof of [12, Theorem 6.4] where the above faithfulness assumption (as well as the assumption of smoothness of $\mathcal{S}$) is omitted.

The following result is similar to (but more special than) [12, Thm. 6.4].

Lemma 2.3.5. *Assume that $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{S}$ are smooth, $\mathcal{S}$ is separated, the morphisms $\mathcal{X} \to \mathcal{S}$ and $\mathcal{Y} \to \mathcal{S}$ are proper, and the morphism $\mathcal{Y} \to \mathcal{S}$ is flat. Let $\Phi_{\mathcal{K}} : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{Y})$ be the $\mathcal{S}$-linear Fourier-Mukai functor associated with a kernel $\mathcal{K}$ in $\mathcal{D}(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y})$, and let $\varphi : \mathcal{T} \to \mathcal{S}$ be a faithful base change for both $\mathcal{X}$ and $\mathcal{Y}$. Assume that the support of $\mathcal{K}$ is proper over $\mathcal{X}$ and that $\Phi_{\mathcal{K}}$ is fully faithful. Then $\Phi_{\mathcal{K}_{\mathcal{T}}} : \mathrm{Perf}(\mathcal{X}_{\mathcal{T}}) \to \mathcal{D}(\mathcal{Y}_{\mathcal{T}})$ is also fully faithful.*

Proof. First, we claim that the functor $\Phi^!_{\mathcal{K}}$ sends $\mathcal{D}(\mathcal{Y})$ to $\mathcal{D}(\mathcal{X})$. To this end we observe that $\Phi_{\mathcal{K}}$ can be computed as the absolute Fourier-Mukai functor $\Phi_{\mathcal{K}'}$, where the kernel $\mathcal{K}'$ is given by the push-forward of $\mathcal{K}$ with respect to the finite morphism $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ (finiteness of this morphism follows from the finiteness of the diagonal morphism for $\mathcal{S}$). Since $\mathcal{X} \times \mathcal{Y}$ is smooth, the right adjoint functor $\Phi^!_{\mathcal{K}'}$ sends $\mathcal{D}(\mathcal{Y})$ to $\mathcal{D}(\mathcal{X})$ (as follows from formula (2.1)).

Thus, the fact that $\Phi_{\mathcal{K}}$ is fully faithful on $\mathcal{D}(\mathcal{X})$ implies that the natural morphism

$$F \to \Phi^!_{\mathcal{K}} \Phi_{\mathcal{K}}(F) \tag{2.3}$$

is an isomorphism for $F \in \mathcal{D}(\mathcal{X})$.

Now to check that $\Phi_{\mathcal{K}_\mathcal{T}}$ is fully faithful on $\mathrm{Perf}(\mathcal{X}_\mathcal{T})$, by Lemma 2.3.2, it is enough to check that the morphism

$$\widetilde{F} \to \Phi^!_{\mathcal{K}_\mathcal{T}} \Phi_{\mathcal{K}_\mathcal{T}}(\widetilde{F})$$

is an isomorphism for objects of the form $\widetilde{F} = \varphi^*_\mathcal{X} F \otimes f^*_\mathcal{T} G$, where $F \in \mathrm{Perf}(\mathcal{X})$ and $G \in \mathrm{Perf}(\mathcal{T})$. But this easily follows from $\mathcal{T}$-linearity of our functors and from Proposition 2.3.3 (note that φ has finite Tor-dimension since $\mathcal{S}$ is smooth):

$$\Phi^!_{\mathcal{K}_\mathcal{T}} \Phi_{\mathcal{K}_\mathcal{T}}(\varphi^*_\mathcal{X} F \otimes f^*_\mathcal{T} G) \simeq \varphi^*_\mathcal{X} \Phi^!_\mathcal{K} \Phi_\mathcal{K}(F) \otimes f^*_\mathcal{T} G,$$

so the needed assertion follows from the fact that (2.3) is an isomorphism. □

Suppose we have an $\mathcal{S}$-linear semiorthogonal decomposition

$$\mathrm{Perf}(\mathcal{X}) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle.$$

Let us define subcategories $\mathcal{A}_{i\mathcal{T}} \subset \mathrm{Perf}(\mathcal{X}_\mathcal{T})$ by the formula

$$\mathcal{A}_{iT} = \langle \varphi^*_\mathcal{X} A \otimes f^*_\mathcal{T} G \rangle_{A \in \mathcal{A}_i, G \in \mathrm{Perf}(\mathcal{T})}.$$

The following two theorems are analogs of [12, Thm. 5.6] and [12, Thm. 6.4].

Theorem 2.3.6. *Suppose $\varphi\colon \mathcal{T} \to \mathcal{S}$ is faithful for $f\colon \mathcal{X} \to \mathcal{S}$. Assume that there is an $\mathcal{S}$-linear semiorthogonal decomposition*

$$\mathrm{Perf}(\mathcal{X}) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle.$$

Then the subcategories $\mathcal{A}_{iT}$ form a $\mathcal{T}$-linear semiorthogonal decomposition of the perfect derived category $\mathrm{Perf}(\mathcal{X}_\mathcal{T})$*,*

$$\mathrm{Perf}(\mathcal{X}_\mathcal{T}) = \langle \mathcal{A}_{1\mathcal{T}}, \ldots, \mathcal{A}_{m\mathcal{T}} \rangle.$$

Proof. As in [12], the semiorthogonality $\langle \mathcal{A}_{i\mathcal{T}}, \mathcal{A}_{j\mathcal{T}} \rangle$ for $i > j$ follows from faithful base change. Now Lemma 2.3.2 implies that the subcategories $\mathcal{A}_{1\mathcal{T}}, \ldots, \mathcal{A}_{m\mathcal{T}}$ generate $\mathrm{Perf}(\mathcal{X}_\mathcal{T})$, and the assertion follows. □

Theorem 2.3.7. *Suppose $\mathcal{X}$ and $\mathcal{S}$ are smooth, $\mathcal{S}$ is separated, the morphism $f\colon \mathcal{X} \to \mathcal{S}$ is flat and proper, and there is an $\mathcal{S}$-linear semiorthogonal* *decomposition*

$$\mathcal{D}(\mathcal{X}) = \langle \mathcal{D}(\mathcal{X}_1), \dots, \mathcal{D}(\mathcal{X}_m)\rangle,$$

where for $i = 1, \dots, m$, *the stacks* $\mathcal{X}_i$ *are smooth, the maps* $\mathcal{X}_i \xrightarrow{f_i} \mathcal{S}$ *are proper, and the embedding functors* $\Phi_i : \mathcal{D}(\mathcal{X}_i) \to \mathcal{D}(\mathcal{X})$ *are given by some kernels* $\mathcal{K}_i$ *in* $\mathcal{D}(\mathcal{X}_i \times_{\mathcal{S}} \mathcal{X})$. *Assume now that* $\varphi\colon \mathcal{T} \to \mathcal{S}$ *is a base change, faithful for* f *and for each* f_i. *Set* $\mathcal{X}_{i\mathcal{T}} = \mathcal{X}_i \times_{\mathcal{S}} \mathcal{T}$. *Then the pullbacks* $\mathcal{K}_{i\mathcal{T}}$ *of* $\mathcal{K}_i$ *to* $\mathcal{X}_{i\mathcal{T}} \times_{\mathcal{T}} \mathcal{X}_{\mathcal{T}}$ *define fully faithful functors*

$$\Phi_{i\mathcal{T}}\colon \operatorname{Perf}(\mathcal{X}_{i\mathcal{T}}) \to \operatorname{Perf}(\mathcal{X}_{\mathcal{T}}).$$

and their images give a $\mathcal{T}$*-linear semiorthogonal decompositon*

$$\operatorname{Perf}(\mathcal{X}_{\mathcal{T}}) = \langle \operatorname{Perf}(\mathcal{X}_{1\mathcal{T}}), \dots, \operatorname{Perf}(\mathcal{X}_{m\mathcal{T}})\rangle.$$

Proof. Let us set $\mathcal{A}_i = \Phi_i \operatorname{Perf}(\mathcal{X}_i)$. By Theorem 2.3.6, we get a semiorthogonal decomposition of $\operatorname{Perf}(\mathcal{X}_{\mathcal{T}})$ into the subcategories $\mathcal{A}_{i\mathcal{T}}$ (note that $\mathcal{D}(\mathcal{X}) = \operatorname{Perf}(\mathcal{X})$ and $\mathcal{D}(\mathcal{X}_i) = \operatorname{Perf}(\mathcal{X}_i)$ by smoothness).

Now we observe that for $F \in \operatorname{Perf}(\mathcal{X}_i)$ and $G \in \operatorname{Perf}(\mathcal{T})$, we have an isomorphism

$$\varphi_{\mathcal{X}}^*(\Phi_i F) \otimes f_{\mathcal{T}}^*(G) \simeq \Phi_{i\mathcal{T}}(\varphi_{\mathcal{X}_i}^* F) \otimes f_{\mathcal{T}}^*(G) \simeq \Phi_{i\mathcal{T}}(\varphi_{\mathcal{X}_i}^* F \otimes f_{i\mathcal{T}}^*(G)),$$

where we used commutation of relative Fourier-Mukai functors with the pullback (see Proposition 2.3.3) and $\mathcal{T}$-linearity of $\Phi_{i\mathcal{T}}$ (and $(f_{i\mathcal{T}}, \varphi_{\mathcal{X}_i})$ have the same meaning for $\mathcal{X}_i$ as (f, φ) for $\mathcal{X}$). Using Lemma 2.3.2 for $\mathcal{X}$ and $\mathcal{X}_i$, we deduce that the image of $\operatorname{Perf}(X_{i\mathcal{T}})$ under $\Phi_{i\mathcal{T}}$ is exactly $\mathcal{A}_{i\mathcal{T}} \subset \operatorname{Perf}(\mathcal{X}_{\mathcal{T}})$. Finally, by Lemma 2.3.5, the functors $\Phi_{i\mathcal{T}}$ are fully faithful. □

An easy example of the faithful base change is restricting to an open subset. In particular, we deduce that condition (MSOD) is preserved when passing to a G-invariant open subset.

Corollary 2.3.8. *Assume that the pair* (X, G) *satisfies (MSOD), and let* $U \subset X$ *be a* G*-invariant open subset. Then the pair* (U, G) *also satisfies (MSOD), and the corresponding kernels on* $\overline{U}_\lambda \times_{\overline{U}} U$ *are obtained as pullbacks of the kernels on* $\overline{X}_\lambda \times_{\overline{X}} X$.

Proof. To deduce this from Theorem 2.3.7, we observe that U is the preimage of the open subset $\overline{U} = U/G \subset \overline{X}$, and $\overline{U}_\lambda \subset \overline{X}_\lambda$ is the preimage of $\overline{U}$ under the map $\overline{X}_\lambda \to \overline{X}$. Note that the natural map of stacks over $[X/G]$,

$$[U/G] \to [X/G] \times_X U \simeq [X/G] \times_{\overline{X}} \overline{U},$$

is an equivalence, since it becomes an isomorphism after the base change $X \to [X/G]$.

Thus, we can apply Theorem 2.3.7 to the faithful base change $\overline{U} \to \overline{X}$. Furthermore, we have

$$(\overline{X}_\lambda \times_{\overline{X}} X) \times_{\overline{X}} \overline{U} \simeq \overline{X}_\lambda \times_{\overline{X}} U \simeq \overline{X}_\lambda \times_{\overline{X}} \overline{U} \times_{\overline{U}} U \simeq \overline{U}_\lambda \times_{\overline{U}} U,$$

so the new kernels live on the correct spaces. □

2.4. Products

We observe that condition (MSOD) is compatible with products (see [12, Corollary 5.10] for a more general result).

Lemma 2.4.1. *Let G (resp., G') be a finite group acting on a smooth variety X (resp., X'), and assume that condition (MSOD) is satisfied for (X, G) (resp., (X', G')). Then condition (MSOD) is also satisfied for the action of $G \times G'$ on $X \times X'$.*

Proof. Let (λ, λ') be a conjugacy class in $G \times G'$. The corresponding scheme

$$\overline{Z}_{\lambda,\lambda'} = (\overline{X}_\lambda \times \overline{X'}_{\lambda'}) \times_{\overline{X} \times \overline{X'}} (X \times X')$$

is naturally identified with $\overline{Z}_\lambda \times \overline{Z}_{\lambda'}$ so we can define the kernel $\mathcal{K}_{\lambda,\lambda'}$ on $\overline{Z}_{\lambda,\lambda'}$ as the exterior tensor product $\mathcal{K}_\lambda \boxtimes \mathcal{K}_{\lambda'}$. The corresponding functor sends $F \boxtimes F'$, where $F \in \mathcal{D}(\overline{X}_\lambda)$, $F' \in \mathcal{D}(\overline{X'}_\lambda)$, to $\Phi_{\mathcal{K}_\lambda}(F) \boxtimes \Phi_{\mathcal{K}_{\lambda'}}(F')$. Thus, computing morphisms between such objects using the Künneth formula, we deduce that the functor $\Phi_{\mathcal{K}_\lambda \boxtimes \mathcal{K}_{\lambda'}}$ is fully faithful on objects of the form $F \boxtimes F'$, and hence on all objects. Similarly, we check semiorthogonality between the images and generation. Thus, we get a semiorthogonal decomposition of $\mathcal{D}([X \times X'/(G \times G')])$ with respect to any total ordering of conjugacy classes in $G \times G'$ compatible with the partial order $(\lambda_1, \lambda'_1) \le (\lambda_2, \lambda'_2)$ if $\lambda_1 \le \lambda_2$ and $\lambda'_1 \le \lambda'_2$ (where we use the total orders on conjugacy classes in G and G' corresponding to the original decompositions). □

3. G-invariant divisors and proof of Theorem 1.2.1

3.1. Smooth G-invariant divisors

Throughout this section we fix a smooth connected variety X with an effective action of a finite group G, such that $\overline{X} = X/G$ is smooth. We denote by $X^{fr} \subset X$ the open subset on which the action of G is free. Recall that $X_\lambda \subset X$ denotes the λ-invariant locus in X (where λ runs over a set of representatives of conjugacy classes in G), and $\overline{X}_\lambda = X_\lambda/C(\lambda)$. Note that the ideal sheaf of X_λ is generated locally by elements of the form $\lambda^*(f) - f$, with $f \in \mathcal{O}_X$, and with this subscheme structure X_λ is smooth.

For each λ and every connected component $Y \subset X_\lambda$, let us denote by $W(Y)$ the quotient of $C(\lambda)$ that acts effectively on Y, and let $Y^{fr} \subset X_\lambda$ denote the open subset on which $W(Y)$ acts freely.

Lemma 3.1.1. *The morphism $X \to \overline{X} = X/G$ is finite flat of degree $|G|$.*

Proof. It is well known that the morphism $X \to X/G$ is finite surjective of degree $|G|$ (see [16, Ch. II.7]). Since X and $\overline{X}$ are smooth, it is flat by the miracle flatness theorem. □

Now let $H \subset X$ be a smooth G-invariant divisor. We have the induced action of G on H, so we can consider varieties $\overline{H} = H/G$, $H_\lambda \subset H$ and $\overline{H}_\lambda = H_\lambda/C(\lambda)$. It is easy to see that

$$H_\lambda = H \cap X_\lambda,$$

the scheme-theoretic intersection.

We start by observing that the smoothness of the geometric quotient is preserved upon passing to a smooth G-invariant divisor.

Proposition 3.1.2. *For a smooth G-invariant divisor $H \subset X$, the quotient $\overline{H} = H/G$ is smooth.*

Proof. Assume first that $x \in H$ is a G-invariant point. We can linearize the action in a formal neighborhood of x in X, so the divisor H will be a G-invariant hyperplane. Since X/G is smooth at x, G is generated by pseudo-reflections. Hence, the same is true for the induced action of G on H, so the quotient H/G is smooth at x.

Now let $x \in H$ be arbitrary, and let $\mathrm{St}_x \subset G$ denote the stabilizer subgroup of x. By Luna's étale slice theorem, [15], the map $X/\mathrm{St}_x \to X/G$

is étale near the image of x in X/St_x. Since X/G is smooth, this implies X/St_x is also smooth at x. Thus, H/St_x is smooth at x, by the previous argument. Since the mapping $H/\mathrm{St}_x \to H/G$ is étale at the image of x in H/St_x, we conclude that H/G is smooth at the image of x (using [8, Theorem 17.11.1]). □

Corollary 3.1.3. *For any λ, if $\overline{X}_\lambda$ is smooth then $\overline{H}_\lambda$ is smooth.*

Proof. The scheme X_λ is smooth as the fixed locus of a finite order automorphism in X. Similarly, H_λ is smooth as H is smooth and H_λ is the fixed locus of λ. Thus, $H_\lambda = H \cap X_\lambda$ is a smooth divisor in X_λ, so we can apply Proposition 3.1.2. □

Lemma 3.1.4. *Assume that $H \cap X^{fr}$ is dense in H. Then the square*

$$\begin{array}{ccc} H & \longrightarrow & \overline{H} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \overline{X} \end{array}$$

is exact cartesian.

Proof. Since $\overline{H}$ (resp., H) is a divisor in $\overline{X}$ (resp., X), and both X and $\overline{X}$ are smooth, by [11, Cor. 2.27], it is enough to check that our square is cartesian. By Lemma 3.1.1, both maps $X \to \overline{X}$ and $H \to \overline{H}$ are finite flat of degree $|G|$ (here we use the assumption that $H \cap X^{fr}$ is dense in H and Proposition 3.1.2 which assures that $\overline{H}$ is smooth). The embedding of H into X factors through $X \times_{\overline{X}} \overline{H}$ which is a closed subscheme of X. Thus $H \subset X \times_{\overline{X}} \overline{H}$ is a closed embedding of schemes, both of which are finite flat of degree $|G|$ over $\overline{H}$, and the assertion follows. □

Proposition 3.1.5. *Assume that for some λ, $\overline{X}_\lambda$ is smooth, and that H satisfies condition $(*)$ introduced before Theorem 1.2.1. Then both squares* *in the diagram*

(3.1)
$$\begin{array}{ccccc} H_\lambda & \longrightarrow & \overline{H}_\lambda & \longrightarrow & \overline{H} \\ \downarrow & & \downarrow & & \downarrow \\ X_\lambda & \longrightarrow & \overline{X}_\lambda & \longrightarrow & \overline{X} \end{array}$$

are exact cartesian.

Proof. By [11, Cor. 2.27], it is enough to check that these squares are cartesian. First, we observe that by Lemma 3.1.4, the right square in the diagram

$$\begin{array}{ccccc} H_\lambda & \longrightarrow & H & \longrightarrow & \overline{H} \\ \downarrow & & \downarrow & & \downarrow \\ X_\lambda & \longrightarrow & X & \longrightarrow & \overline{X} \end{array}$$

is cartesian. Since H_λ is the scheme-theoretic intersection of X_λ with H, the left square in this diagram is also cartesian. Hence, the big rectangle in this diagram, which is the same as the big rectangle in diagram (3.1), is cartesian.

Next, applying Lemma 3.1.4 to every connected component Y of X_λ, the group $W(Y)$ acting on it and the divisor $Y \cap H_\lambda$, we get that the left square is cartesian. Since the map $X_\lambda \to \overline{X}_\lambda$ is flat and surjective, and the big rectangle is cartesian, we derive the same for the right square (by checking that the map $\overline{H}_\lambda \to \overline{X}_\lambda \times_{\overline{X}} \overline{H}$ of $\overline{X}_\lambda$-schemes becomes an isomorphism after the base change $X_\lambda \to \overline{X}_\lambda$). □

3.2. Proof of Theorem 1.2.1

The condition (MSOD) for (X, G) gives an $\overline{X}$-linear semiorthogonal decomposition of $\mathcal{D}([X/G])$. To deduce from this the same condition for (H, G), we want to apply Theorem 2.3.7 to the base change of the morphism $[X/G] \to \overline{X}$ with respect to the morphism $\overline{H} \to \overline{X}$, i.e., to $\mathcal{X} = [X/G]$, $\mathcal{S} = \overline{X}$, $\mathcal{X}_i = \overline{X}_\lambda$ and $\mathcal{T} = \overline{H}$.

Note that by assumption, $[X/G]$, $\overline{X}$ and $\overline{X}_\lambda$ are smooth. Also, the morphisms $[X/G] \to X/G = \overline{X}$ and $\overline{X}_\lambda \to \overline{X}$ are proper, so Theorem 2.3.7 is applicable.

We claim that the corresponding diagram

$$\begin{array}{ccc} [H/G] & \longrightarrow & \overline{H} \\ \big\downarrow & & \big\downarrow \\ [X/G] & \longrightarrow & \overline{X} \end{array}$$

is exact cartesian. Indeed, by Lemma 3.1.4 the natural 1-morphism of stacks $[H/G] \to [X/G] \times_{\overline{X}} \overline{H}$ over $[X/G]$ becomes an isomorphism $H \to X \times_{\overline{X}} \overline{H}$ after the base change $X \to [X/G]$. Hence, it is an equivalence.

Also, by Proposition 3.1.5, the base change of $\overline{X}_\lambda$ gives us $\overline{H}_\lambda$. Note that by Corollary 3.1.3, $[H/G]$ and $\overline{H}_\lambda$ are smooth. Thus, Theorem 2.3.7 gives an $\overline{H}$-linear semiorthogonal decomposition of $\mathrm{Perf}([H/G]) = \mathcal{D}([H/G])$ with the components $\mathrm{Perf}(\overline{H}_\lambda) = \mathcal{D}(\overline{H}_\lambda)$. Furthermore, the kernel on $\overline{H}_\lambda \times_{\overline{H}} H$ giving the functor $\mathcal{D}(\overline{H}_\lambda) \to \mathcal{D}([H/G])$ is given by the pullback of K_λ. Thus, all conditions of Definition 1.1.1 are satisfied for the action of G on H. □

3.3. Subschemes Z_λ in $X_\lambda \times X$

The content of this subsection is not used anywhere else in the paper. Here we discuss certain natural subschemes of $X_\lambda \times_{\overline{X}} X$. In some situations considered in [18] they are related to the equivariant cohomology of the Springer fibers.

Assume that G is a finite group acting on a quasiprojective smooth variety X, such that all $\overline{X}_\lambda = X_\lambda/C(\lambda)$ are smooth.

For each $\lambda \in G/\!\sim$, let us define the closed subscheme $Z_\lambda \subset X_\lambda \times_{\overline{X}} X$ by

$$Z_\lambda := X_\lambda \times_{\overline{X}} X,$$

and let $Z_\lambda^{\mathrm{red}} \subset Z_\lambda$ be the corresponding reduced subscheme. Note that Z_λ^{red} is the union of the graphs of the embeddings $g : X_\lambda \to X$, for g running over $G/C(\lambda)$. [2] It is easy to see that $\overline{Z}_\lambda = Z_\lambda/C(\lambda)$, where $\overline{Z}_\lambda$ is defined by (1.2). Also, we have $\overline{Z}_\lambda^{\mathrm{red}} = Z_\lambda^{\mathrm{red}}/C(\lambda)$ since both are subschemes of $\overline{Z}_\lambda$

[2]Our notation is different from [18] where Z_λ denotes the reduced subscheme.

defined by a nilpotent ideal and since the quotient of a reduced scheme by a finite group is reduced.

The schemes $\overline{Z}_\lambda^{\text{red}}$ play an important role in the work [18]: in the examples considered in that paper (see also Sec. 4.1 below), the kernels of the functors defining the semiorthogonal decompositions of $\mathcal{D}([X/G])$ are given by some vector bundles on $\overline{Z}_\lambda^{\text{red}}$.

The simplest example below shows that Z_λ and $\overline{Z}_\lambda$ are typically nonreduced.

Example 3.3.1. Let $X = \mathbb{A}^1$, $G = \mathbb{Z}_2$ acting on $\mathbb{A}^1$ by $x \mapsto -x$. We can take $t = x^2$ as a coordinate on $X/G \simeq \mathbb{A}^1$. Then for $\lambda \neq 1$, $\overline{Z}_\lambda = Z_\lambda \subset \mathbb{A}^1$ is the subscheme corresponding to the ideal (x^2).

Note that in the special case $\lambda = 1$, we have $\overline{Z}_1 = X$, which is reduced. It turns out that the subscheme $Z_1 = X \times_{\overline{X}} X \subset X \times X$ is still reduced (and is equal to the union of the graphs of all $g \in G$ acting on X) provided the action of G is effective.

Lemma 3.3.2. *Assume that the action of G on X is effective, and the schemes X and X/G are smooth. Then Z_1 is reduced.*

Proof. Since the projection $X \to \overline{X}$ is finite flat, the same is true about the projection $p_1 : Z_1 \to X$. Thus, $p_{1*}\mathcal{O}_{Z_1}$ is locally free over $\mathcal{O}_X$, in particular, it is torsion free as an $\mathcal{O}_X$-module. Furthermore, the fact that the action of G is effective implies that Z_1 is reduced over a generic point of X. Hence, the nilradical of $\mathcal{O}_{Z_1}$ would give a torsion submodule $p_{1*}\mathcal{O}_{Z_1}$, so this nilradical has to be trivial. □

3.4. Passing to the quotient stacks

Note that a general theory of inducing semiorthogonal decompositions on quotients of varieties by actions of reductive groups is considered in [7]. We need an analogous result where instead of varieties we consider stacks of the form $[X/G]$.

Namely, assume that condition (MSOD) holds for a pair (X, G). Assume in addition that there is a reductive algebraic group $\mathbb{G}$ acting on X, such that the actions of G and $\mathbb{G}$ commute and $[X/\mathbb{G}]$ is a DM-stack. In particular, the subvarieties X_λ acquire the action of $C(\lambda) \times \mathbb{G}$ and there is an induced action of $\mathbb{G}$ on $\overline{X}_\lambda = X_\lambda/C(\lambda)$ and on $\overline{Z}_\lambda$. This action is compatible with the projections to $\overline{X}_\lambda$ and to X. Assume also that each kernel $\mathcal{K}_\lambda$ in $\mathcal{D}([\overline{Z}_\lambda/G])$

comes from an object $\widetilde{\mathcal{K}}_\lambda$ in $\mathcal{D}([\overline{Z}_\lambda/(G \times \mathbb{G})])$. In this case each $\widetilde{\mathcal{K}}_\lambda$ defines the Fourier-Mukai functor

$$\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda} : \mathcal{D}([\overline{X}_\lambda/\mathbb{G}]) \to \mathcal{D}([X/(G \times \mathbb{G})])$$

that fits into a commutative square

$$\begin{array}{ccc} \mathcal{D}([\overline{X}_\lambda/\mathbb{G}]) & \xrightarrow{\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda}} & \mathcal{D}([X/(G \times \mathbb{G})]) \\ \downarrow & & \downarrow \\ \mathcal{D}(\overline{X}_\lambda) & \xrightarrow{\Phi_{\mathcal{K}_\lambda}} & \mathcal{D}([X/G]) \end{array}$$

where the vertical arrows are given by forgetting the $\mathbb{G}$-action. In other words, $\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda}$ is defined by the same formula as $\Phi_{\mathcal{K}_\lambda}$, but we view the result as an object of the $G \times \mathbb{G}$-equivariant derived category.

Lemma 3.4.1. *The functors $\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda}$ are fully faithful and their images give a semiorthogonal decomposition*

$$\mathcal{D}([X/(G \times \mathbb{G})]) = \langle \mathcal{D}([\overline{X}_{\lambda_1}/\mathbb{G}]), \ldots, \mathcal{D}([\overline{X}_{\lambda_r}/\mathbb{G}]) \rangle.$$

Proof. For a pair of objects $\mathcal{F}, \mathcal{G} \in \mathcal{D}([\overline{X}_\lambda/\mathbb{G}])$, we have a commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}([\overline{X}_\lambda/\mathbb{G}])}(\mathcal{F}, \mathcal{G}) & \xrightarrow{\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda}} & \mathrm{Hom}_{\mathcal{D}([X/(G\times\mathbb{G})])}(\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda}(\mathcal{F}), \Phi^{\mathbb{G}}_{\mathcal{K}_\lambda}(\mathcal{G})) \\ \mathrm{forg}\downarrow & & \mathrm{forg}\downarrow \\ \mathrm{Hom}_{\mathcal{D}(\overline{X}_\lambda)}(\mathcal{F}, \mathcal{G})^{\mathbb{G}} & \xrightarrow{\Phi_{\mathcal{K}_\lambda}} & \mathrm{Hom}_{\mathcal{D}([X/G])}(\Phi_{\mathcal{K}_\lambda}(\mathcal{F}), \Phi_{\mathcal{K}_\lambda}(\mathcal{G}))^{\mathbb{G}} \end{array}$$

in which the vertical arrows are isomorphisms since $\mathbb{G}$ is reductive. Furthermore, since $\Phi_{\mathcal{K}_\lambda}$ is fully faithful, the bottom horizontal arrow is an isomorphism. Hence, the top horizontal arrow is also an isomorphism, i.e., $\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda}$ is fully faithful.

Similarly, if $\mathrm{Hom}(\Phi_{\mathcal{K}_\lambda}(\cdot), \Phi_{\mathcal{K}_\mu}(\cdot)) = 0$ then by passing to $\mathbb{G}$-invariants, we deduce that $\mathrm{Hom}(\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda}(\cdot), \Phi^{\mathbb{G}}_{\mathcal{K}_\mu}(\cdot)) = 0$. Hence, the semiorthogonality still holds for the images of $\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda}$.

Finally, to see that the images of $\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda}$ generate $\mathcal{D}([X/(G\times\mathbb{G})])$, we observe that the right adjoint functors $(\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda})^!$ and $\Phi^!_{\mathcal{K}_\lambda}$ to $\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda}$ and $\Phi_{\mathcal{K}_\lambda}$ are still compatible with the forgetful functors, i.e., we have a commutative diagram

$$\begin{array}{ccc} \mathcal{D}([X/G\times\mathbb{G}]) & \xrightarrow{(\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda})^!} & \mathcal{D}([\overline{X}_\lambda/\mathbb{G}]) \\ \big\downarrow{\scriptstyle \text{forg}} & & \big\downarrow{\scriptstyle \text{forg}} \\ \mathcal{D}([X/G]) & \xrightarrow{\Phi^!_{\mathcal{K}_\lambda}} & \mathcal{D}(\overline{X}_\lambda) \end{array}$$

Indeed, for any $F\in\mathcal{D}([X/G\times\mathbb{G}])$, we can define $(\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda})^!(F)$ by the same formula for $\Phi^!_{\mathcal{K}_\lambda}$ (see (2.1)) understood in terms of equivariant categories. Then the above commutative diagram holds and the adjunction follows from the chain of isomorphisms

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{D}([X/G\times\mathbb{G}])}(\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda}(F),F')\simeq \mathrm{Hom}_{\mathcal{D}([X/G])}(\Phi_{\mathcal{K}_\lambda}(F),F')^{\mathbb{G}}\\ &\simeq \mathrm{Hom}_{\mathcal{D}(\overline{X}_\lambda)}(F,\Phi^!_{\mathcal{K}_\lambda}(F'))^{\mathbb{G}}\simeq \mathrm{Hom}_{\mathcal{D}([\overline{X}_\lambda/\mathbb{G}])}(F,(\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda})^!(F')). \end{aligned}$$

Now suppose $\mathcal{F}\in\mathcal{D}([X/G\times\mathbb{G}])$ is right orthogonal to the images of $\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda}$. Then by adjointness, $(\Phi^{\mathbb{G}}_{\mathcal{K}_\lambda})^!(\mathcal{F})=0$ for all λ. Thus, using the above commutative diagram, we obtain that $\mathrm{forg}(\mathcal{F})$ is right orthogonal to the images of all $\Phi_{\mathcal{K}_\lambda}$. Using the original semiorthogonal decomposition, we conclude that $\mathrm{forg}(\mathcal{F})=0$. But the forgetful functor is conservative, so $\mathcal{F}=0$. □

Remark 3.4.2. The above lemma can also be deduced from the conservative descent theorem of Bergh and Schnürer [4].

4. Examples of semiorthogonal decompositions obtained from Theorem 1.2.1

4.1. Motivic decomposition for $\mathcal{D}([\mathbb{A}^n/S_n])$

Now we will focus on the case of the standard action of the symmetric group S_n on the affine n-space, $V=\mathbb{A}^n$. In this case $\overline{V}=V/S_n$ is still the affine space $\mathbb{A}^n$ and the morphism $V\to\overline{V}$ is given by the elementary symmetric polynomials.

The conjugacy classes of S_n are labeled by partitions $\lambda=(\lambda_1\ge\lambda_2\ge\cdots)$ of n. Recall that the *dominance partial ordering* $\le$ on partitions of n is defined by $\lambda\ge\mu$ if $\lambda_1+\cdots+\lambda_i\ge\mu_1+\cdots+\mu_i$ for all $i\ge1$. Note that (n) is the biggest partition and (1^n) is the smallest.

For each partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ of n, we choose as a representative of the corresponding conjugacy class the permutation $(1 \cdots \lambda_1)(\lambda_1 + 1 \cdots \lambda_1 + \lambda_2) \cdots$ The corresponding fixed locus $V_\lambda \subset V$ is isomorphic to $\mathbb{A}^{\ell(\lambda)}$, where $\ell(\lambda)$ is the length of λ, and the isomorphism is given by

$$(y_1, \dots, y_\ell) \mapsto (\underbrace{y_1, \dots, y_1}_{\lambda_1}, \underbrace{y_2, \dots, y_2}_{\lambda_2}, \dots, \underbrace{y_\ell, \dots, y_\ell}_{\lambda_\ell}). \tag{4.1}$$

If we write λ exponentially: $\lambda = (1^{r_1}, 2^{r_2}, \dots, p^{r_p})$ (where r_i is the multiplicity of the part i), then we have

$$C(\lambda) \simeq \prod_{i=1}^{p} ((\mathbb{Z}/i\mathbb{Z})^{r_i} \rtimes S_{r_i}),$$

and the quotient of $C(\lambda)$ that acts effectively on V_λ is the group $W_\lambda \simeq \prod_i S_{r_i}$. We have

$$\overline{V}_\lambda = V_\lambda / W_\lambda \simeq \mathbb{A}^{\ell(\lambda)} = \prod_{i=1}^{p} \mathbb{A}^{r_i},$$

and the quotient map $V_\lambda \to \overline{V}_\lambda$ is the product of maps $\mathbb{A}^{r_i} \to \mathbb{A}^{r_i}$ given by the elementary symmetric functions. Under the identiication (4.1), the open subset $V_\lambda^{fr} \subset V_\lambda$, on which W_λ acts freely, is given by

$$V_\lambda^{fr} = \{(y_1, \dots, y_\ell) \mid y_i \neq y_j \text{ whenever } \lambda_i = \lambda_j, i \neq j\}. \tag{4.2}$$

Recall that we have the reduced subscheme $Z_\lambda^{\mathrm{red}} \subset V_\lambda \times V$, invariant under the action of $W_\lambda \times S_n$ (see Sec. 3.3). Explicitly, this is the union of graphs of all maps $V_\lambda \to V : x \mapsto \sigma x$ over $\sigma \in S_n / C(\lambda)$. The quotient $Z_\lambda^{\mathrm{red}} / W_\lambda$ can be identified with the reduced subscheme

$$\overline{Z}_\lambda^{\mathrm{red}} \subset \overline{Z}_\lambda = \overline{V}_\lambda \times_{\overline{V}} V$$

defined as in (1.2). Let us consider the kernels

$$\mathcal{K}_\lambda = \mathcal{O}_{\overline{Z}_\lambda^{\mathrm{red}}}$$

on $\overline{Z}_\lambda$.

Theorem 4.1.1 ([18, Theorem 6.3.1]). *For each λ, $|\lambda| = n$, the functor $\Phi_{\mathcal{K}_\lambda} : \mathcal{D}(\overline{V}_\lambda) \to \mathcal{D}([V/S_n])$ is fully faithful. The images of these functors give* *a semiorthogonal decomposition*

$$\mathcal{D}([\mathbb{A}^n/S_n]) = \langle \mathcal{D}(\overline{V}_{\lambda_1}), \dots, \mathcal{D}(\overline{V}_{\lambda_r}) \rangle.$$

for any total ordering $\lambda_1 < \cdots < \lambda_r$ of partitions of n refining the dominance order. Thus, condition (MSOD) holds for the action of S_n on $\mathbb{A}^n$ and the kernels $(\mathcal{O}_{\overline{Z}_\lambda^{\mathrm{red}}})$.

For each λ, the natural $\mathbb{G}_m$-action on V_λ induces a $\mathbb{G}_m$-action on $\overline{V}_\lambda$ such that the morphism $V_\lambda \to \overline{V}_\lambda$ is $\mathbb{G}_m$-equivariant. We denote by

$$\mathbb{P}\overline{V}_\lambda = [(\overline{V}_\lambda \setminus \{0\})/\mathbb{G}_m]$$

the corresponding weighted projective space stack. More precisely, for each $\lambda = (1^{r_1}, 2^{r_2}, \dots, p^{r_p})$, we get the weighted projective space stack

$$\mathbb{P}\overline{V}_\lambda = \mathbb{P}(1, \dots, r_1, 1, \dots, r_2, \dots, 1, \dots, r_p).$$

Corollary 4.1.2. *There is a semiorthogonal decomposition*

$$\mathcal{D}([\mathbb{P}^{n-1}/S_n]) \cong \langle \mathcal{D}(\mathbb{P}\overline{V}_{\lambda_1}), \dots, \mathcal{D}(\mathbb{P}\overline{V}_{\lambda_r}) \rangle.$$

Proof. This follows from Theorem 4.1.1 by first restricting the semiorthogonal decomposition to the open subset $\mathbb{A}^n \setminus \{0\} \subset \mathbb{A}^n$ using Corollary 2.3.8, and then applying Lemma 3.4.1 to the natural $\mathbb{G}_m$-equivariant structures on the corresponding kernels (which are the structure sheaves of $\mathbb{G}_m$-invariant correspondences). □

4.2. S_n-invariant hypersurfaces

Let $f \in \mathbb{C}[x_1, \dots, x_n]^{S_n}$ be an S_n-invariant polynomial, and let $H(f) \subset V = \mathbb{A}^n$ be the corresponding hypersurface.

Corollary 4.2.1. *Let $T \subset V$ be an S_n-invariant closed subset containing the singular locus of $H(f)$. Assume that for every partition λ, such that $V_\lambda \setminus T \neq \emptyset$, the restriction $f|_{V_\lambda}$ is not identicially zero and the intersection $H(f|_{V_\lambda}) \cap V_\lambda^{fr} \setminus T$ is dense in $H(f|_{V_\lambda}) \setminus T$. Then the pair $(H(f) \setminus T, S_n)$ satisfies (MSOD), so we have a semiorthogonal decomposition*

$$\mathcal{D}([H(f) \setminus T/S_n]) = \langle \mathcal{D}((H(f|_{V_{\lambda_1}}) \setminus T)/W_{\lambda_1}), \dots, \mathcal{D}((H(f|_{V_{\lambda_r}}) \setminus T)/W_{\lambda_r}) \rangle$$

where $\lambda_1 < \cdots < \lambda_r$ is a total order on partitions of n refining the dominance order.

Proof. First, we use Corollary 2.3.8 to prove that condition (MSOD), that holds for the pair (V, S_n) by Theorem 4.1.1, is inherited by the pair $(V \setminus T, S_n)$. Next, we want to check condition $(*)$ for the divisor $H(f) \setminus T$ in $V \setminus T$. Thus, we need to check that for every partition λ such that $(V \setminus T)_\lambda = V_\lambda \setminus T$ is nonempty, we have that $H(f) \setminus T$ does not contain $V_\lambda \setminus T$, and the intersection $H(f) \cap V_\lambda \setminus T = H(f|_{V_\lambda}) \setminus T$ contains a dense open subset on which W_λ acts freely. But this follows from our assumption. Hence, we can apply Theorem 1.2.1 to the smooth divisor $H(f) \setminus T$ in $V \setminus T$ to deduce the result. □

The following simple observation will be useful for us.

Lemma 4.2.2. *For a homogeneous S_n-invariant polynomial f, such that the corresponding hypersurface $\mathbb{P}H(f) \subset \mathbb{P}^{n-1}$ is smooth, one has*

$$f(1, 1, \ldots, 1) \neq 0. \tag{4.3}$$

For any partition λ of n, the restriction $f_\lambda := f|_{V_\lambda}$ is not identically zero and $H(f_\lambda)$ is smooth away from the origin.

Proof. Indeed, consider the morphism

$$\sigma \colon \mathbb{P}^{n-1} \to \mathbb{P}(1, 2, \ldots, n)$$

given by the elementary symmetric polynomials. Then the differential of σ vanishes identically at the S_n-invariant point $(1 : 1 : \cdots : 1)$. But $H(f)$ is the preimage of a hypersurface under σ. Since $\mathbb{P}(H(f))$ is smooth, we deduce that $f(1, \ldots, 1) \neq 0$.

Since $(1, \ldots, 1) \in V_\lambda$, from (4.3) we get that the restriction $f_\lambda := f|_{V_\lambda}$ is not identically zero. Furthermore, $H(f_\lambda) \setminus \{0\}$ is the fixed locus of a permutation acting on $H(f) \setminus \{0\}$, hence, it is smooth. □

Now we are ready to prove Theorem 1.2.2. Recall that in this theorem we assume that f is a *homogeneous* S_n-invariant polynomial such that $\mathbb{P}H(f)$ is smooth.

Proof of Theorem 1.2.2. We would like to apply Corollary 4.2.1 with $T = \{0\}$ to get condition (MSOD) for the pair $(H(f) \setminus \{0\}, S_n)$. Let λ be a partition of n. By Lemma 4.2.2, the restriction $f_\lambda := f|_{V_\lambda}$ is not identically zero. Next, let us check that $H(f_\lambda) \cap V_\lambda^{fr} \setminus \{0\}$ is dense in $H(f_\lambda) \setminus \{0\}$. In the case $\dim V_\lambda \leq 1$, we have $H(f_\lambda) \setminus \{0\} = \emptyset$. Thus, due to the description (4.2) of V_λ^{fr}, it is enough to check that for $\dim V_\lambda \geq 2$, no component of the hypersurface $H(f_\lambda) \subset V_\lambda$ is contained in a hyperplane $y_i = y_j$ for some $i \neq j$ such that $\lambda_i = \lambda_j$.

In the case when the degree of f is 1, it is proportional to $x_1 + \cdots + x_n$, so this is clear. Now assume that $\deg(f) > 1$. By Lemma 4.2.2, $H(f_\lambda)$ is smooth away from the origin. Thus, if $\dim V_\lambda \geq 3$ then $H(f_\lambda)$ is irreducible (since each irreducible component of $H(f_\lambda)$ has dimension ≥ 2 and $H(f_\lambda) \setminus \{0\}$ is smooth), so it cannot be contained in any hyperplane. If $\dim V_\lambda = 2$ then λ has only two parts (λ_1, λ_2). In the case $\lambda_1 \neq \lambda_2$ the statement is empty, so we only have to check the assertion for $\lambda = (n/2, n/2)$ assuming that n is even. But in this case the non-free locus is the line spanned by $(1, \dots, 1)$, so the assertion follows from (4.3).

Thus, we obtain that the pair $(H(f) \setminus \{0\}, S_n)$ satisfies (MSOD). It remains to use Lemma 3.4.1 to pass to the quotients by $\mathbb{G}_m$. $\square$

The semiorthogonal decomposition given by Theorem 1.2.2 is usually not motivic since its components are derived categories of some quotient stacks. The biggest component of the semiorthogonal decomposition of $\mathcal{D}([\mathbb{P}H(f)/S_n])$ corresponds to the partition $\lambda = (1^n)$ and is the image of the pull-back functor with respect to the natural morphism of stacks

$$\pi : [\mathbb{P}H(f)/S_n] \to \mathbb{P}H(\overline{f}) \subset \mathbb{P}(1, 2, \dots, n),$$

where $\overline{f}$ is f viewed as a quasihomogeneous polynomial on $\mathbb{A}^n/S_n$ (so the target of π is the weighted projective stacky hypersurface). The morphism π fits into a Cartesian diagram

$$\begin{array}{ccc} [H(f) \setminus \{0\}/S_n] & \xrightarrow{\widetilde{\pi}} & H(\overline{f}) \setminus \{0\} \\ \Big\downarrow{\scriptstyle \mathbb{G}_m} & & \Big\downarrow{\scriptstyle \mathbb{G}_m} \\ [\mathbb{P}H(f)/S_n] & \xrightarrow{\pi} & \mathbb{P}H(\overline{f}) \end{array}$$

in which the vertical arrows are $\mathbb{G}_m$-torsors and the top horizontal arrow is the coarse moduli map for the action of S_n on $H(f) \setminus \{0\}$. Note that the fact that the pull-back functor under π is fully faithful can be directly deduced from the above diagram. Indeed, by the projection formula, it is enough to check that $R\pi_*\mathcal{O} \simeq \mathcal{O}$. By the base change formula, this reduces to a similar assertion for the morphism $\widetilde{\pi} : [H(f)/S_n] \to H(\overline{f})$, which is the map from a quotient stack by S_n to the corresponding geometric quotient $H(\overline{f})$. For this morphism we have $R^{>0}\widetilde{\pi}_*\mathcal{O} = 0$, and the isomorphism $\widetilde{\pi}_*\mathcal{O} \simeq \mathcal{O}$ follows from the fact that the algebra of functions $\mathcal{O}(H(\overline{f}))$ is identified with the subalgebra of S_n-invariants, $\mathcal{O}(H(f))^{S_n}$.

Example 4.2.3 (S_3-invariant plane curves). Let $C = \mathbb{P}H(f) \subset \mathbb{P}^2$ be an S_3-invariant smooth plane curve of degree d. Since in this case $f(1,1,1) \neq 0$ by Lemma 4.2.2, for the partition (3), we get $H(\bar{f}_{(3)}) = \{0\}$. Hence, the corresponding component in the semiorthogonal decomposition of $\mathcal{D}([C/S_3])$ is zero and can be skipped. Let us consider contributions of the two remaining partitions, (1^3) and $(2,1)$.

- (1^3): We have identifications $V_{(1^3)} = V$, $\overline{V}_{(1^3)} \cong \mathbb{A}^3_{1,2,3}$, where the subscripts indicate the $\mathbb{G}_m$-weights. The vanishing locus of $\bar{f}_{(1^3)}$, $\mathbb{P}H(\bar{f}_{(1^3)})$ will give a smooth stacky curve in $\mathbb{P}(1,2,3)$.
- $(2,1)$: We have identifications $\overline{V}_{(2,1)} = V_{(2,1)} = \{y = z\} \subset V$, and $f_{(2,1)}$ is the restriction of f to this plane. Since $H(f_{(2,1)})$ is smooth away from the origin, it is the union of d lines through the origin, say $l_1, \dots, l_d$. The projectivization is the union of d distinct points $p_1, \dots, p_d$ in the projective line $\mathbb{P}\overline{V}_{2,1}$.

Thus, we have a semiorthogonal decomposition

$$\mathcal{D}([C/S_3]) = \langle \mathcal{D}(\mathbb{P}H(\bar{f}_{(1^3)})), \mathcal{D}(p_1), \dots, \mathcal{D}(p_d) \rangle \tag{4.4}$$

In the case $d = 3$, i.e., when C is an elliptic curve, we can be even more precise about the piece corresponding to (1^3). Namely, in this case

$$f(x,y,z) = \alpha \mathrm{e}_1^3 + \beta \mathrm{e}_1 \mathrm{e}_2 + \gamma \mathrm{e}_3,$$

where $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3$ are elementary symmetric functions in x, y, z. Furthermore, we have $\gamma \neq 0$ (otherwise, C would contain the line $\mathrm{e}_1 = 0$). Thus, the equation $f = 0$ gives a way to express e_3 in terms of e_1 and e_2. Hence, $\mathbb{P}H(\bar{f}_{(1^3)})$ is the weighted projective line stack $\mathbb{P}(1,2)$.

In general, the derived category of $\mathbb{P}H(\bar{f}_{(1^3)})$ has a semiorthogonal decomposition with the main component given by the derived category of the coarse moduli, which is C/S_3, and some exceptional objects supported at the stacky points. Thus, the semiorthogonal decomposition (4.4) can be refined to a decomposition with the main component $\mathcal{D}(C/S_3)$ followed by exceptional objects. The obtained decomposition of $\mathcal{D}([C/S_3])$ matches the one constructed in [17] since the special fibers of the projection $C \to C/S_3$ are either orbits of the points $p_1, \dots, p_d$, corresponding to $\lambda = (2,1)$, or the points of C mapping to the two stacky points of $\mathbb{P}(1,2,3)$.

Note that if $d < 6$ then the geometric quotient of $\mathbb{P}H(\bar{f}_{(1^3)})$ is rational, so in this case the category $\mathcal{D}([C/S_3])$ has a full exceptional collection.

Some features of the above example occur in a more general situation. Below we use the power sum polynomials

$$\mathrm{p}_i(x_1, \ldots, x_n) = x_1^i + \cdots + x_n^i.$$

Proposition 4.2.4. *Let $f(x_1, \ldots, x_n)$ be a generic S_n-invariant homogeneous polynomial of degree $d > 0$.*
(i) Let λ be a partition of n such that all parts of λ are distinct. Then the stack $[\mathbb{P}H(\overline{f}_\lambda)]$ is actually a smooth projective variety.
(ii) Now assume that λ has one part of multiplicity 2 and all the other parts have multiplicity 1. Then the same conclusion as in (i) holds provided the degree d is even.

Proof. (i) For a generic S_n-invariant f, the hypersurface $H(f) \subset V$ is smooth away from the origin, hence, the same is true for $H(f_\lambda)$, the fixed locus of a permutation acting on $H(f)$. Since $\mathbb{P}\overline{V}_\lambda$ is the usual projective space, the assertion follows.
(ii) We have coordinates $(x, y; z_1, \ldots, z_p)$ on V_λ, so that the embedding $\iota_\lambda : V_\lambda \hookrightarrow V$ has form

$$\iota_\lambda : (x, y; z_1, \ldots, z_p) \mapsto (x, y, \ldots, x, y; z_1, \ldots, z_1, \ldots, z_p, \ldots, z_p),$$

where (x, y) is repeated l times, each z_j is repeated m_j times, so that $(l, m_1, \ldots, m_p)$ are all the distinct parts of λ, and l (resp., m_j) occur with multiplicity 2 (resp., 1) in λ. Set $\mathrm{p}_1 = x + y$, $\mathrm{p}_2 = x^2 + y^2$, so that $(\mathrm{p}_1, \mathrm{p}_2)$, (z_j) are the coordinates on $\overline{V}_\lambda$. It is enough to check that $\mathbb{P}H(\overline{f}_\lambda)$ does not contain stacky points of $\mathbb{P}\overline{V}_\lambda$, i.e., the points with $\mathrm{p}_1 = 0$ and all $z_j = 0$. Thus, it is enough that f_λ does not vanish at the point of V_λ with $x = -y = 1$ and $z_j = 0$. Note that

$$\mathrm{p}_2(x_1, \ldots, x_n)|_{\iota_\lambda(1,-1;0,\ldots,0)} \neq 0.$$

Therefore, the same is true for any power of p_2, and hence, for a generic S_n-invariant polynomial of even degree. □

In the next proposition we show that some components of the semiorthogonal decomposition of Theorem 1.2.2 are derived categories of weighted projective space stacks.

Proposition 4.2.5. *Let $f(x_1,\ldots,x_n)$ be an S_n-invariant homogeneous polynomial of degree $d\le n$ such that $\mathbb{P}H(f)$ is smooth, and such that in the expression of f as a polynomial in $\mathrm{p}_1,\ldots,\mathrm{p}_n$ the coefficient of p_d is nonzero. Then for a partition $\lambda=(1^{r_1},2^{r_2},\ldots,p^{r_p})$, such that $r_l\ge d$ for some l, the stack $[\mathbb{P}H(\overline{f}_\lambda)]$ is isomorphic to the weighted projective space stack with the weights obtained by removing one weight d from the sequence*

$$(1,\ldots,r_1,1,\ldots,r_2,\ldots,1,\ldots,r_p).$$

Proof. Let $y_1,\ldots,y_{r_l},z_1,\ldots,z_N$ be the coordinates on V_λ, where $y_1,\ldots,y_{r_l}$ are the coordinates corresponding to the parts of λ equal to l, so that in the embedding $V_\lambda\hookrightarrow V$ each of coordinates $y_1,\ldots,y_{r_l}$ is repeated l times (see (4.1)). Note that the coordinates on $\overline{V}_\lambda$ are given by the functions $(\mathrm{p}_i(y_1,\ldots,y_{r_l}))_{1\le i\le r_l}$, as well as some symmetric functions in other groups of variables. Since $r_l\ge d$, $\mathrm{p}_d(y_1,\ldots,y_{r_l})$ is one of the coordinates on $\overline{V}_\lambda$.

It suffices to check that $\mathrm{p}_d(y_1,\ldots,y_{r_l})$ occurs with nonzero coefficient in f_λ. Indeed, then we can use f_λ to express the coordinate $\mathrm{p}_d(y_1,\ldots,y_{r_l})$ in terms of other coordinates on $\overline{V}_\lambda$, which gives our assertion. We have

$$f(x_1,\ldots,x_n)=\alpha\cdot\mathrm{p}_d(x_1,\ldots,x_n)+g(x_1,\ldots,x_n),$$

where g is a polynomial in $(\mathrm{p}_i(x_1,\ldots,x_n))_{1\le i<d}$. Hence, the restriction of g to V_λ is expressed in terms of coordinates of weight $<d$ on $\overline{V}_\lambda$, so it does not contribute to the coefficient of $\mathrm{p}_d(y_1,\ldots,y_{r_l})$. Furthermore, we have

$$\begin{aligned}&\mathrm{p}_d(\underbrace{y_1,\ldots,y_1}_{l},\ldots,\underbrace{y_{r_l},\ldots,y_{r_l}}_{l},z_1,\ldots)\\&=l\cdot\mathrm{p}_d(y_1,\ldots,y_{r_l})\mod(\mathbb{C}z_1^d+\cdots+\mathbb{C}z_N^d).\end{aligned}$$

Hence, the coefficient of $\mathrm{p}_d(y_1,\ldots,y_{r_l})$ in f_λ is equal to $l\cdot\alpha$. In particular, this coefficient is nonzero, as required. □

Corollary 4.2.6. *The conclusion of Proposition 4.2.5 holds for any S_n-invariant homogeneous polynomial f of degree $d\le 3$ such that $\mathbb{P}H(f)$ is smooth.*

Proof. The case $d=1$ is trivial. In the case $d=2$, we have

$$f=\alpha\mathrm{p}_2+\beta\mathrm{p}_1^2,$$

while in the case $d = 3$, we have

$$f = \alpha \mathrm{p}_3 + \beta \mathrm{p}_1 \mathrm{p}_2 + \gamma \mathrm{p}_1^3,$$

In both cases $\alpha \neq 0$, since otherwise f would be reducible. Hence, we can apply Proposition 4.2.5. □

In the case of cubic forms in ≤ 6 variables, we obtain from Theorem 1.2.2 the following decompositions of S_n-equivariant derived categories.

Proposition 4.2.7. *Let $f(x_1, \ldots, x_n)$ be a generic S_n-invariant homogeneous cubic polynomial, where $n \leq 5$. Then $\mathcal{D}([\mathbb{P}H(f)/S_n])$ has a full exceptional collection. For $n = 6$, there is an exceptional collection in $\mathcal{D}([\mathbb{P}H(f)/S_6])$ such that its right orthogonal is equivalent to $\mathcal{D}(E)$, where E is the elliptic curve given by the cubic $f_{(3,2,1)}$ in $\mathbb{P}V_{(3,2,1)} \simeq \mathbb{P}^2$.*

Proof. First of all, we observe that for $n \leq 5$, a partition λ of n can have at most two distinct parts, while for $n = 6$ the only partition with 3 distinct parts is $(3, 2, 1)$.

By Corollary 4.2.6, if λ has a part of multiplicity ≥ 3 then the corresponding piece in the semiorthogonal decomposition of Theorem 1.2.2 is the derived category of the weighted projective space stack, so it has a full exceptional collection (see [1, Sec. 2]).

Now we claim that all partitions with at most two distinct parts, each of mulitplicity at most 2, lead to subcategories generated by exceptional collections. We prove this case by case. Note that the partition $\lambda = (n)$ does not contribute to the semiorthogonal decomposition since $f(1, \ldots, 1) \neq 0$.

Case $\lambda = (l, l)$. Then V_λ has coordinates x, y and $\overline{V}_\lambda$ has coordinates $\mathrm{p}_1 = x + y$, $\mathrm{p}_2 = x^2 + y^2$. The unique stacky point of the weighted projective line stack $\mathbb{P}\overline{V}_\lambda = \mathbb{P}(1, 2)$ is given by $\mathrm{p}_1 = 0$. Note that $\mathrm{p}_3 = x^3 + y^3$ is divisible by p_1, so $\overline{f}_\lambda$ vanishes at this point. It follows that $\mathbb{P}H(\overline{f}_\lambda)$ is the union of one point and of one stacky point with the automorphism group $\mathbb{Z}/2$. The derived category of such stacky point splits as the direct sum of two derived categories of the usual point.

Case $\lambda = (l_1, l_2)$, where $l_1 > l_2$. Then f_λ is a cubic on the 2-dimensional space V_λ, with isolated singularity at the origin, so $\mathbb{P}H(f_\lambda)$ is the union of three distinct points.

Case $\lambda = (l, 1, 1)$ with $l > 1$ or $\lambda = (2, 2, 1)$. Then V_λ has coordinates x, y, z, where $W_\lambda = S_2$ swaps x and y, so that $\overline{V}_\lambda$ has coordinates $\mathrm{p}_1 = x + y$,

$p_2 = x^2 + y^2$ and z, and $\mathbb{P}\overline{V}_\lambda = \mathbb{P}(1,1,2)$. The cubic $\overline{f}_\lambda$ should have form

$$\overline{f}_\lambda = p_2(\alpha z + \beta p_1) + C(p_1, z),$$

where $C(p_1, z)$ is a binary cubic form. It is easy to see that for generic S_n-invariant f, one has $\alpha \neq 0$, so we can make the change of variables $z_1 = \alpha z + \beta p_1$. Furthermore, $C(p_1, z)$ is not divisible by z_1, since f_λ has an isolated singularity at 0. Thus, rescaling the variables, we can bring f to the form

$$\overline{f}_\lambda = p_2 z_1 + z_1 Q(p_1, z_1) + p_1^3,$$

where Q is a binary quadratic form. Now taking $u = p_2 + Q(p_1, z_1)$ as a new variable of weight 2, we get

$$\overline{f}_\lambda = u z_1 + p_1^3.$$

It is easy to see that $\mathbb{P}H(\overline{f}_\lambda)$ is isomorphic to the weighted projective line stack $\mathbb{P}(1,2)$. Namely, there is an isomorphism given by

$$\mathbb{P}(1,2) \to \mathbb{P}H(\overline{f}_\lambda) : (t : v) \mapsto (u = v^3, z_1 = -t^3, p_1 = vt).$$

Case $\lambda = (2,2,1,1)$. Then we have coordinates x_1, y_1, x_2, y_2 on V_λ, and $W_\lambda = S_2 \times S_2$ permutes x_1 with y_1 and x_2 with y_2. Set $p_1(i) = x_i + y_i$, $p_2(i) = x_i^2 + y_i^2$. Then the cubic $\overline{f}_\lambda$ has form

$$\overline{f}_\lambda = p_2(1) z_1 + p_2(2) z_2 + C(p_1(1), p_1(2)),$$

where z_1 and z_2 are some linear forms in $p_1(1), p_1(2)$. It is easy to see that for generic f, the linear forms z_1 and z_2 will be linearly independent, so we can view $p_2(1), p_2(2), z_1, z_2$ as independent variables. Now adding to $p_2(i)$ appropriate quadratic expressions of z_1, z_2, we can rewrite $\overline{f}_\lambda$ as

$$\overline{f}_\lambda = u_1 z_1 + u_2 z_2,$$

where u_1, u_2, z_1, z_2 are independent variables $(\deg(u_i) = 2,\ \deg(z_i) = 1)$. Thus, we can identify $\mathbb{P}H(\overline{f}_\lambda)$ with $\mathbb{P}(1,2) \times \mathbb{P}^1$ via the isomorphism $\mathbb{P}(1,2) \times \mathbb{P}^1 \to \mathbb{P}H(\overline{f}_\lambda)$,

$$(t : v), (s_1 : s_2) \mapsto (u_1 = v s_1, u_2 = v s_2, z_1 = t s_2, z_2 = -t s_1).$$

Thus, for $n \leq 6$, all of the subcategories corresponding to $\lambda \neq (3,2,1)$ admit full exceptional collections. The remaining subcategory corresponding to $\lambda = (3, 2, 1)$ (for $n = 6$) is equivalent to $\mathcal{D}(E)$, where E is the elliptic curve given by $f_{(3,2,1)}$. □

Remark 4.2.8. Using Corollary 4.2.6 we see that for an S_n-invariant nondegenerate quadric f, the components of the semiorthogonal decomposition of Theorem 1.2.2 are either smooth projective quadrics (for partitions with distinct parts) or weighted projective space stacks (for the remaining partitions). In particular, in this case $\mathcal{D}([\mathbb{P}H(f)/S_n])$ has a full exceptional collection.

4.3. Products of curves

First, let us consider the case of an action of a finite group G on a smooth curve (we assume that the action is effective). Note that in this case the quotient C/G is smooth (since passing to invariants of a finite group preserves normality) and the stabilizer subgroup St_x of every point $x \in C$ is cyclic. Let

$$R = D_1 \sqcup \cdots \sqcup D_r \subset C$$

be the decomposition into G-orbits of the ramification locus of the projection $\pi : C \to C/G$. Then each D_i is a fiber of π and the stabilizer of a point in D_i is isomorphic to $\mathbb{Z}/m_i\mathbb{Z}$. Then the proof of [17, Thm. 1.2] implies that for each i, there is an exceptional collection of G-equivariant sheaves on C,

$$(\omega_C|_{D_i}, \omega_C^{\otimes 2}|_{D_i}, \ldots, \omega_C^{\otimes m_i - 1}|_{D_i}), \tag{4.5}$$

and if $\mathcal{A}_i \subset \mathcal{D}([C/G])$ is the subcategory generated by this collection, then there is a semiorthogonal decomposition

$$\mathcal{D}([C/Y]) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_r, \pi^*\mathcal{D}(C/G)\rangle, \tag{4.6}$$

where $\pi^* : \mathcal{D}(C/G) \to \mathcal{D}([C/G])$ is the pull-back functor. More precisely, in [17] a different decomposition was considered,

$$\mathcal{D}([C/Y]) = \langle \pi^*\mathcal{D}(C/G), \mathcal{B}_1, \ldots, \mathcal{B}_r\rangle, \tag{4.7}$$

with $\mathcal{A}_i = \omega_C \otimes \mathcal{B}_i$, from which (4.6) is obtained by mutation. Also, in [17] it was shown that each $\mathcal{B}_i$ is generated by the exceptional collection

$$(\mathcal{O}_{(m_i-1)D_i}, \ldots, \mathcal{O}_{2D_i}, \mathcal{O}_{D_i}),$$

which can be mutated into $(\mathcal{O}_{D_i}, \mathcal{O}(-D_i)|_{D_i}, \ldots, \mathcal{O}(-(m_i - 2))|_{D_i})$. The collection (4.5) in $\mathcal{A}_i$ is obtained from the latter collection by tensoring with ω_C.

This leads to the following result.

Proposition 4.3.1. *Let G be a finite group acting effectively on a smooth curve C. Then condition (MSOD) is satisfied, where the kernel corresponding to $\lambda = 1$ is the structure sheaf of the graph of π.*

Proof. We claim that the semiorthogonal decomposition (4.6) (or (4.7)) can be restructured to get the decomposition required by (MSOD). Namely, (4.6) consists of the image of the pull-back functor $\pi^* : \mathcal{D}(C/G) \to \mathcal{D}([C/G])$, along with $m_i - 1$ exceptional objects supported on D_i, for $i = 1, \dots, r$. On the other hand, for (MSOD) to hold, for each conjugacy class representative $g \neq 1$, and each $C(g)$-orbit in C^g, we need to have one exceptional object in $\mathcal{D}([C/G])$ supported on the corresponding G-orbit in C. The fact that the numbers of exceptional objects supported on each G-orbit match was proved in [18, Rem. 4.3.2]. □

Using Lemma 2.4.1 we deduce the following

Corollary 4.3.2. *Let $C_1, \dots, C_n$ be smooth curves, and for each i, let G_i be a finite group acting effectively on C_i. Then condition (MSOD) holds for the action of $G_1 \times \cdots \times G_n$ on $C_1 \times \cdots \times C_n$.*

Example 4.3.3. For the standard action of the cyclic group μ_d on $\mathbb{A}^1$, the geometric quotient is isomorphic to $\mathbb{A}^1_d$, where the subscript d indicates the $\mathbb{G}_m$-weight, so that the quotient map $\pi\colon \mathbb{A}^1 \to \mathbb{A}^1_d$ is given by $x \mapsto x^d$. We have a semiorthogonal decomposition

$$\mathcal{D}([\mathbb{A}^1/\mu_d]) = \langle \mathcal{O}_p \otimes \chi^{d-1}, \dots, \mathcal{O}_p \otimes \chi, \pi^*\mathcal{D}(\mathbb{A}^1_d) \rangle,$$

where $\mathcal{O}_p$ denotes the structure sheaf of the origin, and $\chi : \mu_d \to \mathbb{G}_m$ is the character given by the natural embedding.

Now, for positive integers $d_1, \dots, d_k$, let us consider the natural action of $G = \mu_{d_1} \times \cdots \times \mu_{d_k}$ on $\mathbb{A}^k$ (where the ith factor acts on the ith coordinate). By Corollary 4.3.2, we have a motivic semiorthogonal decomposition of $\mathcal{D}([\mathbb{A}^k/G])$. We can describe explicitly the components of this decomposition as follows. The fixed locus of an element of $g = (z_1, \dots, z_k) \in G$ is isomorphic to the affine space $\mathbb{A}^{n_g}$, where n_g is the number of trivial components of g . The geometric quotient by $C(g) = G$ is $\pi_g\colon \mathbb{A}^{n_g} \to \mathbb{A}^{n_g}_{\mathbf{d}_g}$, where $\mathbf{d}_g$ is a multi-index giving weights for the $\mathbb{G}_m$-action ($\mathbf{d}_g$ is the set of d_i for which $z_i = 1$). Let $\iota_g\colon \mathbb{A}^{n_g} \hookrightarrow \mathbb{A}^k$ denote the natural embedding. Then the composite functor

$$\iota_{g*} \circ \pi_g^* \colon \mathcal{D}(\mathbb{A}^{n_g}_{\mathbf{d}_g}) \to \mathcal{D}([\mathbb{A}^k/G])$$

is fully faithful.

For each i, let ζ_{d_i} be a d_ith primitive root of unity. For

$$g = (\zeta_1^{m_1}, \ldots, \zeta_k^{m_k}) \in G,$$

where $0 \le m_i < d_i$, we define the character χ_g of G by setting

$$\chi_g = \chi_1^{m_1} \cdots \chi_k^{m_k},$$

where $\chi_i \colon G \to \mathbb{G}_m$ is given by the ith projection.

Then the functors giving the semiorhogonal decomposition of $\mathcal{D}([\mathbb{A}^k/G])$ (numbered by $g \in G$) are

$$(\iota_{g*} \circ \pi_g^*) \otimes \chi_g \colon \mathcal{D}(\mathbb{A}^{n_g}_{\mathbf{d}_g}) \to \mathcal{D}([\mathbb{A}^k/G]),$$

ordered lexicographically with respect to the reverse order on each set $\{0, \ldots, d_i - 1\}$. Thus, we have a semiorthogonal decomposition

$$\mathcal{D}([\mathbb{A}^k/G]) = \langle \mathcal{D}(pt) \otimes \chi_1^{d_1-1} \cdots \chi_k^{d_k-1}, \ldots, \pi^*\mathcal{D}(\mathbb{A}^k_{d_1,\ldots,d_k}) \rangle.$$

As before, we can delete the origin in all the affine spaces and pass to $\mathbb{G}_m$-equivariant categories yielding a semiorthogonal decomposition of $\mathcal{D}([\mathbb{P}^{k-1}/G])$ indexed by the elements of G. The components of this semiorthogonal decomposition will be the weighted projective space stacks $\mathbb{P}(\mathbf{d}_g)$.

We can also apply Theorem 1.2.1 to get, as in Section 4.2, a semiorthogonal decomposition of $\mathcal{D}([\mathbb{P}H(f)/G])$, where f is a G-invariant homogeneous polynomial on $\mathbb{A}^k$. More precisely, we have to assume that $\mathbb{P}H(f)$ is smooth and that restrictions of f to certain coordinate subspaces are nonzero. Namely, in the case when there exists an index i with $d_i = 1$, we have to assume the nonvanishing of the restriction of f to the subspace where all coordinates with $d_i > 1$ are set to zero. In the case when all $d_i > 1$, we have to assume that the restriction of f to each coordinate line is nonzero.

For example, if $d_1 > 1$, $d_2 = \cdots = d_k = 1$, and $f = x_1^{d_1} - g(x_2, \ldots, x_k)$, then $\mathbb{P}H(f)$ is a cyclic cover of $\mathbb{P}^{k-2}$ and our decomposition of $\mathcal{D}([\mathbb{P}H(f)/\mu_{d_1}])$ matches the one given by Kuznetsov-Perry in [13, Theorem 4.1].

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