Math. Res. Lett.
Volume 27, Number 5, 1513–1563, 2020

Period integrals in nonpositively curved manifolds

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Let M be a compact Riemannian manifold without boundary. We investigate the integrals of L^2-normalized Laplace eigenfunctions over closed submanifolds. General bounds for these quantities were obtained by Zelditch [23], and are sharp in the case where M is the standard sphere. However, as with sup norms of eigenfunctions, there are many interesting settings where improvements can be made to these bounds, e.g. where M is a negatively curved surface and the submanifold is a geodesic (see [6, 18]).

So far, improvements in the nonpositive curvature setting have been confined to the two-dimensional case (see works of Chen and Sogge [6]; Sogge, Xi, and Zhang [18]; and the author [20, 22]). Here, we provide two theorems which extend these results into the higher dimensional setting. First, we provide an improvement of a half power of log over the standard bounds provided the submanifold has codimension 2 and M has strictly negative sectional curvature. Second, we provide the same improvement for hypersurfaces whose second fundamental form differs sufficiently from that of spheres of infinite radius. We use the usual tools, such as the Hadamard parametrix and the method of stationary phase, but critical to our argument is a computation of the Hessian of the distance function on the universal cover of M.

This article was supported in part by NSF grant DMS-1665373.

1. Introduction

1.1. Background

Let (M, g) be a compact n-dimensional Riemannian manifold ($n \geq 2$) without boundary. Let Δ_g denote the Laplace-Beltrami operator, written in local coordinates as

$$\Delta_g = |g|^{-1/2} \sum_{i,j} \partial_i (|g|^{1/2} g^{ij} \partial_j).$$

Let e_j for $j = 0, 1, 2, \ldots$ form a Hilbert basis of eigenfunctions of Δ_g with corresponding eigenvalues λ_j, i.e.

$$-\Delta_g e_j = \lambda_j^2 e_j.$$

We are interested in the relationship between the geometry of M and asymptotic bounds on the means of eigenfunctions over submanifolds as the eigenvalue tends to infinity.

This class of problems has its roots in the theory of automorphic forms, where bounds on the Fourier coefficients of eigenfunctions along closed geodesics are of interest. Using Kuznecov sum formulae, Good [9] and Hejhal [11] independently obtained bounds

$$\int_\gamma e_j \, ds = O(1)$$

where M is a compact hyperbolic surface and γ a closed geodesic. Later Zelditch [23] extended this result to the general Riemannian setting and obtained a Kuznecov sum formula

$$\sum_{\lambda_j \leq \lambda} \left| \int_\Sigma e_j \, d\sigma \right|^2 \sim \lambda^{n-d} + O(\lambda^{n-d-1}) \tag{1.1}$$

for d-dimensional submanifolds Σ, where $d\sigma$ is the surface element on Σ. This provides the general bound,

$$\left| \int_\Sigma e_j \, d\sigma \right| = O(\lambda_j^{\frac{n-d-1}{2}}), \tag{1.2}$$

which is optimal on the sphere for *any*[1] submanifold Σ.

In [13], Reznikov extended the bounds in [9, 11] to geodesic circles and closed horocycles in hyperbolic surfaces of finite area, and put forth a conjecture for optimal bounds.

Conjecture 1.1 ([13]). *Let M be a compact hyperbolic surface and γ a closed geodesic or geodesic circle. Then,*

$$\left| \int_\gamma e_j \, ds \right| = O(\lambda_j^{-1/2+\epsilon})$$

for all $\epsilon > 0$.

It seems the standard techniques will only yield improvements by a power of $\log \lambda_j$ over the standard bounds. The conjecture, let alone any polynomial improvement over the standard bounds, seems a long way off.

The first improvement on (1.2) was obtained by Chen and Sogge [6], who used the Gauss-Bonnet theorem to show

$$\left| \int_\gamma e_j \, ds \right| = o(1)$$

where M is a compact surface with negative sectional curvature and γ is a geodesic. This result was later improved by Sogge, Xi, and Zhang [18] by providing an explicit decay of $O(1/\sqrt{\log \lambda_j})$ under some weaker sectional curvature hypotheses. The author [20] extended [6] and later [18] from geodesics to a wide class of curves satisfying some curvature conditions, albeit without the weakened sectional curvature hypotheses. The result is summarized below.

Theorem 1.2 ([22]). *Let M be a compact Riemannian surface without boundary with nonpositive sectional curvature. For each $p \in M$ and $v \in S_pM$, let $\mathbf{k}(v)$ denote the limit of the curvature of the circular arc at p with center taken out to infinity along the geodesic ray in direction v. Then,*

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[1] Indeed, by (1.1) and the fact that the gaps between successive distinct eigenvalues on S^n approach a constant (see [21])

if γ is a closed curve in M such that

$$\kappa_\gamma \neq \mathbf{k}(v) \qquad \text{for all normal vectors } v \text{ to } \gamma,$$

then

$$\int_\gamma e_j \, ds = O(1/\sqrt{\log \lambda_j}),$$

where here κ_γ denotes the geodesic curvature of γ.

By inspection of the constructions in [20, 22], $\mathbf{k}$ is precisely the curvature of a horosphere as in [10].

There have been a number of recent improvements on the general bounds assuming some dynamical properties of the geodesic flow. Canzani, Galkowski, and Toth [5] provided a little-o improvement on bounds on averages of Cauchy data over hypersurfaces of eigenfunctions belonging to a sequence with defect measure. The author [21] provided a little-o improvement on (1.2) provided the set of looping directions which depart from and arrive at Σ conormally has measure zero. Using quantum defect measures, Canzani and Galkowski [3] recently obtained the little-o improvement for a vast range of situations containing results in [5, 21].

1.2. Statement of results

In this article, we provide a generalization of [22] to nonpositively curved manifolds of arbitrary dimension. Our first result provides an improvement of $1/\sqrt{\log \lambda_j}$ to (1.2) if M has negative sectional curvature and Σ has codimension at least 2.

Theorem 1.3. *Let (M, g) be a compact, n-dimensional Riemannian manifold, without boundary, with negative sectional curvature. Let Σ be a closed d-dimensional submanifold with $d \leq n - 2$. Then,*

$$\left| \int_\Sigma e_j \, d\sigma \right| = O(\lambda_j^{\frac{n-d-1}{2}} / \sqrt{\log \lambda_j}),$$

where $d\sigma$ denotes the induced measure on Σ.

Our second result treats the codimension 1 case and requires a generalization of the limiting curvature $\mathbf{k}$ from the two-dimensional case. First, we define the second fundamental form II_Σ of a submanifold Σ to be the vector-valued bilinear form

$$\mathrm{II}_\Sigma(X, Y) = (\nabla_X Y)^\perp$$

to be the component of the covariant derivative of Y with respect to X which lies normal to Σ. For what follows, we appeal to the formalism of [10], [1], and [8] for stable Jacobi fields, Busemann functions, and horospheres, summarized below.

Let $(\tilde{M}, \tilde{g})$ be a Hadamard manifold with nonpositive sectional curvature (e.g. the universal cover of M). Fix a choice of unit-speed geodesic

$$\begin{aligned} \gamma : \mathbb{R} &\to \tilde{M} \\ t &\mapsto \gamma(t). \end{aligned}$$

Fix a vector X at $\gamma(0)$. A Jacobi field J_X along γ is *stable* if $|J_X(t)|$ is bounded for $t \geq 0$. There exists a unique stable Jacobi field J_X for any choice of γ and X. Our choice of γ also provides a *Busemann function* f, defined as

$$f(x) = \lim_{t\to\infty} (d_{\tilde{g}}(\gamma(t), x) - t).$$

Busemann functions are C^2 and convex, and the flow lines of their gradients are unit-speed geodesics. A *horosphere* is a level set of a Busemann function. They are also the hypersurfaces obtained as a limit of spheres containing some fixed point with centers at $\gamma(t)$ as $t \to \infty$.

Let $p = \gamma(0)$ and $v = \gamma'(0)$. We let $H(v)$ be the horosphere $f^{-1}(f(p))$, where f is the Busemann function associated with γ as above. The second fundamental form of $H(v)$ is then given by

$$\mathrm{II}_{H(v)}(X, Y) = \left\langle -J'_X(0), Y \right\rangle v, \tag{1.3}$$

where J_X is the unique stable Jacobi field along γ with $J_X(0) = X$, and where J'_X denotes the covariant derivative with respect to the parameter of γ. Note $\mathrm{II}_{H(v)}$ is invariant under isometry, and hence is well-defined after quotienting $\tilde{M}$ by a group of deck transformations to obtain a compact manifold M. After accounting for differences in convention, we recover two useful, well-known facts from [1]:

1) $\mathrm{II}_{H(v)}$ is continuous over v in the unit sphere bundle SM.
2) $\langle \mathrm{II}_{H(v)}, v\rangle$ is positive semidefinite.

(1) and (2) follow, respectively, from the facts that Busemann functions are C^2 and convex.

Our second main result, which pertains to period integrals over hypersurfaces, requires hypotheses on the quadratic forms $\langle \mathrm{II}_\Sigma - \mathrm{II}_{H(v)}, v\rangle$ on $T\Sigma$ for each unit vector v normal to Σ.

Theorem 1.4. *Let (M,g) be as in Theorem 1.3 except allow M to have nonpositive sectional curvature, and let Σ be a closed hypersurface. If*

$$\begin{aligned}\operatorname{rank}(\langle \mathrm{II}_\Sigma - \mathrm{II}_{H(v)}, v\rangle) + \operatorname{rank}(\langle \mathrm{II}_\Sigma - \mathrm{II}_{H(-v)}, -v\rangle) \geq n \\ \text{for each } v \in SN\Sigma,\end{aligned} \tag{1.4}$$

then

$$\int_\Sigma e_j \, d\sigma = O(1/\sqrt{\log \lambda_j}). \tag{1.5}$$

Remark 1.5. The results of Theorems 1.3 and 1.4, and of Corollary 1.6 to follow, still hold if we replace the eigenfunctions by quasimodes Ψ_λ with $\|\Psi_\lambda\|_{L^2} \leq 1$ and with spectral support on the frequency band $[\lambda, \lambda + \frac{1}{\log \lambda}]$. The submanifold Σ need not be closed, either, provided the surface element $d\sigma$ is multiplied by some smooth, compactly supported cutoff. This will be made apparent in the next section.

The arguments in Section 4 allow us to pick out some criteria for hypersurfaces which satisfy the hypotheses of Theorem 1.4. As a consequence, we have the following corollaries. (See Proposition 4.3 and Remark 4.4 for details.)

Corollary 1.6. *Let M and Σ be as in Theorem 1.4. Then Σ satisfies* (1.4) *and hence* (1.5) *if any of the following hold.*

1) *At each point in Σ, at least $n/2$ of the principal curvatures of Σ lie outside the interval $[a,b]$, where $0 \leq a \leq b$ are constants such that*

$$0 \geq -a^2 \geq K \geq -b^2$$

on M, where K is the sectional curvature of M.

2) *Σ is a geodesic sphere in M.*

3) *M has strictly negative curvature and Σ is a totally geodesic hypersurface.*

Note Theorem 1.4 not only generalizes Theorem 1.2 to hypersurfaces of arbitrary manifolds, but it is stronger even in the two-dimensional case. In Theorem 1.4, the curvature of γ is *signed*, and in Theorem 1.2 it is not. This lets us apply Theorem 1.4 to all spheres, not just those of some bounded radius as in [22, Corollary 1.6].

Section 2 is dedicated to reducing Theorems 1.3 and 1.4 to bounds on a kernel involving the half wave operator. Following this, we lift our computation to the universal cover $(\tilde{M}, \tilde{g})$ of M, which we identify with $\mathbb{R}^n$ with the pullback metric by the Cartan-Hadamard theorem as in [2, 6, 18, 22]. We then rephrase the kernel as a sum of kernels over the group of deck transformations Γ associated with the covering map. Section 3 is dedicated to writing these summands as oscillatory integrals, roughly

$$\sum_{\alpha \in \Gamma} \int_\Sigma \int_\Sigma a_\alpha(x,y) e^{i\lambda\phi_\alpha(x,y)}\, d\sigma(x)\, d\sigma(y) \tag{1.6}$$

with phase function

$$\phi_\alpha(x,y) = d_{\tilde{g}}(\alpha\tilde{x}, \tilde{y})$$

where $\tilde{x}$ and $\tilde{y}$ are respective lifts of x and y to $\tilde{M}$, and where $d_{\tilde{g}}$ denotes the distance function on the universal cover $\tilde{M}$. Section 4 is dedicated to computing and bounding derivatives of the phase function so that we can use the method of stationary phase in Section 5. Theorems 1.3 and 1.4 follow if we can show each of the non-identity terms of (1.6) is $O(\lambda^{-d/2})$ and $O(\lambda^{-n/2})$, respectively, where the constants implied by the big-O notation are sufficiently uniform.

1.3. Examples and limitations of Theorems 1.3 and 1.4

There are two examples of manifolds which help to illustrate Theorems 1.3 and 1.4: the flat torus and a compact hyperbolic manifold. These two examples help to motivate the statements of Theorems 1.3 and 1.4. At the same time, these specific examples show the deficiencies of our main results and suggest that a more complete picture must be provided with other methods.

The torus. Suppose M is the flat torus $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$. According to Theorem 1.4, since $\mathbb{T}^n$ is flat, we should obtain a decay of $O(1/\sqrt{\log \lambda_j})$ on integrals of eigenfunctions e_j of the torus over hypersurfaces Σ, provided Σ has at least $n/2$ nonzero principal curvatures at each point. In [12], Hezari and Riviere present the following result on the torus, which has both stronger hypotheses and (much) stronger bounds than those in Theorem 1.4.

Theorem 1.7 ([12]). *Let Σ be a smooth, compact, embedded, oriented hypersurface of $\mathbb{T}^n$ without boundary with surface measure σ, and suppose all principal curvatures of Σ are nonzero at each point on Σ. Then,*

$$\int_\Sigma e_\lambda \, d\sigma = O(\lambda^{-1/2+\epsilon})$$

where ϵ is any positive constant, but which is allowed to vanish when $n \geq 5$.

Hezari and Riviere explain that the problem comes down to counting lattice points on spheres, for which there are bounds

$$\#\{m \in \mathbb{Z}^n : |m| = \lambda\} = O(\lambda^{n-2+\epsilon}) \qquad \text{for all } \epsilon > 0 \tag{1.7}$$

but where ϵ is allowed to vanish when $n \geq 5$. At the same time if Σ is a rational hyperplane in $\mathbb{T}^n$, one may pick out a sequence of exponentials with eigenvalues tending to infinity whose restrictions to Σ are a constant. In this sense, some nonvanishing curvature conditions on Σ are necessary to obtain decay.

Now consider the situation where $d \leq n-2$. Theorem 1.3 requires that the sectional curvature of M be strictly negative. However, it is reasonable to ask if a similar result applies in the flat setting. Consider the specific case where $\Sigma = \mathbb{T}^d \times \{0\}^{n-d}$ for $d \leq n-2$. By writing e_λ as a linear combination of exponentials and using Cauchy-Schwarz, we obtain

$$\left| \int_\Sigma e_\lambda \, d\sigma \right| \leq \left(\sum_{|m|=\lambda} |\hat{\sigma}(m)|^2 \right)^{1/2}$$

where

$$\hat{\sigma}(m) = \int_{\mathbb{T}^d} e^{-i\langle x, m' \rangle} \, dx = \begin{cases} (2\pi)^d & m' = 0 \\ 0 & m' \neq 0. \end{cases}$$

where $m' = (m_1, \ldots, m_d)$ are the first d coordinates of m. This and (1.7) yields

$$\left| \int_\Sigma e_\lambda \, d\sigma \right| = O(\lambda^{\frac{d-2+\epsilon}{2}}) \qquad \text{for all } \epsilon > 0,$$

which is better than the bound in Theorem 1.3. Though this computation applies only to a few specific submanifolds, it suggests that Theorem 1.3 may apply to M with merely nonpositive sectional curvature. However, this result is inaccessible with the methods used to prove of our main results. Indeed, we require negative curvature to obtain a uniform constant and finish the proof of Theorem 1.3 in Chapter 5.

A compact hyperbolic manifold. Suppose M is a compact hyperbolic manifold, i.e. the sectional curvature is identically -1. By the corollary, Theorem 1.4 requires that at least $n/2$ of the principal curvatures of Σ not be 1. We ask whether we require the full conditions on Σ to obtain the improved bound (1.5). Consider the extreme example where Σ is precisely a horosphere[2] in M. Is the standard bound

$$\int_\Sigma e_\lambda \, d\sigma = O(1)$$

sharp like it is for rational hyperplanes in the torus? A recent result by Canzani, Galkowski, and Toth [5] shows that if e_λ is a quantum ergodic sequence, its average over *any* hypersurface will be $o(1)$. If we assume the quantum unique ergodicity conjecture, the standard $O(1)$ bound is never sharp regardless of the conditions on the curvature of Σ. Whether or not we obtain an explicit decay of $O(1/\sqrt{\log \lambda})$ for period integrals over horospheres is an open question.

Note. As of the time of publication, this question has been resolved by Canzani and Galkowski. In particular if M has Anosov geodesic flow (e.g. when M has strictly negative curvature) and Σ lifts to a horosphere in $\widetilde{M}$, then we obtain the quantitative bound $O(1/\sqrt{\log \lambda})$ on eigenfunction period integrals over Σ [4, Theorem 4, E].

2. A standard reduction and lift to the universal cover

The following reduction is part of the standard strategy for many problems dealing with the asymptotic distributions of eigenfunctions on manifolds (e.g. [2, 6, 17–19, 23] and many others). We follow the example of [17, 19] and use pseudodifferential operators to microlocalize to cones in T^*M with small support. Afterwards we perform a lift to the universal cover as in [2, 6, 18].

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[2]There are no closed horospheres in a compact hyperbolic manifold. However by Remark 1.5, it suffices to consider a small, embedded piece of a horosphere with surface measure σ which as been multiplied by some smooth, compactly supported bump function.

For both the situations in Theorems 1.3 and 1.4, we will show

$$\sum_{\lambda_j \in [\lambda, \lambda + T^{-1}]} \left| \int_\Sigma e_j \, d\sigma \right|^2 \lesssim T^{-1} \lambda^{n-d-1} + e^{CT} \lambda^\delta \tag{2.1}$$

where δ is some exponent less than $n-d-1$ and where we set

$$T = c \log \lambda \tag{2.2}$$

for some sufficiently small c.

Now we introduce Fermi-type coordinates about Σ. Parametrize a small neighborhood in Σ with geodesic normal coordinates $x' = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Then take a smooth, orthonormal frame $v_{d+1}, \ldots, v_n$ of the normal bundle of Σ. Writing $x = (x', x^\perp) \in \mathbb{R}^n$ where $x^\perp = (x_{d+1}, \ldots, x_n)$ are the remaining $n-d$ coordinates, the coordinate map

$$(x', x^\perp) \mapsto \exp(x_{d+1} v_{d+1}(x') + \cdots + x_n v_n(x')) \tag{2.3}$$

parametrizes a small neighborhood in M containing a piece of Σ. By construction,

$$g(x', 0) = \begin{bmatrix} g_\Sigma(x') & 0 \\ 0 & I_{n-d} \end{bmatrix} \tag{2.4}$$

where g_Σ is the intrinsic metric on Σ and I_{n-d} is the $(n-d) \times (n-d)$ identity matrix. We also note for future use that

$$g_\Sigma(x') = I_d + O(|x'|^2)$$

and that the Christoffel symbols associated with the Levi-Civita connection intrinsic to Σ vanish at $x' = 0$ [7]. In particular, we can take the Christoffel symbols to be as small as desired by shrinking the neighborhood parametrized by our coordinates.

Take a finite open cover of Σ in M of such coordinate charts and with it a subordinate partition of unity

$$\sum_i b_i \equiv 1$$

on Σ. By the Cauchy-Schwarz inequality,

$$\left| \int_\Sigma e_j \, d\sigma \right|^2 = \left| \sum_i \int_\Sigma b_i e_j \, d\sigma \right|^2 \le C_\Sigma \sum_i \left| \int_\Sigma b_i e_j \, d\sigma \right|^2,$$

and so (2.1) follows if we can show

$$\sum_{\lambda_j \in [\lambda, \lambda+T^{-1}]} \left| \int_\Sigma b e_j \, d\sigma \right|^2 \lesssim T^{-1} \lambda^{n-d-1} + e^{CT} \lambda^{\delta} \tag{2.5}$$

where b is a smooth function on Σ with controllably small support, and the constants in the bounds are allowed to depend on b. We will take this a step further and microlocalize to small cones in T^*M. Take a partition of unity

$$\sum_i a_i \equiv 1$$

of the sphere $S^{n-1} \subset \mathbb{R}^n$, and take smooth bump functions β_0 and β_1 both supported on a small interval in $\mathbb{R}$ and for which $\beta_0 \equiv 1$ near 0 and $\beta_1 \equiv 1$ near 1. For each i, we define operators[3]

$$B_{i,\lambda} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y, \xi \rangle} B_{i,\lambda}(x,y,\xi) f(y) \, dy \, d\xi \tag{2.6}$$

with symbol

$$B_{i,\lambda}(x,y,\xi) = \beta_0(|x-y|)\beta_0(|x^\perp|)b(x')\beta_1(|\xi|/\lambda)a_i(\xi/|\xi|),$$

and similarly

$$R_\lambda f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y, \xi \rangle} R(\lambda; x,y,\xi) f(y) \, dy \, d\xi$$

with symbol

$$R_\lambda(x,y,\xi) = \beta_0(|x-y|)\beta_0(|x^\perp|)b(x')(1-\beta_1(|\xi|/\lambda)).$$

Whenever we write, say $B_{i,\lambda}$, without arguments, we are referring to the operator itself. When writing $B_{i,\lambda}(x,y,\xi)$ with arguments, we are referring

[3]The purpose of the operator B_λ is to filter out geodesics which depart y and arrive at x in sufficiently differing directions, as Lemma 3.2 will show in the next section (see also Figure 1 in Section 3). This strategy was used before by Sogge, Toth, and Zelditch [17] who obtained improved sup-norm estimates for eigenfunctions on manifolds provided that, at each point, the set of recurrent directions of geodesics has measure zero.

to its symbol. We will use this convention for similarly constructed operators. Note

$$\int_\Sigma be_j\,d\sigma = \sum_i \int_\Sigma B_{i,\lambda} e_j\,d\sigma + \int_\Sigma R_\lambda e_j\,d\sigma.$$

Note the absence of a discrepancy between the left and right sides of the equation above. Indeed,

$$\sum_i B_{i,\lambda}(x,y,\xi) + R_\lambda(x,y,\xi) = \beta(|x-y|)\beta_0(|x^\perp|)b(x'),$$

which is a symbol constant in ξ. By Fourier inversion, the associated operator acts on a function f by

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle}\beta(|x-y|)\beta_0(|x^\perp|)b(x')f(y)\,dy\,d\xi = \beta_0(|x^\perp|)b(x')f(x),$$

which is precisely bf when $x\in\Sigma$. By the same Cauchy-Schwarz argument as before, (2.5) follows provided we can show

$$\sum_{\lambda_j\in[\lambda,\lambda+T^{-1}]}\left|\int_\Sigma B_\lambda e_j\,d\sigma\right|^2 \lesssim T^{-1}\lambda^{n-d-1} + e^{CT}\lambda^\delta \tag{2.7}$$

where B_λ is defined as in (2.6) with symbol

$$B_\lambda(x,y,\xi) = \beta_0(|x-y|)\beta_0(|x^\perp|)b(x')\beta_1(|\xi|/\lambda)a(\xi/|\xi|) \tag{2.8}$$

where β_0, β_1, a, and of course b all have adjustably small support, and we can show

$$\sum_{\lambda_j\in[\lambda,\lambda+T^{-1}]}\left|\int_\Sigma R_\lambda e_j\,d\sigma\right|^2 = O(\lambda^{-\infty}) \qquad \text{uniformly for } T\geq 1 \tag{2.9}$$

where R_λ is as above. The latter bound follows from Cauchy-Schwarz inequality applied to the integral and the following proposition whose proof we defer until the end of the section.

Proposition 2.1. *Let R_λ be as above.*

$$\sup_{x\in\Sigma}\sum_{\lambda_j\in[\lambda,\lambda+1]}|R_\lambda e_j(x)|^2 = O(\lambda^{-\infty}).$$

We will also use the following generalization of the bound (1.2) to help us contend with (2.7), whose proof we also defer until the end of the section.

Proposition 2.2. *Let B_λ be as above. Then if χ is a (not necessarily nonnegative) Schwartz-class function on $\mathbb{R}$ with* $\mathrm{supp}\hat{\chi}$ *adjustably small,*

$$\sum_j \chi(\lambda_j - \lambda) \left| \int_\Sigma B_\lambda e_j \, d\sigma \right|^2 = O(\lambda^{n-d-1}). \tag{2.10}$$

A careful choice of nonnegative χ yields

$$\sum_{\lambda_j \in [\lambda, \lambda+1]} \left| \int_\Sigma B_\lambda e_j \, d\sigma \right|^2 = O(\lambda^{n-d-1})$$

as a consequence.

Let χ be some nonnegative Schwartz-class function with $\chi(0) = 1$ and $\mathrm{supp}\hat{\chi} \subset [-1, 1]$. Since we can fit some rectangle under the graph of χ, we have (2.7) provided

$$\sum_j \chi(T(\lambda_j - \lambda)) \left| \int_\Sigma B_\lambda e_j \, d\sigma \right|^2 \lesssim T^{-1} \lambda^{n-d-1} + e^{CT} \lambda^\delta. \tag{2.11}$$

To access (2.11), we will make use of the spectrally-defined half-wave operator,

$$e^{it\sqrt{-\Delta_g}} = \sum_j e^{it\lambda_j} E_j$$

where E_j is the orthogonal projection operator onto the e_j-th eigenspace. The half-wave operator has kernel

$$e^{it\sqrt{-\Delta_g}}(x, y) = \sum_j e^{it\lambda_j} e_j(x) \overline{e_j(y)}$$

and so the kernel of the composition $B_\lambda e^{it\sqrt{-\Delta_g}} B_\lambda^*$ is

$$B_\lambda e^{it\sqrt{-\Delta_g}} B_\lambda^*(x, y) = \sum_j e^{it\lambda_j} B_\lambda e_j(x) \overline{B_\lambda e_j(y)},$$

where here B_λ^* denotes the adjoint of B_λ. We use the Fourier inversion formula and the expression above to write the left hand side of (2.11) as

$$\begin{aligned}(2.12)\qquad &\frac{1}{2\pi T}\sum_j \int_{-\infty}^{\infty} \hat\chi(t/T)e^{-it\lambda}e^{it\lambda_j}\\ &\qquad\times \int_\Sigma\int_\Sigma B_\lambda e_j(x)\overline{B_\lambda e_j(y)}\,d\sigma(x)\,d\sigma(y)\,dt\\ &= \frac{1}{2\pi T}\int_\Sigma\int_\Sigma\int_{-\infty}^{\infty}\hat\chi(t/T)e^{-it\lambda}B_\lambda e^{it\sqrt{-\Delta_g}}B_\lambda^*(x,y)\,dt\,d\sigma(x)\,d\sigma(y).\end{aligned}$$

Let β be a smooth bump function on $\mathbb{R}$ such that $\beta(t)=1$ for $|t|\le 2$ and $\beta(t)=0$ for $|t|\ge 3$. At this point we introduce a constant R to be determined later, independent of T and λ, and dependent only on the geometry of M and Σ. We cut the integral (2.12) into $\beta(t/R)$ and $1-\beta(t/R)$ parts and obtain

$$\begin{aligned}(2.13)\qquad &= \frac{1}{2\pi T}\int_\Sigma\int_\Sigma\int_{-\infty}^{\infty}\beta(t/R)\hat\chi(t/T)e^{-it\lambda}\\ &\qquad\times B_\lambda e^{it\sqrt{-\Delta_{\tilde g}}}B_\lambda^*(x,y)\,dt\,d\sigma(x)\,d\sigma(y)\\ &+\frac{1}{2\pi T}\int_\Sigma\int_\Sigma\int_{-\infty}^{\infty}(1-\beta(t/R))\hat\chi(t/T)e^{-it\lambda}\\ &\qquad\times B_\lambda e^{it\sqrt{-\Delta_{\tilde g}}}B_\lambda^*(x,y)\,dt\,d\sigma(x)\,d\sigma(y).\end{aligned}$$

We let X_T denote the function with Fourier transform $\hat X_T(t)=\beta(t/R)\hat\chi(t/T)$. By reversing our argument, we write the first term in (2.13) as

$$\frac{1}{T}\sum_j X_T(\lambda_j-\lambda)\left|\int_\Sigma B_\lambda e_j\,d\sigma\right|^2$$

where X_T is Schwartz-class with bounds

$$|X_T^{(k)}(\lambda_j-\lambda)|\le C_{N,k}(1+|\lambda_j-\lambda|)^{-N}\qquad \text{for } N,k=0,1,2,\dots$$

for constants $C_{N,k}$ uniform for $T\ge 1$. We apply Proposition 2.2 to obtain a bound of $O(T^{-1}\lambda^{n-d-1})$ on the sum above. Indeed, the bounds of the proposition only depends on the support of $\hat X_T$, which is uniform in T, and bounds on finitely many derivatives of X_T.

Hence, we are done if we can show that

$$\Big| T^{-1}\int_\Sigma\int_\Sigma\int_{-\infty}^{\infty}(1-\beta(t/R))\hat\chi(t/T)e^{-it\lambda} \times B_\lambda e^{it\sqrt{-\Delta_g}}B_\lambda^*(x,y)\,dt\,d\sigma(x)\,d\sigma(y)\Big| \lesssim e^{CT}\lambda^\delta. \tag{2.14}$$

As in [2, 6, 18], we will want to replace the half wave operator of (2.14) with the cosine operator so that we have Hügen's principle at our disposal when we lift to the universal cover. By Euler's formula,

$$e^{it\sqrt{-\Delta_g}} = 2\cos(t\sqrt{-\Delta_g}) - e^{-it\sqrt{-\Delta_g}},$$

hence we write what is inside the absolute values in (2.14) as

$$\begin{aligned}&\frac{2}{T}\int_\Sigma\int_\Sigma\int_{-\infty}^{\infty}(1-\beta(t/R))\hat\chi(t/T)e^{-it\lambda}\\ &\qquad\qquad\times B_\lambda\cos(t\sqrt{-\Delta_g})B_\lambda^*(x,y)\,dt\,d\sigma(x)\,d\sigma(y)\\ &+\frac{1}{T}\int_\Sigma\int_\Sigma\int_{-\infty}^{\infty}(1-\beta(t/R))\hat\chi(t/T)e^{-it\lambda}\\ &\qquad\qquad\times B_\lambda e^{-it\sqrt{-\Delta_g}}B_\lambda^*(x,y)\,dt\,d\sigma(x)\,d\sigma(y).\end{aligned}$$

Setting $\hat X_T(t)=\beta(t/R)\hat\chi(t/T)$ as before and reversing our reduction, the latter term is a constant multiple of

$$\sum_j\left(\chi(-T(\lambda_j+\lambda))-\frac{1}{T}X_T(-(\lambda_j+\lambda))\right)\left|\int_\Sigma B_\lambda e_j\,d\sigma\right|^2. \tag{2.15}$$

By (2.8),

$$\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|B_\lambda(x,y,\xi)|d\xi\right)^2 dy = O(\lambda^n)$$

and hence by Cauchy-Schwarz we have operator bounds $\|B_\lambda\|_{L^2\to L^\infty} = O(\lambda^{n/2})$. In particular we have the very cheap, very suboptimal bound

$$\left|\int_\Sigma B_\lambda e_j\right| \lesssim \lambda^{n/2} \qquad \text{uniformly for all } j. \tag{2.16}$$

At the same time, X_T satisfies bounds

$$|X_T(\tau)| \le C_N(1+|\tau|)^{-N} \qquad \text{for } T\ge 1,\ N=1,2,\dots.$$

Putting this together with our cheap polynomial bound (2.16), we find the sum in (2.15) is bounded like $O(\lambda^{-\infty})$. Hence, it suffices to show

$$\begin{aligned}(2.17)\qquad &\bigg| \int_\Sigma \int_\Sigma \int_{-\infty}^\infty (1 - \beta(t/R))\hat\chi(t/T) e^{-it\lambda} \\ &\qquad \times B_\lambda \cos(t\sqrt{-\Delta_g}) B_\lambda^*(x,y)\, dt\, d\sigma(x)\, d\sigma(y) \bigg| \lesssim e^{CT}\lambda^\delta.\end{aligned}$$

We are ready to perform our lift. We now introduce our assumption that M is nonpositively curved. By the Cartan-Hadamard theorem, we identify the universal cover $\tilde M$ of M with $\mathbb{R}^n$ equipped with the pullback $\tilde g$ of the metric g through the covering map. Let Γ denote the group of deck transformations associated with the covering map and let

$$D = \left\{ \tilde x \in \tilde M : d_{\tilde g}(\tilde x, 0) = \inf_{\alpha\in\Gamma} d_{\tilde g}(\alpha \tilde x, 0) \right\}$$

denote a Dirichlet domain in $\tilde M$ with 0 chosen to be a lift of a point on Σ in the support of B_λ. Let $\tilde f$ be a smooth, compactly supported function on $\tilde M$ and set

$$f(x) = \sum_{\alpha\in\Gamma} \tilde f(\alpha \tilde x)$$

where $\tilde x$ is any lift of x to $\tilde M$. Since the covering map is a local isometry,

$$u(t,x) = \sum_{\alpha\in\Gamma} \cos(t\sqrt{-\Delta_{\tilde g}}) \tilde f(\alpha \tilde x)$$

solves the wave equation $(\partial_t^2 - \Delta_g)u = 0$ with initial data $u(0,x) = f(x)$ and $\partial_t u(0,x) = 0$, hence

$$u(t,x) = \cos(t\sqrt{-\Delta_g}) f(x).$$

We conclude

$$(2.18)\qquad \cos(t\sqrt{-\Delta_g}) = \sum_{\alpha\in\Gamma} \alpha^* \cos(t\sqrt{-\Delta_{\tilde g}})$$

where α^* is the pullback operator through α, e.g. $\alpha^* \tilde f(\tilde x) = \tilde f(\alpha\tilde x)$. Hence we will have (2.17) provided

$$(2.19)\qquad \sum_{\alpha\in\Gamma} \left| \int_\Sigma \int_\Sigma K_\alpha(T,\lambda; x, y)\, d\sigma(x)\, d\sigma(y) \right| \lesssim e^{CT}\lambda^\delta$$

where
(2.20)
$$K_\alpha(T,\lambda;x,y)=\int_{-\infty}^{\infty}(1-\beta(t/R))\hat\chi(t/T)e^{-it\lambda}\tilde B_\lambda\alpha^*\cos(t\sqrt{-\Delta_{\tilde g}})\tilde B_\lambda^*(\tilde x,\tilde y)\,dt,$$

where $\tilde B_\lambda$ is the operator on $\tilde M$ associated with the symbol

$$\tilde B_\lambda(\tilde x,\tilde y,\xi)=\begin{cases}B_\lambda(x,y,\xi) & \text{if } \tilde x,\tilde y\in D,\\ 0 & \text{otherwise,}\end{cases} \tag{2.21}$$

where $\tilde x$ and $\tilde y$ are the respective lifts of x and y to the Dirichlet domain D in the universal cover, and where $\tilde B_\lambda\alpha^*\cos(t\sqrt{-\Delta_{\tilde g}})\tilde B_\lambda^*(\tilde x,\tilde y)$ denotes the kernel of the composition $\tilde B_\lambda\alpha^*\cos(t\sqrt{-\Delta_{\tilde g}})\tilde B_\lambda^*$. In (2.21), recall $B_\lambda(x,y,\xi)$ has both x- and y-support on a small neighborhood of some fixed point of Σ, and hence can be uniquely lifted to the Dirichlet domain D in the universal cover. We note now for future reference that, by Hüygen's principle, $K_\alpha(T,\lambda;x,y)$ is supported on $d_{\tilde g}(\tilde x,\tilde y)\le T+1$, after perhaps shrinking the $\tilde x$-support of the symbol $\tilde B_\lambda$. Hence, all except for a finite number of terms in the sum in (2.19) is zero. In fact, by volume comparison [14, Chapter I, Theorem 1.3], the number of Dirichlet domains within a ball of radius T in the universal cover is bounded by a constant times e^{CT}. Hence,

$$\#\{\alpha\in\Gamma:\operatorname{supp}K_\alpha(T,\lambda;\ \cdot\ ,\ \cdot\ )\text{ is nonempty}\}=O(e^{CT}). \tag{2.22}$$

This concludes our reduction, but we still need to prove Propositions 2.1 and 2.2. The proof of Proposition 2.2 is very standard but a bit involved, requiring a parametrix of the half wave operator and two consecutive applications of stationary phase. We refer the reader to [15, 17, 19, 21] for similar arguments.

Proof of Proposition 2.2. With Following the steps in the reduction above, we write (2.10) as

$$\left|\int_\Sigma\int_\Sigma\int_{-\infty}^{\infty}\hat\chi(t)e^{-it\lambda}B_\lambda e^{it\sqrt{-\Delta_g}}B_\lambda^*(x,y)\,dt\,d\sigma(x)\,d\sigma(y)\right|\lesssim\lambda^{n-d-1}.$$

By using Hörmander's parametrix [16, Chapter 4] or by using the Hadamard parametrix and the arguments in section 5.2.2 of [15], we write

$$e^{it\sqrt{-\Delta_g}}(x,y)=\int_{\mathbb{R}^n}e^{i(\varphi(x,y,\xi)+tp(y,\xi))}q(t,x,y,\xi)\,d\xi \tag{2.23}$$

modulo a smooth kernel where q is a zero-order symbol in ξ satisfying

$$|\partial_\xi^\alpha \partial_{t,x,y}^\beta q(t,x,y,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{-|\alpha|}$$

for multiindices α and β, and where since the support of $\hat{\chi}$ is small, $\hat{\chi}(t)q(t,x,y,\xi)$ is supported where $d_g(x,y)$ is near 0. After perhaps further restricting the support of $\hat{\chi}$, the phase function φ is defined on the support of $\hat{\chi}q$, is smooth and homogeneous of degree 1 in ξ, and satisfies

$$\varphi(x,y,\xi) = \langle x-y, \xi\rangle + O(|x-y|^2|\xi|)$$

where here x and y are written in Fermi coordinates (2.3). Finally,

$$p(y,\xi) = \sqrt{\sum_{i,j} g^{ij}(y)\xi_i\xi_j}$$

is the principal symbol associated with the half-Laplacian $\sqrt{-\Delta_g}$. For x and y in Fermi coordinates,

$$\begin{aligned}
&\int_{-\infty}^\infty \hat{\chi}(t)e^{-it\lambda}e^{it\sqrt{-\Delta_g}}(x,y)\,dt \\
&= \int_{\mathbb{R}^n}\int_{-\infty}^\infty \hat{\chi}(t)q(t,x,y,\xi)e^{i(\varphi(x,y,\xi)+t(p(y,\xi)-\lambda))}\,dt\,d\xi \\
&= \lambda^n \int_{\mathbb{R}^n}\int_{-\infty}^\infty \hat{\chi}(t)q(t,x,y,\lambda\xi)e^{i\lambda(\varphi(x,y,\xi)+t(p(y,\xi)-1))}\,dt\,d\xi \\
&= \lambda^n \int_{\mathbb{R}^n}\int_{-\infty}^\infty \hat{\chi}(t)q(t,x,y,\lambda\xi)\beta_1(p(y,\xi))e^{i\lambda(\varphi(x,y,\xi)+t(p(y,\xi)-1))}\,dt\,d\xi \\
&\quad + O(\lambda^{-\infty})
\end{aligned}$$

where β_1 is as before, that is with small support and with $\beta_1 \equiv 1$ near 1. The $O(\lambda^{-\infty})$ bound on the discrepancy is uniform in x and y, and is obtained by integration by parts in t. Hence,

$$\begin{aligned}
&\int_\Sigma\int_\Sigma\int_{-\infty}^\infty \hat{\chi}(t)e^{-it\lambda}B_\lambda e^{it\sqrt{-\Delta_{\tilde{g}}}}B_\lambda^*(x,y)\,dt\,d\sigma(x)\,d\sigma(y) \\
&= \lambda^n \int\cdots\int e^{i\langle x'-w,\eta\rangle}B_\lambda(x',w,\eta)e^{i(\varphi(w,z,\xi)+t(p(z,\xi)-1))}\hat{\chi}(t)q(t,w,z,\xi) \\
&\qquad \times \beta_1(|\xi|)e^{i\langle z-y',\zeta\rangle}\overline{B_\lambda(y',z,\zeta)}\,dt\,dx'\,dy'\,dw\,dz\,d\eta\,d\zeta\,d\xi + O(\lambda^{-\infty}).
\end{aligned}$$

We perform the change of variables $\eta \mapsto \lambda\eta$ and $\zeta \mapsto \lambda\zeta$, and write $\xi = \xi' + r\omega$ in cylindrical coordinates with $r \in (0,\infty)$ and $\omega \in S^{n-d-1}$. The integral on the right hand side is then

$$
\begin{aligned}
(2.24) \qquad = \lambda^{3n} \int \cdots \int & e^{i\lambda\Phi(t,x',y',\xi',r,\omega,w,z,\eta,\zeta)} \\
& \times Q_0(\lambda; t, x', y', \xi', r, \omega, w, z, \eta, \zeta) \\
& \times dt\, dx'\, dy'\, d\xi'\, dr\, d\omega\, dw\, dz\, d\eta\, d\zeta
\end{aligned}
$$

where $d\omega$ denotes the standard volume measure on S^{n-d-1},

$$
\begin{aligned}
\Phi(t, x', y', \xi', r, \omega, w, z, \eta, \zeta) = \langle x' - w, \eta\rangle &+ \varphi(w, z, \xi' + r\omega) \\
&+ t(p(z, \xi' + r\omega) - 1) + \langle z - y', \zeta\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
&Q_0(\lambda; t, x', y', \xi', r, \omega, w, z, \eta, \zeta) \\
&= \hat{\chi}(t) b(x')\overline{b(y')}\beta_0(|x' - w|)\beta_0(|y' - z|)q(t, w, z, \lambda(\xi' + r\omega)) \\
&\times \beta_1(|\eta|)\beta_1(|\zeta|)\beta_1(p(z, \xi' + r\omega))a(\eta/|\eta|)a(\zeta/|\zeta|)r^{n-d-1}.
\end{aligned}
$$

Note all derivatives of a are uniformly bounded for $\lambda \geq 1$.

We will use the method of stationary phase in variables t, x', ξ', r, w, z, η, and ζ. Instead of doing so all at once with eight variables, we break it into two stages – the first involving w, z, η, and ζ, and the second involving the remaining four. We begin by fixing x', y', and ξ and by performing stationary phase with respect to w, z, η, and ζ. The gradient of the phase function in these variables is

$$
\nabla_{w,z,\eta,\zeta}\Phi = \begin{bmatrix} -\eta + \xi + O(|w - z||\xi|) \\ \zeta - \xi + O(|w - z||\xi|) \\ x' - w \\ y' - z \end{bmatrix}
$$

which, when $x' = y'$, has a critical point at $w = z = y'$ and $\eta = \zeta = \xi$. The Hessian matrix at this point is

$$
\nabla^2_{w,z,\eta,\zeta}\Phi = \begin{bmatrix} * & * & -I & 0 \\ * & * & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}
$$

which has determinant -1 and signature 0. There is a smooth curve $(w, z, \eta, \zeta) = (x', y', \eta(x', y', \xi), \zeta(x', y', \xi))$ on a neighborhood of $x' = y'$ on which $\nabla_{w,z,\eta,\zeta}\Phi = 0$ by the implicit function theorem, and $\nabla^2_{w,z,\eta,\zeta}\Phi$ is uniformly nondegenerate by continuity. After perhaps restricting the support of Q_0, by [16, Corollary 1.1.8] the integral (2.24) is

$$= \lambda^n \int \cdots \int e^{i\lambda\Psi(t,x',y',\xi',r,\omega)} Q_1(\lambda; t, x', y', \xi', r, \omega)\, dt\, dx'\, dy'\, d\xi'\, dr \tag{2.25}$$

with phase

$$\begin{aligned}\Psi(t,x',y',r,\omega) &= \Phi(t,x',y',\xi',r,\omega,x',y',\eta(x',y',\xi),\zeta(x',y',\xi)) \\ &= \varphi(x',y',\xi) + t(p(y',\xi)-1)\end{aligned}$$

and where the amplitude Q_1 has compact support and has uniformly bounded derivatives in all variables for $\lambda \geq 1$. Next we fix y' and ω and perform stationary phase in the remaining variables t, r, x', and ξ'. We have

$$\nabla_{t,r,x',\xi'}\Psi = \begin{bmatrix} p(y',\xi) - 1 \\ t\partial_r p(y,\xi) + O(|x'-y'|^2|\xi|) \\ \xi' + O(|x'-y'||\xi|) \\ x' - y' + t\nabla_{\xi'} p(y',\xi) + O(|x'-y'|^2) \end{bmatrix}$$

which has a critical point at $(t, r, x', \xi') = (0, 1, y', 0)$ whereat we have the Hessian

$$\nabla^2_{t,r,x',\xi'}\Psi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & * & I \\ 0 & 0 & I & 0 \end{bmatrix}$$

where in the computations we use

$$p(y',\xi) = \sqrt{r^2 + \sum_{i,j=0}^{d} g_\Sigma^{ij}(y')\xi_i'\xi_j'},$$

a consequence of the construction of our Fermi coordinates (2.4). After perhaps further restricting the support of Q_1 (and in particular $\hat{\chi}$), this is our only critical point for fixed y' and ω. [16, Corollary 1.1.8] in $2d+2$ variables yields the desired bound of $O(\lambda^{n-d-1})$ for (2.25). $\square$

Proof of Proposition 2.1. Let χ be as in the proof of Proposition 2.2. It suffices to show

$$\sum_j \chi(\lambda_j - \lambda)|R_\lambda e_j(x')|^2 \le C_N \lambda^{-N} \qquad N = 1, 2, \ldots$$

uniformly for $x' \in \Sigma$. Using a similar reduction as before, the sum on the left is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\chi}(t) e^{-it\lambda} R_\lambda e^{it\sqrt{-\Delta_g}} R_\lambda^*(x', x')\, dt.$$

Using the argument in the proof of Proposition 2.2, the expression above is

$$= \lambda^{3n} \int \cdots \int e^{i\lambda\Phi(t,x',w,z,\eta,\zeta,\xi)} a(\lambda; t, x', w, z, \eta, \zeta, \xi)\, dt\, dw\, dz\, d\eta\, d\zeta\, d\xi$$

where

$$\Phi(t, x', w, z, \eta, \zeta, \xi) = \langle x' - w, \eta\rangle + \varphi(w, z, \xi) + t(p(z, \xi) - 1) + \langle z - x', \zeta\rangle$$

and

$$\begin{aligned} a(\lambda; t, x', w, z, \eta, \zeta, \xi) = \hat{\chi}(t)|b(x')|^2 \beta_0(|x' - w|)\beta_0(|x' - z|) q(t, w, z, \lambda\xi) \\ \times (1 - \beta_1(|\eta|))(1 - \beta_1(|\zeta|))\beta_1(p(z, \xi)). \end{aligned}$$

As before, the critical points of Φ occur only where $\eta = \zeta = \xi$. By the construction of our coordinates,

$$p(x', \xi) = (1 + O(|x'|^2))|\xi|$$

and so we may adjust the support of b so that $(1 - \beta_1(|\xi|))\beta_1(p(x', \xi)) \equiv 0$. Hence, the critical points of Φ lie outside the support of the amplitude and the desired bound follows from nonstationary phase [16, Lemma 0.4.7]. $\square$

3. Kernel bounds

We require a characterization of the kernels K_α defined in (2.20) to proceed. Note first that if x and y are expressed in our Fermi coordinates (2.3) about Σ,

$$K_\alpha(T,\lambda;x,y) = \frac{1}{(2\pi)^{2n}} \iiiint e^{i\langle x-w,\eta\rangle} B_\lambda(x,w,\eta) \times K(T,\lambda;\alpha\tilde{w},\tilde{z}) e^{i\langle z-y,\zeta\rangle} \overline{B_\lambda(y,z,\zeta)}\, dw\, dz\, d\eta\, d\zeta \tag{3.1}$$

where $\tilde{w}$ and $\tilde{z}$ are the respective lifts of w and z to the Dirichlet domain D and

$$K(T,\lambda;\tilde{x},\tilde{y}) = \int (1-\beta(t/R))\hat{\chi}(t/T) e^{-it\lambda} \cos(t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x},\tilde{y})\, dt. \tag{3.2}$$

We begin by developing a characterization of the kernel $K(T,\lambda;\tilde{x},\tilde{y})$ for $\tilde{x},\tilde{y} \in \tilde{M}$ with $d_{\tilde{g}}(\tilde{x},\tilde{y})$ bounded away from zero as in [2, 6, 18]. In what follows, we draw liberally from Sogge's text, *Hangzhou Lectures on Eigenfunctions of the Laplacian* [15], for its arguments and notation, and also Bérard's article [2] for asymptotic bounds on derivatives of the distance function and the coefficients of the Hadamard parametrix. In what follows, we still impose $T = c\log\lambda$ as in (2.2).

Lemma 3.1. *Fix a positive integer m. There exist functions $a_\pm(T,\lambda;\tilde{x},\tilde{y})$ and $R(T,\lambda;\tilde{x},\tilde{y})$ depending on m such that*

$$K(T,\lambda;\tilde{x},\tilde{y}) = \lambda^{\frac{n-1}{2}} \sum_{\pm} a_\pm(T,\lambda;\tilde{x},\tilde{y}) e^{\pm i\lambda d_{\tilde{g}}(\tilde{x},\tilde{y})} + R(T,\lambda;\tilde{x},\tilde{y})$$

where if $d_{\tilde{g}}(\tilde{x},\tilde{y}) \geq 1$,

$$|\Delta_x^j \Delta_y^k a_\pm(T,\lambda;\tilde{x},\tilde{y})| \leq C_{j,k} e^{C_{j,k} d_{\tilde{g}}(\tilde{x},\tilde{y})} \qquad j,k = 0,1,2,\dots \tag{3.3}$$

and

$$|R(T,\lambda;\tilde{x},\tilde{y})| \lesssim e^{CT}\lambda^{-m}. \tag{3.4}$$

Moreover if $d_{\tilde{g}}(\tilde{x},\tilde{y}) \leq R$,

$$|K(T,\lambda;\tilde{x},\tilde{y})| \lesssim e^{CT}\lambda^{-m}. \tag{3.5}$$

Proof. By Theorem 2.4.1 and Remark 1.2.5 of [15],

$$\cos(t\sqrt{-\Delta_{\tilde g}})(\tilde x,\tilde y)=\sum_{\nu=0}^{N}\alpha_\nu(\tilde x,\tilde y)\partial_t E_\nu(t,d_{\tilde g}(\tilde x,\tilde y))+R_N(t,\tilde x,\tilde y) \tag{3.6}$$

where $\partial_t E_\nu(t,r)$ is some distribution supported on $r\le|t|$, and if $\tilde x$ is expressed in geodesic normal coordinates about $\tilde y$ with metric $\tilde g$, the coefficients α_ν are defined inductively by

$$\alpha_0(\tilde x,\tilde y)=|\tilde g(\tilde x)|^{-1/4}$$

and

$$\alpha_\nu(\tilde x,\tilde y)=\alpha_0(\tilde x,\tilde y)\int_0^1 t^{\nu-1}\frac{\Delta_{\tilde g}\alpha_{\nu-1}(\exp_{\tilde y}(t\log_{\tilde y}\tilde x),\tilde y)}{\alpha_0(\exp_{\tilde y}(t\log_{\tilde y}\tilde x),\tilde y)}\,dt, \tag{3.7}$$
$$\nu=1,2,3,\dots$$

where here $\Delta_{\tilde g}$ operates in the $\tilde x$ variable and where $\log_{\tilde y}$ is the inverse of the exponential map at $\tilde y$. Note α_ν are defined on all of $\tilde M$ since $|\tilde g(\tilde x)|$ is nonvanishing. Finally the remainder term satisfies

$$(\partial_t^2-\Delta_{\tilde g})R_N(t,\tilde x,\tilde y)=\Delta_{\tilde g}\alpha_N(\tilde x,\tilde y)\partial_t E_N(t,d_{\tilde g}(\tilde x,\tilde y)).$$

where $\Delta_{\tilde g}$ operates in the $\tilde x$ variable. In addition, the appendix of [2] provides us with exponential bounds,

$$|\Delta_{\tilde y}^j\alpha_\nu(\tilde x,\tilde y)|\le C_j e^{C_j d_{\tilde g}(\tilde x,\tilde y)}\qquad j=0,1,2,\dots,$$

which, with the fact that $\cos(t\sqrt{-\Delta_{\tilde g}})$ is self-adjoint, provide us with the same bounds on derivatives in $\tilde x$

$$|\Delta_{\tilde x}^j\alpha_\nu(\tilde x,\tilde y)|\le C_j e^{C_j d_{\tilde g}(\tilde x,\tilde y)}\qquad j=0,1,2,\dots$$

(see [18]). Proposition 6.1 in the appendix provides us with exponential bounds on the mixed derivatives,

$$|\Delta_{\tilde x}^j\Delta_{\tilde y}^k\alpha_\nu(\tilde x,\tilde y)|\le C_{j,k}e^{C_{j,k}d_{\tilde g}(\tilde x,\tilde y)}\qquad j,k=0,1,2,\dots. \tag{3.8}$$

The same proposition and Bérard's exponential bounds on derivatives of the distance function provide

$$|\Delta_{\tilde x}^j\Delta_{\tilde y}^k d_{\tilde g}(\tilde x,\tilde y)|\le C_{j,k}e^{C_{j,k}d_{\tilde g}(\tilde x,\tilde y)}\qquad j,k=0,1,2,\dots. \tag{3.9}$$

From (3.8), (3.9), an energy estimate argument in [15, §3.1], and the fact that $\partial_t E_\nu(t,r)$ is supported on $r \le |t|$, we have that R_N is C^m and satisfies bounds

$$|\partial_t^j R_N(t,\tilde{x},\tilde{y})| \le C_j e^{C_j d_{\tilde{g}}(\tilde{x},\tilde{y})} |t|^{2N+2-n-j} \qquad \text{for } j = 0,1,\dots,m$$

provided $N > m + \frac{n+1}{2}$. Integration by parts m times yields the bound

$$\left| \int_{\infty}^{\infty} (1-\beta(t/R))\hat{\chi}(t/T) e^{-it\lambda} R_N(t,\tilde{x},\tilde{y})\,dt \right| \lesssim e^{C_{N,m}T} \lambda^{-m} \tag{3.10}$$

as desired by (3.4).

In light of (3.6), (3.8), and (3.9), it suffices to show

$$\int_{-\infty}^{\infty} (1-\beta(t/R))\hat{\chi}(t/T) e^{-it\lambda} \partial_t E_\nu(t,r)\,dt = \lambda^{\frac{n-1}{2}} \sum_{\pm} a_\pm^\nu(T,\lambda;r) e^{\pm i\lambda r} \tag{3.11}$$

modulo terms whose contributions can be absorbed by the remainder $R(T,\lambda;\tilde{x},\tilde{y})$ since $T = c\log\lambda$, where $a_\pm^\nu$ satisfy bounds

$$\begin{aligned} &|\partial_r^\ell a_\pm^\nu(T,\lambda;r)| \le C_{\nu,\ell} \lambda^{-\nu} P_{\ell,\nu,k,j}(r) \\ &\text{for } \ell = 0,1,2,\dots,\ T \ge 1,\ r \ge 1 \end{aligned} \tag{3.12}$$

where $P_{\ell,\nu,k,j}$ is some polynomial. By [15, Remark 1.2.5], $\partial_t E_\nu(t,r)$ is a finite linear combination of distributions

$$t^j \int_{|\xi|\ge 1} e^{ir\xi_1 \pm it|\xi|} |\xi|^{-\nu-k}\,d\xi \qquad \text{for } j+k=\nu,\ j,k = 0,1,2,\dots \tag{3.13}$$

modulo smooth terms whose derivatives grow at most polynomially in t and r. The contribution of these discrepancy terms hence satisfy the same bounds as (3.5) and may be absorbed by the remainder. The contribution of each term (3.13) to the integral in (3.11) is

$$\int_{|\xi|\ge 1} \int_{-\infty}^{\infty} t^j (1-\beta(t/R))\hat{\chi}(t/T) e^{-it\lambda} e^{ir\xi_1 \pm it|\xi|} |\xi|^{-\nu-k}\,dt\,d\xi.$$

If the sign in the exponent is negative, the integral satisfies good bounds by integrating by parts in t and may be absorbed into the remainder, so it suffices only to consider the situation where the sign in the exponent is positive. In this case, we perform a change of variables $\xi \mapsto \lambda\xi$ and obtain

$$\begin{aligned}&\int_{|\xi|\geq 1}\int_{-\infty}^{\infty} t^j(1-\beta(t/R))\hat{\chi}(t/T)e^{i(r\xi_1+t(|\xi|-\lambda))}|\xi|^{-\nu-k}\,dt\,d\xi\\ &=\lambda^{n-\nu-k}\int_{|\xi|\geq\lambda^{-1}}\int_{-\infty}^{\infty} t^j(1-\beta(t/R))\hat{\chi}(t/T)e^{i\lambda(r\xi_1+t(|\xi|-1))}|\xi|^{-\nu-k}\,dt\,d\xi.\end{aligned}$$

Let $\beta_1 \in C_0^\infty(\mathbb{R},[0,1])$ be equal to 1 near 1 and have small support. We cut the integral in the second line into $\beta_1(|\xi|)$ and $(1-\beta_1(|\xi|))$ parts. The latter cut contributes a $O(T^{j-m+1}\lambda^{-m})$ term by integrating by parts in the t variable m times, and we let it be absorbed into the remainder. The $\beta_1(|\xi|)$ cut comes to

$$\lambda^{n-\nu-k}\int_{\mathbb{R}^n}\int_{-\infty}^{\infty} t^j(1-\beta(t/R))\hat{\chi}(t/T)e^{i\lambda(r\xi_1+t(|\xi|-1))}\beta_1(|\xi|)|\xi|^{-\nu-k}\,dt\,d\xi.$$

We take a moment to note that the integrand is supported on $|t|\geq 2R$, and hence if $r\leq R$, the gradient in ξ of the phase satisfies

$$|\nabla_\xi(r\xi_1+t(|\xi|-1))| = |re_1+t\xi/|\xi|| \geq R$$

for all t in the support of the integrand by the triangle inequality. Nonstationary phase and the bounds on our remainder term thus far yields (3.5).

From now on, we take $r\geq R$. By a change of coordinates $t\mapsto rt$, we write the integral as

$$\begin{aligned}&\lambda^{n-\nu-k}\int_{\mathbb{R}^n}\int_{-\infty}^{\infty} t^j(1-\beta(t/R))\hat{\chi}(t/T)e^{i\lambda(r\xi_1+t(|\xi|-1))}\beta_1(|\xi|)|\xi|^{-\nu-k}\,dt\,d\xi\\ &=\lambda^{n-\nu-k}r^{j+1}\int_{\mathbb{R}^n}\int_{-\infty}^{\infty} t^j(1-\beta(rt/R))\hat{\chi}(rt/T)\\ &\qquad\qquad\times e^{i\lambda r(\xi_1+t(|\xi|-1))}\beta_1(|\xi|)|\xi|^{-\nu-k}\,dt\,d\xi.\end{aligned}$$

We cut the integral one last time into $\beta_1(|t|)$ and $(1-\beta_1(|t|))$ components. By Hüygen's principle, we only consider the situation where $r\leq T$, and hence $\beta_1(|t|)(1-\beta(rt/R))\hat{\chi}(rt/T)$ and $(1-\beta_1(|t|))(1-\beta(rt/R))\hat{\chi}(rt/T)$ have bounded derivatives in t and r of all orders. The norm of the ξ-gradient of the phase function is

$$|\nabla_\xi(\xi_1+t(|\xi|-1))| = |e_1+t\xi/|\xi||$$

which is again bounded away from 0 on the support of $(1-\beta_1(|t|))$ and so contributes a term to be absorbed by the remainder by nonstationary phase.

We write the $\beta_1(|t|)$ cut as $I_+(T,\lambda;r) + I_-(T,\lambda;r)$ where

$$I_\pm(T,\lambda;r) = \lambda^{n-\nu-k} r^{j+1} \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} t^j (1-\beta(rt/R))\beta_1(\pm t)\hat{\chi}(rt/T) \times e^{i\lambda r(\xi_1 + t(|\xi|-1))} \beta_1(|\xi|)|\xi|^{-\nu-k}\, dt\, d\xi.$$

The phase function of $I_\pm$ has a critical point at $(t,\xi) = \pm(1,-e_1)$ at which the Hessian of the phase function,

$$\pm \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & I \end{bmatrix},$$

is nondegenerate. We write $s = \lambda r$ and subsequently write $I_\pm$ as

$$\lambda^{n-\nu-k-j-1} s^{j+1} \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} t^j (1-\beta(st/R\lambda))\beta_1(\pm t)\hat{\chi}(st/T\lambda) \times e^{is(\xi_1 + t(|\xi|-1))} \beta_1(|\xi|)|\xi|^{-\nu-k}\, dt\, d\xi.$$

Note for $T \geq 1$, the amplitude of the integrand satisfies bounds

$$\left| \partial_\xi^\alpha \partial_t^m \partial_s^\ell \left[t^j (1-\beta(st/R\lambda))\beta_1(\pm t)\hat{\chi}(st/T\lambda)\beta_1(|\xi|)|\xi|^{-\nu-k} \right] \right| \leq C_{\nu,\alpha,\ell,m} \lambda^{-\ell}.$$

Taking s as the frequency parameter and using the method of stationary phase [15, Proposition 4.1.2] yields

$$|\partial_s^\ell (e^{\pm is} I_\pm)| \leq C_{\ell,\nu,k,j} \lambda^{n-\nu-k-j-1} s^{j-\ell-\frac{n-1}{2}}$$

from which we obtain

$$|\partial_r^\ell (e^{\pm ir\lambda} I_\pm(T,\lambda;r))| \leq C_{\ell,\nu,k,j} \lambda^{\frac{n-1}{2}-\nu-k} r^{j-\ell-\frac{n-1}{2}}.$$

(3.11) and (3.12) follow. □

Set

$$\Gamma_R = \left\{ \alpha \in \Gamma : \sup_{x,y \in \operatorname{supp} b} d_{\tilde{g}}(\alpha\tilde{x},\tilde{y}) \leq R \right\}.$$

The contribution of the terms of Γ_R to the sum (2.19) are $O(e^{CT}\lambda^{-m})$ by (3.5) of the lemma, which is better than we need. Moreover by restricting the support of b, we ensure that

$$\inf_{x,y\in \operatorname{supp} b} d_{\tilde g}(\alpha\tilde x, \tilde y) \geq R-1 \qquad \text{if } \alpha \in \Gamma\setminus\Gamma_R. \tag{3.14}$$

In light of this, what remains is to show that

$$\sum_{\Gamma\setminus\Gamma_R} \left| \int_\Sigma \int_\Sigma K_\alpha(T,\lambda;x,y)\, d\sigma(x)\, d\sigma(y) \right| \lesssim e^{CT}\lambda^\delta. \tag{3.15}$$

The next lemma uses the previous to characterize the conjugated kernel K_α. Here the function of the operators B_λ begins to surface. Conjugating K by B_λ filters out points $\tilde x$ and $\tilde y$ in $\tilde M$ for which the geodesic connecting $\tilde y$ to $\alpha\tilde x$ departs and arrives in dissimilar directions (see Figure 1). This will be very useful in Section 5, when we need to control the gradient of the phase function $d_{\tilde g}(\alpha\tilde x, \tilde y)$. As usual, $\tilde x$ and $\tilde y$ denote the respective lifts of x and y to the Dirichlet domain D.

Lemma 3.2. *We have*

$$K_\alpha(T,\lambda;x,y) = \lambda^{\frac{n-1}{2}} \sum_\pm a_{\alpha,\pm}(T,\lambda;x,y) e^{\pm i\lambda d_{\tilde g}(\alpha\tilde x,\tilde y)} + O(e^{CT}\lambda^\delta) \tag{3.16}$$

where the amplitude $a_{\alpha,\pm}$ satisfies bounds

$$|\Delta_x^j \Delta_y^k a_{\alpha,\pm}(T,\lambda;x,y)| \leq C_{i,j} e^{C_{i,j} d_{\tilde g}(\alpha\tilde x,\tilde y)} \tag{3.17}$$

and is supported on $\operatorname{supp}_x B_\lambda \times \operatorname{supp}_x B_\lambda$. *Moreover, there exists an open conical neighborhood $U \subset T^*\tilde M$ which can be made small by restricting the support of B_λ such that*

$$|a_{\alpha,\pm}(T,\lambda;x,y)| \leq C_{U,N} e^{C_{U,N} d_{\tilde g}(\alpha\tilde x,\tilde y)} \lambda^{-N} \qquad N = 1,2,\ldots \tag{3.18}$$

for all x and y for which neither of

$$\begin{aligned} &(\gamma'(0), \alpha^*\gamma'(1)) \in U\times U \qquad \textit{nor} \\ &(-\gamma'(0), -\alpha^*\gamma'(1)) \in U\times U \end{aligned}$$

hold, where γ is the constant-speed geodesic with $\gamma(0) = \tilde y$ and $\gamma(1) = \alpha\tilde x$, and where γ' is understood as an element in $T^*\tilde M$, and where α^* is the pullback on the cotangent bundle through α.

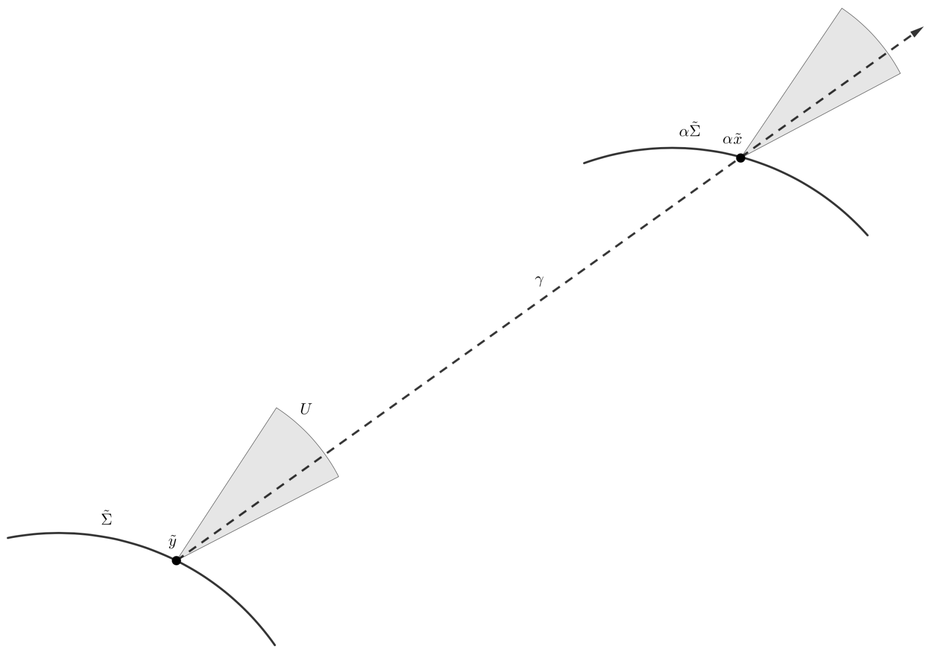

Figure 1: Lemma 3.2 tells us the only relevant terms in the sum are those with geodesics (the dashed line) starting in U (resp. $-U$) and ending in αU (resp. $\alpha(-U)$).

Proof. By Lemma 3.1, we have

$$\begin{aligned}
&K_\alpha(T,\lambda;x,y)\\
&= \frac{\lambda^{\frac{n-1}{2}}}{(2\pi)^{2n}} \sum_{\pm} \iiiint e^{i\langle x-w,\eta\rangle} B_\lambda(x,w,\eta) a_\pm(T,\lambda;\alpha\tilde{w},\tilde{z}) e^{\pm i\lambda d_{\tilde{g}}(\alpha\tilde{w},\tilde{z})}\\
&\qquad\qquad \times e^{i\langle z-y,\zeta\rangle} \overline{B_\lambda(y,z,\zeta)}\, dw\, dz\, d\eta\, d\zeta\\
&\quad + \frac{1}{(2\pi)^{2n}} \iiiint e^{i\langle x-w,\eta\rangle} B_\lambda(x,w,\eta) R(T,\lambda;\alpha\tilde{w},\tilde{z}) e^{i\langle z-y,\zeta\rangle}\\
&\qquad\qquad \times \overline{B_\lambda(y,z,\zeta)}\, dw\, dz\, d\eta\, d\zeta.
\end{aligned}$$

The second integral on the right hand side is $O(e^{CT}\lambda^\delta)$ by taking m in (3.4) greater than $2n-\delta$ and the fact that, by (2.8),

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |B_\lambda(x,w,\eta)|\, dw\, d\eta = O(\lambda^n).$$

It suffices then to equate the first term to the right hand side of (3.16). Using a change of variables $\eta \mapsto \lambda\eta$ and $\zeta \mapsto \lambda\zeta$, this is

$$\frac{\lambda^{2n+\frac{n-1}{2}}}{(2\pi)^{2n}} \sum_{\pm} \iiiint e^{i\lambda\Phi_\pm(x,y,w,z,\eta,\zeta)} A(T,\lambda;x,y,w,z,\eta,\zeta)\, dw\, dz\, d\eta\, d\zeta$$

where

$$\Phi_\pm(x,y,w,z,\eta,\zeta) = \langle x-w,\eta\rangle \pm d_{\tilde g}(\alpha\tilde w,\tilde z) + \langle z-y,\zeta\rangle$$

and by (2.8),

$$\begin{aligned} &A(T,\lambda;x,y,w,z,\eta,\zeta) \\ &= \beta_0(|x-w|)\beta_0(|z-y|)\beta_0(|x^\perp|)\beta_0(|y^\perp|)b(x')\overline{b(y')} \\ &\quad\times a_\pm(T,\lambda;\alpha\tilde w,\tilde z)a(\eta/|\eta|)a(\zeta/|\zeta|)\beta_1(|\eta|)\beta_1(|\zeta|) \end{aligned} \tag{3.19}$$

For clarity, we focus only on the Φ_+ component; the argument for the alternate sign is the same. The Euclidean gradient of the phase function with respect to the variables of integration is

$$\nabla_{w,z,\eta,\zeta}\Phi_\pm = \begin{bmatrix} -\eta + \nabla_{\tilde w} d_{\tilde g}(\alpha\tilde w,\tilde z) \\ \zeta + \nabla_{\tilde z} d_{\tilde g}(\alpha\tilde w,\tilde z) \\ x-w \\ z-y \end{bmatrix}$$

which has a critical point at $(w,z,\eta,\zeta) = (x,y,\nabla_{\tilde x} d_{\tilde g}(\alpha\tilde x,\tilde y), -\nabla_{\tilde y} d_{\tilde g}(\alpha\tilde x,\tilde y))$ at which the phase takes the value $d_{\tilde g}(\alpha\tilde x,\tilde y)$ and has Hessian

$$\nabla^2_{w,z,\eta,\zeta}\Phi_\pm = \begin{bmatrix} * & * & -I & 0 \\ * & * & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix},$$

which has determinant -1. We have (3.16) and (3.17) by (3.9), (3.3), and [16, Corollary 1.1.8]. Consider a conic neighborhood V of $\mathbb{R}^n$ containing the support of a in (3.19) (also (2.8)). In local coordinates (2.3), let U' be the set of (x,ξ) for which x lies in a small open neighborhood of the support of b in (2.8) and ξ lies in V, and let $U \subset T^*\tilde M$ be the set of the duals of the vectors of U'. If $\nabla_{\tilde{x}} d_{\tilde{g}}(\alpha\tilde{x}, \tilde{y})$ lies in the complement of U', then

$$|-\eta + \nabla_{\tilde{x}} d_{\tilde{g}}(\alpha\tilde{x}, \tilde{y})| \geq c > 0 \tag{3.20}$$

on the support of A for some constant c depending on U'. Hence,

$$|-\eta + \nabla_{\tilde{w}} d_{\tilde{g}}(\alpha\tilde{w}, \tilde{z})| \geq c - |\nabla_{\tilde{x}} d_{\tilde{g}}(\alpha\tilde{x}, \tilde{y}) - \nabla_{\tilde{w}} d_{\tilde{g}}(\alpha\tilde{w}, \tilde{z})|.$$

In the next section, we will show that the Hessian of the distance function is uniformly bounded on the entirety of $\tilde{M} \times \tilde{M}$ minus a neighborhood of the diagonal (see Remark 4.2 and (3.14)). Moreover since $\tilde{x}, \tilde{y}, \tilde{w}$, and $\tilde{z}$ are all in the same local coordinates, the Christoffel symbols of the metric are bounded. Hence, the Euclidean Hessian of $d_{\tilde{g}}(\alpha\tilde{x}, \tilde{y})$ in both variables is uniformly bounded[4] in α and

$$|\nabla_{\tilde{x}} d_{\tilde{g}}(\alpha\tilde{x}, \tilde{y}) - \nabla_{\tilde{w}} d_{\tilde{g}}(\alpha\tilde{w}, \tilde{z})| \leq C(|x - w| + |y - z|)$$

by the mean value theorem. We restrict the support of β_0 in (3.19) so that $|-\eta + \nabla_{\tilde{w}} d_{\tilde{g}}(\alpha\tilde{w}, \tilde{z})|$ is bounded away from 0 uniformly in α. We remark that the covector $\langle\ \cdot\ , \nabla_{\tilde{x}} d_{\tilde{g}}(\alpha\tilde{x}, \tilde{y})\rangle$ with the Euclidean inner product is precisely the dual of $\gamma'(1)/|\gamma'(1)|$ pulled back by α. In particular, if $\tilde{x}$ and $\tilde{y}$ are such that the dual of $\gamma'(1)/|\gamma'(1)|$ is in the complement of U, we have (3.20). The desired bound (3.18) then follows from (3.9), (3.3), and nonstationary phase [16, Lemma 0.4.7] in the w variable. The argument is similar if $-\nabla_{\tilde{y}} d_{\tilde{g}}(\alpha\tilde{x}, \tilde{y})$ is in the complement of U. □

Let Γ_U denote the subset of Γ for which there exist x and y in the support of $a_{\alpha,\pm}$ such that the geodesic $\gamma : [0, 1] \to \tilde{M}$ with $\gamma(0) = \tilde{y}$ and $\gamma(1) = \alpha\tilde{x}$ has both $\gamma'(0) \in U$ and $\alpha^*\gamma'(1) \in U$. Lemma 3.2 and (2.22) show us

$$\sum_{\alpha \in (\Gamma \setminus \Gamma_U) \setminus \Gamma_R} \left| \int_\Sigma \int_\Sigma K_\alpha(T, \lambda; x, y)\, d\sigma(x)\, d\sigma(y) \right| \lesssim e^{CT} \lambda^{-m}$$

for some m which can be made large. So, (3.15) would follow from

$$\sum_{\alpha \in \Gamma_U \setminus \Gamma_R} \left| \int_\Sigma \int_\Sigma K_\alpha(T, \lambda; x, y)\, d\sigma(x)\, d\sigma(y) \right| \lesssim e^{CT} \lambda^{\delta}. \tag{3.21}$$

It is now time to specify the statements we require to prove Theorems 1.3 and 1.4. Recall from (2.1) that the only requirement for the exponent δ

[4]See (4.2) for the relationship between the Hessian on a manifold and the Euclidean Hessian in local coordinates.

is that it is less than $n-d-1$. Propositions 3.3 and 3.4 along with Lemma 3.2 and (2.22) imply (3.21) under the hypotheses of Theorem 1.3 and Theorem 1.4, respectively.

Proposition 3.3. *Under the hypotheses of Theorem 1.3, we have*

$$\left|\int_\Sigma \int_\Sigma a_{\alpha,\pm}(T,\lambda;x,y) e^{\pm i\lambda d_{\tilde g}(\alpha\tilde x,\tilde y)}\, d\sigma(x)\, d\sigma(y)\right| \lesssim e^{CT}\lambda^{-d/2}$$
for $\alpha \in \Gamma_U \setminus \Gamma_R$,

where the constant C is uniform in α.

Proposition 3.4. *Assume the hypotheses of Theorem 1.4. If $\alpha \in \Gamma_U \setminus \Gamma_R$,*

$$\left|\int_\Sigma \int_\Sigma a_{\alpha,\pm}(T,\lambda;x,y) e^{\pm i\lambda d_{\tilde g}(\alpha\tilde x,\tilde y)}\, d\sigma(x)\, d\sigma(y)\right| \lesssim e^{CT}\lambda^{-n/2}$$

where the constant C is uniform in α.

Remark 3.5. If M has strictly negative curvature, then the hypotheses of Theorem 1.4 contain the hypotheses of Theorem 1.3, and Proposition 3.3 applies for hypersurfaces and gives us a bound of $e^{CT}\lambda^{-n/2+1/2}$. This is not enough to obtain the δ we need in (2.1). The added hypotheses on the curvature of Σ in Theorem 1.4 allow us use the method of stationary phase in Section 5 over one more variable, improving the bound by a factor of $\lambda^{-1/2}$.

4. Geometry and phase function bounds

We will need some information about the first and second derivatives of the phase functions in Propositions 3.3 and 3.4. This section will provide the tools necessary to do so. Specifically, we will compute the Hessian of the phase function using the second fundamental form of Σ and of spheres in $\tilde M$. We then verify Definition 1.3 and prove some useful properties of the second fundamental form of circles of large radius. Finally, we use these properties to provide good bounds on the Hessian of the phase function. DoCarmo's text [7] is our primary reference for this section.

We outline some basic facts before we begin. For a general Riemannian manifold (M,g) with Levi-Civita connection ∇, the Hessian of $f \in C^\infty(M)$

is the quadratic form

$$\mathrm{Hess} f(X,Y) = X(Yf) - (\nabla_X Y)f \tag{4.1}$$

where X and Y are vector fields on M. For future use we note, in local coordinates $x = (x_1, \dots, x_n)$,

$$\mathrm{Hess} f\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_k \Gamma_{ij}^k \frac{\partial f}{\partial x_k} \tag{4.2}$$

and so if the Christoffel symbols Γ_{ij}^k are small and the first derivatives of f are bounded, the Hessian of f is nearly the Euclidean Hessian. Suppose Σ is a submanifold of M with the induced metric $\overline{g}$ and Levi-Civita connection $\overline{\nabla}$. By (4.1),

$$\mathrm{Hess}_\Sigma f(X,Y) = \mathrm{Hess}_M f(X,Y) + \mathrm{II}_\Sigma(X,Y)f \tag{4.3}$$

where X, Y are vectors in Σ and where II_Σ is the second fundamental form of Σ in M, given by

$$\mathrm{II}_\Sigma(X,Y) = \nabla_X Y - \overline{\nabla}_X Y = (\nabla_X Y)^\perp, \tag{4.4}$$

the orthogonal projection of $\nabla_X Y$ onto the normal bundle $N\Sigma$. The Hessians and the second fundamental form are tensorial and only depend on the value of X and Y at a point. (For details see [7, Section 6.2].)

4.1. Computing the Hessian of the phase function

We will want to compute the Hessian of the phase functions from Propositions 3.3 and 3.4, that is the function $\phi : \Sigma \times \Sigma \to \mathbb{R}$ given by

$$\phi(x,y) = d_{\tilde{g}}(\alpha\tilde{x}, \tilde{y})$$

where $\Sigma \times \Sigma$ is endowed with the product metric, where $\tilde{x}$ and $\tilde{y}$ are the respective lifts of x and y to our Dirichlet domain D in the universal cover, and where α is a fixed, non-identity deck transformation. By (4.3),

$$\begin{aligned} \mathrm{Hess}_{\Sigma\times\Sigma}\phi(X,Y) &= \mathrm{Hess}_{\alpha\tilde{\Sigma}\times\tilde{\Sigma}} d_{\tilde{g}}(X,Y) \\ &= \mathrm{Hess}_{\tilde{M}\times\tilde{M}} d_{\tilde{g}}(X,Y) + \mathrm{II}_{\alpha\tilde{\Sigma}\times\tilde{\Sigma}}(X,Y) d_{\tilde{g}} \end{aligned} \tag{4.5}$$

where X and Y are both vectors in $\Sigma \times \Sigma$ with the same base point, but are also understood to be their respective lifts to $\alpha\tilde{\Sigma} \times \tilde{\Sigma}$ where appropriate.

To compute the Hessian of the phase function, it suffices to compute the Hessian of $d_{\tilde{g}}$ on $\tilde{M} \times \tilde{M}$ and the second fundamental form of $\alpha\tilde{\Sigma} \times \tilde{\Sigma}$. To this end, we write

$$X = X_1 \oplus X_2 \qquad \text{and} \qquad Y = Y_1 \oplus Y_2$$

where X_1 and Y_1 are vectors on $\alpha\tilde{\Sigma}$ and X_2 and Y_2 are vectors on $\tilde{\Sigma}$ and write

$$\text{Hess}_{\tilde{M}\times\tilde{M}} d_{\tilde{g}}(X,Y) = \sum_{i,j=1,2} \text{Hess}_{\tilde{M}\times\tilde{M}} d_{\tilde{g}}(X_i,Y_j) \qquad \text{and} \tag{4.6}$$

$$\mathrm{II}_{\alpha\tilde{\Sigma}\times\tilde{\Sigma}}(X,Y) d_{\tilde{g}} = \sum_{i,j=1,2} \mathrm{II}_{\alpha\tilde{\Sigma}\times\tilde{\Sigma}}(X_i,Y_j) d_{\tilde{g}}. \tag{4.7}$$

Note the $i \neq j$ terms of (4.7) vanish and we are left with

$$\mathrm{II}_{\alpha\tilde{\Sigma}\times\tilde{\Sigma}}(X,Y) = \mathrm{II}_{\alpha\tilde{\Sigma}}(X_1,Y_1) d_{\tilde{g}} + \mathrm{II}_{\tilde{\Sigma}}(X_2,Y_2) d_{\tilde{g}}. \tag{4.8}$$

The next lemma helps us compute the terms in (4.6).

Lemma 4.1. *Let $\tilde{x}$ and $\tilde{y}$ be* any *points in $\tilde{M}$, let $r = d_{\tilde{g}}(\tilde{x},\tilde{y})$, and let $X_1, Y_1 \in T_{\tilde{x}}\tilde{M}$ and $X_2, Y_2 \in T_{\tilde{y}}\tilde{M}$, and we understand*

$$X_1 d_{\tilde{g}} = X_1 d_{\tilde{g}}(\,\cdot\,,\tilde{y}) \qquad \textit{and} \qquad X_2 d_{\tilde{g}} = X_2 d_{\tilde{g}}(\tilde{x},\,\cdot\,),$$

and similarly for Y_i, $i = 1,2$. Then, the following are true.

1) *$X_1 d_{\tilde{g}} = |X_1|\cos\theta$ where θ is the angle between X_1 and the first derivative of the geodesic adjoining $\tilde{y}$ to $\tilde{x}$. In particular, $X_1 d_{\tilde{g}} = 0$ if and only if X_1 is perpendicular to this geodesic. This holds similarly for $X_2 d_{\tilde{g}}$.*
2) *We have absolute bounds*

$$|\text{Hess}_{\tilde{M}\times\tilde{M}} d_{\tilde{g}}(X_1,Y_2)| \leq 2|X_1||Y_2|/r \qquad \textit{and}$$
$$|\text{Hess}_{\tilde{M}\times\tilde{M}} d_{\tilde{g}}(X_2,Y_1)| \leq 2|X_2||Y_1|/r.$$

3) *Let $S_{\tilde{y}}(r)$ denote the sphere in $\tilde{M}$ with center $\tilde{y}$ and radius r. Then,*

$$\text{Hess}_{\tilde{M}\times\tilde{M}} d_{\tilde{g}}(X_1,Y_1) = -\mathrm{II}_{S_{\tilde{y}}(r)}(X_1',Y_1') d_{\tilde{g}}$$

where X_1' and Y_1' are the orthogonal projections of X_1 and Y_1 onto $T_{\tilde{x}}S_{\tilde{y}}(r)$, respectively. We similarly have

$$\mathrm{Hess}_{\tilde{M}\times\tilde{M}} d_{\tilde{g}}(X_2, Y_2) = -\mathrm{II}_{S_{\tilde{x}}(r)}(X_2', Y_2')d_{\tilde{g}}.$$

Proof. Fix X_1 and Y_2 as above and let $\sigma_1, \sigma_2 : (-\epsilon, \epsilon) \to \tilde{M}$ be curves with

$$\sigma_1'(0) = X_1 \in T_{\tilde{x}}\tilde{M} \qquad \text{and} \qquad \sigma_2'(0) = Y_2 \in T_{\tilde{y}}\tilde{M}.$$

We then define a map

$$\gamma : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \times [0, 1] \to \tilde{M}$$

such that for all $u, v \in (-\epsilon, \epsilon)$,

$$\gamma(u, v, 1) = \sigma_1(u) \qquad \text{and} \qquad \gamma(u, v, 0) = \sigma_2(v),$$

and where $t \mapsto \gamma(u, v, t)$ traces out the constant-speed geodesic connecting $\sigma_2(v)$ to $\sigma_1(u)$. Since ∂_u, ∂_v, and ∂_t are coordinate vector fields in the domain of γ, the Lie brackets

$$[\partial_u, \partial_t] = 0, \qquad [\partial_v, \partial_t] = 0, \qquad \text{and} \qquad [\partial_u, \partial_v] = 0$$

all vanish. Hence,

$$0 = [\partial_u, \partial_t]\gamma = [\partial_u\gamma, \partial_v\gamma] = \nabla_u\partial_t\gamma - \nabla_t\partial_u\gamma,$$

where ∇ is the Levi-Civita connection on $\tilde{M}$ and where ∇_u and ∇_t are shorthand for the covariant derivative with respect to the vector fields $\partial_u\gamma$ and $\partial_t\gamma$. This and similar calculations yield the identities

$$\nabla_u\partial_t\gamma = \nabla_t\partial_u\gamma, \qquad \nabla_v\partial_t\gamma = \nabla_t\partial_v\gamma, \qquad \text{and} \qquad \nabla_u\partial_v\gamma = \nabla_v\partial_u\gamma$$

which we will use repeatedly and without reference. Next, we write

$$d_{\tilde{g}}(\sigma_1(u), \sigma_2(v))^2 = \int_0^1 |\partial_t\gamma(u, v, t)|^2 \, dt.$$

Taking a derivative in u of $\frac{1}{2}d_{\tilde{g}}^2$ yields

$$
\begin{aligned}
d_{\tilde{g}}\partial_u d_{\tilde{g}} &= \int_0^1 \langle \partial_t\gamma(u,v,t), \nabla_u\partial_t\gamma(u,v,t)\rangle\, dt \\
&= \int_0^1 \langle \partial_t\gamma(u,v,t), \nabla_t\partial_u\gamma(u,v,t)\rangle\, dt \\
&= \int_0^1 \partial_t\langle \partial_t\gamma(u,v,t), \partial_u\gamma(u,v,t)\rangle\, dt \\
&= \langle \partial_t\gamma(u,v,1), \partial_u\gamma(u,v,1)\rangle
\end{aligned}
$$

where the third line is due to the geodesic equation $\nabla_t\partial_t\gamma = 0$ and the fourth to the fundamental theorem of calculus. Part (1) of the lemma follows after noting $\partial_t\gamma(0,0,1)/d_{\tilde{g}}$ is the unit vector in the direction of the geodesic $\gamma(0,0,t)$ at $t=1$, and after recalling $\partial_u\gamma(0,0,1) = X_1$. Next, we take a derivative in v and obtain

$$
\begin{aligned}
d_{\tilde{g}}\partial_u\partial_v d_{\tilde{g}} + \partial_u d_{\tilde{g}}\partial_v d_{\tilde{g}} &= \langle \nabla_v\partial_t\gamma(u,v,1), \partial_u\gamma(u,v,1)\rangle \\
&\quad + \langle \partial_t\gamma(u,v,1), \nabla_v\partial_u\gamma(u,v,1)\rangle.
\end{aligned}
$$

Note $\nabla_v\partial_u\gamma(u,v,1) = \nabla_u\partial_v\gamma(u,v,1) = 0$, since $\gamma(u,v,1)$ is constant in v. Hence,

$$
d_{\tilde{g}}\partial_u\partial_v d_{\tilde{g}} + \partial_u d_{\tilde{g}}\partial_v d_{\tilde{g}} = \langle \nabla_t\partial_v\gamma(0,0,1), X_1\rangle. \tag{4.9}
$$

We pause here to make a couple observations. First, $t \mapsto \partial_v\gamma(0,0,t)$ is a Jacobi field along $t \mapsto \gamma(0,0,t)$ with boundary data

$$
\partial_v\gamma(0,0,0) = Y_2 \qquad \text{and} \qquad \partial_v\gamma(0,0,1) = 0.
$$

Observe that $\partial_u\partial_v d_{\tilde{g}}$ is independent of our choice of curves σ_1 and σ_2, and that

$$
\operatorname{Hess}_{\tilde{M}\times\tilde{M}} d_{\tilde{g}}(X_1, Y_2) = X_1(Y_2 d_{\tilde{g}}) = \partial_u\partial_v d_{\tilde{g}}(\sigma_1(u), \sigma_2(v))
$$

at $u = v = 0$. To get part (2) of the lemma, we claim the right side of (4.9) is bounded by $|X_1||Y_2|$. Let $h(t)$ denote the inner product of $\partial_v\gamma(0,0,t)$ with the parallel vector field $w(t)$ obtained by a translation of $\pm X_1$ along γ, with the sign chosen so that $h(0) \geq 0$. Note since $\partial_v\gamma(0,0,t)$ is a scalar multiple of $w(t)$ for $t = 0$ and $t = 1$, it is a scalar multiple of $w(t)$ for all t. In fact, $\partial_v\gamma(0,0,t) = h(t)w(t)$. Let R denote the Riemann curvature tensor as in [7], and by an abuse of notation, set

$$R(t) = \langle R(\gamma'(t), \partial_v \gamma(0,0,t))\gamma'(t), w(t)\rangle.$$

Then by the Jacobi equation,

$$h''(t) + R(t)h(t) = \left\langle \frac{D^2}{dt^2}\partial_v\gamma + R(\gamma', \partial_v\gamma)\gamma', w(t) \right\rangle = 0.$$

Now,

$$R(t)h(t)^2 = \langle R(\gamma', \partial_v\gamma)\gamma', \partial_v\gamma\rangle \leq 0$$

since the sectional curvature of $\tilde{M}$ is nonpositive. In particular, $R(t) \leq 0$. If h is nontrivial, it vanishes only at 1 and hence is nonnegative on $[0,1]$. Then,

$$h''(t) \geq 0 \qquad \text{for } t \in [0,1].$$

By convexity,

$$0 \leq h(t) \leq h(0)(1-t),$$

and hence

$$0 \geq h'(1) \geq -h(0).$$

We know $h'(1)$ is equal to the right hand side of (4.9) up to a sign, and that $|h(0)| \leq |X_1||Y_2|$ by Cauchy-Schwarz. Furthermore, $|X_1 d_{\tilde{g}}| \leq |X_1|$ and $|Y_2 d_{\tilde{g}}| \leq |Y_2|$ by the triangle inequality. Hence,

$$|\partial_u \partial_v d_{\tilde{g}}| = \frac{|\langle \nabla_t \partial_v \gamma(0,0,1), X_1\rangle - (\partial_u d_{\tilde{g}})(\partial_v d_{\tilde{g}})|}{d_{\tilde{g}}} \leq \frac{2|X_1||Y_2|}{d_{\tilde{g}}},$$

as desired.

Finally we prove part (3) of the lemma. Consider geodesic normal coordinates $(x_2, \dots, x_n)$ at $\tilde{x}$ of the sphere $S_{\tilde{y}}(r)$. We take an extension $(x_1, x_2, \dots, x_n)$ of these coordinates to a neighborhood of $\tilde{M}$, where x_1 is the radial coordinate. By the geodesic equation $\nabla_1 \partial_1 = 0$,

$$\mathrm{Hess}_{\tilde{M}\times\tilde{M}} d_{\tilde{g}}(\partial_1, \partial_1) = \partial_1(\partial_1 x_1) - (\nabla_1 \partial_1)x_1 = 0.$$

Moreover if $i \neq 1$,

$$\mathrm{Hess}_{\tilde{M}\times\tilde{M}} d_{\tilde{g}}(\partial_i, \partial_1) = \partial_i(\partial_1 x_1) - (\nabla_i \partial_1)x_1 = -\nabla_1 \partial_i x_1,$$

where $\nabla_i \partial_1 = \nabla_1 \partial_i$ by a similar argument as in the proof of part (1). Notice that ∂_i is a perpendicular Jacobi field along the x_1 coordinate geodesic.

Hence, $\nabla_1\partial_i$ is also perpendicular to the x_1 coordinate geodesic, and $\nabla_1\partial_i x_1 = 0$. Then,

$$\mathrm{Hess}_{\tilde{M}\times\tilde{M}} d_{\tilde{g}}(X_1, Y_1) = \mathrm{Hess}_{\tilde{M}} d_{\tilde{g}}(X_1', Y_1')$$

where X_1' and Y_1' are the orthogonal projections of X_1 and Y_1 onto $T_{\tilde{x}}S_{\tilde{y}}(r)$. It suffices then to show

$$\mathrm{Hess}_{\tilde{M}\times\tilde{M}} d_{\tilde{g}}(X_1, Y_1) = -\mathrm{II}_{S_{\tilde{y}}(r)}(X_1, Y_1) d_{\tilde{g}}$$

in the situation where X_1 and Y_1 are vectors tangent to the sphere $S_{\tilde{y}}(r)$. In this situation we have $Y_1 d_{\tilde{g}} \equiv 0$, whence

$$\mathrm{Hess}_{\tilde{M}\times\tilde{M}} d_{\tilde{g}}(X_1, Y_1) = -(\nabla_{X_1} Y_1) d_{\tilde{g}} = -(\nabla_{X_1} Y_1)^{\perp} d_{\tilde{g}} = -\mathrm{II}_{S_{\tilde{y}}(r)}(X_1, Y_1) d_{\tilde{g}},$$

as desired. □

Remark 4.2. By comparison with the Euclidean case, the Hessian of the distance function $d_{\tilde{g}}$ in one variable is uniformly bounded for $d_{\tilde{g}} \geq 1$ (see [14, Theorem 1.1]). This, part (2) of Lemma 4.1, and (4.6) show that the Hessian of $d_{\tilde{g}}$ in both variables is uniformly bounded for $d_{\tilde{g}} \geq 1$.

Lemma 4.1, (4.5), and (4.8) combined provide us with the crucial computation

$$\begin{aligned} \text{(4.10)}\quad \mathrm{Hess}_{\Sigma\times\Sigma}\phi(X, Y) = {} & \mathrm{II}_{\alpha\tilde{\Sigma}}(X_1, Y_1) d_{\tilde{g}} - \mathrm{II}_{S_{\tilde{y}}(d_{\tilde{g}})}(X_1', Y_1') d_{\tilde{g}} \\ & + \mathrm{II}_{\tilde{\Sigma}}(X_2, Y_2) d_{\tilde{g}} - \mathrm{II}_{S_{\alpha\tilde{x}}(d_{\tilde{g}})}(X_2', Y_2') d_{\tilde{g}} + R(X, Y) \end{aligned}$$

where

$$|R(X, Y)| \leq 2(|X_1||Y_2| + |X_2||Y_1|)/d_{\tilde{g}}.$$

4.2. The second fundamental form of spheres

To provide any useful bounds on $\mathrm{Hess}_{\Sigma\times\Sigma}\phi$, we need to understand the behavior of the second fundamental form of spheres of large radius. Here we provide quantitative estimates for the perhaps obvious fact that the second fundamental forms of large spheres are very nearly the second fundamental forms of horospheres.

The second fundamental forms of spheres and horospheres both satisfy a revealing ordinary differential equation. Let γ be a geodesic in $\tilde{M}$ and let X be a unit normal parallel vector field along γ. Moreover suppose J is a stable Jacobi field along γ for which $J(0) = X$. By (1.3) and the observation that J is a scalar multiple of the parallel transport of X along γ,

$$\langle \mathrm{II}_{H(-\gamma')}(X,X), -\gamma' \rangle = \frac{\langle \frac{D}{dr} J, X \rangle}{\langle J, X \rangle}.$$

Differentiating the right hand side shows that $\langle \mathrm{II}_{H(-\gamma')}(X,X), -\gamma' \rangle$ satisfies the Riccati equation

$$\frac{d}{dr} u + K(X, \gamma'(r)) + u^2 = 0 \tag{4.11}$$

where K is the sectional curvature of $\tilde{M}$. The same equation is satisfied if we replace $\mathrm{II}_{H(-\gamma'(r))}$ with $\mathrm{II}_{S_{\gamma(0)}(r)}$. To see this, let J and Y be respective angular and radial coordinate vector fields of some spherical coordinates about $\gamma(0)$, defined on a neighborhood of $\gamma(r)$ for $r > 0$. In particular, we choose Y so that $\gamma' = Y$, J restricts to a Jacobi field along γ with $J(0) = 0$ and $\frac{D}{dr} J(0) = X(0)$, and

$$0 = [J, Y] = \nabla_J Y - \nabla_Y J.$$

Since J is parallel to X and vanishes uniquely at $\gamma(0)$, $X = J/|J|$. Hence,

$$\langle \mathrm{II}_{S_{\gamma(0)}(r)}(X,X), -\gamma' \rangle = -\frac{\langle \nabla_J X, \gamma' \rangle}{\langle J, X \rangle} = \frac{\langle X, \nabla_J Y \rangle}{\langle J, X \rangle} = \frac{\langle X, \nabla_Y J \rangle}{\langle J, X \rangle} = \frac{\langle X, \frac{D}{dr} J \rangle}{\langle J, X \rangle},$$

so similarly satisfies (4.11). This ordinary differential equation provides us with means to bound $\langle \mathrm{II}_{H(v)}, v \rangle$ and to compare the second fundamental forms of spheres of large radius to those of horocycles.

Proposition 4.3. *The following are true.*

1) *If the sectional curvature K of M satisfies bounds $-a^2 \geq K \geq -b^2$ for some nonnegative constants a and b, then*

$$a|X|^2 \leq \langle \mathrm{II}_{H(v)}(X,X), v \rangle \leq b|X|^2$$

for all v.

2) *For all $r > 0$,*

$$0 < \langle \mathrm{II}_{S_{\gamma(0)}(r)}(X,X), -\gamma'(r) \rangle - \langle \mathrm{II}_{H(-\gamma'(r))}(X,X), -\gamma'(r) \rangle \leq r^{-1}|X|^2.$$

Proof. Let X be a unit length, parallel vector field normal to γ and set

$$u(r) = \langle \mathrm{II}_{H(-\gamma'(r))}(X(r), X(r)), -\gamma'(r) \rangle \qquad r \in \mathbb{R}$$

and

$$v(r) = \langle \mathrm{II}_{S_{\gamma(0)}(r)}(X(r), X(r)), -\gamma'(r) \rangle \qquad r > 0.$$

Both u and v satisfy (4.11) as argued above.

$u \geq 0$ and is uniformly bounded by continuity of $\mathrm{II}_{H(v)}$ and compactness of SM. If $u(r_0) > b$ for some $r_0 \in \mathbb{R}$, then

$$u'(r) \leq b^2 - u^2(r_0) < 0 \qquad \text{for } r \leq r_0,$$

which contradicts boundedness. If $u(r_0) < a$, then

$$u'(r) \geq a^2 - u^2(r_0) > 0 \qquad \text{for } r \leq r_0,$$

which contradicts nonpositivity. (1) follows.

(2) Note,

$$v'(r) - u'(r) = -(v^2(r) - u^2(r)).$$

Since u is bounded and the curvature of small spheres is large, $v(r) - u(r) > 0$ for small r. Since $v' - u' = 0$ where $v = u$, $v(r) - u(r) > 0$ for all $r > 0$, hence the lower bound in (2). Then,

$$v'(r) - u'(r) = -\frac{v(r) + u(r)}{v(r) - u(r)}(v(r) - u(r))^2 \leq -(v(r) - u(r))^2,$$

which implies the upper bound by an elementary computation. □

Remark 4.4. Part (2) of the proposition above implies the difference between the second fundamental form of a sphere and that a tangential horocycle is always nondegenerate. This provides us with part (2) of Corollary 1.6 from Theorem 1.4. Part (1) shows that if K is strictly negative, $\langle \mathrm{II}_{H(v)}(X, X), v \rangle$ is strictly positive definite. Hence, part (3) of the corollary. Part (1) also shows that the (unsigned) principal curvatures of $H(v)$ fall in the interval $[a, b]$. Hence if at each point at least $n/2$ of the principal curvatures of Σ fall outside of the interval $[a, b]$,

$$\langle \mathrm{II}_\Sigma - \mathrm{II}_{H(v)}, v \rangle \qquad \text{and} \qquad \langle \mathrm{II}_\Sigma - \mathrm{II}_{H(-v)}, -v \rangle$$

both have rank at least $n/2$ for each $v \in SN\Sigma$, from which follows part (1) of Corollary 1.6.

5. The conclusion of the proofs of Theorems 1.3 and 1.4

In Section 2, we reduced the problem of bounding integrals of an eigenfunction over our submanifold Σ to bounds on the integral of a cosine wave kernel. We also microlocalized to small cones covering the cosphere bundle with base points in Σ and lifted the kernel to a sum over the deck group in the universal cover. In Section 3 we used the Hadamard parametrix to described this lifted kernel as an oscillatory integral with geometric phase function

$$\begin{aligned} \phi_\alpha : \Sigma \times \Sigma &\to \mathbb{R} \\ (x, y) &\mapsto d_{\tilde{g}}(\alpha \tilde{x}, \tilde{y}) \end{aligned}$$

where $\tilde{x}$ and $\tilde{y}$ are the respective lifts of x and y to the same Dirichlet domain in the universal cover, $d_{\tilde{g}}$ is the distance function on $\tilde{M}$, and α is an element of the group of deck transformations. Moreover in Lemma 3.2, we established that the only relevant terms in the sum over the deck group are those for which there exists a geodesic intersecting both $\tilde{\Sigma}$ and $\alpha\tilde{\Sigma}$ nearly in the normal direction. In Section 4, we computed the first derivatives and the Hessian of the phase function ϕ_α as a function on $\Sigma \times \Sigma$.

In this section we adapt the tools we developed in Section 4 to local coordinates to prove Propositions 3.3 and 3.4. The respective main results, Theorems 1.3 and 1.4, follow. Recall we are trying to bound an oscillatory integral of the form

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x, y) e^{\pm i\lambda\phi(x,y)} \, dx \, dy \tag{5.1}$$

where

$$\phi(x, y) = d_{\tilde{g}}(\alpha\tilde{x}, \tilde{y})$$

and

$$a(x, y) = a_{\alpha,\pm}(T, \lambda; x, y).$$

We have determined much of the behavior of ϕ in the last section, and in Section 3, we determined that $\operatorname{supp} a \subset \operatorname{supp} b \times \operatorname{supp} b$, and

$$|\partial_x^{\beta_1} \partial_y^{\beta_2} a(x, y)| \le C_\beta e^{C_\beta T}$$

for multiindices β, among other things.

After taking the supports of b to be small and perhaps taking a smooth extension of Σ in M, we assume $\operatorname{supp} b$ is contained inside a ball $B \subset \mathbb{R}^d$

centered at 0 in our Fermi local coordinates (2.3). Furthermore, we assume the phase function ϕ is defined on $2B \times 2B$ with the same center but twice the radius. Fix $(x, y) \in 2B \times 2B$ and let $v_1(x, y)$ and $v_2(x, y)$ are the unit vectors denoting the arriving and departing directions, respectively, of the geodesic in $\tilde{M}$ starting at $\tilde{y} \in \tilde{\Sigma}$ and ending at $\alpha\tilde{x} \in \alpha\tilde{\Sigma}$. By abuse of notation, we will also use v_1 and v_2 to denote their push-forwards to M through the covering map where appropriate.

We fix a constant $\epsilon > 0$ and consider $\alpha \in \Gamma_U \setminus \Gamma_R$ for which

$$|\nabla\phi(x, y)| > \epsilon \qquad \text{for some } (x, y) \in 2B \times 2B, \tag{5.2}$$

where here ∇ is the gradient with respect to the product metric on $\Sigma \times \Sigma$. By Remark 4.2 and (4.10), the Hessian of $\mathrm{Hess}_{\Sigma\times\Sigma}\phi$ is a uniformly bounded quadratic form for non-identity α. Hence by the mean value theorem, we may restrict B so that

$$|\nabla\phi(x, y)| \geq \epsilon/2 \qquad \text{for all } (x, y) \in 2B \times 2B$$

for all α satisfying (5.2). Since the metric tensor of $\Sigma \times \Sigma$ is nearly the identity at $(0, 0)$, by taking B small we ensure that the Euclidean gradient of ϕ in local coordinates is bounded below by $\epsilon/4$. The oscillatory integral (5.1) is then bounded by a constant multiple of $e^{C_N T}\lambda^{-N}$ for any suitably large N by Part (1) of Lemma 6.2.

All that remains is the situation where

$$|\nabla\phi| \leq \epsilon \qquad \text{on } 2B \times 2B. \tag{5.3}$$

Now is when we really capitalize on our ability to take R large and restrict B and U. Recall that $\nabla^2_{x,y}\phi$ is the Euclidean Hessian matrix of ϕ in the variables x and y. We eventually want to show

$$\nabla^2_{x,y}\phi(x, y) = \begin{bmatrix} \nabla^2_x\phi(0, 0) & 0 \\ 0 & \nabla^2_y\phi(0, 0) \end{bmatrix} + E(x, y) \tag{5.4}$$

for all $\alpha \in \Gamma_U \setminus \Gamma_R$, where E is an error matrix whose entries are controlled by an adjustably small constant uniform in α. By (4.2) and since the Christoffel symbols of the product metric on $\Sigma \times \Sigma$ vanish at $(0, 0)$, we may restrict the support of b so that

$$\begin{aligned}
\partial_{x_i}\partial_{x_j}\phi(x,y) &= \mathrm{Hess}_{\Sigma\times\Sigma}\phi(\partial_{x_i},\partial_{x_j}), \\
\partial_{y_i}\partial_{y_j}\phi(x,y) &= \mathrm{Hess}_{\Sigma\times\Sigma}\phi(\partial_{y_i},\partial_{y_j}), \qquad \text{and} \\
\partial_{x_i}\partial_{y_j}\phi(x,y) &= \mathrm{Hess}_{\Sigma\times\Sigma}\phi(\partial_{x_i},\partial_{y_j})
\end{aligned}$$

modulo some small, controllable error terms for $i,j = 1,\dots,d$. Hence, it suffices to show

$$\begin{aligned}
\mathrm{Hess}_{\Sigma\times\Sigma}\phi(x,y)(\partial_{x_i},\partial_{x_j}) &= \mathrm{Hess}_{\Sigma\times\Sigma}\phi(0,0)(\partial_{x_i},\partial_{x_j}), \\
\mathrm{Hess}_{\Sigma\times\Sigma}\phi(x,y)(\partial_{y_i},\partial_{y_j}) &= \mathrm{Hess}_{\Sigma\times\Sigma}\phi(0,0)(\partial_{y_i},\partial_{y_j}), \qquad \text{and} \\
\mathrm{Hess}_{\Sigma\times\Sigma}\phi(x,y)(\partial_{x_i},\partial_{y_j}) &= 0
\end{aligned} \tag{5.5}$$

modulo small, controllable error terms which are bounded independently of α. Note the third line follows by taking R in (3.14) large and invoking part (2) of Lemma 4.1.

Fix indices i and j. We claim that the diameter of the set

$$\{\mathrm{Hess}_{\Sigma\times\Sigma}\phi(x,y)(\partial_{x_i},\partial_{x_j}) : x,y\in B\}$$

can be controlled by taking B and ϵ small and R large. Recall $v_1 = v_1(x,y)$ and $v_2 = v_2(x,y)$ are the unit vectors denoting the arriving and departing directions, respectively, of the geodesic in $\tilde{M}$ starting at $\tilde{y}\in\tilde{\Sigma}$ and ending at $\alpha\tilde{x}\in\alpha\tilde{\Sigma}$. By part (2) of Proposition 4.3

$$\langle \mathrm{II}_{S_{\tilde{y}}(d_{\tilde{y}})}(\partial'_{x_i},\partial'_{x_j}), -v_1\rangle = \langle \mathrm{II}_{H(-v_1)}(\partial'_{x_i},\partial'_{x_j}), -v_1\rangle$$

modulo an error term controllable by taking R large. Hence by (4.10), we have

$$\mathrm{Hess}_{\Sigma\times\Sigma}\phi(\partial_{x_i},\partial_{x_j}) = \langle \mathrm{II}_{\Sigma}(\partial_{x_i},\partial_{x_j}), v_1\rangle - \langle \mathrm{II}_{H(-v_1)}(\partial'_{x_i},\partial'_{x_j}), v_1\rangle$$

modulo controllable error terms. The diameter of the set of values achieved by the first term on the right is controlled by taking v_1 close to normal, i.e. by taking ϵ small and using Lemma 3.2 and part (1) of Lemma 4.1, and similarly for the second term. The first line of (5.5) follows. The second line follows similarly. We now have (5.4) and are ready to prove our propositions.

Proof of Proposition 3.3. We will select d coordinates in which to use the method of stationary phase in order to obtain the desired bound

$$\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x,y)e^{\pm i\lambda\phi(x,y)}\,dx\,dy\right| \lesssim e^{CT}\lambda^{-d/2} \tag{5.6}$$

for (5.1). By an orthogonal change of variables on $\mathbb{R}^d$, we may take ∂_{x_i} for $i = 1,\dots,d$ to align with the principal directions of Σ at 0. Now,

$$\partial_{x_i}\partial_{x_j}\phi(0,0) = \begin{cases} \langle \kappa_i, v_1\rangle - \langle \mathrm{II}_{H(-v_1)}(\partial'_{x_i},\partial'_{x_i}), v_1\rangle & i=j \\ -\langle \mathrm{II}_{H(-v_1)}(\partial'_{x_i},\partial'_{x_j}), v_1\rangle & i\neq j \end{cases} \tag{5.7}$$

and

$$\partial_{y_i}\partial_{y_j}\phi(0,0) = \begin{cases} \langle \kappa_i, -v_2\rangle - \langle \mathrm{II}_{H(v_2)}(\partial'_{y_i},\partial'_{y_i}), -v_2\rangle & i=j \\ -\langle \mathrm{II}_{H(v_2)}(\partial'_{y_i},\partial'_{y_j}), -v_2\rangle & i\neq j \end{cases}$$

modulo controllable errors, where $\kappa_i = \mathrm{II}_\Sigma(\partial_{x_i},\partial_{x_i})$ is the ith principal curvature vector. We can take ϵ small to keep v_1 and v_2 within a small, bounded deviation from a normal direction to Σ, and in particular the projections ∂'_{x_i} form a nearly orthonormal subset of the tangent space of $H(-v_1)$. Since M is compact and has negative curvature per the assumptions of Theorem 1.3, the sectional curvature of $\tilde{M}$ is bounded above by some negative constant. Then, by part (1) of Proposition 4.3 and the discussion above,

$$\sum_{i,j=1}^d \langle \mathrm{II}_{H(-v_1)}(\partial'_{x_i},\partial'_{x_j}), -v_1\rangle \xi_i\xi_j \geq c|\xi|^2 \qquad \text{and} \tag{5.8}$$

$$\sum_{i,j=1}^d \langle \mathrm{II}_{H(v_2)}(\partial'_{y_i},\partial'_{y_j}), v_2\rangle \xi_i\xi_j \geq c|\xi|^2$$

for all $\xi \in \mathbb{R}^d$ and for some positive constant c at $(x,y) = (0,0)$. We have trivially that for each $i = 1,\dots,d$, either

$$\langle \kappa_i, v_1\rangle \geq 0 \qquad \text{or} \qquad \langle \kappa_i, -v_1\rangle \geq 0.$$

Hence by taking U in Lemma 3.2 small, we can ensure v_1 and v_2 are close in $T\tilde{M}$ and that

$$\langle \kappa_i, v_1\rangle \geq -c/2 \qquad \text{or} \qquad \langle \kappa_i, -v_2\rangle \geq -c/2 \tag{5.9}$$

for each $i = 1, \dots, d$. We pick coordinates $z = (z_1, \dots, z_d)$ where $z_i = x_i$ if $\langle \kappa_i, v_1 \rangle \geq -c/2$ and $z_i = y_i$ if $\langle \kappa_i, -v_2 \rangle \geq -c/2$. By reordering, assume that

$$z = (x_1, \dots, x_\ell, y_{\ell+1}, \dots, y_d)$$

for some $\ell \in \{0, 1, \dots, d\}$, and let $w = (y_1, \dots, y_\ell, x_{\ell+1}, \dots, x_d)$ be the complimentary coordinates. We bound the left side of (5.6) by

$$\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} a(x,y) e^{\pm i\lambda\phi(x,y)} \, dz \right| \, dw$$

and use the method of stationary phase on the inner integral to obtain the desired bound. By (5.4),

$$\nabla_z^2 \phi(x,y) = \begin{bmatrix} \nabla^2_{x_1,\dots,x_\ell} \phi(0,0) & 0 \\ 0 & \nabla^2_{y_{\ell+1},\dots,y_d} \phi(0,0) \end{bmatrix} + E(x,y).$$

Now by (5.7), (5.8), and our selection of coordinates by (5.9),

$$|\nabla^2_{x_1,\dots,x_\ell} \phi(0,0)\xi| \geq \frac{c}{2}|\xi| \qquad \text{for all } \xi \in \mathbb{R}^\ell$$

and similarly

$$|\nabla^2_{y_{\ell+1},\dots,y_d} \phi(0,0)\xi| \geq \frac{c}{2}|\xi| \qquad \text{for all } \xi \in \mathbb{R}^{d-\ell}.$$

Hence if $E(x,y)$ is made small enough,

$$|\nabla_z^2 \phi(x,y)\xi| \geq \frac{c}{4}|\xi| \qquad \text{for all } \xi \in \mathbb{R}^d, \ x, y \in 2B.$$

The proposition follows after an application of Lemma 6.2. □

Proof of Proposition 3.4. Let v be the normal vector to Σ which points in a similar direction to v_1 and v_2. By the hypotheses (1.4), we select two subspaces V_1 and V_2 of $T\Sigma$, with respective dimensions ℓ_1 and ℓ_2 with $\ell_1 + \ell_2 = n$, and on which the restriction of the quadratic form $\langle \mathrm{II}_\Sigma - \mathrm{II}_{H(-v)}, v \rangle$ to V_1 and the restriction of $\langle \mathrm{II}_\Sigma - \mathrm{II}_{H(v)}, -v \rangle$ to V_2 are nondegenerate. In particular, select local coordinates $(x_1, \dots, x_{\ell_1})$ of V_1 such that $\partial_{x_1}, \dots, \partial_{x_{\ell_1}}$

forms an orthonormal basis at 0 at which

$$|\langle \mathrm{II}_\Sigma(\partial_{x_i}, \partial_{x_i}) - \mathrm{II}_{H(-v)}(\partial_{x_i}, \partial_{x_i}), v\rangle| \geq 4c \qquad \text{for } i = 1, \dots, \ell_1$$

for some positive constant c and

$$\langle \mathrm{II}_\Sigma(\partial_{x_i}, \partial_{x_j}) - \mathrm{II}_{H(-v)}(\partial_{x_i}, \partial_{x_j}), v\rangle = 0 \qquad \text{for } i \neq j.$$

By ensuring ϵ in (5.3) is sufficiently small, we take

$$|\partial_{x_i}\partial_{x_i}\phi(0,0)| = |\langle \mathrm{II}_\Sigma(\partial_{x_i}, \partial_{x_i}) - \mathrm{II}_{H(-v_1)}(\partial'_{x_i}, \partial'_{x_i}), v_1\rangle| \geq 2c$$
$$\text{for } i = 1, \dots, \ell_1$$

and

$$|\partial_{x_i}\partial_{x_j}\phi(0,0)| = |\langle \mathrm{II}_\Sigma(\partial_{x_i}, \partial_{x_j}) - \mathrm{II}_{H(-v_1)}(\partial'_{x_i}, \partial'_{x_j}), v_1\rangle| \leq c/8n \quad \text{for } i \neq j.$$

We similarly select a parametrization $(y_1, \dots, y_{\ell_2})$ of V_2 for which

$$|\partial_{y_i}\partial_{y_i}\phi(0,0)| = |\langle \mathrm{II}_\Sigma(\partial_{y_i}, \partial_{y_i}) - \mathrm{II}_{H(v_2)}(\partial'_{y_i}, \partial'_{y_i}), -v_2\rangle| \geq 2c$$
$$\text{for } i = 1, \dots, \ell_2$$

and

$$|\partial_{y_i}\partial_{y_j}\phi(0,0)| = |\langle \mathrm{II}_\Sigma(\partial_{y_i}, \partial_{y_j}) - \mathrm{II}_{H(v_2)}(\partial'_{y_i}, \partial'_{y_j}), -v_2\rangle| \leq c/8n \quad \text{for } i \neq j.$$

By bounding each of the entries of $E(x, y)$ in (5.4) by $c/8n$, the $n \times n$ Hessian matrix $\nabla^2_{x_1,\dots,x_{\ell_1},y_1,\dots,y_{\ell_2}}\phi(x, y)$ has diagonal terms whose absolute values are bounded below by c, and off-diagonal terms bounded by $c/4n$. It follows that

$$|\nabla^2_{x_1,\dots,x_{\ell_1},y_1,\dots,y_{\ell_2}}\phi(x, y)\xi| \geq \frac{c}{2}|\xi| \qquad \text{for all } \xi \in \mathbb{R}^n,\ x, y \in 2B.$$

This and Lemma 6.2 show us

$$\left|\int \cdots \int a(x, y)e^{i\pm\lambda\phi(x,y)}\, dx_1 \cdots dx_{\ell_1} dy_1 \cdots dy_{\ell_2}\right| \lesssim e^{CT}\lambda^{-n/2}$$

uniformly over the remaining variables $x_{\ell+1}, \dots, x_{n-1}$ and $y_{\ell+1}, \dots, y_{n-1}$. The integral in (5.1) hence satisfies the same bounds. □

6. Appendix

6.1. Exponential bounds on mixed derivatives

The following proposition allows us to obtain exponential bounds on mixed derivatives of functions $f(x, y)$ in $C^\infty(\tilde{M} \times \tilde{M})$ if we are only provided with exponential bounds on pure derivatives in both variables. We use this to obtain bounds on the mixed derivatives of the amplitudes in Lemma 3.1.

Proposition 6.1. *Let (M, g) be a compact, n-dimensional, boundaryless Riemannian manifold with nonpositive sectional curvature and let $(\tilde{M}, \tilde{g})$ denote the universal cover of M equipped with the pullback metric. Let $f : \tilde{M} \times \tilde{M} \to \mathbb{R}$ be a function satisfying bounds*

$$|\Delta_x^j f(x, y)| \le C_j e^{C_j d_{\tilde{g}}(x,y)} \qquad \text{and} \qquad |\Delta_y^k f(x, y)| \le C_k e^{C_j d_{\tilde{g}}(x,y)}$$

where $d_{\tilde{g}}(x, y) \ge 1$. Then,

$$|\Delta_x^j \Delta_y^k f(x, y)| \le C_{j,k} e^{C_{j,k} d_{\tilde{g}}(x,y)} \qquad \text{for } d_{\tilde{g}}(x, y) \ge 1,$$

where the constants $C_{j,k}$ depend only on the constants C_j and C_k and the manifold.

Proof. Fix x_0 and y_0 in $\tilde{M}$ and fix a smooth function $\beta \in C_0^\infty(\mathbb{R}, [0, 1])$ equal to 1 near 0 and supported in $(-\mathrm{inj}M, \mathrm{inj}M)$. Then let

$$F(x, y) = \beta(d_{\tilde{g}}(x, x_0))\beta(d_{\tilde{g}}(y, y_0))f(x, y).$$

Note

$$|\Delta_x^j F(x, y)| \le C_j' e^{C_j' d_{\tilde{g}}(x_0,y_0)} \quad \text{and} \quad |\Delta_y^k F(x, y)| \le C_k' e^{C_k' d_{\tilde{g}}(x_0,y_0)} \tag{6.1}$$

by (3.9) for constants C_j' and C_k' which are independent of x, y, x_0, and y_0. The cutoffs allow us to interpret F as a function on $M \times M$. By Sobolev embedding,

$$\begin{aligned} |\Delta_x^j \Delta_y^k f(x_0, y_0)| &\le \|\Delta_x^j \Delta_y^k F(x, y)\|_{L^\infty(M\times M)} \\ &\le C\|(I - \Delta_x - \Delta_y)^{n+1} \Delta_x^j \Delta_y^k F(x, y)\|_{L^2(M\times M)} \end{aligned} \tag{6.2}$$

where we understand $\Delta_x + \Delta_y$ as the Laplace-Beltrami operator on the product manifold $M \times M$. It follows $e_p(x)e_q(y)$ for $p, q = 0, 1, 2, \ldots$ form an orthonormal basis of eigenfunctions on $M \times M$ with

$$(\Delta_x + \Delta_y)e_p(x)e_q(y) = -(\lambda_p^2 + \lambda_q^2)e_p(x)e_q(y).$$

We use the shorthand

$$\hat{F}(p,q) = \int_M \int_M F(x,y)\overline{e_p(x)e_q(y)}\, dx\, dy$$

and write

$$\begin{aligned}
&\|(I - \Delta_x - \Delta_y)^{n+1}\Delta_x^j \Delta_y^k F(x,y)\|_{L^2(M\times M)}^2 \\
&= \sum_{p,q} (1 + \lambda_p^2 + \lambda_q^2)^{2n+2} \lambda_p^{4j}\lambda_q^{4k} |\hat{F}(p,q)|^2 \\
&\le \sum_{p,q} (1 + \lambda_p^{4(n+j+k+1)} + \lambda_q^{4(n+j+k+1)}) |\hat{F}(x,y)|^2 \\
&= \|F\|_{L^2(M\times M)}^2 + \|\Delta_x^{n+j+k+1} F\|_{L^2(M\times M)}^2 + \|\Delta_y^{n+j+k+1} F\|_{L^2(M\times M)}^2.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\|F\|_{L^2(M\times M)}^2 + \|\Delta_x^{n+j+k+1} F\|_{L^2(M\times M)}^2 + \|\Delta_y^{n+j+k+1} F\|_{L^2(M\times M)}^2 \\
&\le \mathrm{vol}(M)^2 \Big(\|F\|_{L^\infty(M\times M)}^2 + \|\Delta_x^{n+j+k+1} F\|_{L^\infty(M\times M)}^2 \\
&\qquad\qquad + \|\Delta_y^{n+j+k+1} F\|_{L^\infty(M\times M)}^2 \Big),
\end{aligned}$$

and the proposition follows from (6.1). □

6.2. A stationary phase lemma

The following stationary phase lemma helps us obtain uniform bounds on (5.1) in both the proofs of Propositions 3.3 and 3.4.

Lemma 6.2. *Let*

$$I(\lambda) = \int_{\mathbb{R}^n} a(x) e^{i\lambda\phi(x)}\, dx$$

where a is a smooth function on $\mathbb{R}^n$ with support contained in the unit ball $B = \{x : |x| \le 1\}$, and where ϕ is a smooth function on $\sqrt{2}B = \{x : |x| \le \sqrt{2}\}$.

1) *If* $|\nabla\phi(x,y)| \geq c$ *on* B *for some* $c > 0$*, then*

$$|I(\lambda)| \leq C_N \lambda^{-N} \qquad \text{for } \lambda \geq 1$$

for $N = 1, 2, \ldots$*.*

2) *If*

$$\big|(\nabla^2\phi)\xi\big| \geq c|\xi| \qquad \text{for all } \xi \in \mathbb{R}^n$$

on $\sqrt{2}B$ *for some* $c > 0$*, then*

$$|I(\lambda)| \leq C\lambda^{-n/2} \qquad \text{for } \lambda \geq 1.$$

In both situations (1) and (2), the constants C *and* C_N *are polynomials in* c^{-1} *and* $\sup_B |\partial_x^\beta a|$ *and* $\sup_B |\partial_x^\beta \phi|$ *for finitely many multiindices* β*.*

Proof. (1) follows by careful inspection of the nonstationary phase argument [15, Lemma 4.1.1].

For (2), let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a unit speed curve in $\sqrt{2}B$ where

$$\nabla\phi(\gamma(t)) \neq 0 \qquad \text{for } t \in (0, \ell)$$

and

$$\gamma'(t) = \frac{\nabla|\nabla\phi|}{|\nabla|\nabla\phi||}.$$

Setting $\gamma(0) = x_0$ and $\gamma(\ell) = x_1$, the mean value theorem gives us a time $t \in (0, \ell)$ at which

$$\begin{aligned}
|\nabla\phi(x_1)| - |\nabla\phi(x_0)| &= \ell \frac{d}{dt} |\nabla|\nabla\phi(\gamma(t))|| \\
&= \ell \frac{d}{dt} \left| \nabla^2\phi(\gamma(t)) \frac{\nabla\phi(\gamma(t))}{|\nabla\phi(\gamma(t))|} \right| \\
&\geq \ell c \\
&\geq c|x_1 - x_0|.
\end{aligned} \tag{6.3}$$

If ϕ has a critical point at some x_0 in $\sqrt{2}B$, since $\nabla^2\phi(x_0)$ is a linear isomorphism from $\mathbb{R}^n \to \mathbb{R}^n$, there exist such flow lines of $\nabla|\nabla\phi|$ in every direction starting at x_0. Moreover by (6.3), $|\nabla\phi| \neq 0$ on this neighborhood minus the point x_0. By an open-closed argument, there exists such a flow line connecting x_0 to any other point $x \in \sqrt{2}B$, and we conclude

$$|\nabla\phi(x)| \geq c|x - x_0| \qquad \text{for all } x \in \sqrt{2}B$$

from (6.3). The desired bound on $I(\lambda)$ follows from this estimate of $|\nabla\phi(x)|$ and careful inspection of the proof of [15, Proposition 4.1.2].

On the other hand, if there are no critical points of ϕ in $\sqrt{2}B$, we have

$$|\nabla|\nabla\phi|| = \left|\nabla^2\phi\frac{\nabla\phi}{|\nabla\phi|}\right| \geq c > 0,$$

and hence $|\nabla\phi|$ has no critical points on $\sqrt{2}B$. In particular, $|\nabla\phi|$ attains a minimum on B only on the boundary. Select such a point x_0 on ∂B and take a unit-speed curve γ with $\gamma(0) = x_0$ and

$$\gamma'(t) = -\frac{\nabla|\nabla\phi|}{|\nabla|\nabla\phi||}.$$

By the same argument as before,

$$|\nabla\phi(x_0)| - |\nabla\phi(\gamma(t))| \geq ct \qquad \text{for all } t > 0.$$

Hence, $\gamma(t)$ never intersects B for $t > 0$. Moreover since $|\nabla\phi|$ is bounded below on $\sqrt{2}B$, γ must intersect the boundary $\partial(\sqrt{2}B)$ at some point x_1 at some time ℓ. Hence,

$$\inf_B |\nabla\phi| = |\nabla\phi(x_0)| \geq c\ell \geq c(\sqrt{2} - 1),$$

and hence we have reduced the problem back to (1). □

Acknowledgements

I would like to extend my thanks to my advisor, Chris Sogge, for his ongoing support. I also extend my deepest gratitude to the anonymous referee for his or her generous, thorough, and invaluable feedback which greatly improved the quality and readability of this manuscript.

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Received September 19, 2018
Accepted May 7, 2019