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Levi–Kähler reduction of CR structures, products of spheres, and toric geometry

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We introduce a process, which we call Levi–Kähler reduction, for constructing Kähler manifolds and orbifolds from CR manifolds (of arbitrary codimension) with a transverse torus action. Most of the paper is devoted to the study of Levi–Kähler reductions of toric CR manifolds, and in particular, products of odd dimensional spheres. We obtain explicit descriptions and characterizations of the orbifolds obtained by such reductions, and find that the Levi–Kähler reductions of products of 3-spheres are extremal in a weighted sense introduced by G. Maschler and the first author [11], and further studied by A. Futaki and H. Ono [34].

Introduction

In recent years there has been considerable interest in the interaction between Kähler geometry and its odd-dimensional younger cousin, Sasaki geometry [19]. On the one hand, ideas in Kähler geometry, such as toric methods or extremal metrics, have led to the development of analogues in Sasaki geometry. On the other hand, Sasaki manifolds have a canonical 1-dimensional foliation generated by the Reeb vector field, which both provides a construction of Kähler metrics on the leaf space when the latter is a manifold or orbifold, as well as a "horizontal" generalization of such quotients when it is not.

Our thesis herein is that these ideas need not be limited to 1-dimensional foliations. Indeed, any Sasaki manifold has an underlying codimension one CR structure, whereas CR manifolds arise naturally in arbitrary codimension [17]. This prompts us to introduce transverse "Reeb foliations" on arbitrary CR manifolds $(N, \mathcal{D}, J)$. A theory of such foliations has recently been developed in [46], but here we focus on the *horizontal Kähler geometry* of $(\mathcal{D}, J)$, i.e., the Kähler structures induced on the space of (local) leaves of the Reeb foliation. However, whereas in codimension one, the exterior derivative of the contact form equips the horizontal distribution $\mathcal{D}$ with a nondegenerate 2-form (which, together with the complex structure J on $\mathcal{D}$, defines the horizontal Kähler structure on $\mathcal{D}$), in higher codimension, the non-integrability of $\mathcal{D}$ is measured by a 2-form on $\mathcal{D}$ with values in $TN/\mathcal{D}$, called the *Levi form* $L_{\mathcal{D}}$. In order to construct a Kähler metric on the leaf space of the Reeb foliation, we therefore need also to choose a nondegenerate component of $L_{\mathcal{D}}$. This construction, which we call a *Levi–Kähler quotient* or a *Levi–Kähler reduction* (see Definition 6 for more details), is our main topic of study.

In particular, the Levi form of $\mathcal{D}$ must *have* a nondegenerate component. Rank $2m$ distributions $\mathcal{D}$ of this type, on manifolds N of dimension $2m + \ell$, were studied in a companion paper [8], to which the present work may be viewed as a sequel, although we do not here rely upon intimate familiarity with that paper, or its main theorem. Indeed, whereas in [8] we study the general theory of toric contact manifolds in higher codimension, the applications in Kähler geometry we develop herein use only the simplest examples: toric CR submanifolds of flat space, and in particular, products of spheres. We review the necessary ingredients from [8] in Theorem 1 below.

Our prime contribution in this paper is the construction of new compact toric Kähler orbifolds M, i.e., Kähler orbifolds of real dimension $2m$ admitting an isometric hamiltonian action of a real m-torus $\mathfrak{t}/2\pi\Lambda$, which have nice curvature properties in the sense pioneered by Calabi [24], as well as weighted extensions studied in [11, 34]. As we recall in Section 2.1, any symplectic toric orbifold (M, ω) is classified [14, 29, 44] by the image Δ of its momentum map, which is a convex polytope in an m-dimensional real affine space $\mathcal{A}$ modelled on $\mathfrak{t}^*$. Explicitly, Δ is an intersection of half-spaces $L_s \geqslant 0$, where $s \in \mathcal{S}$ indexes the facets (codimension one faces) of Δ, and

$L_s \in \mathfrak{h}$, the $(m+1)$-dimensional vector space of affine functions on $\mathcal{A}$. The linear parts $u_s \in \mathfrak{t}$ of L_s for $s \in \mathcal{S}$ generate the normal rays to Δ, defining a fan Y in $\mathfrak{t}$ which in turn determines [26, 32] the equivariant biholomorphism type of any toric ω-compatible complex structure on M. It is convenient to encode these data in linear maps

$$\begin{aligned} \mathbf{L}\colon \mathbb{R}_{\mathcal{S}} &\to \mathfrak{h} \\ e_s &\mapsto L_s \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathbf{u}\colon \mathbb{R}_{\mathcal{S}} &\to \mathfrak{t} \\ e_s &\mapsto u_s \end{aligned}$$

where $\mathbb{R}_{\mathcal{S}}$ is the standard real vector space with basis $e_s : s \in \mathcal{S}$. The kernel $\mathfrak{g}$ of $\mathbf{u}$ is the Lie algebra of a subtorus G of $\mathbb{R}_{\mathcal{S}}/2\pi\mathbb{Z}_{\mathcal{S}}$, and we let $\lambda \in \mathfrak{g}^*$ be the restriction of $\mathbf{L}$ to $\mathfrak{g}$ (which takes values in the constant affine functions, i.e., $\mathbb{R}$). Equipping the complexification $\mathbb{C}_{\mathcal{S}}$ of $\mathbb{R}_{\mathcal{S}}$ with its standard flat Kähler structure, M is then equivariantly symplectomorphic to the symplectic quotient of $\mathbb{C}_{\mathcal{S}}$ by G at momentum level λ [29, 44], whereas the equivariant biholomorphism type of M defined by Y is given by the GIT quotient of $\mathbb{C}_{\mathcal{S}}$ by the complexification of G [25].

However, the metric on M which comes from the Kähler reduction by G of the flat metric on $\mathbb{C}_{\mathcal{S}}$ — and which has an elegant expression given by Guillemin [37] — has not been found to have particularly interesting curvature properties, except in the simplest case of the complex projective space (when it defines a Fubini–Study metric). We are thus motivated to make the following observations. First, the level set $N_{\mathfrak{g},\lambda} := \mu_{\mathfrak{g}}^{-1}(\lambda)$ of the momentum map of G is a CR submanifold of $\mathbb{C}_{\mathcal{S}}$, in fact an intersection of quadric hypersurfaces, preserved by the natural action of the torus $\mathbb{T}_{\mathcal{S}} := \mathbb{R}_{\mathcal{S}}/2\pi\mathbb{Z}_{\mathcal{S}}$ on $\mathbb{C}_{\mathcal{S}}$ (i.e. $N_{\mathfrak{g},\lambda}$ is a *toric* CR manifold). Secondly, since $G \leq \mathbb{T}_{\mathcal{S}}$ acts on $N_{\mathfrak{g},\lambda}$ with orbits transverse to the CR distribution $\mathcal{D}$, we may regard the Levi form $L_{\mathcal{D}}$ as a $\mathfrak{g}$-valued 2-form on $\mathcal{D}$. We find that the λ component is nondegenerate and induces a Kähler metric on $M \cong N_{\mathfrak{g},\lambda}/G$ — in other words, the data $(\mathfrak{g}, \lambda)$ is exactly what we need (see Definition 6) to define a Levi–Kähler quotient of $N_{\mathfrak{g},\lambda}$. While the Levi–Kähler quotient metric on M associated to the data $(\mathfrak{g}, \lambda)$ is in general different from the Guillemin metric, it may not have nice curvature properties either. There is, however, a third observation that proves to be decisive. To explain this, note that the pair $(\mathfrak{g}, \lambda)$ plays a dual role in the above construction: it is used both to define $N_{\mathfrak{g},\lambda}$ and also to define the transverse group action and momentum level for the Levi–Kähler quotient. We do not need to use the same pair for these independent roles: if we introduce a new pair $(\mathfrak{g}_o, \lambda_o)$, corresponding to another polytope Δ_o which has the *same combinatorial type* as Δ (see Definition 8), and such that $(\mathfrak{g}, \lambda)$ defines a Levi–Kähler quotient of the toric CR submanifold $N_{\mathfrak{g}_o,\lambda_o}$, then we find that the resulting toric orbifold is equivariantly symplectomorphic to (M, ω) (see Theorem 2), i.e. the pair $(\mathfrak{g}_o, \lambda_o)$ gives rise to a ω-compatible toric Kähler metric on M.

We take advantage of these observations by following the idea that a Levi–Kähler reduction of a CR manifold N can be expected to have nice curvature properties if N does. The simplest examples, in codimension one, are round CR $(2m+1)$-spheres, which are the toric CR submanifolds associated to m-simplices, and are circle orbibundles over complex weighted projective spaces. The Levi–Kähler quotients in this case give rise to the natural *Bochner-flat* Kähler metrics on weighted complex projective spaces, studied in [2, 22, 28, 48], which are also *extremal* in the sense of [24]. Similarly, we show in this paper that the Levi–Kähler quotients of *products* of CR spheres provide a natural extension of the Bochner-flat Kähler metrics in higher codimension: if Δ is a polytope with the same combinatorial type as the product of simplices Δ_o, we construct (see Theorem 4 and Corollary 2) a distinguished toric Kähler metric on the toric symplectic orbifold associated to Δ, obtained as a Levi–Kähler quotient of the toric CR manifold $N_{\mathfrak{g}_o,\lambda_o}$ associated to Δ_o (which as a CR manifold is the product of CR spheres). The investigation of the curvature properties of this class of toric Kähler metrics is the main focus of the article.

The product structure on a product N of CR spheres induces distributions on M and we show that the curvature of M has vanishing Bochner component on each such distribution, simply because CR spheres have vanishing Chern–Moser tensor. For 3-dimensional CR manifolds, the vanishing of the Chern–Moser tensor is automatic and we compute instead the scalar curvature of a Levi–Kähler quotient of a product of 3-spheres and observe that, when the polytope is projectively equivalent to a cube, the Levi–Kähler quotient metric can be characterized as being extremal in a weighted sense that was introduced (in a special case) in [11]. More precisely, given a "conformal dimension" $p \in \mathbb{R}$ and a positive function w on a compact symplectic orbifold (M, ω) whose hamiltonian vector field is quasiperiodic (i.e., it belongs to the Lie algebra of a torus $\mathbb{T}$ in $\mathrm{Ham}(M, \omega)$), we can generalize the approach of Donaldson [30] and Fujiki [33] to Calabi's extremal Kähler metrics [24] by using $w^{-(p-1)}$ as a weight for the formal Fréchet symplectic structure on the space of $\mathbb{T}$-invariant compatible complex structures. Then the action of $\mathrm{Ham}^{\mathbb{T}}(M, \omega)$ on this space is hamiltonian, and if we weight the inner product on its Lie algebra of by $w^{-(p+1)}$ then the momentum map at J may be identified with a modification $s_{J,w,p}$ of the scalar curvature of $g_J = \omega(\cdot, J\cdot)$, which is what the scalar curvature of the conformally related metric $w^2 g_J$ would be if M had dimension p. (The special case when $p = 2m$ is studied in [11, 34].)

If the polytope Δ of M is projectively equivalent to a cube, the intersections of opposite facets of Δ lie in a common hyperplane (and this is a characterization of projective cubes), so there is a unique affine-linear function w up to scale which is positive on Δ and vanishes on the intersections of opposite facets. We prove in Theorem 6 that the unique (up to equivariant symplectomorphism) toric metric on (M, ω) for which $s_{J,w,m+2}$ is an affine-linear function is the one arising as a Levi–Kähler quotient of a product of 3-spheres. We can summarize some of our findings (Theorems 2, 5 and Corollary 2) as follows.

Theorem A. *Let (M, ω) be a compact $2m$-dimensional symplectic toric orbifold with Delzant image $\Delta \subseteq \mathfrak{t}^*$. Then Δ has the combinatorics of a product of simplices if and only if (M, ω) can be obtained as a Levi–Kähler reduction of a product of odd dimensional CR spheres. In particular, such an orbifold admits a distinguished compatible Kähler metric $h_{\mathbf{L}}$ whose symplectic potential is explicitly given in Theorem 5 below. If, furthermore, the Delzant image Δ is a projectively equivalent to a product of 1-dimensional simplices (i.e., is a projective cube), then $h_{\mathbf{L}}$ is an $(m+2, w)$-extremal Kähler metric, where w is the unique (up to scale) positive affine-linear function on $\mathfrak{t}^*$ which vanishes at the intersections of all pairs of opposite facets of Δ.*

An important source of (smooth) toric Kähler manifolds M whose Delzant polytope has the combinatorics of a product of simplices — and to which Theorem A applies — is obtained from the generalized Calabi construction (see [10]), where both the base and the fibre are toric manifolds with Delzant polytopes having the combinatorics of product of simplices. This includes the complex Hirzebruch surfaces, holomorphic projective bundles over a projective space, the Bott manifolds recently studied in [21], and, inductively, rigid toric fibrations where the base and the fibre are one of the aforementioned smooth complex manifolds. We show that in this special setting, the Kähler metric corresponding to the Levi–Kahler quotient of the product of spheres associated to M is obtained from the generalized Calabi construction, where the metrics on the base and on the fibre are themselves Levi–Kahler quotients of product of spheres.

Theorem A yields the existence of a canonical weighted extremal metric on any toric 4-orbifold whose Delzant polytope is a quadrilateral (i.e. with second Betti number $b_2(M) = 2$) whereas by the results in [6] the existence of genuine (unweighted) extremal Kähler metrics is generally obstructed. This motivates a further investigation of the class of $(m + 2, w)$-extremal

Kähler metrics, in the general framework developed in [11]. Notice that in complex dimension 2, this coincides with the class of weighted extremal metrics appearing in [11], but in higher dimensions, our approach suggests a different weight $m+2$ than the conformal generalization $2m$ proposed in [11]. This observation has inspired subsequent work [3, 12] which has elucidated the natural role that the weight $m+2$ plays in Kähler and Sasaki geometry.

As another novel aspect, the explicit form of the Levi–Kähler quotient metrics on the toric Kähler manifolds whose Delzant polytope is projectively equivalent to a cube gives rise to an extension of the ambitoric ansatz of Segre type [5, 6] to arbitrary dimension. This has also been further studied in [3].

The Levi–Kähler quotient metrics of products of $\ell \leqslant 2$ odd dimensional spheres can also lead to extremal Kähler metrics (in the classical sense of [24]). As we have already mentioned above, in the case $\ell = 1$, the Levi–Kähler quotients of a CR sphere give rise to the Bochner-flat Kähler metrics on weighted complex projective spaces, which are extremal [22]. We find some new extremal Kähler orbifold examples, obtained as the Levi–Kähler quotient of a product of two CR spheres ($\ell = 2$). In particular, the construction in Section 4.4.2 yields the following result.

Theorem B. *There exists a countable family of compact constant scalar curvature Kähler* 6*-orbifolds obtained as Levi–Kähler reductions of* $\mathbb{S}^5 \times \mathbb{S}^3$.

The main results are presented in three courses, which we serve up in Sections 2, 3 and 4, after presenting some background and preliminary results in Section 1. The preliminary material reviews the notion of a CR $(2m+\ell)$-manifold of codimension ℓ and studies infinitesimal CR torus actions transverse to the CR distribution. These have associated Kähler cones of dimension $2(m+\ell)$ and so may be viewed as a natural generalization of Sasaki structures. Such a transverse infinitesimal action of an ℓ-dimensional abelian Lie algebra $\mathfrak{g}$, together with an element $\lambda \in \mathfrak{g}^*$ is called a *positive Levi pair* if $\lambda \circ L_{\mathcal{D}}$ is a horizontal Kähler structure on the CR distribution. A Levi–Kähler reduction is thus obtained when the action of $\mathfrak{g}$ integrates to an action of a Lie group G.

Section 2 presents our general results on toric CR submanifolds of $\mathbb{C}_{\mathcal{S}}$ and their Levi–Kähler quotients. We first review the elements of toric geometry and combinatorics, and make precise the notion of combinatorial type. Then we show in Theorem 2 that if $(\mathfrak{g}, \lambda)$ is a positive Levi pair associated to polytope Δ and N is a toric submanifold of $\mathbb{C}_{\mathcal{S}}$ with the same combinatorial type, then the momentum map of the horizontal Levi–Kähler structure (or the Levi–Kähler quotient when that exists) has image Δ. In the case that N is a toric intersection of quadric hypersurfaces contained in a round hypersphere, the Levi–Kähler structure can be made explicit, as we show in Theorem 3.

In Section 3, we study the construction of toric Kähler metrics as Levi–Kähler quotients of products of spheres. Building on Theorem 2, we characterize such quotients in Theorem 4 as those associated to a polytope with the combinatorial type of a product of simplices. In turn, Theorem 5 builds on Theorem 3 by giving explicit formulae for the Levi–Kähler quotients of ℓ-fold products of spheres and their symplectic and Kähler potentials, extending the results of [22] in the case $\ell = 1$ to arbitrary ℓ. In the remainder of the section, we explore relations between this construction and other explicit methods in Kähler geometry such as the generalized Calabi construction [9, 10].

The final Section 4 investigates the curvature properties of a Levi–Kähler quotient M of a product of spheres N. We discuss here the curvature characterization of such Levi–Kähler quotients, the weighted extremal geometry they define when the Delzant polytope is a projective cube, as well as the special cases when the Levi–Kähler quotient is extremal in the usual sense of [24].

1. Levi–Kähler quotients of CR manifolds

1.1. CR structures of arbitrary codimension

Definition 1. A *CR structure* $(\mathcal{D}, J)$ *of rank* m *and codimension* ℓ on a real $(2m+\ell)$-dimensional manifold N is a real rank $2m$ distribution $\mathcal{D} \subseteq TN$ equipped with an almost complex structure $J\colon \mathcal{D} \to \mathcal{D}$, which satisfies the integrability conditions

$$\begin{aligned} &[X,Y] - [JX,JY] \in C^\infty(\mathcal{D}),\\ &[X,JY] + [JX,Y] = J([X,Y] - [JX,JY]), \quad \forall\, X,Y \in C^\infty(\mathcal{D}), \end{aligned} \tag{1}$$

where $C^\infty(\mathcal{D})$ denotes the sheaf of smooth sections of $\mathcal{D}$; equivalently,

$$[C^\infty(\mathcal{D}^{1,0}), C^\infty(\mathcal{D}^{1,0})] \subseteq C^\infty(\mathcal{D}^{1,0}),$$

where $\mathcal{D}^{1,0} \subseteq TN \otimes \mathbb{C}$ is the subbundle of $(1,0)$ vectors in $\mathcal{D} \otimes \mathbb{C}$.

$(N, \mathcal{D}, J)$ is then called a *CR manifold* (of codimension ℓ).

The underlying rank $2m$ distribution $\mathcal{D}$ on N may be viewed as a codimension ℓ generalization of a contact structure on N [8]. The fundamental invariant of $\mathcal{D}$ is its *Levi form* $L_{\mathcal{D}}\colon \wedge^2\mathcal{D} \to TN/\mathcal{D}$, defined, via $X, Y \in C^\infty(\mathcal{D})$, by the tensorial expression

$$L_{\mathcal{D}}(X,Y) = -q_{\mathcal{D}}([X,Y]) \tag{2}$$

where $q_{\mathcal{D}}\colon TN \to TN/\mathcal{D}$ is the quotient map. The transpose of $q_{\mathcal{D}}$ identifies $(TN/\mathcal{D})^*$ canonically with the annihilator $\mathcal{D}^0$ of $\mathcal{D}$, which is a rank ℓ subbundle of T^*N. We denote throughout by $p\colon T^*N \to N$ the cotangent bundle projection or its restriction to any subbundle of T^*N such as $\mathcal{D}^0$. The normalization convention for $L_{\mathcal{D}}$ is chosen so that for any section α of $\mathcal{D}^0$, the restriction of $d\alpha$ to $\wedge^2\mathcal{D} \subseteq \wedge^2 TN$ is $\alpha \circ L_{\mathcal{D}}$.

The *nondegeneracy locus* of $\mathcal{D}$ is the open subset

$$U_{\mathcal{D}} := \{\alpha \in \mathcal{D}^0 \cong (TN/\mathcal{D})^* \mid \alpha \circ L_{\mathcal{D}} \text{ is nondegenerate}\}$$

of $\mathcal{D}^0$. If $U_{\mathcal{D}} \cap \mathcal{D}^0_z$ is nonempty then, since nondegeneracy is an open condition, $\mathcal{D}^0_z$ has a basis $\alpha_1, \dots, \alpha_\ell$ in $U_{\mathcal{D}}$ and so $U_{\mathcal{D}} \cap \mathcal{D}^0_z$ is the complement of the set where $(\sum_{i=1}^{\ell} t_i\alpha_i) \circ L_{\mathcal{D}}$ degenerates, which is the cone over a projective hypersurface $V_{\mathcal{D},z}$ of degree m (the zero set of a homogeneous degree m polynomial in the ℓ variables $t_1, \dots, t_\ell$). In [8], $V_{\mathcal{D}} \subseteq \mathrm{P}(\mathcal{D}^0)$ is called the *degeneracy variety* of $\mathcal{D}$. Therein it is shown that $U_{\mathcal{D}}$ is a canonical "symplectization" of $(N, \mathcal{D})$: $U_{\mathcal{D}}$ is the open subset of $\mathcal{D}^0$ over which the pullback of the canonical symplectic form Ω on T^*N to $\mathcal{D}^0$ is nondegenerate.

Definition 2. Let $(N, \mathcal{D}, J)$ be a *CR manifold* (of codimension ℓ). We say $\mathcal{D}$ is *Levi nondegenerate* if $U_{\mathcal{D}}$ has nonempty intersection with each fibre of $p\colon \mathcal{D}^0 \to N$. A (local) section of $U_{\mathcal{D}}$ is called a (local) *contact form* on N.

Note that the Levi form $L_{\mathcal{D}}$ satisfies

$$L_{\mathcal{D}}(X,Y) = -\tfrac{1}{2} q_{\mathcal{D}}([X,Y] + [JX,JY])$$

and hence is J-invariant or "type (1,1)" on $\mathcal{D}$. It follows that $h_{\mathcal{D}}(X,Y) := L_{\mathcal{D}}(X,JY)$ is a section of $S^2\mathcal{D}^* \otimes TN/\mathcal{D}$. We say $(N,\mathcal{D},J)$ is *Levi definite* if at each $z \in N$ there exists $\alpha \in \mathcal{D}^0_z$ such that $\alpha \circ h_{\mathcal{D}} \in S^2\mathcal{D}^*_z$ is positive definite.

Clearly Levi definite CR manifolds are Levi nondegenerate: more generally $U^+_{\mathcal{D}} := \{\alpha \in \mathcal{D}^0 \mid \alpha \circ h_{\mathcal{D}} \text{ is positive definite}\}$ is an open and closed submanifold of $U_{\mathcal{D}}$.

Examples 1. (i) A maximally real codimension ℓ submanifold of $\mathbb{C}^{m+\ell}$ is a smooth submanifold $N \subseteq \mathbb{C}^{m+\ell}$ for which $\mathcal{D} := TN \cap JTN$, where J is the standard complex structure of $\mathbb{C}^{m+\ell}$, has rank $2m$ (i.e., corank ℓ in TN). Then $(N, \mathcal{D})$, with the induced action of J on $\mathcal{D}$, is a CR manifold of rank m and codimension ℓ. A model example, in codimension one, is the unit sphere $\mathbb{S}^{2m+1}$ in $\mathbb{C}^{m+1}$.

(ii) If $(N_i, \mathcal{D}_i, J_i)$ are CR manifolds, with codimensions ℓ_i, for $i \in \{1, \dots, n\}$, then so is $(\prod_{i=1}^n N_i, \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_n, J_1 \oplus \cdots \oplus J_n)$, with codimension $\ell = \ell_1 + \cdots + \ell_n$ and $U_{\mathcal{D}} = \prod_{i=1}^n U_{\mathcal{D}_i}$. In particular, the product of $n = \ell$ codimension one CR spheres $\mathbb{S}^{2m_1+1} \times \cdots \times \mathbb{S}^{2m_\ell+1}$ is a CR manifold with codimension ℓ.

Remark 1. The Levi form of a CR manifold $(N, \mathcal{D}, J)$ is traditionally defined to be the hermitian form $h_{\mathcal{D}} + \sqrt{-1} L_{\mathcal{D}} \colon \mathcal{D} \times \mathcal{D} \to \mathbb{C} \otimes TN/\mathcal{D}$; however it is uniquely determined (given J) by its real or imaginary part, and the imaginary part is an invariant of the underlying (real) distribution $\mathcal{D}$. The rank is usually called the *CR dimension*.

Levi nondegeneracy implies that $X \mapsto L_{\mathcal{D}}(X, \cdot)$ is an injective bundle homomorphism from $\mathcal{D}$ to $\mathrm{Hom}(\mathcal{D}, TN/\mathcal{D})$. This condition, together with the assumption that $L_{\mathcal{D}}$ is surjective onto $TN/\mathcal{D}$, appears in the study [15] of CR automorphisms of real quadrics $N_\sigma := \{(z, w) \in \mathbb{C}^{m+\ell} \mid \Im(w) = \Im\sigma(z, z)\}$, where $\sigma \colon \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^\ell$ is hermitian and $\Im$ denotes the imaginary part. Such quadrics are homogeneous CR manifolds of rank m and codimension ℓ, with Levi form isomorphic to σ (or $\Im\sigma$ in our sense).

Levi definiteness extends the codimension one notion of strict pseudoconvexity.

1.2. Infinitesimal CR actions, generalized Sasaki structures and Kähler cones

Definition 3. Let $(N, \mathcal{D}, J)$ be a CR manifold; then the space $\mathfrak{cr}(N, \mathcal{D}, J)$ of *CR vector fields* is the Lie subalgebra of vector fields X on N such that

$$\mathcal{L}_X C^\infty(\mathcal{D}) \subseteq C^\infty(\mathcal{D}) \quad \text{and} \quad \mathcal{L}_X J = 0. \tag{3}$$

An (*infinitesimal, effective*) *CR action* of a Lie algebra $\mathfrak{g}$ on $(N, \mathcal{D}, J)$ is a Lie algebra monomorphism $\mathbf{K} \colon \mathfrak{g} \to \mathfrak{cr}(N, \mathcal{D}, J)$. For $v \in \mathfrak{g}$, we write K_v for the induced vector field $\mathbf{K}(v)$, and we define

$$\kappa^{\mathfrak{g}} \colon N \times \mathfrak{g} \to TN; (z, v) \mapsto K_{v,z} := (K_v)_z.$$

Let $\mathcal{K}^{\mathfrak{g}} \subseteq TN$ be the image of $\kappa^{\mathfrak{g}}$, i.e., ${\mathcal{K}^{\mathfrak{g}}}_z := \operatorname{span}\{K_{v,z} \mid v \in \mathfrak{g}\}$. Since $\mathbf{K}\colon \mathfrak{g} \to \mathfrak{cr}(N, \mathcal{D}, J)$ is a Lie algebra morphism, $\mathcal{K}^{\mathfrak{g}}$ is an integrable distribution.

Example 2. Let $\pi\colon N \to M$ be a principal G-bundle with connection $\eta : TN \to \mathfrak{g}$, where $\dim G = \ell$ and $\dim M = 2m$. Then $\mathcal{D} := \ker \eta$ is a rank $2m$ distribution on N, and η induces a bundle isomorphism of $TN/\mathcal{D}$ with $N \times \mathfrak{g}$. In this trivialization, the Levi form of $\mathcal{D}$ is $d\eta + \frac{1}{2}[\eta \wedge \eta]_{\mathfrak{g}}$, the pullback to N of the curvature F^η of η. If M has an (integrable) complex structure J for which F^η is J-invariant, the horizontal lift of J to $\mathcal{D}$ equips N with a G-invariant CR structure. For G abelian, this is the principal torus bundle construction of [47].

Suppose $\mathbf{K}\colon \mathfrak{g} \to \mathfrak{cr}(N, \mathcal{D}, J)$ is an infinitesimal CR action, where $(N, \mathcal{D}, J)$ is CR of codimension $\ell = \dim \mathfrak{g}$. Abstracting the infinitesimal geometry of Example 2, we say the action of $\mathfrak{g}$ is *transversal* if the following condition holds.

Condition 1. At every point of N, $\mathcal{D} + \mathcal{K}^{\mathfrak{g}} = TN$. Equivalently:

(i) $\operatorname{rank} \mathcal{K}^{\mathfrak{g}} = \ell$ everywhere on N;

(ii) $\mathcal{D} \cap \mathcal{K}^{\mathfrak{g}}$ is the zero section of TN (and thus $TN = \mathcal{D} \oplus \mathcal{K}^{\mathfrak{g}}$).

The composite $q_{\mathcal{D}} \circ \kappa^{\mathfrak{g}}\colon N \times \mathfrak{g} \to \mathcal{K}^{\mathfrak{g}} \to TN/\mathcal{D}$ is a bundle isomorphism and so there is a canonically defined 1-form $\eta^{\mathfrak{g}}\colon TN \to \mathfrak{g}$, characterized by

$$\ker \eta^{\mathfrak{g}} = \mathcal{D} \quad \text{and} \quad \forall\, v \in \mathfrak{g}, \quad \eta^{\mathfrak{g}}(K_v) = v.$$

We also denote by $\eta^{\mathfrak{g}}$ the induced map from $TN/\mathcal{D}$ to $\mathfrak{g}$. For any $\lambda \in \mathfrak{g}^*$, define

$$\eta^{\mathfrak{g},\lambda}\colon N \to \mathcal{D}^0 \quad \text{by} \quad \eta^{\mathfrak{g},\lambda}_z(X) = \langle \eta^{\mathfrak{g}}_z(X), \lambda\rangle := \lambda(\eta^{\mathfrak{g}}_z(X)),$$

so that $\eta^{\mathfrak{g},\lambda}(K_v) = \lambda(v)$ and $d\eta^{\mathfrak{g},\lambda}|_{\mathcal{D}} = \langle d\eta^{\mathfrak{g}}|_{\mathcal{D}}, \lambda\rangle = \eta^{\mathfrak{g},\lambda} \circ L_{\mathcal{D}}$ is the λ-component of the Levi form of $\mathcal{D}$. When there is no danger of confusion, we shall omit the index $\mathfrak{g}$ and denote

$$\eta := \eta^{\mathfrak{g}}, \qquad \eta^\lambda := \eta^{\mathfrak{g},\lambda}.$$

If $\mathbf{K}$ integrates to an action of a connected Lie group G on N, then Condition 1(i) implies that the G-action is locally free, so that $M := N/G$

is a compact orbifold. Condition 1(ii) then ensures that $\mathcal{D}$ is isomorphic to the pullback of TM to N, and hence G-invariant data on $\mathcal{D}$ descend to M. Invariant components of the Levi form provide examples of such data.

In codimension one, a transversal CR action is essentially a CR Reeb vector field, or equivalently, a compatible Sasaki structure, which makes the symplectic cone Kähler.

To generalize this to arbitrary codimension, note that the total space $\mathcal{D}^0$ of $p\colon \mathcal{D}^0 \to N$ inherits from T^*N a *tautological* 1*-form* τ: using the exact sequence

$$0 \to p^*\mathcal{D}^0 \to T\mathcal{D}^0 \xrightarrow{p_*} p^*TN \to 0, \tag{4}$$

we have that $\tau_\alpha = \alpha \circ p_*\colon T_\alpha\mathcal{D}^0 \to \mathbb{R}$ for any $\alpha \in \mathcal{D}^0$, where p_* is the derivative of the cotangent bundle projection $p\colon T^*N \to N$, here restricted to $\mathcal{D}_0$. Thus τ is the restriction to $\mathcal{D}^0$ of the Liouville 1-form on T^*N. We further set $\Omega^{\mathcal{D}} = d\tau$, which is the pullback of the tautological symplectic form on T^*N to $\mathcal{D}^0$.

Any $X \in C^\infty(TN)$ has a lift to a hamiltonian vector field $\tilde{X}$ on T^*N with $p_*(\tilde{X}) = X$ and hamiltonian $f_X = \tau(\tilde{X})$, i.e., $f_X(\alpha) = \tau_\alpha(\tilde{X}) = \alpha(X)$; furthermore $\{f_X, f_Y\} = f_{[X,Y]}$. (Explicitly, $df_X = -\Omega^{\mathcal{D}}(\tilde{X}, \cdot)$, where $\tilde{X}_\alpha = \alpha_*(X_z) - (\mathcal{L}_X\alpha)_z$ for any extension of $\alpha \in T_z^*N$ to a local section.) If $X \in \mathfrak{cr}(N, \mathcal{D}, J)$, then $\tilde{X}$ is tangent to $\mathcal{D}^0 \subseteq T^*N$.

Observation 1. *Let* $\mathbf{K}\colon \mathfrak{g} \to \mathfrak{cr}(N, \mathcal{D}, J)$ *be an infinitesimal CR action of* $\mathfrak{g}$ *on* $(N, \mathcal{D})$ *and define*

$$\mu_{\mathfrak{g}}\colon \mathcal{D}^0 \to \mathfrak{g}^* \quad \textit{by} \quad \langle \mu_{\mathfrak{g}}(\alpha), v\rangle = \alpha(K_v) \quad \forall\, \alpha \in \mathcal{D}^0,\ v \in \mathfrak{g}.$$

Then the lift of $\mathbf{K}$ *to* T^*N *preserves* $\mathcal{D}^0$*, and the induced infinitesimal action* $\tilde{\mathbf{K}}$ *is hamiltonian on* $U_{\mathcal{D}}$ *with momentum map* $\mu_{\mathfrak{g}}|_{U_{\mathcal{D}}}$*; in particular* $\langle d\mu_{\mathfrak{g}}(\tilde{K}_v), w\rangle = -\langle \mu_{\mathfrak{g}}, [v, w]_{\mathfrak{g}}\rangle$ *for all* $v, w \in \mathfrak{g}$*.*

This is immediate. Now $(p, \mu_{\mathfrak{g}})\colon \mathcal{D}^0 \to N \times \mathfrak{g}^*$ is a bundle isomorphism with inverse $\psi_{\mathfrak{g}}(z, \lambda) := \langle \eta_z, \lambda\rangle$: if $\alpha = \langle \eta_z, \lambda\rangle$ for some $\lambda \in \mathfrak{g}^*$ and $z \in N$, then $\mu_{\mathfrak{g}}(\alpha) = \lambda$.

Lemma 1. *Let* τ *be the tautological* 1*-form on* $\mathcal{D}^0$*. Then*

$$(\psi_{\mathfrak{g}}^*\tau)_{(z,\lambda)}(X + a) = \langle \eta(X), \lambda\rangle, \tag{5}$$

and hence

$$(\psi_{\mathfrak{g}}^*\Omega^{\mathcal{D}})_{(z,\lambda)}(X+a,Y+b) = \langle a,\eta(Y)\rangle - \langle b,\eta(X)\rangle + \langle d\eta(X,Y),\lambda\rangle. \tag{6}$$

Proof. Since $\tau_\alpha(Z) = \alpha(p_*(Z))$,

$$(\psi_{\mathfrak{g}}^*\tau)_{(z,\lambda)}(X+a) = \tau_{\langle\eta_z,\lambda\rangle}\big((\psi_{\mathfrak{g}})_*(X+a)\big) = \langle\eta(X),\lambda\rangle.$$

Hence $\psi_{\mathfrak{g}}^*\tau = \langle p_2, p_1^*\eta\rangle$, where p_1 and p_2 are the first and second projections of $N\times\mathfrak{g}^*$, which yields (5). Now

$$\psi_{\mathfrak{g}}^*\Omega^{\mathcal{D}} = \psi_{\mathfrak{g}}^* d\tau = d(\psi_{\mathfrak{g}}^*\tau) = \langle dp_2\wedge p_1^*\eta\rangle + \langle p_2, p_1^* d\eta\rangle,$$

which yields (6). □

For $(z,v)\in N\times\mathfrak{g}$, let $\mathcal{J}_{(z,v)}$ be the complex structure on $T_{(z,v)}(N\times\mathfrak{g}) = T_zN\oplus\mathfrak{g}$ defined by

$$\mathcal{J}_{(z,v)}(X+w) = JX^{\mathcal{D}} + K_{w,z} - \eta(X)$$

where $X^{\mathcal{D}}$ denotes the $\mathcal{D}_z$-component of $X\in T_zN = \mathcal{D}_z\oplus(\mathcal{K}^{\mathfrak{g}})_z$.

Lemma 2. *The almost complex structure $\mathcal{J}$ is integrable if and only if $\mathfrak{g}$ is abelian.*

Proof. Let X, Y be vector fields on N with $\eta(X)$ and $\eta(Y)$ constant and let $u, v\in\mathfrak{g}^*$; as vector fields on $N\times\mathfrak{g}$, these are constant in the $\mathfrak{g}$ direction. First observe that

$$\begin{aligned} N_{\mathcal{J}}(X,Y) &= [X,\mathcal{J}Y] + [\mathcal{J}X,Y] - \mathcal{J}([X,Y]-[\mathcal{J}X,\mathcal{J}Y]) \\ &= [X,JY^{\mathcal{D}}] + [JX^{\mathcal{D}},Y] - J[X,Y]^{\mathcal{D}} \\ &\quad + \eta([X,Y]) + \mathcal{J}[JX^{\mathcal{D}},JY^{\mathcal{D}}] \\ &= 0, \end{aligned}$$

since $[X,Y]^{\mathcal{D}} = [X^{\mathcal{D}},Y^{\mathcal{D}}] + [K_{\eta(X)},Y^{\mathcal{D}}] + [X^{\mathcal{D}},K_{\eta(Y)}]$. Next

$$\begin{aligned} \mathcal{J}N_{\mathcal{J}}(u,Y) &= \mathcal{J}([u,\mathcal{J}Y] + [\mathcal{J}u,Y]) + [u,Y] - [\mathcal{J}u,\mathcal{J}Y] \\ &= J[K_u,Y]^{\mathcal{D}} - \eta([K_u,Y]) - [K_u, JY^{\mathcal{D}} - \eta(Y)] = 0. \end{aligned}$$

Finally,

$$\begin{aligned} \mathcal{J}N_{\mathcal{J}}(u,v) &= \mathcal{J}([u,\mathcal{J}v] + [\mathcal{J}u,v]) + [u,v] - [\mathcal{J}u,\mathcal{J}v] \\ &= [K_u,K_v] = K_{[u,v]}, \end{aligned}$$

which vanishes for all $u, v \in \mathfrak{g}$ iff $\mathfrak{g}$ is abelian. $\square$

Using η to identify $\mathcal{K}^{\mathfrak{g}}$ with $N \times \mathfrak{g}$, we observe that

$$T_{(z,\lambda)}(N \times \mathfrak{g}^*) \cong \mathcal{D}_z \oplus \mathfrak{g} \oplus \mathfrak{g}^* \quad \text{and} \quad T_{(z,v)}(N \times \mathfrak{g}) \cong \mathcal{D}_z \oplus \mathfrak{g} \oplus \mathfrak{g}; \tag{7}$$

in these terms, $(\psi_{\mathfrak{g}}^* \Omega^{\mathcal{D}})_{(z,\lambda)}$ is the sum of $\langle d\eta_z, \lambda \rangle$ and the standard symplectic structure on $\mathfrak{g} \oplus \mathfrak{g}^*$, while $\mathcal{J}_{(z,v)}$ is the sum of the complex structure J on $\mathcal{D}_z$ and the standard complex structure on $\mathfrak{g} \oplus \mathfrak{g}$. Thus if $\mathfrak{g}$ is abelian and we identify $N \times \mathfrak{g}$ with $N \times \mathfrak{g}^*$ using a symmetric positive definite bilinear form ζ on $\mathfrak{g}$, we obtain a Kähler structure on the open subset of $(z, \lambda) \in N \times \mathfrak{g}^*$ with $\psi_{\mathfrak{g}}(z, \lambda) \in U_{\mathcal{D}}^+$.

Proposition 1. *Let $(N, \mathcal{D}, J)$ be a codimension ℓ CR manifold, and $\mathfrak{g}$ a transversal CR action of an (ℓ-dimensional) abelian Lie algebra $\mathfrak{g}$ with a (positive definite) inner product. Then $U_{\mathcal{D}}^+ \subseteq N \times \mathfrak{g}^*$ has a canonical Kähler metric on which $\mathfrak{g}$ defines an infinitesimal isometric hamiltonian action whose momentum map is the projection $p_2 \colon N \times \mathfrak{g}^* \to \mathfrak{g}^*$.*

Proof. Lemmas 1 and 2 imply that $U_{\mathcal{D}}^+$ is Kähler. The hamiltonian vector field generated by the component $\langle p_2, v \rangle$ of p_2 is the pullback of K_v, which clearly preserves the complex structure $\mathcal{J}$, hence the metric h on $N \times \mathfrak{g}^*$. $\square$

Definition 4. Let $(N, \mathcal{D}, J)$ be Levi definite, and let $\mathfrak{g}$ be an (ℓ-dimensional) abelian Lie algebra. Then a $\mathfrak{g}$-*Sasaki structure* on N is a transversal infinitesimal CR action of $\mathfrak{g}$ together with an inner product ζ on $\mathfrak{g}$; given such an action, we say $(N, \mathcal{D}, J, \mathfrak{g}, \zeta)$ is a *codimension ℓ Sasaki manifold* with *generalized Kähler cone $U_{\mathcal{D}}^+$*.

Example 3. A basic example of codimension ℓ Sasaki manifold $(N, \mathcal{D}, J, \mathfrak{g}, \zeta)$ is obtained by taking $N = H$ where H is a compact simple Lie group and $G \leqslant H$ a maximal ℓ-dimensional torus with Lie algebra $\mathfrak{g}$. In this case, the root decomposition

$$\mathfrak{h} \otimes \mathbb{C} = (\mathfrak{g} \otimes \mathbb{C}) \oplus \bigoplus_{\alpha \in R_+} (\mathfrak{h}_\alpha \oplus \mathfrak{h}_{-\alpha}),$$

where $\mathfrak{g} \otimes \mathbb{C}$ is the corresponding Cartan subalgebra of $\mathfrak{h} \otimes \mathbb{C}$ and R_+ is a set of positive roots, gives rise to a G-invariant CR structure J on N defined by $(\mathcal{D}\otimes\mathbb{C})^{1,0}_e \cong \bigoplus_{\alpha\in R_+}\mathfrak{h}_\alpha$. Equivalently, we have that

$$\mathcal{D}\cong N\times\bigoplus_{\alpha\in R_+}\mathfrak{m}_\alpha,$$

where $\mathfrak{m}_\alpha$ is the J-invariant 2-dimensional real part of $\mathfrak{h}_{-\alpha}\oplus\mathfrak{h}_\alpha$. Furthermore, we can take ζ to be, up to sign, the restriction of the Killing form of $\mathfrak{h}$ to $\mathfrak{g}$. The general theory of simple Lie groups yields that for any $x\in\mathfrak{g}$, and $u,v\in\mathcal{D}$, the Levi form $L_{\mathcal{D}}$ satisfies

$$\zeta(L_{\mathcal{D}}(u,v),x) = -\zeta([u,v],x) = \sum_{\alpha\in R_+} -i\alpha(x)\zeta(Ju_\alpha,v_\alpha),$$

where u_α and v_α are the projections of u and v to the subspace $\mathfrak{m}_\alpha$, respectively. Thus, $U^+_{\mathcal{D}}\cong N\times\mathfrak{g}^*_+$ where

$$\mathfrak{g}_+ := \{x\in\mathfrak{g}\,|\,\forall\,\alpha\in R_+,\ -i\alpha(x)>0\}$$

is the positive Weyl chamber, and $\mathfrak{g}^*_+$ is its image under ζ.

1.3. CR torus actions and Levi–Kähler quotients

Definition 5. An infinitesimal CR action $\mathbf{K}\colon \mathfrak{t}_N\to\mathfrak{cr}(N,\mathcal{D},J)$ of an abelian Lie algebra $\mathfrak{t}_N$ on a CR manifold $(N,\mathcal{D},J)$ is called an *infinitesimal CR torus action*, and is said to be a *CR torus action* if it integrates to an effective (i.e., faithful) action of a compact torus $\mathbb{T}_N=\mathfrak{t}_N/2\pi\Lambda_N$, where Λ_N is the lattice of generators of the circle subgroups of $\mathbb{T}_N$. If $\mathfrak{g}\leqslant\mathfrak{t}_N$ is a subalgebra such that $\mathbf{K}\colon\mathfrak{g}\to\mathfrak{cr}(N,\mathcal{D},J)$ satisfies Condition 1, we refer to $\mathcal{K}^{\mathfrak{g}}$ and its integral submanifolds as the associated *Reeb distribution* and *Reeb foliation* transverse to $\mathcal{D}$.

Given an infinitesimal CR torus action $\mathbf{K}\colon\mathfrak{t}_N\to\mathfrak{cr}(N,\mathcal{D},J)$, let

$$\begin{aligned}
&\kappa := \kappa^{\mathfrak{t}_N}\colon N\times\mathfrak{t}_N\to TN, &&\text{with} &&\kappa(z,v)=K_{v,z}, \quad\text{and}\\
&\mu := \mu_{\mathfrak{t}_N}\colon \mathcal{D}^0\to\mathfrak{t}^*_N, &&\text{with} &&\langle\mu(\alpha),v\rangle=\alpha(K_v),
\end{aligned}$$

so that $(p,\mu)\colon\mathcal{D}^0\to N\times\mathfrak{t}^*_N$ is the pointwise transpose of $q_{\mathcal{D}}\circ\kappa\colon N\times\mathfrak{t}_N\to TN/\mathcal{D}$.

Definition 6. An ℓ-dimensional subalgebra $\iota\colon\mathfrak{g}\hookrightarrow\mathfrak{t}_N$ and an element $\lambda\in\mathfrak{g}^*\setminus 0$ together form a *Levi pair* $(\mathfrak{g},\lambda)$ for an infinitesimal CR torus action $\mathbf{K}$ if:

- $\mathfrak{g}$ acts transversally on N via $\mathbf{K}$, i.e., $\mathcal{K}^{\mathfrak{g}} := \mathrm{span}\{K_{v,z} \mid v \in \mathfrak{g}\}$ satisfies Condition 1.

Let $\eta\colon TN \to \mathfrak{g}$ be the connection 1-form of $\mathfrak{g}$, $\eta^\lambda := \langle \eta, \lambda\rangle$, and $h_{\mathcal{D},\lambda} := d\eta^\lambda|_{\mathcal{D}}(\cdot, J\cdot)$. Then $(\mathcal{D}, h_{\mathcal{D},\lambda}, J)$ is called a *Levi structure* and we say that $(\mathfrak{g}, \lambda)$ or $(\mathcal{D}, h_{\mathcal{D},\lambda}, J)$ is

- *nondegenerate* if η^λ is a contact form, i.e., $h_{\mathcal{D},\lambda}$ is nondegenerate on $\mathcal{D}$;
- *positive* if $h_{\mathcal{D},\lambda}$ is positive definite on $\mathcal{D}$.

We say $(N, \mathcal{D}, J, \mathbf{K})$ is *Reeb type* if it admits a nondegenerate Levi pair, and if positive, we say that $(N, \mathcal{D}, J, \mathfrak{g}, \lambda)$ or $(N, \mathcal{D}, J, h_{\mathcal{D},\lambda})$ is *Levi–Kähler.*

If $\mathbf{K}$ is a CR torus action of $\mathbb{T}_N$ and $\mathfrak{g}$ is the Lie algebra of a closed subgroup G of $\mathbb{T}_N$, then N/G, with the Kähler metric induced by $(h_{\mathcal{D},\lambda}, J, d\eta^\lambda|_{\mathcal{D}})$ is called the *Levi–Kähler quotient* of $(N, \mathcal{D}, J)$ by $(\mathfrak{g}, \lambda)$.

If N is compact, $\{\lambda \in \mathfrak{g}^* \setminus 0 \mid (\mathfrak{g}, \lambda)\,\text{is a Levi pair}\}$ is an open cone $\mathcal{C}_{\mathfrak{g}} \subseteq \mathfrak{g}^*$.

Example 4. Going back to Example 3, we take $G = \mathbb{T}_N$. Then for any $\lambda \in \mathfrak{g}^*_+$, $(\mathfrak{g}, \lambda)$ defines a nondegenerate positive Levi pair on N and the corresponding Levi–Kähler quotient is the flag manifold $M = H/G$ endowed with the H-invariant Kähler structure (g_λ, J) with constant scalar curvature. The Kähler–Einstein structure on M corresponds to the special choice $\lambda = \sum_{\alpha\in R_+} -i\alpha$.

Let $(\mathfrak{g}, \lambda)$ be a Levi pair. For any $v \in \mathfrak{t}_N$, $(d\eta^\lambda)(K_v, \cdot) = -d(\eta^\lambda(K_v))$. We may thus view $\eta^\lambda(K_v) = \langle \mu(\eta^\lambda), v\rangle$ as the "horizontal momentum" of K_v with respect to the Levi structure $(\mathcal{D}, d\eta^\lambda|_{\mathcal{D}})$. Observe that if $v \in \mathfrak{g}$, $\eta^\lambda_z(K_v) = \langle v, \lambda\rangle$, which vanishes for $v \in \ker\lambda \subseteq \mathfrak{g}$. Hence $z \mapsto \mu(\eta^\lambda_z) \in \mathfrak{t}^*_N$ takes values in $(\ker\lambda)^0 \cong (\mathfrak{t}_N/\ker\lambda)^*$.

Stratagem 1. For any pair $(\mathfrak{g}, \lambda)$ with $\mathfrak{g} \subseteq \mathfrak{t}_N$ and $\lambda \in \mathfrak{g}^* \setminus 0$, the quotient $\mathfrak{t}_N/\ker\lambda$ is an extension by $\mathbb{R}$ of the quotient $\mathfrak{t}_N/\mathfrak{g}$. To allow $(\mathfrak{g}, \lambda)$ to vary, it is convenient to fix this extension $\mathfrak{h} \to \mathfrak{t}$ (where $\mathfrak{h}$ and $\mathfrak{t}$ are abelian Lie algebras of dimensions $m + 1$ and m); then the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{g} \arrow[r, "\iota"] \arrow[d, "\lambda"'] & \mathfrak{t}_N \arrow[r, "\mathbf{u}_N"] \arrow[d, "\mathbf{L}_N"'] & \mathfrak{t} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{R} \arrow[r, "\varepsilon"] & \mathfrak{h} \arrow[r, "d"] & \mathfrak{t} \arrow[r] & 0. \end{tikzcd}$$

of short exact sequences associates pairs $(\mathfrak{g}, \lambda)$, with $\mathfrak{t}_N/\ker\lambda \cong \mathfrak{h}$, to surjective linear maps $\mathbf{L}_N\colon \mathfrak{t}_N \to \mathfrak{h}$ (thus $\mathfrak{g}$ is the kernel of $\mathbf{u}_N := d\circ \mathbf{L}_N$, and λ is induced by $\mathbf{L}_N|_{\mathfrak{g}}$).

Let $\varepsilon^\top\colon \mathfrak{h}^* \to \mathbb{R}$ be the transpose of ε, and let $\mathcal{A} \subseteq \mathfrak{h}^*$ be the affine subspace $(\varepsilon^\top)^{-1}(1)$ of $\mathfrak{h}^*$, modelled on $\mathfrak{t}^*$; then $\mathfrak{h}$ may be identified with the affine linear functions $\ell\colon \mathcal{A} \to \mathbb{R}$, whence $d\ell \in \mathfrak{t}$ is the linear part of $\ell \in \mathfrak{h}$.

By Observation 1, $\mathbf{K}$ lifts to an infinitesimal hamiltonian action on $U_{\mathcal{D}}$ with momentum map $\mu|_{U_{\mathcal{D}}}$. If $(\mathfrak{g}, \lambda)$, defined by $\mathbf{L}_N\colon \mathfrak{t}_N \to \mathfrak{h}$, is a Levi pair, then the map $\mu^\lambda\colon N \to \mathcal{A} \subseteq \mathfrak{h}^*$, determined uniquely by the formula

$$\langle \mu^\lambda(z), \mathbf{L}_N(v)\rangle = \eta^\lambda_z(K_v) \tag{8}$$

for all $z \in N$ and $v \in \mathfrak{t}_N$, will be called the *horizontal momentum map* of $(\mathcal{D}, d\eta^\lambda|_{\mathcal{D}})$. Equivalently the diagram

$$\begin{tikzcd} N \arrow[r, "\eta^\lambda"] \arrow[d, "\mu^\lambda"'] & U_{\mathcal{D}} \arrow[d, "\mu"] \\ \mathfrak{h}^* \arrow[r, "\mathbf{L}_N^\top"] & \mathfrak{t}_N^* \end{tikzcd}$$

commutes, i.e., $\mathbf{L}_N^\top \circ \mu^\lambda = \mu \circ \eta^\lambda$.

Observation 2. *Let $(N, \mathcal{D}, J, \mathbf{K})$ be a CR manifold of Reeb type, and let $(\mathfrak{g}, \lambda)$ be a nondegenerate Levi pair, where $\mathfrak{g}$ is the Lie algebra of a subtorus G of $\mathbb{T}_N$. Then $M := N/G$, equipped with the 2-form induced by $d\eta^\lambda|_{\mathcal{D}}$, is the symplectic quotient ${\mu_{\mathfrak{g}}}^{-1}(\lambda)/G$ of $U_{\mathcal{D}}$ by the lifted G-action; it is therefore a compact symplectic orbifold with a hamiltonian action of $\mathbb{T} = \mathbb{T}_N/G$ whose momentum map is induced by the G-invariant map $\mu^\lambda\colon N \to \mathcal{A} \subseteq \mathfrak{h}^*$ defined in* (8).

Let $\mathcal{K} = \mathcal{K}^{\mathfrak{t}_N} = \operatorname{im}\kappa$,

$$\mathcal{E} := \operatorname{im}(p, \mu) \subseteq N \times \mathfrak{t}_N^* \quad \text{and} \quad \Theta := \mathcal{E}^0 = \ker(q_{\mathcal{D}} \circ \kappa) = \kappa^{-1}(\mathcal{D}) \subseteq N \times \mathfrak{t}_N.$$

If $\operatorname{rank}\mathcal{D} = 2m$, then $\operatorname{rank}\mathcal{K}\cap\mathcal{D} \leqslant m$, and hence $\dim\mathfrak{t}_N \leqslant m + \ell$.

Proposition 2. *Let $\mathbf{K}\colon \mathfrak{t}_N \to \mathfrak{cr}(N, \mathcal{D}, J)$ be an infinitesimal CR torus action. Then $\mathcal{K}\cap\mathcal{D}$ is an integrable distribution, i.e., $L_{\mathcal{D}}(X, Y) = 0$ for all $X, Y \in \mathcal{K}\cap\mathcal{D}$.*

Proof. For any $v, w \in \mathfrak{t}_N$ and any section α of $\mathcal{D}^0$,

$$d\alpha(K_v, K_w) = (\mathcal{L}_{K_v}\alpha)(K_w) - (\mathcal{L}_{K_w}\alpha)(K_v).$$

Hence if $X = \sum_i f_i K_{v_i}$ and $Y = \sum_j g_j K_{w_j}$ are sections of $\mathcal{K} \cap \mathcal{D}$ (for functions f_i and g_j on N), then

$$(\alpha \circ L_{\mathcal{D}})(X, Y) = d\alpha(X, Y) = \textstyle\sum_i f_i(\mathcal{L}_{K_{v_i}}\alpha)(Y) - \sum_j g_j(\mathcal{L}_{K_{w_j}}\alpha)(X) = 0$$

since K_v preserves $\mathcal{D}^0$ for any $v \in \mathfrak{t}_N$. $\square$

Remark 2. If $(\mathfrak{g}, \lambda)$ is a nondegenerate Levi pair then $(p, \mu_{\mathfrak{g}}) \colon \mathcal{D}^0 \to N \times \mathfrak{g}^*$, with $\mu_{\mathfrak{g}} := \iota^\top \circ \mu$, is an isomorphism. In particular, (p, μ) injects, i.e., $\mathcal{E}$ is a rank ℓ subbundle of $N \times \mathfrak{t}_N^*$ (with $\mathcal{E}^* \cong (N \times \mathfrak{t}_N)/\Theta \cong TN/\mathcal{D}$). Equivalently, the transpose $q_{\mathcal{D}} \circ \kappa$ surjects (pointwise), i.e., $\mathcal{K} \cap \mathcal{D}$ has codimension ℓ in $\mathcal{K}$. Conversely, this suffices for the local existence (i.e., on an open neighbourhood of each point of N) of a transversal subalgebra $\iota \colon \mathfrak{g} \hookrightarrow \mathfrak{t}_N$, hence also a Levi pair: $U_{\mathcal{D}}$ is open with nonempty fibres, so we can find $\lambda \in \mathfrak{g}^*$ such that η^λ is locally a contact form.

This remark prompts the following.

Definition 7. We say $(N, \mathcal{D}, J, \mathbf{K})$ is *locally Reeb type* if $q_{\mathcal{D}} \circ \kappa$ surjects and *infinitesimally toric CR* if it is locally Reeb type with $\dim \mathfrak{t}_N = m + \ell$.

On any open set where $\operatorname{rank} \mathcal{K} = m + \ell$, $(N, \mathcal{D}, \mathbf{K})$ is locally Reeb type.

Proposition 3. *Let $(N, \mathcal{D}, J, \mathbf{K})$ be an infinitesimally toric CR manifold and let N^o be the open subset where $\operatorname{rank} \mathcal{K} = m + \ell$, and $U^o_{\mathcal{D}}$ its inverse image in $U_{\mathcal{D}} \subseteq \mathcal{D}^0$.*

Then $U_{\mathcal{D}}$ is an infinitesimal toric symplectic manifold under the lift $\tilde{\mathbf{K}}$, with momentum map $\mu \colon U_{\mathcal{D}} \to \mathfrak{t}_N^*$ defined by $\langle \mu(\alpha), v \rangle = \alpha(K_v)$, where $\alpha \in U_{\mathcal{D}} \subseteq T^*N$ and $K_v := \mathbf{K}(v)$ for $v \in \mathfrak{t}_N$. Further, there are angular coordinates $\boldsymbol{\varphi} \colon U^o_{\mathcal{D}} \to \mathfrak{t}_N$, defined up to additive constants, with $\Omega^{\mathcal{D}} = \langle d\mu \wedge d\boldsymbol{\varphi} \rangle$ and $\ker d\boldsymbol{\varphi} = p_*^{-1}(J(\mathcal{K} \cap \mathcal{D}))$ (restricting $p \colon \mathcal{D}^0 \to N$ to $U^o_{\mathcal{D}}$).

Proof. The first part is immediate from Observation 1. By Proposition 2, $\mathcal{K} \cap \mathcal{D}$ is an integrable rank m subbundle of $\mathcal{D}$, hence so is $J(\mathcal{K} \cap \mathcal{D})$ by the J-invariance of the Levi form, and $TN^o = \mathcal{K} \oplus J(\mathcal{K} \cap \mathcal{D})$. The 1-form $\beta \colon TN^o \to \mathfrak{t}_N$ defined by $\ker \beta = J(\mathcal{K} \cap \mathcal{D})$ and $\beta(K_v) = v$ is therefore closed. It is not exact, but by definition of N^o, the local primitives (which are defined up to additive constants) pull back to $U^o_{\mathcal{D}}$ to give $\boldsymbol{\varphi}$. $\square$

In terms of the short exact sequence (4) restricted to $U^o_{\mathcal{D}}$, we have

$$\begin{tikzcd} 0 \arrow[r] & p^*\mathcal{D}^0 \arrow[r] \arrow[d, equal] & \ker d\boldsymbol{\varphi} \arrow[r] \arrow[d] & p^*J(\mathcal{K}\cap\mathcal{D}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & p^*\mathcal{D}^0 \arrow[r] & TU^o_{\mathcal{D}} \arrow[r, "p_*"] & p^*TN^o \arrow[r] & 0, \end{tikzcd}$$

where for any $\alpha \in U^o_{\mathcal{D}}$, $(d\mu)_\alpha(\ker d\boldsymbol{\varphi}) = \mathfrak{t}^*_N$, and $(d\mu)_\alpha(p^*\mathcal{D}^0) = \mu(\mathcal{D}^0_{p(\alpha)}) = \mathcal{E}_{p(\alpha)}$. This identifies $(d\mu)_\alpha(p^*J(\mathcal{K}\cap\mathcal{D}))$ with $\Theta^*_{p(\alpha)}$.

A subalgebra $\iota\colon \mathfrak{g} \hookrightarrow \mathfrak{t}_N$ satisfying Condition 1 splits the exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow N^o \times \mathfrak{t}^*_N \longrightarrow \Theta^* \longrightarrow 0$$

i.e., $\ker\iota^\top \cong \mathfrak{t}^*$ is transverse to $\mathcal{E}_z$ for all $z \in N^o$. Thus $\mu(U^o_{\mathcal{D}}) \subseteq \mathfrak{t}^*_N$ is foliated by its intersection with the m-dimensional family $\mathcal{E}_z$ of ℓ-dimensional linear subspaces of $\mathfrak{t}^*_N$.

Remark 3. The Levi–Kähler quotient of N by $(\mathfrak{g},\lambda)$ is the Kähler quotient of any $\mathbb{T}_N$-invariant, $\Omega^{\mathcal{D}}$ compatible metric $\hat{g}$ on $U^+_{\mathcal{D}}$ whose pullback to $\mu_{\mathfrak{g}}^{-1}(\lambda) \cong N$ is the orthogonal sum of a metric on $\mathcal{K}^{\mathfrak{g}}$ and the metric $h_{\mathcal{D},\lambda}$ on $\mathcal{D}$. We may assume $\hat{g}$ has angular coordinates $d\boldsymbol{\varphi}$ on $U^{o,+}_{\mathcal{D}} = U^o_{\mathcal{D}} \cap U^+_{\mathcal{D}}$, so that it is determined uniquely there by the induced $\mathbb{T}_N$-invariant metric on $\ker d\boldsymbol{\varphi} \cong N^o \times \mathfrak{t}^*_N$, which descends to a metric $\mathbf{G}$ on $\mu(U^{o,+}_{\mathcal{D}}) \subseteq \mathfrak{t}^*_N$. Since $h_{\mathcal{D},c\lambda} = c\,h_{\mathcal{D},\lambda}$ for $c \in \mathbb{R}^+$, we assume $\mathbf{G}$ is homogeneous of degree 1 on $\mathfrak{t}^*_N$, i.e., as an $S^2\mathfrak{t}_N$-valued function on $\mu(U^{o,+}_{\mathcal{D}})$, it is homogeneous of degree -1. Examples of such metrics include the generalized Kähler cone metrics (Definition 4).

If $(\mathfrak{g},\lambda)$ is given by $\mathbf{L}_N\colon \mathfrak{t}_N \to \mathfrak{h}$, then the Levi–Kähler quotient metric depends only on the pullback of $\mathbf{G}$ to $\mathbf{L}^\top_N(\mathcal{A}) = (\iota^\top)^{-1}(\lambda)$, an affine subspace transverse to $\mathcal{E}$.

2. Levi–Kähler reduction in toric geometry

2.1. Polytopes, fans, combinatorics, and toric contact manifolds

Suppose that $(N,\mathcal{D},J,\mathbf{K})$ is a toric CR manifold of rank m and codimension ℓ, under a (real) torus $\mathbb{T}_N = \mathfrak{t}_N/2\pi\Lambda_N$ with (abelian) Lie algebra $\mathfrak{t}_N = \Lambda_N \otimes_{\mathbb{Z}} \mathbb{R}$.

The theory of effective actions of tori [14, 36] implies that for any subtorus $H \leqslant \mathbb{T}_N$,

$$N_{(H)} := \{z \in N \mid H = \mathrm{Stab}_{\mathbb{T}_N}(z)\} \subseteq N^H := \{z \in N \mid H \subseteq \mathrm{Stab}_{\mathbb{T}_N}(z)\}$$

is an open submanifold of a closed submanifold of N, and if $N_{(H)}$ is nonempty then $N_{(H)}$ is dense in N^H. The connected components of $N_{(H)}$ and their closures in N are called *open* and *closed orbit strata* of $(N, \mathbf{K})$. Let Φ_N be the set of closed orbit strata, partially ordered by inclusion, and let $N_s : s \in \mathcal{S}$ index the closed orbit strata stabilized by a circle. The 1-dimensional Lie algebra of N_s has a primitive generator $v_s \in \Lambda_N \subseteq \mathfrak{t}_N$, unique up to sign. We refer to Φ_N as the *combinatorics* of N; it is a "poset over $\mathcal{S}$".

Definition 8. A *poset* (partially ordered set) *over a set* $\mathcal{S}$ is a set Φ equipped with a partial ordering (reflexive antisymmetric transitive relation) and a map $\mathcal{S} \to \Phi$. A morphism $\Phi \to \Phi'$ of posets over $\mathcal{S}$ is an order preserving map whose composite with the map $\mathcal{S} \to \Phi$ is the map $\mathcal{S} \to \Phi'$. We say Φ and Φ' have the *same combinatorial type* if they are isomorphic as posets over $\mathcal{S}$. The combinatorics arising in toric geometry are typically isomorphic to subposets of the power set $P(\mathcal{S})$ or its opposite $P(\mathcal{S})^{op}$, which are posets over $\mathcal{S}$ under inclusion or reverse inclusion respectively, with the map from $\mathcal{S}$ being the singleton map $s \mapsto \{s\}$.

To illustrate this, we start, as in Stratagem 1, with an exact sequence

$$0 \to \mathbb{R} \xrightarrow{\varepsilon} \mathfrak{h} \xrightarrow{d} \mathfrak{t} \to 0$$

of vector spaces, viewed as an extension of abelian Lie algebras with $\dim \mathfrak{t} = m$, and let $\mathcal{A} := (\varepsilon^\top)^{-1}(1)$ be the corresponding m-dimensional affine subspace of $\mathfrak{h}^*$. Recall that a convex polytope Δ in $\mathcal{A}$ is a subset of the form

$$\Delta := \{\xi \in \mathcal{A} \mid \forall s \in \mathcal{S},\ \ L_s(\xi) \geqslant 0\}$$

where $\mathcal{S}$ is a finite set, and $L_s \in \mathfrak{h}$ (an affine function on $\mathcal{A}$) for each $s \in \mathcal{S}$.

Remark 4. The *combinatorics* Φ_Δ of Δ is the poset over $\mathcal{S}$ of closed faces of Δ. More precisely, for $\xi \in \mathcal{A}$, let $S_\xi = \{s \in \mathcal{S} \mid L_s(\xi) = 0\}$; we assume that $\Delta \subseteq \mathcal{A}$ has nonempty interior $\mathring{\Delta}$ (so for $\xi \in \mathring{\Delta}$, $S_\xi = \varnothing$) and that for any $s \in \mathcal{S}$, there exists $\xi \in \Delta$ with $S_\xi = \{s\}$ (otherwise we may discard s without changing Δ). The map sending $S \subseteq \mathcal{S}$ to

$$F_S := \{\xi \in \Delta \mid S \subseteq S_\xi\} = \{\xi \in \Delta \mid \forall s \in S,\ \ L_s(\xi) = 0\}$$

restricts to an isomorphism from $\{S_\xi \in P(\mathcal{S})^{op} \mid \xi \in \Delta\}$ to Φ_Δ over $\mathcal{S}$. Any closed face is thus the intersection of the *facets* $F_s := F_{\{s\}}$ containing it: $F_S = \bigcap_{s \in S} F_s$. We assume that the empty face is an element of Φ_Δ, so that $F_S \in \Phi_\Delta$ for all $S \in P(\mathcal{S})$.

Given a compact convex polytope $\Delta \subseteq \mathcal{A}$, the positive span $\mathbb{R}^+\Delta$ is a cone in $\mathfrak{h}^*$; the *dual cone* to Δ is

$$\Delta^* := \{L \in \mathfrak{h} \mid \forall \xi \in \Delta,\ \ L(\xi) \geqslant 0\},$$

and its projection onto $\mathfrak{t}$ defines a decomposition Y of $\mathfrak{t}$, called the associated (complete) *fan*, into a union of polyhedral cones

$$\begin{aligned}\mathcal{C}_S := \{&dL \in \mathfrak{t} \mid \forall \xi \in \Delta,\ \ L(\xi) \geqslant 0\\ &\text{with equality for all } \xi \in F_S\} = \operatorname{span}\{u_s \mid s \in S\}\end{aligned}$$

corresponding to the faces F_S of Δ. These cones form a poset Φ_{Y} over $\mathcal{S}$ under inclusion, and a (complete) fan Y is uniquely determined by its combinatorics Φ_{Y} and its rays (one dimensional cones) $\mathcal{C}_s := \mathcal{C}_{\{s\}}$. When Y is constructed from Δ as above then Φ_{Y} is canonically isomorphic to Φ_Δ^{op} over $\mathcal{S}$, and in particular there is a canonical bijection between the facets of Δ and the rays $\mathcal{C}_s := \mathcal{C}_{\{s\}}$ of Y.

The rays of Y determine $u_s : s \in \mathcal{S}$ up to positive scale, and similarly Δ determines $L_s : s \in \mathcal{S}$ up to positive scale. Given a choice of these scales, we say Δ is a *labelled polytope* with *affine normals* $L_s \in \mathfrak{h}$ and *inward normals* $u_s := dL_s \in \mathfrak{t}$ for $s \in \mathcal{S}$, and that Y is a *labelled fan* with generators $u_s \in \mathfrak{t}$ for $s \in \mathcal{S}$.

Definition 9. A (complete) fan Y is *simplicial* if the rays in any cone are linearly independent. A (compact) convex polytope Δ is *simple* if its fan is simplicial. In terms of a labelling, this means for all $S \in \Phi_{\mathsf{Y}}$, $u_s : s \in S$ is linearly independent, or (for polytopes) $\forall \xi \in \Delta$, $\mathcal{B}_\xi := (u_s : s \in S_\xi)$ is linearly independent. This condition only depends on the vertices ξ of Δ, where it means that $\mathcal{B}_\xi$ is a basis; in particular, each vertex is m-valent.

Returning to $(N, \mathcal{D}, J, \mathbf{K})$, the underlying contact manifold $(N, \mathcal{D})$ is toric under $(\mathbf{K}, \mathbb{T}_N)$ in the sense of [8], where the following result is obtained, following the methods of [13, 29, 38, 42–44].

Theorem 1. *Let $(N, \mathcal{D}, \mathbf{K})$ be a (compact, connected) toric contact manifold under $\mathbb{T}_N$. Then the stabilizers in $\mathbb{T}_N$ of points in N are connected (i.e., subtori) and the fibres of the momentum map μ on $U_{\mathcal{D}}$ are $\mathbb{T}_N$-orbits. For any nondegenerate Levi pair $(\mathfrak{g}, \lambda)$ the signs of the primitive generators $v_s \in \mathfrak{t}_N$ of the circle stabilizers $N_s : s \in \mathcal{S}$ may be chosen uniquely such that the image of the horizontal momentum map $\mu^\lambda \colon N \to \mathcal{A} \subseteq \mathfrak{h}^*$ is the compact* *simple convex polytope* Δ *in* $\mathcal{A}$ *defined by the affine functions* $L_s := \mathbf{L}_N(v_s)$ *and* μ^λ *is a submersion over the interior of each face.*

In particular, μ^λ induces a poset isomorphism over $\mathcal{S}$ of Φ_N with Φ_Δ.

Corollary 1. N *and* Δ *(i.e.,* Φ_N *and* Φ_Δ*) have the same combinatorial type.*

Remark 5. The primitive generators $v_s : s \in \mathcal{S}$ need not be linearly independent in $\mathfrak{t}_N$, nor even distinct, although after taking a quotient of N by a subtorus acting freely, we may assume that they span.

Suppose now that G is a closed subgroup of $\mathbb{T}_N$ with Lie algebra $\mathfrak{g}$. Such a subgroup exists if and only if the lattice of circle subgroups Λ_N in $\mathfrak{t}_N$ is mapped to a (rank m) lattice Λ in $\mathfrak{t}$, which holds if and only if $u_s : s \in \mathcal{S}$ span a lattice in $\mathfrak{t}$.

Definition 10. If $\Lambda \subseteq \mathfrak{t}$ is a lattice, then a polytope Δ or fan Y labelled by L_s (with $u_s = dL_s$) or $u_s : s \in \mathcal{S}$ is *rational* with respect to Λ if for all $s \in \mathcal{S}$, $u_s \in \Lambda$.

We are now ready to study the Levi–Kähler quotient N/G, which is a compact toric Kähler orbifold M of real dimension $2m$ under an m-torus $\mathbb{T}$ with Lie algebra $\mathfrak{t}$ and hamiltonian generators $\mathfrak{h}$. Indeed, with respect to the symplectic form on N/G induced by $d\eta^\lambda|_{\mathcal{D}}$, $\mu^\lambda \colon N \to \mathcal{A} \subseteq \mathfrak{h}^*$ descends to a (natural) momentum map for the action of $\mathbb{T} = \mathbb{T}_N/G$ on N/G, whose image is the rational simple convex polytope $\Delta \subseteq \mathcal{A}$. Rational Delzant theory [29, 44] asserts that any such toric Kähler orbifold M is determined up to symplectomorphism or biholomorphism by its labelled polytope or fan. The construction of M from these data is relevant here, so we review it now.

Let $\mathbb{Z}_{\mathcal{S}}$ be the free abelian group generated by $\mathcal{S}$, let $\mathfrak{t}_{\mathcal{S}} = \mathbb{Z}_{\mathcal{S}} \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{C}_{\mathcal{S}} = \mathbb{Z}_{\mathcal{S}} \otimes_{\mathbb{Z}} \mathbb{C}$ be the corresponding free vector spaces over $\mathbb{R}$ and $\mathbb{C}$, and let $\mathbb{T}_{\mathcal{S}} = \mathfrak{t}_{\mathcal{S}}/2\pi\mathbb{Z}_{\mathcal{S}}$ and $\mathbb{T}^{\mathbb{C}}_{\mathcal{S}} = \mathbb{C}_{\mathcal{S}}/2\pi\mathbb{Z}_{\mathcal{S}} \cong \mathbb{C}^{\times}_{\mathcal{S}}$ be the corresponding real and complex tori. Denote the generators of $\mathbb{Z}_{\mathcal{S}} \subseteq \mathfrak{t}_{\mathcal{S}} \subseteq \mathbb{C}_{\mathcal{S}}$ by $e_s : s \in \mathcal{S}$, and observe that $\mathbb{T}_{\mathcal{S}}$ and $\mathbb{T}^{\mathbb{C}}_{\mathcal{S}}$ act diagonally on $\mathbb{C}_{\mathcal{S}}$, via $[\sum_s t_s e_s] \cdot (\sum_s z_s e_s) = \sum_s \exp(i t_s) z_s e_s$, where $z_s \in \mathbb{C}$ and $t_s \in \mathbb{R}$ or $\mathbb{C}$. The action of $\mathbb{T}_{\mathcal{S}}$ on $\mathbb{C}_{\mathcal{S}}$ is hamiltonian (with respect to the standard symplectic form $\omega_{\mathcal{S}}$ on $\mathbb{C}_{\mathcal{S}}$) and has a momentum map $\boldsymbol{\sigma} \colon \mathbb{C}_{\mathcal{S}} \to \mathfrak{t}^*_{\mathcal{S}}$ defined by

$$\langle \boldsymbol{\sigma}(z), e_s \rangle = \sigma_s(z) := \tfrac{1}{2}|z_s|^2, \tag{9}$$

where $z_s \colon \mathbb{C}_{\mathcal{S}} \to \mathbb{C}$ denote the standard (linear) complex coordinates on $\mathbb{C}_{\mathcal{S}}$.

The labellings $s \mapsto L_s$ and $s \mapsto u_s$ of Δ and Y induce, and are defined by (without loss, surjective) linear maps $\mathbf{L}\colon \mathfrak{t}_{\mathcal{S}} \to \mathfrak{h}$ and $\mathbf{u}\colon \mathfrak{t}_{\mathcal{S}} \to \mathfrak{t}$ with $\mathbf{L}(e_s) = L_s$ and $\mathbf{u}(e_s) = u_s$ for all $s \in \mathcal{S}$. Let $\tilde{\mathfrak{g}}$ be the kernel of $\mathbf{u}$; then $\mathbf{L}$ determines a linear form $\tilde{\lambda} \in \tilde{\mathfrak{g}}^*$ completing the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & \tilde{\mathfrak{g}} \arrow[r, "\tilde{\iota}"] \arrow[d, "\tilde{\lambda}"'] & \mathfrak{t}_{\mathcal{S}} \arrow[r, "\mathbf{u}"] \arrow[d, "\mathbf{L}"'] & \mathfrak{t} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r, "\varepsilon"] & \mathfrak{h} \arrow[r, "d"] & \mathfrak{t} \arrow[r] & 0.
\end{tikzcd}
\tag{10}
$$

When Δ (or Y) is rational, then $\mathbf{u}$ maps $\mathbb{Z}_{\mathcal{S}}$ to Λ and hence defines a map from $\mathbb{T}_{\mathcal{S}} = \mathfrak{t}_{\mathcal{S}}/2\pi\mathbb{Z}_{\mathcal{S}}$ to $\mathfrak{t}/2\pi\Lambda$ whose kernel is a closed subgroup $\tilde{G}$ of $\mathbb{T}_{\mathcal{S}}$ with Lie algebra $\tilde{\mathfrak{g}}$. We let $\tilde{G}^{\mathbb{C}} \leqslant \mathbb{T}_{\mathcal{S}}^{\mathbb{C}}$ be the kernel of the complexification of this map.

The combinatorics Φ_{Y} of Y (or Δ) define an open subset $\mathbb{C}_{\mathcal{S}}^{\circ} \subseteq \mathbb{C}_{\mathcal{S}}$ as the union of $z \in \mathbb{C}_{\mathcal{S}}$ for which $S_z := \{s \in \mathcal{S} \mid z_s = 0\}$ is in Φ_{Y}. In other words, $\mathbb{C}_{\mathcal{S}}^{\circ}$ is the union of the $\mathbb{T}_{\mathcal{S}}^{\mathbb{C}}$-orbits

$$\mathbb{C}_{\mathcal{S},S} := \{z \in \mathbb{C}_{\mathcal{S}} \mid z_s = 0 \text{ iff } s \in S\}$$

over $S \in \Phi_{\mathsf{Y}}$. Thus the set of $\mathbb{T}_{\mathcal{S}}^{\mathbb{C}}$ orbits in $\mathbb{C}_{\mathcal{S}}^{\circ}$ is isomorphic to Φ_{Y}, and $S \subseteq S'$ iff the closure of $\mathbb{C}_{\mathcal{S},S}$ contains $\mathbb{C}_{\mathcal{S},S'}$.

Lemma 3. *Let $(\mathsf{Y}, \mathbf{u})$ be a simplicial fan with combinatorics Φ_{Y}. Then $\tilde{\mathfrak{g}} \subseteq \mathfrak{t}_{\mathcal{S}}$ acts locally freely on $\mathbb{C}_{\mathcal{S}}^{\circ}$. If in addition $(\mathsf{Y}, \mathbf{u})$ is rational, and $\tilde{G}$ is the corresponding closed subgroup of $\mathbb{T}_{\mathcal{S}}$, then for any $S \in \Phi_{\mathsf{Y}}$, the stabilizer in $\tilde{G}$ of any $z \in \mathbb{C}_{\mathcal{S},S}$ is*

$$\mathrm{Stab}_{\tilde{G}}(z) \cong (\Lambda \cap \mathrm{span}_{\mathbb{R}}\{u_s \mid s \in S\})/\mathrm{span}_{\mathbb{Z}}\{u_s \mid s \in S\}. \tag{11}$$

Proof. Let $S \in \Phi_{\mathsf{Y}}$ and $z \in \mathbb{C}_{\mathcal{S},S}$, so that $z_s = 0$ iff $s \in \mathcal{S}$. Then any element of the stabilizer of z in $\mathfrak{t}_{\mathcal{S}}$ has the form $v = \sum_{s\in S} t_s e_s$, which belongs to $\tilde{\mathfrak{g}}$ iff $\sum_{s\in S} t_s u_s = 0$. However, since $(\mathsf{Y}, \mathbf{u})$ is simplicial, $u_s : s \in S$ is linearly independent, hence $v = 0$.

For the second part, an element $[v] = [\sum_{s\in S} t_s e_s]$ of the stabilizer of z in $\mathbb{T}_{\mathcal{S}} = \mathfrak{t}_{\mathcal{S}}/2\pi\mathbb{Z}_{\mathcal{S}}$ is in $\tilde{G}$ iff $\sum_{s\in S} t_s u_s \in \Lambda$, and is the identity element iff $t_s \in \mathbb{Z}$ for all $s \in S$. The result follows. □

The Delzant–Lerman–Tolman correspondence and the relation between symplectic and complex (GIT) quotients [29, 44] now assert that:

(i) as a complete toric variety, M is a complex (GIT) quotient $\mathbb{C}_{\mathcal{S}}^{\circ}/\tilde{G}^{\mathbb{C}}$ of $\mathbb{C}_{\mathcal{S}}$ by $\tilde{G}^{\mathbb{C}}$;

(ii) as a compact toric symplectic orbifold, M is a symplectic quotient $\tilde{N}/\tilde{G}$ — where $\tilde{N} = (\tilde{\iota}^\top \boldsymbol{\sigma})^{-1}(\tilde{\lambda})$ — of $\mathbb{C}_{\mathcal{S}}$ by $\tilde{G}$ at momentum level $\tilde{\lambda} \in \tilde{\mathfrak{g}}^*$.

The orbifold structure groups of points in $M = \tilde{N}/\tilde{G}$ are given by the stabilizers in $\tilde{G}$ of corresponding points in $\tilde{N}$, hence are related to the labellings of the polytope and fan by (11). Note that the complex structure in (i) is biholomorphic to the complex structure of the Kähler quotient in (ii), which is the quotient of the induced CR structure on $\tilde{N} \subseteq \mathbb{C}_{\mathcal{S}}$. However, M is not typically *isometric* to the Kähler quotient of $\mathbb{C}_{\mathcal{S}}$ by $\tilde{G}$, which is called the *Guillemin metric* [1, 2, 37] of Δ.

In particular, the Levi–Kähler quotient $(N/G, d\eta^\lambda|_{\mathcal{D}}, J|_{\mathcal{D}})$ is symplectomorphic to the toric symplectic orbifold obtained from $(\Delta, \mathbf{L})$ by the Delzant–Lerman–Tolman construction, but also biholomorphic to the Levi–Kähler quotient of the toric CR submanifold $\tilde{N}$ of flat space by $(\tilde{\mathfrak{g}}, \tilde{\lambda})$. In general N and $\tilde{N}$ have different dimensions, but there is a map from Λ to Λ_N sending e_s to v_s, and if the latter form a basis for Λ_N, then we may identify $\mathbb{T}_N$ with $\mathbb{T}_{\mathcal{S}}$, $\mathbf{L}_N$ with $\mathbf{L}$, and hence $(\tilde{\mathfrak{g}}, \tilde{\lambda})$ with $(\mathfrak{g}, \lambda)$. This motivates the study of Levi–Kähler quotients of toric CR submanifolds of flat space, which will occupy us for the remainder of the paper.

2.2. Toric CR submanifolds of flat space

Let $\mathcal{S}$ be a d element set, and define $\mathbb{Z}_{\mathcal{S}}$, $\mathfrak{t}_{\mathcal{S}}$, $\mathbb{C}_{\mathcal{S}}$, $\mathbb{T}_{\mathcal{S}}$ and $\mathbb{T}^{\mathbb{C}}_{\mathcal{S}}$ as in §2.1. Let $\mathbf{K}\colon \mathfrak{t}_{\mathcal{S}} \to \mathfrak{ham}(\mathbb{C}_{\mathcal{S}}, \omega_{\mathcal{S}})$ be the infinitesimal hamiltonian action, and let $\boldsymbol{\vartheta}\colon \mathbb{C}^\times_{\mathcal{S}} \to \mathbb{T}_{\mathcal{S}}$ be angular coordinates conjugate to the momentum components (9) on the open set $\mathbb{C}^\times_{\mathcal{S}}$ of $\mathbb{C}_{\mathcal{S}}$ where $z_s \neq 0$ for all $s \in \mathcal{S}$; thus $d\boldsymbol{\vartheta} : T\mathbb{C}^\times_{\mathcal{S}} \to \mathfrak{t}_{\mathcal{S}}$ satisfies $d\boldsymbol{\vartheta}(K_v) = v$ and $d\boldsymbol{\vartheta}(JK_v) = 0$ for all $v \in \mathfrak{t}_{\mathcal{S}}$. The flat Kähler metric in action-angle coordinates on $\mathbb{C}^\times_{\mathcal{S}}$ is then

$$\begin{aligned} g_{\mathcal{S}} &= \sum_{s\in\mathcal{S}} \left(\frac{d\sigma_s{}^2}{2\sigma_s} + 2\sigma_s d\boldsymbol{\vartheta}_s{}^2 \right), \\ \omega_{\mathcal{S}} &= \sum_{s\in\mathcal{S}} d\sigma_s \wedge d\boldsymbol{\vartheta}_s, \quad d^c\sigma_s := J d\sigma_s = 2\sigma_s d\boldsymbol{\vartheta}_s. \end{aligned} \tag{12}$$

In particular, the metric $\mathbf{H}(v, w) := g_{\mathcal{S}}(K_v, K_w)$ on the $\mathbb{T}_{\mathcal{S}}$-orbits is given by the smooth function $\mathbf{H} = 2\delta\boldsymbol{\sigma}\colon \mathbb{C}_{\mathcal{S}} \to S^2\mathfrak{t}^*_{\mathcal{S}}$, where $\delta\colon \mathfrak{t}^*_{\mathcal{S}} \to S^2\mathfrak{t}^*_{\mathcal{S}} \subseteq \mathfrak{t}^*_{\mathcal{S}} \otimes \mathfrak{t}^*_{\mathcal{S}}$ is the coproduct dual to componentwise multiplication in $\mathfrak{t}_{\mathcal{S}}$; thus if we write $v = \sum_{s\in\mathcal{S}} v_s e_s$ and $w = \sum_{s\in\mathcal{S}} w_s e_s$ then $\mathbf{H}_z(v, w) = \sum_{s\in\mathcal{S}} 2\sigma_s(z) v_s w_s$, which is positive definite for $z \in \mathbb{C}^\times_{\mathcal{S}}$. Note the following crucial property of the flat Kähler metric on $\mathbb{C}_{\mathcal{S}}$: $d^c\sigma_s(K_v) = 2\sigma_s v_s$ for all $v \in \mathfrak{t}_{\mathcal{S}}$, i.e.,

$$d^c\boldsymbol{\sigma}(K_v) = \mathbf{H}(v) = 2(\delta\boldsymbol{\sigma})(v), \tag{13}$$

where we use the natural inclusion $S^2\mathfrak{t}^*_{\mathcal{S}} \subseteq \mathrm{Hom}(\mathfrak{t}_{\mathcal{S}}, \mathfrak{t}^*_{\mathcal{S}})$ to evaluate $\mathbf{H} = 2\delta\boldsymbol{\sigma}$ on v.

We now restrict attention to Levi–Kähler quotients in the following setting.

Definition 11. A *toric CR submanifold* of $\mathbb{C}_{\mathcal{S}}$ is a compact connected CR submanifold $(N, \mathcal{D}, J)$ which is invariant and locally Reeb type under the action of $\mathbb{T}_{\mathcal{S}}$.

We assume that for any $S \subseteq \mathcal{S}$, the intersection of N with the $\mathbb{T}^{\mathbb{C}}_{\mathcal{S}}$ orbit $\mathbb{C}_{\mathcal{S},S}$ is connected; these intersections are then the orbit strata, and the combinatorics Φ_N of N may be identified with the poset of those $S \in P(\mathcal{S})$ such that $N \cap \mathbb{C}_{\mathcal{S},S}$ is nonempty. We also assume that for all $s \in \mathcal{S}$, $\{s\} \in \Phi_N$, i.e., $N_s := N \cap \mathbb{C}_{\mathcal{S},\{s\}} = \{z \in N \mid z_s = 0\}$ is nonempty, with generic stabilizer $<\exp(te_s)>$ (if this did not hold, the $<\exp(te_s)>$ circle action on N would be free, and we could take a quotient).

We refer to a Levi–Kähler quotient M of a toric codimension ℓ CR submanifold $(N, \mathcal{D}, J)$ in $\mathbb{C}_{\mathcal{S}}$ by a positive Levi pair $(\mathfrak{g}, \lambda)$, where $\mathfrak{g}$ is the Lie algebra of an ℓ-dimensional subtorus $G \subseteq \mathbb{T}_{\mathcal{S}}$, as a (codimension ℓ) *Levi–Kähler reduction* of $\mathbb{C}_{\mathcal{S}}$.

The data $(N, \mathcal{D}, J)$ and $(\mathfrak{g}, \lambda)$ are linked by Condition 1, which may be viewed as a constraint on $(N, \mathcal{D}, J)$ given $(\mathfrak{g}, \lambda)$ or vice versa. We specify the choice of $(\mathfrak{g}, \lambda)$ as in §1.3 via a surjective linear map $\mathbf{L}\colon \mathfrak{t}_{\mathcal{S}} \to \mathfrak{h}$, or equivalently, an indexed family $L_s : s \in \mathcal{S}$ of vectors in $\mathfrak{h}$ which span (where $L_s = \mathbf{L}(e_s)$). In other words, for toric CR submanifolds $(N, \mathcal{D}, J)$ of $\mathbb{C}_{\mathcal{S}}$, a pair $(\mathfrak{g}, \lambda)$ is associated canonically, via the set-up (10), with a (not necessarily compact or nonempty) convex polytope

$$\Delta_{\mathfrak{g},\lambda} = \{\xi \in \mathcal{A} \mid \forall\, s \in \mathcal{S},\;\; L_s(\xi) \geqslant 0\}$$

labelled (formally) by $L_s : s \in \mathcal{S}$ (although some facets F_s could be empty a priori). We denote the combinatorics of $\Delta_{\mathfrak{g},\lambda}$ by $\Phi_{\mathfrak{g},\lambda}$.

Lemma 4. *Let N be a toric CR submanifold of $\mathbb{C}_{\mathcal{S}}$ satisfying Condition* 1 *relative to* $(\mathfrak{g}, \lambda)$*. Then there is a smooth pointwise surjective function $\chi_{N,\mathfrak{g}}\colon N \to \mathrm{Hom}(\mathfrak{t}^*_{\mathcal{S}}, \mathfrak{g})$ such that for all $z \in N$, $\eta_z = \chi_{N,\mathfrak{g}}(z) \circ d^c\boldsymbol{\sigma}_z$ and $d\eta_z|_{\mathcal{D}_z} = \chi_{N,\mathfrak{g}}(z) \circ dd^c\boldsymbol{\sigma}_z|_{\mathcal{D}_z}$.*

Proof. The CR submanifold N may be written (at least locally, and in our examples globally) $N = (F \circ \boldsymbol{\sigma})^{-1}(0)$ where $F\colon \mathfrak{t}_{\mathcal{S}}^* \to W$ is a smooth function with values in an ℓ-dimensional vector space W, for which 0 is a regular value. Hence $T_z N = \ker dF_{\boldsymbol{\sigma}(z)} \circ d\boldsymbol{\sigma}_z$ and so $\mathcal{D}$ is the kernel of the pullback $\boldsymbol{\nu}$ of $dF \circ d^c\boldsymbol{\sigma}$ to N, with $\boldsymbol{\nu}_z(K_v) = dF_{\boldsymbol{\sigma}(z)}(\mathbf{H}_z(v))$ for $v \in \mathfrak{t}_{\mathcal{S}}$ and $z \in N$. By Condition 1, $dF \circ \mathbf{H} \circ \iota\colon N \to \mathrm{Hom}(\mathfrak{g}, W)$ is a pointwise isomorphism, and $\eta = (dF \circ \mathbf{H} \circ \iota)^{-1}\boldsymbol{\nu}$.

We may now set $\chi_{N,\mathfrak{g}} = (dF \circ \mathbf{H} \circ \iota)^{-1} \circ dF$; this formula may only be valid locally, but the result is independent of the choice of $F\colon \mathfrak{t}_{\mathcal{S}}^* \to W$, so $\chi_{N,\mathfrak{g}}$, with $\eta = \chi_{N,\mathfrak{g}} \circ d^c\boldsymbol{\sigma}$ is globally defined. Since $\boldsymbol{\nu}|_{\mathcal{D}} = 0$, $d\eta|_{\mathcal{D}} = (dF \circ \mathbf{H} \circ \iota)^{-1} d\boldsymbol{\nu}|_{\mathcal{D}}$. Now

$$d(dF \circ d^c\boldsymbol{\sigma}) = (\mathrm{Hess}\, F)(d\boldsymbol{\sigma} \wedge d^c\boldsymbol{\sigma}) + dF \circ dd^c\boldsymbol{\sigma}.$$

Pulling back to N and restricting to $\mathcal{D}$, the first term vanishes (since $F \circ \boldsymbol{\sigma}$ is constant on N). Hence $d\eta|_{\mathcal{D}} = \chi_{N,\mathfrak{g}} \circ dd^c\boldsymbol{\sigma}|_{\mathcal{D}}$. $\square$

The *characteristic function* of $(N, \mathfrak{g}, \lambda)$ is the (nowhere vanishing) function $\chi = \chi_{N,\mathfrak{g},\lambda}\colon N \to \mathfrak{t}_{\mathcal{S}}$ with $\chi_{N,\mathfrak{g},\lambda}(z) = \lambda \circ \chi_{N,\mathfrak{g}}(z)$. Hence $\eta^\lambda := \langle \eta, \lambda\rangle = \langle d^c\boldsymbol{\sigma}, \chi_{N,\mathfrak{g},\lambda}\rangle$ and the horizontal momentum map $\mu^\lambda\colon N \to \mathcal{A} \subseteq \mathfrak{h}^*$ satisfies:

$$\begin{aligned} \langle \mu^\lambda(z), \mathbf{L}(v)\rangle = \eta_z^\lambda(K_v) &= \mathbf{H}_z(v, \chi_{N,\mathfrak{g},\lambda}(z)) \\ &= 2\langle \delta\boldsymbol{\sigma}(z), v \otimes \chi_{N,\mathfrak{g},\lambda}(z)\rangle. \end{aligned} \tag{14}$$

Thus $\eta^\lambda = \sum_{s\in\mathcal{S}} \chi_s\, d^c\sigma_s$ and $\langle \mu^\lambda, \mathbf{L}(v)\rangle = \sum_{s\in\mathcal{S}} 2\sigma_s\chi_s v_s$, i.e., $L_s(\mu^\lambda) = 2\sigma_s\chi_s$. Since $dd^c\sigma_s = 2d\sigma_s \wedge d\boldsymbol{\vartheta}_s$, the induced metric on $\mathcal{D}$ (over $\mathbb{C}_{\mathcal{S}}^\times \cap N$) is

$$\begin{aligned} h_{\mathcal{D},\lambda} &= \sum_{s\in\mathcal{S}} \frac{L_s(\mu^\lambda)}{\sigma_s} \left(\frac{d\sigma_s{}^2}{2\sigma_s} + 2\sigma_s d\boldsymbol{\vartheta}_s{}^2\right)\bigg|_{\mathcal{D}} \\ &= \sum_{s\in\mathcal{S}} 2\chi_s \left(\frac{d\sigma_s{}^2}{2\sigma_s} + 2\sigma_s d\boldsymbol{\vartheta}_s{}^2\right)\bigg|_{\mathcal{D}}. \end{aligned} \tag{15}$$

Theorem 1 shows that if $(\mathfrak{g}, \lambda)$ is a nondegenerate Levi pair, then the image of the horizontal momentum map μ^λ is the compact simple convex polytope Δ, defined by $\pm L_s : s \in \mathcal{S}$ for some choice of signs, and also that Δ has the same combinatorial type as N. Now if $\Delta = \Delta_{\mathfrak{g},\lambda}$ (i.e., all signs are positive) then equation (15) shows that $(\mathfrak{g}, \lambda)$ is a *positive* Levi pair. This motivates the introduction of the following constraint.

Condition 2. $\Delta_{\mathfrak{g},\lambda}$ is a compact convex polytope with the same combinatorial type as N (as a subset of $P(\mathcal{S})$).

Theorem 2. *Let N be a toric submanifold of $\mathbb{C}_{\mathcal{S}}$ and suppose $(\mathfrak{g},\lambda)$ is a Levi pair. Then $\operatorname{im}\mu^{\lambda}=\Delta_{\mathfrak{g},\lambda}$ if and only if $(\mathfrak{g},\lambda)$ is a positive Levi pair satisfying Condition 2.*

Proof. If $\operatorname{im}\mu^{\lambda}=\Delta_{\mathfrak{g},\lambda}$, then (15), applied to each orbit stratum, shows that $h_{\mathcal{D},\lambda}$ is positive definite over the interior of each face of $\Delta_{\mathfrak{g},\lambda}$, hence everywhere. Thus $(\mathfrak{g},\lambda)$ is a positive Levi pair. Under this assumption, Theorem 1 shows that $\operatorname{im}\mu^{\lambda}=\Delta_{\mathfrak{g},\lambda}$ if and only if Condition 2 holds. □

2.3. Levi–Kähler reduction for quadrics

We now specialize to the case that N is an intersection of quadrics. For N to be a toric CR manifold with codimension ℓ, it is then the level set of an ℓ-dimensional family of components of $\boldsymbol{\sigma}\colon \mathbb{C}_{\mathcal{S}}\to\mathfrak{t}_{\mathcal{S}}^*$, hence of the form $(F\circ\boldsymbol{\sigma})^{-1}(0)$, with $F=\iota_o^{\top}-\lambda_o\colon \mathfrak{t}_{\mathcal{S}}^*\to\mathfrak{g}_o^*$, where $\iota_o\colon\mathfrak{g}_o\hookrightarrow\mathfrak{t}_{\mathcal{S}}$ is an inclusion of an ℓ-dimensional subspace, and $\lambda_o\in\mathfrak{g}_o^*$ is in the image of the positive quadrant of $\mathfrak{t}_{\mathcal{S}}^*$.

Thus $N=\mu_o^{-1}(\lambda_o)$, where $\mu_o=\iota_o^{\top}\boldsymbol{\sigma}$, is defined by the same sort of data $(\mathfrak{g}_o,\lambda_o)$ as the data $(\mathfrak{g},\lambda)$ which determines the Levi–Kähler structure on N. These data may therefore be fixed in the same way as $(\mathfrak{g},\lambda)$ using a diagram of linear maps

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{g}_o \arrow[r, "\iota_o"] \arrow[d, "\lambda_o"'] & \mathfrak{t}_{\mathcal{S}} \arrow[r, "\mathbf{u}^o"] \arrow[d, "\mathbf{L}^o"'] & \mathfrak{t} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{R} \arrow[r, "\varepsilon"'] & \mathfrak{h} \arrow[r, "d"'] & \mathfrak{t} \arrow[r] & 0. \end{tikzcd}$$

We write $N_{\mathfrak{g}_o,\lambda_o}$ or $N_{\mathbf{L}^o}$ for the CR submanifold corresponding to these data. We shall assume that $\Delta_{\mathfrak{g}_o,\lambda_o}$ is a compact convex polytope, so that it satisfies Condition 2: the image of $\boldsymbol{\sigma}\colon N\to\mathfrak{t}_{\mathcal{S}}^*$ thus lies in the nonnegative quadrant of

$$\mathfrak{t}_{\mathcal{S}}^o := \{\xi\in\mathfrak{t}_{\mathcal{S}}^* \mid \forall\, v\in\mathfrak{g}_o,\ \ \xi_s v_s=0\ \forall\, s\in\mathcal{S}\Rightarrow v=0\},$$

and $u_s^o : s\in\mathcal{S}$ are the normals of a complete fan.

Since F is affine linear, dF is constant, equal to $\iota_o^{\top}$, and so $\boldsymbol{\nu}_z(K_v)=\iota_o^{\top}(\mathbf{H}_z(v))$. Hence $\mathcal{E}_z=\operatorname{im}(\mathbf{H}_z\circ\iota_o\colon\mathfrak{g}_o\to\mathfrak{t}_{\mathcal{S}}^*)$ and so $\mathcal{E}=\boldsymbol{\sigma}^*\mathcal{E}^o$ where $\mathcal{E}^o\to\mathfrak{t}_{\mathcal{S}}^o$ has fibre $\mathcal{E}_\xi^o=\{(\xi_s v_s)_{s\in\mathcal{S}}\in\mathfrak{t}_{\mathcal{S}}^*\mid v\in\mathfrak{g}_o\}$.

Proposition 4. *On $N_{\mathfrak{g}_0,\lambda_0}$, $(\mathfrak{g}_o,\lambda_o)$ is a positive Levi pair.*

Proof. Since $u_s^o : s \in \mathcal{S}$ are the normals of a complete fan, $\{\alpha \in \mathfrak{t}^* \mid \forall s \in \mathcal{S},\ \alpha(u_s^o) \geqslant 0\} = \{0\}$. Hence $(\mathbf{u}^o)^\top(\mathfrak{t}^*)$ meets the positive quadrant of $\mathfrak{t}_\mathcal{S}^*$ only at 0, so the image in $\mathfrak{g}_o^*$ of this positive quadrant is a strictly convex cone $\mathcal{C}$ whose dual cone $\mathcal{C}_*$ is the intersection of $\mathfrak{g}_o$ with the inverse image (under ι_o) of the positive quadrant in $\mathfrak{t}_\mathcal{S}$. Since $\mathbf{H}_z$ is diagonal and positive definite, it maps the positive quadrant of $\mathfrak{t}_\mathcal{S}$ onto the positive quadrant of $\mathfrak{t}_\mathcal{S}^*$. Thus $\iota_o^\top \circ \mathbf{H}_z \circ \iota_o$ maps $\mathcal{C}_*$ onto $\mathcal{C}$. Since $\lambda_o \in \mathcal{C}$, $\chi^o(z) := \iota_o(\iota_o^\top \circ \mathbf{H}_z \circ \iota_o)^{-1}(\lambda_o)$ has positive components, and hence $h_{\mathcal{D},\lambda_o}$ is positive definite. □

Thus we can satisfy Condition 1 by letting $(\mathfrak{g}, \lambda)$ equal $(\mathfrak{g}_o, \lambda_o)$.

If the fan associated to $(\mathfrak{g}_o, \lambda_o)$ is rational, (M, J) is the underlying complex orbifold of the Delzant–Guillemin Kähler quotient of $\mathbb{C}_\mathcal{S}$ by $(\mathfrak{g}_o, \lambda_o)$. Its Kähler form belongs to the same Kähler class as the Levi–Kähler quotient, but will not be the same in general.

Remark 6. By continuity, we also obtain a positive definite metric for $(\mathfrak{g}, \lambda)$ in an open neighbourhood of $(\mathfrak{g}_o, \lambda_o)$. In particular, we can fix $\mathfrak{g} = \mathfrak{g}_o$ and vary λ to obtain an ℓ-dimensional family of Levi–Kähler quotients on the same complex orbifold. As $H^2_{dR}(M)$ is ℓ-dimensional, it is natural to ask if all Kähler classes are obtained in this way.

The characteristic function $\chi = \chi_{N,\mathfrak{g},\lambda}$ of $N = N_{\mathfrak{g}_o,\lambda_o}$ with respect to an arbitrary Levi pair $(\mathfrak{g}, \lambda)$ is given by $\chi(z) = (\iota^\top \circ \mathbf{H}_z \circ \iota_o)^{-1}(\lambda) \in \mathfrak{g}_o$, where we tacitly omit the inclusion $\iota_o \colon \mathfrak{g}_o \subseteq \mathfrak{t}_\mathcal{S}$.

Remark 7. The function $\chi \colon N \to \mathfrak{g}_o$ is determined by $\mathbf{H}(\chi)|_\mathfrak{g} = \lambda$; since $\mathbf{H} = 2\delta\boldsymbol{\sigma}$, it is linear in $\boldsymbol{\sigma}$, which implies χ is a rational function of $\boldsymbol{\sigma}$. Now for any $v \in \mathfrak{g}_o$ and $z \in N$, $\frac{1}{2}\mathbf{H}(\sum_{s\in\mathcal{S}} e_s) = \sum_{s\in\mathcal{S}} \langle \sigma_s(z), v\rangle = \lambda_o(v)$, and so the characteristic function χ^o of the canonical Levi pair $(\mathfrak{g}_o, \lambda_o)$ is characterized by $\iota_o^\top \mathbf{H}_z\big(\chi^o(z) - \frac{1}{2}\sum_{s\in\mathcal{S}} e_s\big) = 0$.

Proposition 5. *If $\sum_{s\in\mathcal{S}} u_s^o = 0$ then $2\chi_s^o = 1$ for all $s \in \mathcal{S}$.*

Proof. Since $\sum_{s\in\mathcal{S}} e_s \in \mathfrak{g}_o$, for all $z \in N$, $\chi^o(z) = \frac{1}{2}\sum_{s\in\mathcal{S}} e_s$ by the characterization. □

If this assumption holds, we say N is *spherical*: N is then contained in a hypersphere in $\mathbb{C}_\mathcal{S}$. (The equivariant topology of such manifolds has been studied in [18], but here we focus on the geometry of their Levi–Kähler quotients.) In the spherical case, $\sigma_s = \langle \mu^{\lambda_o}, L_s^o\rangle$, and the canonical Levi–Kähler quotient metric agrees with the Delzant–Guillemin Kähler quotient. In particular, the reduced metric on $\mathcal{A} \subseteq \mathfrak{h}^*$ is $\sum_{s\in\mathcal{S}} (dL_s^o)^2/2L_s^o$, which is the pullback by $(\mathbf{L}^o)^\top$ of the metric $h_o = \sum_{s\in\mathcal{S}} d\zeta_s{}^2/2\zeta_s$ on $\mathfrak{t}_{\mathcal{S}}^*$, where we write ζ_s for the linear function $\zeta_s(\xi) = \xi_s$ on $\mathfrak{t}_{\mathcal{S}}^*$ corresponding to $e_s \in \mathfrak{t}_{\mathcal{S}}$ (thus $d\zeta_s$ is e_s, viewed as a constant 1-form).

In order to compute the reduced metric for any Levi pair $(\mathfrak{g}, \lambda)$, not just $(\mathfrak{g}_o, \lambda_o)$ we observe that the bundle $\mathcal{E}^o \subseteq T\mathfrak{t}_{\mathcal{S}}^o = \mathfrak{t}_{\mathcal{S}}^o \times \mathfrak{t}_{\mathcal{S}}^*$ (with $\mathcal{E} = \boldsymbol{\sigma}^*\mathcal{E}^o$) is the orthogonal complement to $\mathfrak{t}_{\mathcal{S}}^o \times (\mathbf{u}^o)^\top(\mathfrak{t}^*)$ with respect to h_o:

$$\sum_{s\in\mathcal{S}} \frac{d\zeta_s((\xi_s v_s)_{s\in\mathcal{S}})d\zeta_s(w\circ\mathbf{u}^o)}{\zeta_s(\xi)} = \sum_{s\in\mathcal{S}} v_s(w\circ\mathbf{u}^o)_s = w(\mathbf{u}^o(v)) = 0.$$

Hence by Remark 3, we have the following result.

Theorem 3. *Let $(\mathfrak{g}, \lambda)$ be a positive Levi pair on a spherical quadric $N = N_{\mathfrak{g}_o,\lambda_o}$. Then the reduced metric of the Levi–Kähler structure is the pullback by $\mathbf{L}^\top : \Delta_{\mathfrak{g},\lambda} \to \mathfrak{t}_{\mathcal{S}}^o$ of the restriction of h_o to $\mathfrak{t}_{\mathcal{S}}^o \times (\mathbf{u}^o)^\top(\mathfrak{t}^*) \subseteq T\mathfrak{t}_{\mathcal{S}}^o$ (extended by zero on $\mathcal{E}^o \subseteq T\mathfrak{t}_{\mathcal{S}}^o$).*

Example 5. The weighted projective space $\mathbb{C}P^m_{\mathbf{a}}$ of weight $\mathbf{a} = (a_0, \ldots, a_m) \in \mathbb{N}^{m+1}$ has the structure of a toric symplectic orbifold whose Delzant polytope is a labelled simplex (Δ, u). The corresponding momentum level set $N_\lambda \subseteq \mathbb{C}^{m+1}$ is CR G-equivariantly isometric to the sphere $\mathbb{S}^{2m+1} \subseteq \mathbb{C}^{m+1}$, acted by the $\mathbf{a}$-weighted diagonal $\mathbb{S}^1$ action. By a result of S. Webster [48], Levi–Kähler reduction defines on $\mathbb{C}P^m_{\mathbf{a}}$ a homothety class of *Bochner-flat* Kähler metrics [22, 28], which are *extremal* Kähler metrics [24]. The Bochner-flat metric coincides with the Guillemin symplectic-Kähler reduction if and only if $\mathbf{a} = (1, \ldots, 1)$, i.e., only on $\mathbb{C}P^m$, see e.g., [2]. Thus one can obtain Levi–Kähler quotients of the same (flat) CR structure on $\mathbb{S}^{2m+1} \subseteq \mathbb{C}^{m+1}$ on any labelled rational simplex, by varying the subgroup $G \cong \mathbb{S}^1$ within a fixed maximal torus $\mathbb{T}^{m+1}$ in the group $\mathrm{Aut}_{\mathrm{CR}}(\mathbb{S}^{2m+1}) = \mathrm{PU}(m+1, 1)$ of CR transformations of $\mathbb{S}^{2m+1}$.

Locally, the construction is defined by a one-dimensional subspace $\mathfrak{g} \subseteq \mathfrak{t}_{\mathcal{S}}$, generated by a non-zero element $v \in \mathfrak{t}_{\mathcal{S}}$, with corresponding vector field K_v transverse to the CR distribution on $\mathbb{S}^{2m+1}$, and the choice of a contact form η^v with $\eta^v(K_v) = 1$ and $\ker \eta^v = \mathcal{D}$. In this case $(K_v, \eta_v, \mathcal{D}, J)$ defines a *Sasaki structure* compatible with the standard CR structure $(\mathcal{D}, J)$ on $\mathbb{S}^{2m+1}$, see e.g. [20]. The horizontal Kähler geometry $(d\eta^v, \mathcal{D}, J)$ may be described by a compatible toric metric over a (perhaps not rational) labelled simplex (see [40]). The fact that $\mathfrak{g}$ is the Lie algebra of a subgroup $G \leqslant \mathbb{T}^{m+1}$ implies a rationality condition on $\mathfrak{g}$, hence on the corresponding labelled simplex.

3. Levi–Kähler reduction for products of spheres

Our main motivation for the study of toric Levi–Kähler quotients is the construction of Kähler metrics on toric varieties with "nice" curvature properties. For this we observe that CR submanifolds of $\mathbb{C}_{\mathcal{S}}$ have local invariants, and so one approach to constructing Levi–Kähler quotients with nice curvature is to start from a nice CR submanifold N of $\mathbb{C}_{\mathcal{S}}$. In particular, when N is a product of spheres, it is flat as a CR manifold. Hence we might hope that Levi–Kähler quotients of products of spheres have interesting curvature properties.

3.1. Products of simplices and products of spheres

We specialize the set-up of §2 as follows. Fix positive integers ℓ and $m_1, m_2, \ldots, m_\ell$, and let

$$\mathcal{I} = \{1, 2, \ldots, \ell\}, \quad I_i = \{0, 1, \ldots, m_i\}, \quad \mathcal{S} = \{(i, r) \mid i \in \mathcal{I} \text{ and } r \in I_i\}.$$

Let $m = \sum_{i\in\mathcal{I}} m_i$ and $d = m + \ell$ as usual. Thus $\mathbb{C}_{\mathcal{S}} \cong \mathbb{C}^{m_1+1} \times \mathbb{C}^{m_2+1} \times \cdots \times \mathbb{C}^{m_\ell+1} \cong \mathbb{C}^d$ and $\mathfrak{t}_{\mathcal{S}}$ has a natural subspace $\mathfrak{g}_o = \{x \in \mathfrak{t}_{\mathcal{S}} \mid x_{iq} = x_{ir}$ for all $i \in \mathcal{I}$ and $q, r \in I_i\}$. We denote by x_i the common value of the x_{ir} and thus identify $\mathfrak{g}_o$ with $\mathbb{R}^\ell$. On $\mathfrak{g}_o$ we have a natural linear form λ_o sending $(x_1, x_2, \ldots, x_\ell)$ to $x_1 + x_2 + \cdots + x_\ell \in \mathbb{R}$, and we let $\mathbf{L}^o\colon \mathfrak{t}_{\mathcal{S}} \to \mathfrak{h} = \mathfrak{t}_{\mathcal{S}}/\ker\lambda_o$ and $\mathbf{u}^o\colon \mathfrak{t}_{\mathcal{S}} \to \mathfrak{t} = \mathfrak{t}_{\mathcal{S}}/\mathfrak{g}_o$ be the quotient maps.

Under the canonical identification of $\mathfrak{t}_{\mathcal{S}}^*$ with $\mathfrak{t}_{\mathcal{S}}$, $\mathfrak{h}^*$ is isomorphic to the subspace of $\xi = (\xi_s)_{s\in\mathcal{S}}$ such that $\sum_{r\in I_i} \xi_{ir}$ is independent of i, this constant being the natural projection $\mathfrak{h}^* \to \mathbb{R}$. Hence $\mathfrak{t}^*$ is a product (over $i \in \mathcal{I}$) of the codimension one linear subspaces of $\mathbb{R}^{m_i+1}$ where the coordinate sum is zero, and $\mathcal{A}$ is the corresponding product of affine subspaces $\mathcal{A}_i$ where the coordinate sum is one.

Notation 1 (The faces of Σ). The polytope Σ in $\mathfrak{h}^*$ defined by $\mathbf{L}^o$ is a product of simplices Σ_i in the affine spaces $\mathcal{A}_i$. In the following, we sometimes write $i(s)$ and $r(s)$ for the components of $s \in \mathcal{S}$, i.e., $s = (i(s), r(s))$.

- The facets of Σ_i are $F_r^i = \{\xi_{ir} = 0\} \cap \Sigma_i$ for $r \in I_i$.
- The vertices of Σ_i may also be indexed by $r \in I_i$: we let p_r^i be the unique vertex of Σ_i that is not in F_r^i.

- The vertices of Σ are thus indexed by $(r_1, \ldots, r_\ell) \in I_1 \times \cdots \times I_\ell$:

$$p_{(r_1,\ldots,r_\ell)} = (p^1_{r_1}, \ldots, p^\ell_{r_\ell}).$$

- Each facet F^i_r (for $r \in I_i$) of the simplex Σ_i determines a facet

$$F_{ir} = \Sigma_1 \times \cdots \times \Sigma_{i-1} \times F^i_r \times \Sigma_{i+1} \times \cdots \times \Sigma_\ell$$

of Σ and a corresponding inward normal $u^o_{ir} = dL^o_{ir}$.

The corresponding CR submanifold of $\mathbb{C}_{\mathcal{S}}$ is

$$N = N_{\mathbf{L}^o} = \{z \in \mathbb{C}_{\mathcal{S}} \mid \textstyle\sum_{r\in I_i} \sigma_{ir} = 1 \text{ for all } i \in \mathcal{I}\},$$

where $\sigma_{ir} = \frac{1}{2}|z_{ir}|^2$. Thus $N \cong \mathbb{S}^{2m_1+1} \times \cdots \times \mathbb{S}^{2m_\ell+1}$. As in §2.3, N is the level set, at the regular value λ_o, of the momentum map $\mu_o = \iota_o^\top \boldsymbol{\sigma}$. Thus

$$\mu_o(z) = (\boldsymbol{\sigma}^1(z), \ldots, \boldsymbol{\sigma}^\ell(z)), \quad \text{where} \quad \boldsymbol{\sigma}^i(z) = \sum_{r\in I_i} \sigma_{ir}(z) = \sum_{r\in\ I_i} \tfrac{1}{2}|z_{ir}|^2,$$

and we denote $z = (z^1, \ldots, z^\ell)$ with $z^i = (z_{i0}, \ldots, z_{im_i})$ the linear coordinates of $\mathbb{C}_{\mathcal{S}}$. These data are associated to the Delzant construction for the product $\Sigma = \Sigma_1 \times \cdots \times \Sigma_\ell \subseteq \mathcal{A}$ of standard Delzant simplices $\Sigma_i \subseteq \mathcal{A}_i$. More specifically, $\mathfrak{g}_o$ is the Lie algebra of a subtorus G_o of $\mathbb{T}_{\mathcal{S}}$, which acts freely on N preserving the CR structure $(\mathcal{D}, J)$, with quotient space $(M, J) = \mathbb{C}P^{m_1} \times \cdots \times \mathbb{C}P^{m_\ell}$. The Lie algebra $\mathfrak{g}_o$ of G_o defines a Reeb foliation on $N \subseteq \mathbb{C}_{\mathcal{S}}$ with induced horizontal Levi structure consisting of scales of product of Fubini–Study metrics.

Theorem 4. *Let $N = \mathbb{S}^{2m_1+1} \times \cdots \times \mathbb{S}^{2m_\ell+1} \subseteq \mathbb{C}_{\mathcal{S}}$ be a product of standard CR spheres. Then for a pair $(\mathfrak{g}, \lambda)$, defined by $\mathbf{L}\colon \mathfrak{t}_{\mathcal{S}} \to \mathfrak{h}$, with associated polytope $\Delta_{\mathfrak{g},\lambda} \subseteq \mathcal{A}$, the following are equivalent.*

(i) *$(\mathfrak{g}, \lambda)$ is a positive Levi pair, i.e., defines a Levi–Kähler structure on N.*

(ii) *$(\mathfrak{g}, \lambda)$ is a Levi pair (i.e., $\mathfrak{g}$ satisfies Condition* 1*) whose horizontal momentum map $\mu^\lambda\colon N \to \mathcal{A}$ has $\operatorname{im}\mu^\lambda = \Delta_{\mathfrak{g},\lambda}$.*

(iii) *$(\mathfrak{g}, \lambda)$ satisfies Condition* 2*, i.e., $\Delta_{\mathfrak{g},\lambda}$ is a compact convex polytope with the same combinatorial type as Σ.*

The proof makes use a couple of Lemmas.

Lemma 5. *If* $(\mathfrak{g}, \lambda)$ *satisfies Condition* 2*, then it satisfies Condition* 1*.*

Proof. Condition 1(i) holds since it only depends on the combinatorics of Δ.

Suppose that $\mathfrak{g}$ does not satisfy Condition 1(ii). Then there exist $z \in N$ and $v = (x_s) \in \mathfrak{t}_{\mathcal{S}} \setminus 0$ such that $K_v(z) \in \mathcal{D} = \bigcap_{i\in\mathcal{I}} \ker d^c\boldsymbol{\sigma}^i$ and $v \in \mathfrak{g}$, that is

$$\sum_{s\in\mathcal{S}} x_s u_s = 0 \in \mathfrak{t} \quad \text{and} \quad \sum_{r\in I_i} x_{ir}\sigma_{ir} = 0 \qquad \text{for } i \in \mathcal{I}, \tag{16}$$

where $\sigma_s = \frac{1}{2}|z_s|^2$. As equations on $v = (x_s)$ for fixed z, this system is a linear map from $\mathfrak{t}_{\mathcal{S}}$ to $\mathbb{R}^\ell \oplus \mathfrak{t}$, which both have dimension d. We may write the $d \times d$-matrix $A = A_z$ of this linear map as follows: for $j \in \mathcal{I}$ the j-th row is $\sigma_s\delta_{i(s)j}$, while the lower part B of the matrix is the $m \times d$-matrix, whose s-th column is u_s for $s \in \mathcal{S}$ (written with respect to some basis of $\mathfrak{t}$). We compute the determinant of $A = A_z$ by expanding along the first ℓ rows. The nonzero terms are all obtained by choosing, for each $j \in \mathcal{I}$, $r_j \in I_j$ to obtain a minor

$$\pm\sigma_{1r_1}\sigma_{2r_2}\cdots\sigma_{\ell r_\ell} \det B_{(r_1,\ldots,r_\ell)},$$

where $B_{(r_1,\ldots,r_\ell)}$ is the submatrix of B obtained by removing the columns $u_{1r_1}, \ldots, u_{\ell r_\ell}$. Up to an overall sign (depending on $|I_j|$) each such minor contributes to $\det A$ with sign $(-1)^{\sum_{j\in\mathcal{I}} r_j}$. Hence to show $\det A \neq 0$, it suffices to show that (for a fixed basis of $\mathfrak{t}$) $(-1)^{\sum_{j\in\mathcal{I}} r_j} \det B_{(r_1,\ldots,r_\ell)} > 0$ for all $(r_1, \ldots, r_\ell) \in I_1 \times \cdots \times I_\ell$, because $\sigma_s \geqslant 0$ and the products $\prod_{j\in\mathcal{I}} \sigma_{jr_j}$ do not all vanish at the same time.

Since Δ has the same combibatorial type as Σ, we know that the coloms of $B_{(r_1,\ldots,r_\ell)}$ are inward normals of the facets meeting at the vertex $p_{(r_1,\ldots,r_\ell)}$, see Notation 1. These form a basis by the Delzant condition on Δ, and so it suffices to show that $(-1)^{\sum_{j=1}^{l} r_j}$ times the wedge product of the columns of $B_{(r_1,\ldots,r_\ell)}$ has sign independent of $(r_1, \ldots, r_\ell)$. This will hold for Δ if it holds for Σ, so it suffices to check that for each $j \in \mathcal{I}$, $(-1)^{r_j} u^o_{j0} \wedge \cdots \hat{u}^o_{jr_j} \wedge \cdots u^o_{jm_j}$ (with the $u^o_{jr_j}$ factor omitted) is independent of $r_j \in I_j$. Since $\sum_{r\in I_j} u^o_{jr} = 0$, this is a triviality. $\square$

Lemma 6. *Suppose* $(\mathfrak{g}, \lambda)$ *satisfies Condition* 1 *and let* $\chi\colon N \to \mathfrak{g}_o \cong \mathbb{R}^\ell$ *be the characteristic function of* $(N, \mathfrak{g}, \lambda)$*. Then* $(\mathfrak{g}, \lambda)$ *is a positive Levi pair if and only if* $\chi_1, \ldots, \chi_\ell$ *are positive.*

Proof. First observe that since $\mathcal{D} = \bigcap_{i\in\mathcal{I}} \ker d^c\boldsymbol{\sigma}^i$, we have

$$\eta^\lambda = \sum_{i\in\mathcal{I}} \chi_i\, d^c\boldsymbol{\sigma}^i, \quad d\eta^\lambda|_{\mathcal{D}} = \sum_{i\in\mathcal{I}} \chi_i\, dd^c\boldsymbol{\sigma}^i|_{\mathcal{D}}. \tag{17}$$

Moreover, $\mathcal{D}$ splits as a sum $\mathcal{D} = \bigoplus_{i\in\mathcal{I}} \mathcal{D}_i$ where $\mathcal{D}_i$ is tangent the i-th sphere in the product $N = \prod_{i\in\mathcal{I}} \mathbb{S}^{2m_i+1}$ (that is $\mathcal{D}_i = T\mathbb{S}^{2m_i+1} \cap JT\mathbb{S}^{2m_i+1}$). For $i \in \mathcal{I}$, $dd^c\boldsymbol{\sigma}^i$ is nondegenerate on $\mathcal{D}_i$, and if $j \neq i$, $\mathcal{D}_j \subseteq \ker dd^c\boldsymbol{\sigma}^i$. Thus $d\eta^\lambda|_{\mathcal{D}}$ defines a positive definite metric iff $\chi_i > 0$ for all $i \in \mathcal{I}$. □

Before proving the theorem we need a bit more notation. For each $j \in \mathcal{I}$, we define

$$C_j^\pm := \{\xi \in \mathcal{A} \mid \forall r \in I_j,\ \pm L_{jr}(\xi) \geqslant 0\} \quad \text{and} \quad \Delta = \bigcap_{j\in\mathcal{I}} C_j^+.$$

Note that C_j^- is potentially empty but C_j^+ is not.

Proof of Theorem 4. By Lemma 5, we may assume Condition 1 holds. Then the formula (14) for the induced momentum map μ^λ here reduces to

$$L_{ir}(\mu^\lambda(z)) = 2\chi_i(z)\sigma_{ir}(z) \tag{18}$$

where $L_{ir} = \mathbf{L}(e_{ir})$ for the standard basis e_{ir} of $\mathfrak{t}_{\mathcal{S}}$.

If $(\mathfrak{g}, \lambda)$ is a positive Levi pair, the functions χ_i are positive by Lemma 6 and then equation (18) and Theorem 1 imply that $\operatorname{im}\mu^\lambda = \Delta_{\mathfrak{g},\lambda}$; thus (i)⇒(ii).

Now (ii)⇒(i)&(iii) by Theorem 2, which also shows (i)⇒(iii).

Finally, to prove (iii)⇒(i), it suffices, by Lemma 6, to show that the functions χ_i are positive. First note the following consequences of equation (18).

(a) $\operatorname{im}\mu^\lambda$ contains all the vertices of $\Delta = \Delta_{\mathfrak{g},\lambda}$. Indeed, each $z \in N$ having only one nonzero coordinates in each spherical factor is sent to a vertex of Δ. Moreover, on the vertices of Δ, $L_{s'} \geqslant 0$ for each $s' \in \mathcal{S}$; thus equation (18) implies that $\chi_i(z) \geqslant 0$ for any $z \in N$ such that $\mu^\lambda(z)$ is a vertex of Δ.

(b) If $L_{ir}(\mu^\lambda(z)) \geqslant 0$ (resp. $L_{ir}(\mu^\lambda(z)) \leqslant 0$) then for all $q \in I_i$, $L_{iq}(\mu^\lambda(z)) \geqslant 0$ (resp. $L_{iq}(\mu^\lambda(z)) \leqslant 0$). That is $\mu^\lambda(z) \in \bigcap_{j\in\mathcal{I}}(C_j^+ \cup C_j^-)$.

Thanks to the statement (a) above, it is sufficient to prove that none of the χ_i's vanishes on N. From statement (b) we have the following inclusion

$$\operatorname{im}\mu^\lambda \subseteq \bigcap_{j\in\mathcal{I}}(C_j^+ \cup C_j^-) = \bigcup_{\mathcal{J}\subseteq\mathcal{I}} \Delta_{\mathcal{J}},$$

where $\Delta_{\mathcal{J}} = \left(\bigcap_{j\in\mathcal{J}} C_j^-\right) \cap \left(\bigcap_{j\in\mathcal{J}^c} C_j^+\right)$ with $\mathcal{J}^c := \mathcal{I}\backslash\mathcal{J}$ (so $\Delta_\varnothing = \Delta$).

Statement (a) implies that $\Delta \cap \operatorname{im}\mu^\lambda$ is not empty, but this image is connected and for $\mathcal{J}$ nonempty, $\Delta_{\mathcal{J}}$ does not meet Δ. Hence $\operatorname{im}\mu^\lambda$ is contained in Δ. However, if $\chi_i(z) = 0$ for some $i \in \mathcal{I}$ and $z \in N$, then $L_{ir}(\mu^\lambda(z)) = 0$ for all $r \in I_i$, contradicting the combinatorial type of Δ. $\square$

Corollary 2. *Any toric symplectic orbifold whose rational Delzant polytope* $(\Delta, \mathbf{L})$ *has the combinatorics of a product of simplices admits a compatible toric Kahler metric* $h_{\mathbf{L}}$ *which is a Levi–Kähler reduction of a product of spheres.*

Remark 8. It seems an interesting question to classify the *smooth* compact toric manifolds whose Delzant polytope has the combinatorics of a product of simplices. In dimension $2m = 2$ the only such example is $\mathbb{C}P^1$ whereas if $2m = 4$ the only smooth examples are $M = \mathbb{C}P^1 \times \mathbb{C}P^1$ and the Hirzebruch complex surfaces $M = \mathrm{P}(\mathcal{O} \oplus \mathcal{O}(k)) \to \mathbb{C}P^1$, $k \geq 1$. More generally, one can apply the *generalized Calabi construction* (see [9, 10] and Section 3.5 below) with the base and the fibre being compact toric manifolds with Delzant polytopes having the combinatorics of product of simplices in order to obtain, inductively, higher dimensional examples of such smooth toric manifolds. Thus, the Hirzebruch complex surfaces are obtained by the generalized Calabi constriction in which the base and the fibre are $\mathbb{C}P^1$, and the corresponding Delzant image has the combinatorics of the product of two 1-dimensional simplices (i.e. is a 2-cuboid). More generally, the *Bott towers* are higher dimensional generalizations of Hirzebruch surfaces in which the base is a Bott tower of dimension $2(m-1)$ and the fibre is $\mathbb{C}P^1$ and corresponding Delzant polytope is an m-cuboid (cf. [21]). Conversely, by a result in [45], any smooth compact toric manifold whose Delzant polytope is a cuboid is a Bott tower. The generalized Calabi construction mentioned above provides scope to extend this classification result to the larger class of toric manifolds whose Delzant images have the combinatorics of product of simplices.

We now give a closed formula for the symplectic potential of $h_{\mathbf{L}}$.

Theorem 5. *Let $N = \mathbb{S}^{2m_1+1} \times \cdots \times \mathbb{S}^{2m_\ell+1} \subseteq \mathbb{C}_{\mathcal{S}}$ be a product of standard CR spheres, and suppose that the kernel $\mathfrak{g}$ of $\mathbf{u} = d \circ \mathbf{L}$ satisfies Condition* 1. *Then*

$$G_{\mathbf{L}} = \frac{1}{2} \sum_{i \in \mathcal{I}} \sum_{r \in I_i \cup \{\infty\}} L_{ir} \log |L_{ir}| = \frac{1}{2} \sum_{i \in \mathcal{I}} \sum_{r \in I_i} L_{ir} \log \left| \frac{L_{ir}}{L_{i\infty}} \right|$$

is a symplectic potential for the Levi–Kähler metric, where $L_{ir} \in \mathfrak{h}$ is viewed as a linear function on $\mathfrak{h}^*$, hence an affine function on $\mathcal{A} \subseteq \mathfrak{h}^*$, and $L_{i\infty} = -\sum_{r \in I_i} L_{ir}$. Equivalently, the reduced metric on the image of the horizontal momentum map μ^λ is given by

$$h^{\mathrm{red}}_{\mathbf{L}} = \frac{1}{2} \sum_{i \in \mathcal{I}} \sum_{r \in I_i \cup \{\infty\}} \frac{dL_{ir}{}^2}{L_{ir}} = \frac{1}{2} \sum_{i \in \mathcal{I}} \sum_{0 \leqslant r < s \leqslant m_i} \frac{L_{ir} L_{is}}{\sum_{t=0}^{m_i} L_{it}} \left(\frac{dL_{ir}}{L_{ir}} - \frac{dL_{is}}{L_{is}} \right)^2.$$

Proof. The hessian of the stated potential $G_{\mathbf{L}}$ evaluates readily to the stated reduced metric, which we can compute in two ways. Using Theorem 3, we decompose

$$h_o := \sum_{s \in \mathcal{S}} \frac{d\zeta_s{}^2}{2\zeta_s} = \frac{1}{2} \sum_{i \in \mathcal{I}} \left(\sum_{r \in I_i} \frac{d\zeta_{ir}{}^2}{\zeta_{ir}} - \frac{(\sum_{r \in I_i} d\zeta_{ir})^2}{\sum_{r \in I_i} \zeta_{ir}} \right) + \frac{1}{2} \sum_{i \in \mathcal{I}} \frac{(\sum_{r \in I_i} d\zeta_{ir})^2}{\sum_{r \in I_i} \zeta_{ir}}$$

into components orthogonal and parallel to $\mathcal{E}^o$, and the reduced metric is the pullback of the first term by $\mathbf{L}$. Alternatively, by (15), the horizontal metric on $J(\mathcal{K} \cap \mathcal{D})$ is

$$\begin{aligned} \sum_{i \in \mathcal{I}} 2\chi_i \sum_{r \in I_i} \frac{d\sigma_{ir}{}^2}{2\sigma_{ir}} \bigg|_{\mathcal{D}} &= \boldsymbol{\sigma}^* \left[\frac{1}{2} \sum_{i \in \mathcal{I}} \rho_i \left(\sum_{r \in I_i} \frac{d\zeta_{ir}{}^2}{\zeta_{ir}} - \frac{(\sum_{r \in I_i} d\zeta_{ir})^2}{\sum_{r \in I_i} \zeta_{ir}} \right) \right] \bigg|_{\mathcal{D}} \\ &= \frac{1}{2} \sum_{i \in \mathcal{I}} \boldsymbol{\sigma}^* \left(\sum_{r \in I_i} \frac{d(\rho_i \zeta_{ir})^2}{\rho_i \zeta_{ir}} - \frac{\left(\sum_{r \in I_i} d(\rho_i \zeta_{ir})\right)^2}{\sum_{r \in I_i} \rho_i \zeta_{ir}} \right) \bigg|_{\mathcal{D}} \end{aligned}$$

where $\boldsymbol{\sigma}^* \rho_i = 2\chi_i$ and we use that $\sum_{r \in I_i} \sigma_{ir}$ is constant on N, so that

$$\boldsymbol{\sigma}^* \left(\sum_{r \in I_i} d\zeta_i \right) = 0,$$

and then exploit rescaling invariance. The result is the pullback by μ^λ of the stated reduced metric, since $L_{ir}(\mu^\lambda) = 2\chi_i \sigma_{ir}$ by (18). □

As shown by Guillemin [37], a Kähler potential may be computed as a Legendre transform of the symplectic potential $G_{\mathbf{L}}$ with respect to some basepoint $p \in \mathcal{A}$:

$$\begin{aligned} &H_{\mathbf{L}} = \langle \mu^\lambda - \mu^\lambda(p), dG_{\mathbf{L}} \rangle - G_{\mathbf{L}} \\ &= \sum_{i\in\mathcal{I}} \sum_{r\in I_i\cup\{\infty\}} \Big(\tfrac{1}{2}\big(\langle \mu^\lambda - \mu^\lambda(p), dL_{ir}\rangle - L_{ir}\big) \log|L_{ir}| + \tfrac{1}{2}\langle \mu^\lambda - \mu^\lambda(p), dL_{ir}\rangle \Big) \\ &= \sum_{i\in\mathcal{I}} \sum_{r\in I_i\cup\{\infty\}} \tfrac{1}{2} L_{ir}(p)\log|L_{ir}|, \qquad \text{since} \qquad \sum_{r\in I_i\cup\{\infty\}} L_{ir} = 0. \end{aligned}$$

3.2. Products of 3-spheres

As a special case of Theorem 4, consider an ℓ-fold product $N = \mathbb{S}^3 \times \cdots \times \mathbb{S}^3$ of 3-spheres as a codimension ℓ submanifold of $\mathbb{C}^{2\ell} \cong \mathbb{C}^\ell \otimes \mathbb{C}^2$ with momentum coordinates $\sigma_{ir}(z)$ ($i \in \mathcal{I} = \{1, \dots, \ell\}$, $r \in \{0, 1\}$). Thus $\mathfrak{t}_{\mathbb{S}} = \mathbb{R}^{2\ell}$ is the Lie algebra of $\mathbb{T}_{\mathbb{S}} = \mathbb{T}^{2\ell}$ acting diagonally on $\mathbb{C}^{2\ell}$. We study the Levi–Kähler metric on the open subset where the quotient torus $\mathbb{T}$ acts freely, ignoring rationality conditions.

3.2.1. Geometry of the Levi–Kähler metric. Consider, for any ℓ-dimensional subspace $\mathfrak{g} \subseteq \mathfrak{t}_{\mathbb{S}} = \mathbb{R}^{2\ell}$, the integrable distribution $\mathcal{K}^{\mathfrak{g}} = \operatorname{span}\{K_v \mid v \in \mathfrak{g}\}$ on N. Then, around each point of N such that $\mathcal{K}^{\mathfrak{g}}$ is transversal to $\mathcal{D}$, the local quotient space M of leaves of $\mathcal{K}^{\mathfrak{g}}$ has induced complex structure J. We will further assume that $\lambda \in \mathfrak{g}^*$ is such that $d\eta^\lambda$ induces a Kähler metric $(h_{\mathbf{L}}, J, \omega_{\mathbf{L}})$ on M.

The reduced metric of Theorem 5 specializes to

$$h^{\mathrm{red}}_{\mathbf{L}} = \frac{1}{2}\sum_{i\in\mathcal{I}} \frac{L_{i0}L_{i1}}{L_{i0}+L_{i1}}\Big(\frac{dL_{i0}}{L_{i0}} - \frac{dL_{i1}}{L_{i1}}\Big)^2, \quad \text{i.e.,} \quad (\mu^\lambda)^* h^{\mathrm{red}}_{\mathbf{L}} = \sum_{i\in\mathcal{I}} \frac{\chi_i\, d\sigma_i{}^2}{\sigma_i(1-\sigma_i)},$$

where $L_{ir}(\mu^\lambda) = 2\chi_i\sigma_{ir}$, $2\chi_i = -L_{i\infty}(\mu^\lambda)$, $\sigma_i := \sigma_{i0} = -L_{i0}(\mu^\lambda)/L_{i\infty}(\mu^\lambda)$, $L_{i0} + L_{i1} = -L_{i\infty}$ and hence $\sigma_{i1} = 1 - \sigma_i$ on N. In other words, the characteristic functions χ_i and the orthogonal coordinates σ_i are affine and birational functions (respectively) of the momentum coordinates μ^λ. Thus the momentum images of the coordinate hypersurfaces with σ_i constant are hyperplanes in $\mathcal{A}$ through the codimension two affine subspace where $L_{i0}(\mu^\lambda) = L_{i1}(\mu^\lambda) = 0$, and σ_i is inverse to the unique affine coordinate on this pencil of hyperplanes sending 0, 1 and ∞ to the facets $L_{i0}(\mu^\lambda) = 0$, $L_{i1}(\mu^\lambda) = 0$ (of $\Delta_{\mathfrak{g},\lambda}$) and the "characteristic hyperplane" $\chi_i = -\frac{1}{2}L_{i\infty}(\mu^\lambda) = 0$ (respectively). These pencils introduce a factorization structure (in the sense of [6]) which is adapted to the class of polytopes in $\mathcal{A}$ with the combinatorics of the product of intervals.

To allow for more general coordinates, set $\mathcal{H} = \{0, 1, \infty\}$ (so that $\sum_{r\in\mathcal{H}} L_{ir} = 0$) and introduce an arbitrary affine coordinate

$$\xi_i = -N_{i0}(\mu^\lambda)/N_{i\infty}(\mu^\lambda)$$

on the pencil which takes the values α_{ir} at the points $[L_{ir}]$, meaning

$$\begin{aligned}
\sigma_i &= \frac{(\xi_i - \alpha_{i0})(\alpha_{i1} - \alpha_{i\infty})}{(\xi_i - \alpha_{i\infty})(\alpha_{i1} - \alpha_{i0})}, \quad 1 - \sigma_i = \frac{(\xi_i - \alpha_{i1})(\alpha_{i0} - \alpha_{i\infty})}{(\xi_i - \alpha_{i\infty})(\alpha_{i0} - \alpha_{i1})},\\
L_{ir} &= \frac{2(\xi_i - \alpha_{ir})N_{i\infty}}{A_i'(\alpha_{ir})}
\end{aligned}$$

for affine functions N_{ir} of μ^λ with $\sum_{r\in\mathcal{H}} N_{ir} = 0$, and $A_i(y) = a_i \prod_{r\in\mathcal{H}}(y - \alpha_{ir})$. Note that we can also allow $\alpha_{i\infty} = \infty$, in which case A_i is of degree 2. (This latter case can be derived from the generic case $\deg A_i = 3$ by letting $\alpha_{i\infty} = 1/\varepsilon$ and taking a limit of εA_i as $\varepsilon \to 0$.) Thus

$$\begin{aligned}
d\sigma_i &= \frac{(\alpha_{i1} - \alpha_{i\infty})(\alpha_{i0} - \alpha_{i\infty})}{(\xi_i - \alpha_{i\infty})(\xi_i - \alpha_{i\infty})^2},\\
\frac{\chi_i\, d\sigma_i{}^2}{\sigma_i(1-\sigma_i)} &= \frac{L_{i\infty}(\mu^\lambda)\, A_i'(\alpha_{i\infty}) d\xi_i{}^2}{2(\xi_i - \alpha_{i\infty})A_i(\xi_i)} = N_{i\infty}(\mu^\lambda)\, \frac{d\xi_i{}^2}{A_i(\xi_i)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
d\xi_i &= \frac{N_{i0}(\mu^\lambda) dN_{i\infty} - N_{i\infty}(\mu^\lambda) dN_{i0}}{N_{i\infty}(\mu^\lambda)^2} \circ d\mu^\lambda = -\frac{(\xi_i dN_{i\infty} + dN_{i0}) \circ d\mu^\lambda}{N_{i\infty}(\mu^\lambda)}\\
\therefore \quad d\mu^\lambda &= -\sum_{j\in\mathcal{I}} N_{j\infty}(\mu^\lambda) d\xi_j \otimes \mathbf{Q}_j, \quad \text{where} \quad \mathbf{Q}_i \colon M \to \mathfrak{t}^* \quad \text{satisfy}\\
&\sum_{i\in\mathcal{I}} (\xi_i dN_{i\infty} + dN_{i0}) \otimes \mathbf{Q}_i = Id_{\mathfrak{t}} \quad \text{and} \quad \langle (\xi_i dN_{i\infty} + dN_{i0}), \mathbf{Q}_j \rangle = \delta_{ij}.
\end{aligned}$$

Hence $d\mathbf{Q}_i = -\sum_j \langle \mathbf{Q}_i, dN_{j\infty}\rangle d\xi_j \otimes \mathbf{Q}_j$ and the Levi–Kähler metric is

$$\begin{aligned}
h_{\mathbf{L}} &= \sum_{i\in\mathcal{I}} N_{i\infty}(\mu^\lambda)\Big(\frac{d\xi_i{}^2}{A_i(\xi_i)} + A_i(\xi_i)\theta_i{}^2\Big),\\
\omega_{\mathbf{L}} &= \langle d\mu^\lambda \wedge d\mathbf{t}\rangle = -\sum_{i\in\mathcal{I}} N_{i\infty}(\mu^\lambda) d\xi_i \wedge \theta_i,
\end{aligned} \tag{19}$$

with $\theta_i = \langle \mathbf{Q}_i, d\mathbf{t}\rangle$ for angular coordinates $\mathbf{t}$ on M such that $d\mathbf{t}^{(1,0)}$ is holomorphic:

$$dd^c\mathbf{t} = d\left(\sum_{i\in\mathcal{I}} \frac{\xi_i dN_{i\infty} + dN_{i0}}{A_i(\xi_i)} d\xi_i\right) = 0.$$

The canonical affine coordinates may be obtained by setting $A_i(t) = 2t(1-t)(1-\varepsilon t)$ in the limit $\varepsilon \to 0$; we then have $N_{ir} = -L_{ir}$ and $\xi_i = \sigma_i$ so that $N_{i\infty}(\mu^\lambda) = 2\chi_i$ and

$$\begin{aligned} h_{\mathbf{L}} &= \sum_{i\in\mathcal{I}} 2\chi_i\left(\frac{d\sigma_i{}^2}{2\sigma_i(1-\sigma_i)} + 2\sigma_i(1-\sigma_i)\theta_i{}^2\right),\\ \omega_{\mathbf{L}} &= -\sum_{i\in\mathcal{I}} 2\chi_i\, d\sigma_i\wedge\theta_i. \end{aligned} \tag{20}$$

3.2.2. Explicit normal form. Recall that $\mathfrak{g}_o \subseteq \mathfrak{t}_{\mathcal{S}}$ has a canonical basis $e_i := e_{i0} + e_{i1}$ for $i \in \mathcal{I}$. We obtain a similar basis $w_i : i \in \mathcal{I}$ for $\mathfrak{g}$ by observing that the vectors e_{i0} in $\mathfrak{t}_{\mathcal{S}}$ project to $\mathfrak{t} = \mathfrak{t}_{\mathcal{S}}/\mathfrak{g}_o \cong \mathfrak{t}_{\mathcal{S}}/\mathfrak{g}$ to the normals $u_{i0} : i \in \mathcal{I}$ at a vertex of $\Delta_{\mathfrak{g},\lambda}$. Hence $\mathfrak{g}$ is transversal to $\operatorname{span}\{e_{i0} \mid i \in \mathcal{I}\}$ and there are canonical $w_i \in \mathfrak{g}$ of the form

$$w_i = e_i + \sum_{j\in\mathcal{I}} C_{ji} e_{j0} = e_{i0} + e_{i1} + \sum_{j\in\mathcal{I}} C_{ji} e_{j0} \tag{21}$$

for an $\ell\times\ell$ matrix of real numbers C. We denote by $K_i^o = K_{e_i}$ and $K_i = K_{w_i}$ the induced vector fields in $\mathcal{K}^{\mathfrak{g}_o}$ and $\mathcal{K}^{\mathfrak{g}}$ respectively. We shall also write K_{ir} as a shorthand for $K_{e_{ir}} = \partial/\partial\boldsymbol{\vartheta}_{ir}$. Thus

$$\begin{aligned} d^c\boldsymbol{\sigma}^i(K_j^o) &= 2(\sigma_{i0}d\boldsymbol{\vartheta}_{i0} + \sigma_{i1}d\boldsymbol{\vartheta}_{i1})(K_{j0}+K_{j1}) = 2\delta_{ij}\\ d^c\boldsymbol{\sigma}^i(K_j) &= 2(\sigma_{i0}d\boldsymbol{\vartheta}_{i0} + \sigma_{i1}d\boldsymbol{\vartheta}_{i1})\big(K_j^o + \textstyle\sum_k C_{kj}K_{k0}\big) = 2(\delta_{ij} + \sigma_i C_{ij}). \end{aligned}$$

on N, since $\sigma_{i0} + \sigma_{i1} = 1$. Now if $\eta_i : i \in \mathcal{I}$ are the 1-forms on N defined by $\eta_i(K_j) = \delta_{ij}$ and $\bigcap_{i\in\mathcal{I}} \ker\eta_i = \mathcal{D} = \bigcap_{i\in\mathcal{I}} \ker d^c\boldsymbol{\sigma}^i$, we may write

$$\tfrac12 d^c\boldsymbol{\sigma}^i = \sum_{j\in\mathcal{I}} P_{ij}\eta_j \quad\text{where}\quad P_{ij} := \tfrac12 d^c\boldsymbol{\sigma}^i(K_j) = \delta_{ij} + \sigma_i C_{ij}. \tag{22}$$

We have noted that $e_{i0} : i \in \mathcal{I}$ project onto a bases for $\mathfrak{t} = \mathfrak{t}_{\mathcal{S}}/\mathfrak{g}_o \cong \mathfrak{t}_{\mathcal{S}}/\mathfrak{g}$. In order to compute the toral part of the quotient metric, we need to project the corresponding vector fields K_{i0} onto $\mathcal{D}$. We thus define projections

$$\begin{aligned} X_i^o &= K_{i0} - \sum_{j\in\mathcal{I}} \tfrac{1}{2} d^c \boldsymbol{\sigma}^j(K_{i0}) K_j^o = K_{i0} - \sigma_i(K_{i0} + K_{i1}) \\ &= (1-\sigma_i)K_{i0} - \sigma_i K_{i1}, \\ X_i &= K_{i0} - \sum_{j\in\mathcal{I}} \eta_j(K_{i0}) K_j, \end{aligned}$$

along $\mathcal{K}^{\mathfrak{g}_o}$ and $\mathcal{K}^{\mathfrak{g}}$ respectively. Let $I_\sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_\ell)$.

Lemma 7. *$X_i^o = \sum_{j\in\mathcal{I}} X_j \tilde{P}_{ji}$ with $\tilde{P}_{ji} = \delta_{ji} + C_{ji}\sigma_i$, i.e., $PI_\sigma = I_\sigma \tilde{P}$.*

Proof. Since $\sum_{k\in\mathcal{I}} P_{jk}\eta_k(K_{i0}) = \frac{1}{2} d^c\boldsymbol{\sigma}^j(K_{i0}) = \sigma_i\delta_{ij}$, it follows from the relation between P and $\tilde{P}$ that $\frac{1}{2}d^c\boldsymbol{\sigma}^j(K_{i0}) = \sum_{k\in\mathcal{I}} \eta_j(K_{k0})\tilde{P}_{ki}$. We now have

$$\begin{aligned} X_i^o &= K_{i0} - \sum_{j\in\mathcal{I}} \tfrac{1}{2} d^c\boldsymbol{\sigma}^j(K_{i0}) \left(K_j - \sum_{k\in\mathcal{I}} C_{kj} K_{k0} \right) \\ &= K_{i0} - \sum_{j,k\in\mathcal{I}} \Big(\eta_j(K_{k0}) \tilde{P}_{ki} K_j - \sigma_j \delta_{ij} C_{kj} K_{k0} \Big) \\ &= \sum_{k\in\mathcal{I}} K_{k0}(\delta_{ki} + C_{ki}\sigma_i) - \sum_{j,k\in\mathcal{I}} \eta_j(K_{k0}) K_j \tilde{P}_{ki} = \sum_{k\in\mathcal{I}} X_k \tilde{P}_{ki} \end{aligned}$$

as required. $\square$

Writing $\lambda = (c_1, \ldots, c_\ell)$ with respect to the dual basis to $w_i : i \in \mathcal{I}$, $\eta^\lambda = \sum_{i\in\mathcal{I}} c_i\eta_i$. The corresponding momentum coordinates $\mu_i : i \in \mathcal{I}$ are given by (cf. (14))

$$\mu_i = \langle \mu^\lambda, L_{i0}\rangle = \sum_{j\in\mathcal{I}} c_j \eta_j(K_{i0}) = \sum_{j\in\mathcal{I}} c_j\sigma_j \tilde{Q}_{ji} = \sum_{j\in\mathcal{I}} c_j Q_{ji}\sigma_i = 2\chi_i\sigma_i, \tag{23}$$

where $\tilde{Q} = \tilde{P}^{-1}$, $Q = P^{-1}$ and $2\chi_i = \sum_{j\in\mathcal{I}} c_j Q_{ji}$. If we rewrite (23) as

$$c_j\sigma_j = \sum_{i\in\mathcal{I}} \mu_i \tilde{P}_{ij} = \mu_j + \sum_{i\in\mathcal{I}} \mu_i C_{ij}\sigma_j,$$

then we can specialize (19) with $N_{jr} = L_{jr}$ (up to overall scale) using

$$L_{j0}(\mu^\lambda) = \mu_j, \qquad L_{j\infty}(\mu^\lambda) = -(L_{j0} + L_{j1})(\mu^\lambda) = \sum_{i\in\mathcal{I}} \mu_i C_{ij} - c_j.$$

We may also compute directly that the toral part of the metric on $\mathcal{D}$ is

$$
\begin{aligned}
(24)\qquad h^{\mathrm{tor}}_{\mathbf{L}} &:= \sum_{i,j,k\in\mathcal{I}} c_i d\eta_i(X_j, JX_k)\, dt_j\, dt_k \\
&= \sum_{i,j,k,p\in\mathcal{I}} \tfrac{1}{2} c_i Q_{ip}\, dd^c\boldsymbol{\sigma}^p(X_j, JX_k)\, dt_j\, dt_k \\
&= \sum_{i,j,k,p,q,r\in\mathcal{I}} 2c_i Q_{ip}\left(\sigma_p d\boldsymbol{\vartheta}_{p0}{}^2 + (1-\sigma_p) d\boldsymbol{\vartheta}_{p1}{}^2\right) \\
&\qquad\qquad \times (X^o_q \tilde{Q}_{qj}, X^o_r \tilde{Q}_{rk})\, dt_j\, dt_k \\
&= \sum_{i\in\mathcal{I}} 4\chi_i \sigma_i(1-\sigma_i)\left(\sum_{j\in\mathcal{I}} \tilde{Q}_{ij}\, dt_j\right)^2,
\end{aligned}
$$

where we use $\sigma_i(1-\sigma_i)^2 + (1-\sigma_i)\sigma_i^2 = \sigma_i(1-\sigma_i)$. This agrees with (19) since

$$
\begin{aligned}
d\mu_i &= \sum_j c_j (d\sigma_j\, \tilde{Q}_{ji} + \sigma_j\, d\tilde{Q}_{ji}) = \sum_{j,k,l} c_j (Q_{jk} P_{kl} - \sigma_j \tilde{Q}_{jk} C_{kl})\, d\sigma_l\, \tilde{Q}_{li} \\
&= \sum_{j,k,l} c_j Q_{jk} (P_{kl} - \sigma_k C_{kl})\, d\sigma_l\, \tilde{Q}_{li} = \sum_k 2\chi_k\, d\sigma_k\, \tilde{Q}_{ki}.
\end{aligned}
$$

3.3. Projective cubes

For Levi–Kähler quotients of an ℓ-fold product of 3-spheres, the polytope $\Delta_{\mathfrak{g},\lambda}$ is an ℓ-cuboid, i.e., it has the combinatorics of a product of $m = \ell$ intervals (an m-cube). Such a polytope is *projectively equivalent to a cube* if the intersections of pairs of opposite facets lie in a hyperplane: transforming this hyperplane projectively to infinity, opposite facets become parallel and meet the hyperplane at infinity in the facets of an $(m-1)$-simplex, and all simplices are projectively equivalent. Projective equivalence to an m-cube is automatic when $m = 2$, but is restrictive for $m \geqslant 3$ (when $m = 3$, opposite facets of a generic cuboid meet in skew lines, not coplanar lines).

The assumption of projective equivalence to an m-cube simplifies the previous analysis, because we may take $N_{i\infty}$ to be the equation of the hyperplane common to the pencils spanned by opposite facets, independent of $i \in \mathcal{I} = \{1, \ldots, m\}$. Concretely, let $b_j \in \mathbb{R}$ for $0 \leqslant j \leqslant m$ and $\mu_j : 0 \leqslant j \leqslant m$ be affine coordinates on the affine space $\mathcal{A} = \{(\mu_0, \mu_1, \ldots, \mu_m) : \sum_{j=0}^m b_j \mu_j = 1\}$.

We now set $N_{i\infty} = \mu_0$, $N_{i0} = -\mu_i$ and $N_{i1} = \mu_i - \mu_0$ for $1 \leqslant i \neq m$. We thus have

$$\xi_i = \frac{\mu_i}{\mu_0} \quad \text{and hence} \quad b_0 + \sum_{i=1}^{n} b_i \xi_i = \sum_{i=0}^{m} \frac{b_i \mu_i}{\mu_0} = \frac{1}{\mu_0}, \tag{25}$$

so that the inverse transformation is

$$\mu_0 = \frac{1}{b_0 + b_1\xi_1 + \cdots + b_m\xi_m}, \quad \mu_i = \xi_i \mu_0 = \frac{\xi_i}{b_0 + b_1\xi_1 + \cdots + b_m\xi_m}. \tag{26}$$

Differentiating μ_i, we may then write the symplectic form as

$$\omega_{\mathbf{L}} = \sum_{i=1}^{m} d\mu_i \wedge dt_i = \mu_0 \sum_{i=1}^{m} d\xi_i \wedge \theta_i, \tag{27}$$

where

$$\theta_i = dt_i - \mu_0 b_i \sum_{j=1}^{m} \xi_j dt_j = dt_i - b_i \sum_{j=1}^{m} \mu_j dt_j. \tag{28}$$

In particular $d\theta_i = -b_i \omega_{\mathbf{L}}$. Letting

$$J d\xi_i := A_i(\xi_i)\theta_i, \quad 1 \leqslant i \leqslant m, \tag{29}$$

J defines an integrable almost complex structure and the t_i are pluriharmonic. We thus have the following diagonal form of $(h_{\mathbf{L}}, \omega_{\mathbf{L}})$

$$\begin{aligned} h_{\mathbf{L}} &= \frac{1}{b_0 + b_1\xi_1 + \cdots + b_m\xi_m} \sum_{i=1}^{m} \left(\frac{d\xi_i{}^2}{A_i(\xi_i)} + A_i(\xi_i)\theta_i{}^2 \right) \\ \omega_{\mathbf{L}} &= \frac{1}{b_0 + b_1\xi_1 + \cdots + b_m\xi_m} \sum_{i=1}^{m} d\xi_i \wedge \theta_i, \end{aligned} \tag{30}$$

where the 1-forms θ_i are given by (28).

The product of intervals $\xi_i \in [\alpha_{i0}, \alpha_{i1}]$, $1 \leqslant i \leqslant m$ transforms to the compact convex polytope Δ determined by the hyperplanes $(\xi_i - \alpha_{ir})\mu_0 = 0$, (for $r \in \{0, 1\}$, $1 \leqslant i \leqslant n$). As before, we set

$$A_i(y) := a_i \prod_{r \in \mathcal{H}} (y - \alpha_{ir}), \qquad L_{ir}(\mu) := \frac{2(\xi_i - \alpha_{ir})\mu_0}{A_i'(\alpha_{ir})} = \frac{2(\mu_i - \alpha_{ir}\mu_0)}{A_i'(\alpha_{ir})},$$

where $r \in \mathcal{H} = \{0, 1, \infty\}$. Note that $L_{ir} \geqslant 0$ on Δ for $1 \leqslant i \leqslant m$ and $r \in \{0, 1\}$, and that $\sum_{r \in \mathcal{H}} L_{ir} = 0$. We can compute a Kähler potential from the symplectic potential $G_{\mathbf{L}}$ by Legendre transform based at $\mu_j = 0$ to get, modulo constants,

$$H_{\mathbf{L}} = \sum_{i,r} \frac{\log |L_{ir}|}{A_i'(\alpha_{ir})} = \sum_{i,r} \frac{\log |\xi_i - \alpha_{ir}|}{A_i'(\alpha_{ir})} = \sum_{i=1}^{m} \int^{\xi_i} \frac{ds}{A_i(s)}. \tag{31}$$

3.4. Levi–Kähler metrics of convex quadrilaterals

We now specialize to the case $m = 2$, i.e.,

$$N = \mathbb{S}^3 \times \mathbb{S}^3 = \{z \in \mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \mid (\sigma_{10} + \sigma_{11})(z) = 1, (\sigma_{20} + \sigma_{21})(z) = 1\}.$$

By Theorem 4, the compact Kähler 4-orbifolds $(M, h_{\mathbf{L}}, \omega_{\mathbf{L}})$ obtained as a Levi–Kähler quotient of $\mathbb{S}^3 \times \mathbb{S}^3$ by an abelian subgroup $G \subseteq \mathbb{T}^4$ are the compact toric 4-orbifolds whose rational Delzant polytope is a quadrilateral. Note that from its very construction, $h_{\mathbf{L}}$ is compatible with a second complex structure $\tilde{J}$ on M, coming from the quotient of the product CR structure $(\mathcal{D}, J_1 - J_2)$ on $N = \mathbb{S}^3 \times \mathbb{S}^3$, where $(\mathcal{D}, J) = (\mathcal{D}_1, J_1) \oplus (\mathcal{D}_2, J_2)$ is the direct sum of the CR distributions of each $\mathbb{S}^3$ factor. Thus $J = J_1 + J_2$, and $\tilde{J} = J_1 - J_2$ defines a second CR structure on N, associated to the same distribution $\mathcal{D} \subseteq TN$, which commutes with J and induces the opposite orientation on $\mathcal{D}$. In the terminology of [5], $h_{\mathbf{L}}$ is Kähler with respect to J and *ambihermitian* with respect to commuting complex structures $(J, \tilde{J})$ on M. We are going to show that $(h_{\mathbf{L}}, \tilde{J})$ is, in fact, conformal to another $\mathbb{T}$-invariant Kähler metric $(\tilde{h}_{\mathbf{L}}, \tilde{J})$ (which induces the opposite orientation of (M, J)), i.e., that $(h_{\mathbf{L}}, J)$ and $(\tilde{h}_{\mathbf{L}}, \tilde{J})$ define an *ambitoric* structure on M in the sense of [5]. These structures have been extensively studied and classified, both locally [5] and globally [6].

As any quadrilateral is a projective cube, the general form of the Levi–Kähler metric $h_{\mathbf{L}}$ is described by (30) but we shall also describe below how this form is derived from the choice of the subgroup G. Following the notation in §3.2, we specialize (24) to $\ell = 2$, and set

$$C = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix},$$

so that

$$A = \begin{pmatrix} 1 + \alpha\sigma_1 & \gamma\sigma_1 \\ \beta\sigma_2 & 1 + \delta\sigma_2 \end{pmatrix} \quad \text{and} \quad \tilde{A} = \begin{pmatrix} 1 + \alpha\sigma_1 & \gamma\sigma_2 \\ \beta\sigma_1 & 1 + \delta\sigma_2 \end{pmatrix};$$

hence

$$B = \frac{1}{Z}\begin{pmatrix} 1+\delta\sigma_2 & -\gamma\sigma_1 \\ -\beta\sigma_2 & 1+\alpha\sigma_1 \end{pmatrix} \quad \text{and} \quad \tilde{B} = \frac{1}{Z}\begin{pmatrix} 1+\delta\sigma_2 & -\gamma\sigma_2 \\ -\beta\sigma_1 & 1+\alpha\sigma_1 \end{pmatrix},$$

where $Z = (1+\alpha\sigma_1)(1+\delta\sigma_2) - \beta\gamma\sigma_1\sigma_2 = 1 + \alpha\sigma_1 + \delta\sigma_2 + (\alpha\delta - \beta\gamma)\sigma_1\sigma_2$. Then

$$\mu_1 = \frac{c_1\sigma_1(1+\delta\sigma_2) - c_2\beta\sigma_1\sigma_2}{Z}, \quad \mu_2 = \frac{-c_1\gamma\sigma_1\sigma_2 + c_2(1+\alpha\sigma_1)\sigma_2}{Z}, \tag{32}$$

while the toral part of the metric on $\mathcal{D}$ is

$$\begin{aligned} h^{\mathrm{tor}}_{\mathbf{L}} &= \sigma_1(1-\sigma_1)\big((1+\delta\sigma_2)c_1 - \beta\sigma_2 c_2\big)\big((1+\delta\sigma_2)dt_1 - \gamma\sigma_2 dt_2\big)^2/Z^3 \\ &\quad + \sigma_2(1-\sigma_2)\big((1+\alpha\sigma_1)c_2 - \gamma\sigma_1 c_1\big)\big((1+\alpha\sigma_1)dt_2 - \beta\sigma_1 dt_1\big)^2/Z^3. \end{aligned} \tag{33}$$

We now transform this expression into the ansatz (30) for projective cubes (all quadrilaterals are projectively equivalent). To do this we first find the base loci of the families of lines with σ_1 or σ_2 constant. We find that for $\sigma_1 = c_2/(c_1\gamma - c_2\alpha)$, $1+\alpha\sigma_1 = c_1\gamma/(c_1\gamma - c_2\alpha)$ and hence $(\mu_1,\mu_2) = (c_2/\gamma, 0)$, independent of σ_2. Similarly for $\sigma_2 = c_1/(c_2\beta - c_1\delta)$, $(\mu_1,\mu_2) = (0, c_1/\beta)$, independent of σ_1. These coordinates are examples of *Segre factorization structures*, as defined in [6]. We transform the coordinate singularity to infinity by setting

$$\begin{aligned} \xi_1 &= \sigma_1\Delta_1, \quad \Delta_1 = \frac{1}{c_2(1+\alpha\sigma_1) - c_1\gamma\sigma_1} = \frac{1 + (c_1\gamma - c_2\alpha)\xi_1}{c_2}, \\ \xi_2 &= \sigma_2\Delta_2, \quad \Delta_2 = \frac{1}{c_1(1+\delta\sigma_2) - c_2\beta\sigma_2} = \frac{1 + (c_2\beta - c_1\delta)\xi_2}{c_1} \end{aligned}$$

We then compute that $1 + c_1\gamma\xi_1 + c_2\beta\xi_2 = c_1c_2\Delta_1\Delta_2 Z$ and

$$\begin{gathered} \Delta_1 + \alpha\xi_1 = (1 + c_1\gamma\xi_1)/c_2, \quad \Delta_2 + \delta\xi_2 = (1 + c_2\beta\xi_2)/c_1, \\ (1+\delta\sigma_2)dt_1 - \gamma\sigma_2 dt_2 = \big((1 + c_2\beta\xi_2)dt_1 - c_1\gamma\xi_2 dt_2\big)/(c_1\Delta_2) \\ (1+\alpha\sigma_1)dt_2 - \beta\sigma_1 dt_1 = \big((1 + c_1\gamma\xi_1)dt_2 - c_2\beta\xi_1 dt_1\big)/(c_2\Delta_1), \end{gathered}$$

so that, setting $\mu_0 = 1/(1 + c_1\gamma\xi_1 + c_2\beta\xi_2)$, we have

$$\begin{aligned} h^{\mathrm{tor}}_{\mathbf{L}} &= c_1c_2^3\mu_0\Delta_1\xi_1(\Delta_1 - \xi_1)\big(dt_1 - \mu_0 c_1\gamma(\xi_1 dt_1 + \xi_2 dt_2)\big)^2 \\ &\quad + c_1^3c_2\mu_0\Delta_2\xi_2(\Delta_2 - \xi_2)\big(dt_2 - \mu_0 c_2\beta(\xi_1 dt_1 + \xi_2 dt_2)\big)^2, \end{aligned} \tag{34}$$

$$\mu_1 = \frac{\sigma_1}{\Delta_2 Z} = \frac{c_1c_2\xi_1}{1 + c_1\gamma\xi_1 + c_2\beta\xi_2}, \quad \mu_2 = \frac{\sigma_2}{\Delta_1 Z} = \frac{c_1c_2\xi_2}{1 + c_1\gamma\xi_1 + c_2\beta\xi_2} \tag{35}$$

This has the form (30) with $A_1(\xi_1) = c_1c_2^3\Delta_1\xi_1(\Delta_1 - \xi_1)$ and $A_2(\xi_2) = c_1^3c_2\Delta_2\xi_2(\Delta_2 - \xi_2)$.

We now relate the Levi–Kähler metrics to the local forms of ambitoric metrics studied in [5]. The results depend crucially on whether $\beta = 0$ (when the curves ξ_1 =constant pass through the point at infinity), $\gamma = 0$ (when the curves ξ_2 =constant pass through the point at infinity), or both (when we have a product structure). We break this down into three cases as follows.

3.4.1. The product case. This is the case when $\beta = 0, \gamma = 0$. Letting $x = \xi_1, y = \xi_2$ the metric becomes

$$h_{\mathbf{L}} = \frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + A(x)dt_1^2 + B(y)dt_2^2$$

with $A(x), B(y)$ positive valued polynomials of degree 2 or 3, i.e., a local product of extremal toric Riemann surfaces. The construction yields (up to an equivariant isometry corresponding to affine transformations of x and y) all such products for which $A(x)$ and $B(y)$ are of degree 2 or 3 with distinct real roots.

3.4.2. The Calabi case. Without loss of generality, this is the case $\beta = 0$, $\gamma \neq 0$, so that the curves ξ_1 =constant pass through the point at infinity. We now let $x = 1 + c_1\gamma\xi_1, y = -c_1\gamma\xi_2$ so that the metric becomes

$$h_{\mathbf{L}} = \frac{1}{x}\left(\frac{dy^2}{B(y)} + B(y)dt_2^2 + \frac{dx^2}{A(x)} + \frac{A(x)}{x^2}\big(dt_1 + y\,dt_2\big)^2\right) \tag{36}$$

i.e., given by the Calabi construction with respect to the variables $\bar{x} = 1/x, y$ (see e.g. [4, 39]) starting from a toric extremal Riemann surface $(\Sigma, g_\Sigma = \frac{dy^2}{B(y)} + B(y)dt_2^2)$, and taking an extremal toric metric on the fibre associated to the profile function $\Theta(\bar{x}) := \bar{A}(\bar{x})/\bar{x}$ with $\bar{A}(\bar{x}) = \bar{x}^4A(1/\bar{x})$. Once again, up to affine changes of x and y, one covers all toric metrics of Calabi type for which the functions $A(x), B(y)$ are polynomials of degree 2 or 3 with distinct real roots.

3.4.3. The negative orthotoric case. This is the generic case when $\beta\gamma \neq 0$. We can therefore let $x = 1 + c_1\gamma\xi, y = -c_2\beta\xi_2$ so that the metric becomes

$$\begin{aligned} h_{\mathbf{L}} = {} & \frac{1}{x-y}\left(\frac{dx^2}{A(x)} + \frac{dy^2}{B(y)}\right) \\ & + \frac{A(x)(d\theta_1 + yd\theta_2)^2 + B(y)(d\theta_1 + xd\theta_2)^2}{(x-y)^3}, \end{aligned} \tag{37}$$

where $A(x)$ and $B(y)$ are both polynomials of degree 2 or 3 with distinct real roots. (In terms of [5], the conformal oppositely oriented Kähler metric $(\tilde{h}_{\mathbf{L}}, \tilde{J})$ is orthotoric.)

Writing $\tilde{h}_{\mathbf{L}} = (1/w^2)h_{\mathbf{L}}$, a case by case inspection shows that the conformal factor w is respectively given by $w = 1, 1/x$ or $1/(x-y)$, according to whether we are in the product, Calabi-type or negative orthotoric case. We observe that in each case w is an affine function in the momenta with respect to $\omega_{\mathbf{L}}$, which vanishes at the (possibly infinite) intersection points of the pair of opposite facets of $(\Delta, \mathbf{L})$. We summarize the discussion as follows.

Proposition 6. *Let $(h_{\mathbf{L}}, \omega_{\mathbf{L}})$ be a Levi–Kähler quotient of $\mathbb{S}^3 \times \mathbb{S}^3 \subseteq \mathbb{C}^2 \times \mathbb{C}^2$, corresponding to a subspace $\mathfrak{g} \subseteq \mathfrak{t}_S$, and some $\lambda \in \mathfrak{g}^*$. Then $(h_{\mathbf{L}}, \omega_{\mathbf{L}})$ is ambitoric in the sense of [5], and is either a product, of Calabi-type or conformal and oppositely oriented to an orthotoric metric, depending on whether $\mathfrak{g}$ intersects nontrivially two, one or zero of the 2-dimensional subspaces $(\mathfrak{t}_1, \mathfrak{t}_2)$, where $\mathfrak{t}_i = \mathbb{R}^2 \subseteq \mathbb{R}^4$ is the Lie algebra of the 2-torus $\mathbb{T}^2_i \subseteq \mathbb{T}^4$ naturally acting on the i-th factor $(i = 1, 2)$ of $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$. Furthermore, in all cases, $h_{\mathbf{L}}$ is expressed in terms of two arbitrary polynomials of degree 2 or 3, each with real distinct roots, whereas the oppositely oriented Kähler metric is $\tilde{h}_{\mathbf{L}} = (1/w^2)h_{\mathbf{L}}$ for a positive affine function w on the quadrilateral $\Delta_{\mathfrak{g},\lambda}$, vanishing at the intersection points of its opposite facets. Conversely, any ambitoric metric of the above mentioned types determined by two polynomials of degree 2 or 3 with distinct real roots[1] arises as a Levi–Kähler quotient of $\mathbb{S}^3 \times \mathbb{S}^3$.*

3.5. Toric bundles and Levi–Kähler quotients of products of spheres

The Calabi and the product cases appearing in the analysis in the previous subsection have a natural generalization to higher dimensions in the framework of *semisimple rigid toric bundle construction* of [9, 10], where it is also referred to as the *generalized Calabi construction*. Let us first recall briefly the setting of these works.

Let $\pi\colon M \to S$ be a bundle of toric kählerian manifolds or orbifolds of the form $M = P \times_T V$, for an ℓ-torus T, a principal T-bundle P over a kählerian manifold S of dimension $2d$, and a toric 2ℓ-manifold (or orbifold) V with

[1] This constraint comes from the fact that $\mathfrak{g}$ is a subspace of the Cartan subalgebra $\mathfrak{t}_S$ consisting of diagonal elements of the Lie algebra $\mathfrak{su}(1,2) \oplus \mathfrak{su}(1,2)$ of the CR automorphisms of $\mathbb{S}^3 \times \mathbb{S}^3$.

Delzant polytope $\Delta \subseteq \mathfrak{t}^*$; thus M has dimension $2m = 2(d+\ell)$. We let F_i $(i = 1, \dots, n)$ denote the co-dimension one faces of Δ and u_i the primitive inward normals (with respect to a lattice Λ). Let be the dimension of M. Let $\theta \in \Omega^1(M, \mathfrak{t})$ be the connection 1-form induced by a principal T-connection on P, with curvature $\Omega \in \Omega^1(S, \mathfrak{t})$. Suppose that Ω^0 is a closed 2-form on S. Then the rigid toric bundle construction on M is a Kähler metric of the form

$$
\begin{aligned}
g &= g_0 + \langle x, g_\Omega \rangle + \langle dx, (\mathbf{H}^V)^{-1}, dx \rangle + \langle \theta, \mathbf{H}^V, \theta \rangle, \\
\omega &= \Omega_0 + \langle x, \Omega \rangle + \langle dx \wedge \theta \rangle, \qquad d\theta = \Omega,
\end{aligned} \tag{38}
$$

where:

- $x \in C^\infty(M, \mathfrak{t}^*)$ is the momentum map of the T action with image Δ;
- $\mathbf{H}^V \in C^\infty(\Delta, S^2\mathfrak{t}^*)$ is a matrix valued function which, firstly, satisfies the boundary conditions that on any co-dimension one face F_i, there is a function h_i with

  $$\sum_t \mathbf{H}^V_{st}(x)(u_i)_t = 0, \qquad \sum_t \frac{\partial \mathbf{H}^V_{st}}{\partial x_r}(x)(u_i)_t = (u_i)_r h_i(x)_s$$

  and $\langle h_i(x), u_i \rangle := \sum_s h_i(x)_s (u_i)_s = 2$ for all $x \in F_i$; secondly the inverse $(\mathbf{H}^V)^{-1} \in C^\infty(\Delta, S^2\mathfrak{t})$ of $\mathbf{H}^V$ is the hessian of a function G_V on Δ; thirdly, $\mathbf{H}^V$ induces a positive definite metric on the interior of each face F of Δ (as an element of $S^2(\mathfrak{t}/\mathfrak{t}_F)^*$, where $\mathfrak{t}_F$ is the isotropy algebra of F);
- the metric $g_0 + \langle x, g_\Omega \rangle$ associated to $\Omega_0 + \langle x, \Omega \rangle$ via the complex structure on S is positive definite for all $x \in \Delta \subseteq \mathfrak{t}^*$.

Throughout, angle brackets denote natural contractions of $\mathfrak{t}$ with $\mathfrak{t}^*$, and we omit pullbacks by x and π. In particular x itself will denote the standard $\mathfrak{t}^*$-valued coordinate on Δ, as well as its pullback to M.

The function G_V is called a *symplectic potential* for $\mathbf{H}^V$ and is determined up to an affine function on $\mathfrak{t}^*$. According to [2, Thm. 2] and [9, Rem. 4.2], the boundary and positivity conditions above can be equivalently formulated in terms of G_V, by requiring that G_V is smooth and strictly convex on the interior $\mathring{\Delta}$ of Δ, such that

$$
\begin{aligned}
& G_V - \frac{1}{2}\sum_{i=1}^k L_i \log L_i \in C^\infty(\Delta) \\
& \det(\operatorname{Hess} G_V) \prod_{i=1}^k L_k \in C^\infty(\Delta) \quad \text{and is strictly positive,}
\end{aligned} \tag{39}
$$

where $L_i = \langle u_i, x\rangle - v_i, i = 1, \dots, k$ are the labels defining Δ.

Let X be a holomorphic vector field on S which is hamiltonian with respect to $\Omega_0 + \langle x, \Omega\rangle$ for all $x \in \Delta$. Thus $-\imath_X(\Omega_0 + \langle x, \Omega\rangle) = df_0 + \langle x, df_\Omega\rangle$ for functions $f_0 \in C^\infty(S, \mathbb{R})$ and $f_\Omega \in C^\infty(S, \mathfrak{t})$. Generalizing an observation from the proof of [10, Lemma 5], any such X can be lifted to a hamiltonian Killing vector field of (M, g, ω):

$$\hat{X} = X^H + \langle f_\Omega, K\rangle, \tag{40}$$

where $K := \mathrm{grad}_\omega x \in C^\infty(M, TM) \otimes \mathfrak{t}^*$ is the family of hamiltonian vector fields generated by the principal T bundle P, and X^H denotes the horizontal lift of X (to the kernel of θ). Indeed,

$$-\imath_{\hat{X}}\omega = -\imath_X(\Omega_0 + \langle x, \Omega\rangle) + \langle f_\Omega, dx\rangle = d(f_0 + \langle x, f_\Omega\rangle),$$

so $\hat{X}$ has hamiltonian $f_0 + \langle x, f_\Omega\rangle$ (omitting pullbacks of f_0 and f_Ω to M).

Suppose now that the family $g_0 + \langle x, g_\Omega\rangle$ of Kähler metrics on S is toric with respect to a fixed torus action of a torus T_S with Lie algebra $\mathfrak{t}_S$. For each fixed x, the momentum map of T_S may be written $\xi_0 + \langle x, \xi_\Omega\rangle$, where $\xi \in C^\infty(S, \mathfrak{t}_S^*)$ and $\xi_\Omega \in C^\infty(S, \mathfrak{t}_S^* \otimes \mathfrak{t})$, and pulling back these functions to M, $\xi_0 + \langle x, \xi_\Omega\rangle$ is the momentum map for the T_S action on M defined by lifting the generators to M using (40). Since these lifts commute with K, M is toric under the combined action of $T \times T_S$.

Lemma 8. *Let (M, g, ω) be a Kähler manifold or orbifold given by* (38) *with fibrewise symplectic potential G_V, and suppose that $(S, \Omega_0 + \langle x, \Omega\rangle)$ is toric with respect to a fixed action of T_S for all $x \in \Delta$. Then, (M, g, ω) is toric with respect to the lifted $T \times T_S$ action and has a symplectic potential*

$$G_M = G_V + G_0 + \langle x, G_\Omega\rangle, \tag{41}$$

where G_0 and G_Ω are functions of $\xi \in \mathfrak{t}_S$ such that for each fixed x, $G_0 + \langle x, G_\Omega\rangle$ is a symplectic potential for $g_0 + \langle x, g_\Omega\rangle$ on S.

Proof. The torus action T on (M, g, ω) is *rigid*, meaning that the metric on the torus orbits depends only on the value of the momentum map x [7]. Hence for each fixed $\xi \in \mathfrak{t}_S$, the restriction of G_M to $\mathfrak{t}^* + \xi \cong \mathfrak{t}^*$ differs from G_V by an affine function of x, hence has the form (41). By construction, for each fixed $x \in \Delta$, $(S, g_0 + \langle x, g_\Omega\rangle)$ is the Kähler quotient of M by T at momentum level x. Hence by [23], the restriction of G_M to $x + \mathfrak{t}_S^* \cong \mathfrak{t}_S^*$ is a symplectic potential for $g_0 + \langle x, g_\Omega\rangle$, hence so is $G_0 + \langle x, G_\Omega\rangle$. $\square$

A rigid toric bundle metric (38) is *semisimple* if S is a product of Kähler manifolds (S_j, ω_j) and there exist $c_j \in \mathbb{R}$ and $p_j \in \mathfrak{t}$ (for $j \in \{1, 2, \ldots, N\}$) such that $\Omega_0 + \langle x, \Omega\rangle = \sum_{j=1}^N (c_j + \langle p_j, x\rangle)\omega_j$. If each (S_j, ω_j) is toric under a torus T_j with Lie algebra $\mathfrak{t}_j$, then $(S, \Omega_0 + \langle x, \Omega\rangle)$ is toric for all x under the product action of $T_S = \prod_{j=1}^N T_j$ with $\mathfrak{t}_S = \bigoplus_{j=1}^N \mathfrak{t}_j$. We refer to this special case as the *toric generalized Calabi ansatz.*

Proposition 7. *Suppose that (g, ω) is a Kähler metric obtained by the toric generalized Calabi ansatz, where the fibre V is a compact toric orbifold with labelled polytope $(\Delta, L_1, \ldots, L_k)$, and each factor (S_j, ω_j) of the base $S = \prod_{j=1}^N S_j$ is a compact toric orbifold with labelled Delzant polytope $(\Delta_j, L_1^j, \ldots, L_{k_j}^j)$, respectively. Then (g, ω) is a toric Kähler metric on the compact symplectic orbifold with labelled Delzant polytope*

$$\hat{\Delta} = \{L_i(x) \geq 0,\ \hat{L}^j_{r_j} := (\langle p_j, x\rangle + c_j)L^j_{r_j} \geq 0\}.$$

Moreover, if the fibrewise toric metric determined by G^V and the Kähler metrics ω_j are all Levi–Kähler quotients of product spheres, then the resulting metric (38) *is a Levi–Kähler quotient of the overall product of spheres.*

Proof. We check that G_M satisfies the conditions (39). Lemma 8 and the fact that $\langle p_j, x\rangle + c_j$ is strictly positive on Δ imply that G_M differs by a smooth function

$$G_M^0 := \frac{1}{2}\left(\sum_{i=1}^k L_i \log L_i + \sum_{j=1}^N \left(\sum_{r_j=1}^{k_j} \hat{L}^j_{r_j} \log \hat{L}^j_{r_j}\right)\right).$$

It remains to see that $\det(\mathrm{Hess}(G_M))\big(\prod_{i=1}^k L_i\big)\big(\prod_{j,r_j} \hat{L}^j_{r_j}\big)$ is smooth and positive on $\hat{\Delta}$. The determinant $\det(\mathrm{Hess}(G_M)^{-1})$ is, up to a positive scale, the norm with respect to g of the wedge product of the Killing vector fields $(K_1, \ldots, K_\ell, \hat{K}^j_{r_j})$ for $j = 1, \ldots, N$, $r_j = 1, \ldots, d_j$. Using (40) and the specific form (38) of the metric g, one sees that

$$\det(\mathrm{Hess}(G_M)^{-1}) = C\left(\prod_{j=1}^N (\langle p_j, x\rangle + c_j)^{d_j}\right) \\ \times \det(\mathrm{Hess}(G_V)^{-1}) \det(\mathrm{Hess}(G_j)^{-1}),$$

where $C > 0$ is constant. Using the compactification criteria (39) for each G_V and G_j, and the fact that $\hat{\Delta}$ is a simple polytope, near any point y on a face $\hat{F} \subseteq \hat{\Delta}$ we have:

$$
\begin{aligned}
(42) \qquad \det(\mathrm{Hess}(G_M)^{-1}) &= \delta \left(\prod_{i=1}^{k} L_k\right)\left(\prod_{j=1}^{N}(\langle p_j, x\rangle + c_j)^{d_j}\left(\prod_{r_j=1}^{k_j} L^j_{r_j}\right)\right)\\
&= \delta' \left(\prod_{i=1}^{k} L_k\right)\prod_{j=1}^{N}\left(\prod_{r_j=1}^{k_j} \hat{L}^j_{r_j}\right),
\end{aligned}
$$

where δ and δ' are smooth positive functions around $y \in \hat{\Delta}$.

For the second part, we assume by Theorem 5 that $G_V = \frac{1}{2}\sum_{ir} L_{ir}\log L_{ir}$ and $G_j = \frac{1}{2}\sum_{q_j r_j} L^j_{q_j r_j}\log L^j_{q_j r_j}$ with $\sum_r L_{ir} = 0$ and $\sum_{r_j} L^j_{q_j r_j} = 0$. Then, by Lemma 8,

$$
\begin{aligned}
G_M &= \frac{1}{2}\left(\sum_{ir} L_{ir}\log L_{ir} + \sum_{j=1}^{N}(\langle p_j, x\rangle + c_j)\left(\sum_{q_j r_j} L^j_{q_j r_j}\log L_{q_j r_j}\right)\right)\\
&= \frac{1}{2}\left(\sum_{ir} L_{ir}\log L_{ir} + \sum_{j=1}^{N}\left(\sum_{q_j r_j} \hat{L}^j_{q_j r_j}\log L_{q_j r_j}\right)\right)\\
&= \frac{1}{2}\left(\sum_{ir} L_{ir}\log L_{ir}\right.\\
&\qquad \left. + \sum_{j=1}^{N}\left(\sum_{q_j r_j}\left(\hat{L}^j_{q_j r_j}\log \hat{L}_{q_j r_j} - \sum_{q_j r_j} L_{q_j p_j}\log(\langle p_j, x\rangle + c_j)\right)\right)\right)\\
&= \frac{1}{2}\left(\sum_{ir} L_{ir}\log L_{ir} + \sum_{j=1}^{N}\left(\sum_{q_j r_j} \hat{L}^j_{q_j r_j}\log \hat{L}_{q_j r_j}\right)\right)
\end{aligned}
$$

We conclude by using Theorem 5 again. □

Remark 9. The toric Calabi construction provides a practical method for constructing new toric metrics from old ones. Suppose (g_j, ω_j) and (g^V, ω_V) are toric Kähler metrics on $\mathring{\Delta}_j \times \mathbb{T}^{d_j}$ and $\mathring{\Delta} \times \mathbb{T}^{\ell}$ respectively, for labelled (simple convex, compact) polytopes $\Delta_j = \{x^j \in \mathbb{R}^{d_j} : L^j_{r_j}(x^j) \geq 0,\ \ r_j = 1, \dots, k_j\}$ $(j = 1, \dots, N)$ and $\Delta = \{x \in \mathbb{R}^{\ell} : L_i(x) \geq 0,\ \ i = 1, \dots, N\}$. Let $p_j = (p_{j1}, \dots, p_{j\ell})$, $\theta = (\theta_1, \dots, \theta_\ell)$ with

$$
\theta_i = dt_i + \sum_{j=1}^{N} p_{ji}\left(\sum_{r_j=1}^{d_j} x^j_{r_j} dt^j_{r_j}\right),
$$

where the affine functions $\sum_{i=1}^{\ell} p_{ji}x_i + c_j$ are positive on Δ. Then, as in Propostion 7, we get a toric Kähler metric on the interior of $\hat{\Delta}$ times $\mathbb{T}^\ell \times \mathbb{T}^{d_1} \times \cdots \times \mathbb{T}^{d_N}$, whose symplectic potential G_M is given by Lemma 8. Taking all the affine functions $L^j_{r_j}$ and L_i, and $(\sum_{i=1}^{\ell} p_{ji}x_i + c_j)$ be all with rational coefficients, one gets a rational labelled polytope $\hat{\Delta}$, and if the ingredient metrics (g_j, ω_j) and (g^V, ω_V) satisfty the Abreu boundary conditions (39) for the corresponding labellings $L^j_{r_j}$ and L_i, then so does the metric given (locally) by the toric generalized Calabi Ansatz, with the labelling defined in Proposition 7. Hence the metric compactifies on the compact toric orbifold M given by $\hat{\Delta}$ and these labels.

4. Curvature of Levi–Kähler quotients of products of spheres

4.1. Bochner condition

Recall (see e.g. [5, 22]) that if $R \in \wedge^{1,1}\mathbb{C}^{m*} \otimes \mathfrak{u}(m)$ is a *formal Kähler curvature tensor*, i.e., $R_{u,v}(w) + R_{v,w}(u) + R_{w,u}(v) = 0$, then the *Bochner part* of R is its orthogonal projection $\mathcal{B}(R)$ onto the $U(m)$-submodule of formal Kähler curvature tensors with vanishing Ricci trace. The *Bochner tensor* B^g of a Kähler manifold is then the (pointwise) Bochner part $\mathcal{B}(R^g)$ of its riemannian curvature $R^g \in C^\infty(\wedge^{1,1}T^*M \otimes \mathfrak{u}(TM))$. One can extend this definition to more general hermitian curvature tensors $R \in \wedge^2\mathbb{C}^{m*} \otimes \mathfrak{u}(m)$ (which do not a priori satisfy the Bianchi identity) where one still denotes by $\mathcal{B}(R)$ the orthogonal projection of an element R onto the $U(m)$-submodule of formal Kähler curvature tensors with vanishing Ricci trace.

In our language, S. Webster [48] showed that in codimension one the Bochner tensor of a Levi–Kähler quotient M of CR manifold N pulls back to the Chern–Moser tensor of N, which vanishes when the N is locally CR diffeomorphic to a standard CR sphere $(\mathbb{S}^{2m+1}, \mathcal{D}, J)$. In particular, every Levi–Kähler quotient (M^{2m}, g, J) of $\mathbb{S}^{2m+1}$ is Bochner-flat. We generalize this to arbitrary codimension, by describing the curvature of Kähler metrics arising as Levi–Kähler quotients of a product of CR-spheres.

Let $N = \mathbb{S}^{2m_1+1} \times \cdots \times \mathbb{S}^{2m_\ell+1} \subseteq \mathbb{C}_{\mathcal{S}}$ be a product of standard CR spheres and $\big(\mathcal{D} = \bigoplus_{i\in\mathcal{I}} \mathcal{D}_i, J = \bigoplus_{i\in\mathcal{I}} J_i\big)$ be the product CR structure and denote by $N_i \cong \mathbb{S}^{2m_i+1}$ the i-th factor of N, with projection $p_i\colon N \to N_i$. The bundle $p_i^*TN_i$ is identified with the subbundle $E^i := \bigcap_{j\neq i} \ker(p_{j*})$ of TN via the restriction $p_{i*}\colon E^i_z \xrightarrow{\simeq} T_{p_i(z)}N_i$. We denote the projection $r_i\colon TN \to E^i$.

Let $(\mathfrak{g}, \lambda)$ be a positive Levi pair (corresponding to $\mathbf{L}$) and assume that $\mathfrak{g}$ is the Lie algebra of a subtorus G of $\mathbb{T}_{\mathcal{S}}$; denote by $M = N/G$ the Levi–Kähler quotient and by $\pi\colon N \to M$ the quotient map. The global assumption on $\mathfrak{g}$ is made purely to make statements about M rather than local quotients. In addition to a Kähler structure $(\check{g} = h_{\mathbf{L}}, J, \omega_{\mathbf{L}})$, M inherits a $\check{g}$-orthogonal splitting of its tangent space, namely

$$TM = \bigoplus_{i\in\mathcal{I}} \check{\mathcal{D}}_i \tag{43}$$

where $\check{\mathcal{D}}_i = \pi_*(\mathcal{D}_i)$. We denote by $\check{\nabla}^i$ the connection on $\check{\mathcal{D}}_i$ induced by the Levi–Civita connection $\check{\nabla}$ of $\check{g}$, by $R^{\check{\nabla}^i}$ the corresponding curvature tensor (a section of $\wedge^2 T^*M \otimes \mathfrak{gl}(\check{\mathcal{D}}_i, J)$, where $\mathfrak{gl}(\check{\mathcal{D}}_i, J)$ denotes the bundle of J-commuting endomorphisms of $\check{\mathcal{D}}_i$), and by $B^i := \mathcal{B}^i(R^{\check{\nabla}^i})$ the Bochner projection of $R^{\check{\nabla}^i}|_{\check{\mathcal{D}}_i} \in \wedge^2\check{\mathcal{D}}_i^* \otimes \mathfrak{gl}(\mathcal{D}_i, J)$.

Proposition 8. *For each $i \in \mathcal{I}$, $B^i = 0$.*

We prove this result using the observation (see [27] and [48]) that the Chern–Moser tensor of $(N_i, J_i, \mathcal{D}_i)$ may be computed from the horizontal part of the curvature of the *Tanaka connection* (see e.g. [16]) associated to any contact form α compatible with the (codimension 1) CR structure $(\mathcal{D}, J)$. The Chern–Moser tensor does not depend upon the chosen compatible contact structure (it is a CR invariant). If the Reeb vector field V of α (determined by $\alpha(V) = 1$ and $\mathcal{L}_V\alpha = 0$) is a transverse CR vector field, the horizontal part of the Tanaka connection (i.e., its restriction on $\mathcal{D}$) is the pullback to N of the Levi-Civita connection of the Levi–Kähler quotient (M, g, J, ω) of N by V using the identification $\mathcal{D} \cong \pi^*TM$.

In our situation, for each $w \in \prod_{j\neq i} N_j$, there is an embedding $\iota_w\colon N_i \hookrightarrow N$. We shall (slightly abusively) still denote by $\mathcal{D}_i$ the pullback bundle $\iota_w^*\mathcal{D}_i$ and by J_i the corresponding (standard) CR-structure on $(N_i, \mathcal{D}_i)$. Recall that N is endowed with a 1-form η^λ with $d\eta^\lambda = \pi^*\omega_{\mathbf{L}}$, where $\lambda \in \mathfrak{g}^*$ is the value defining the Levi–Kähler quotient $\omega_{\mathbf{L}}$. The pullback $\alpha_{w,i} := \iota_w^*\eta^\lambda$ thus defines a 1-form on N_i, and the next Lemma implies that $\alpha_{w,i}$ is a contact 1-form compatible with the CR structure $(N_i, J_i, \mathcal{D}_i)$.

Lemma 9. *For any $i \in \{1, \ldots, \ell\}$ and $z \in N$, the subspace of E^i_z defined by*

$$\mathcal{R}^{\mathfrak{g},i}_z := r_i(K_{\eta(E^i_z)})$$

has dimension 1 and is transverse to $\mathcal{D}_i$. In particular, $\mathcal{R}^{\mathfrak{g},i} \to N$ is a real line bundle and there exists a unique vector field $V_i \in C^\infty(\mathcal{R}^{\mathfrak{g},i})$ such that

$\eta^\lambda(V_i) = 1$. Furthermore, for each $w \in \prod_{j\neq i} N_j$, $\alpha_{w,i}$ is a contact form on N_i, which induces $\iota_{w,i}^* \circ \pi^*\omega_{\mathbf{L}}$ as a transversal symplectic structure and the vector field $V_{w,i}$ on $(N_i, \mathcal{D}_i)$ defined by $\iota_{w*}V_{w,i} = V_i$ is a Reeb vector field for $\alpha_{w,i}$.

Proof. Since $\eta(E^i_z) = \eta\big((E^i/\mathcal{D}_i)_z\big)$, $K_{\eta(E^i_z)} \subseteq \mathcal{K}^{\mathfrak{g}}$ is at most 1-dimensional. Since $\sum_{i=1}^{\ell} K_{\eta(E^i_z)} = K_{\eta(T_zN)} = \mathcal{K}^{\mathfrak{g}}_z$ is ℓ-dimensional, it follows that $K_{\eta(E^i_z)}$ is 1-dimensional and $\dim(\mathcal{R}^{\mathfrak{g},i}_z) \leqslant 1$. As $\mathcal{K}^{\mathfrak{g}}$ is transversal to $\mathcal{D}$ ($(\mathfrak{g},\lambda)$ being Levi pair), each $X \in E^i_z$ decomposes as $X = K_{\eta(X)} + X^{\mathcal{D}}$; applying r_i (and using $r_i(X) = X$, $r_i(\mathcal{D}) = \mathcal{D}_i$), we obtain $X = r_i(K_{\eta(X)}) + X^{\mathcal{D}_i}$, showing $E^i = \mathcal{R}^{\mathfrak{g},i} \oplus \mathcal{D}_i$.

For any $Y, Z \in \mathcal{D}_i$,

$$d\alpha_{w,i}(Y, Z) = -\alpha_{w,i}([Y, Z]) = -\eta^\lambda([Y, Z]) = \pi^*\omega_{\mathbf{L}}(Y, Z).$$

In order to show that $V_{w,i}$ is Reeb field for $\alpha_{w,i}$ we need to check that for any $Z \in \mathcal{D}_i$, $d\alpha_{w,i}(V_{w,i}, Z) = d\eta^\lambda(V_i, Z) = 0$ (where we have used that E^i is integrable and $\iota_w^*(E_i) \cong TN_i$). Writing $V_i = r_i(K_{\eta(X)}) = X - X^{\mathcal{D}_i}$ for $X \in E^i$, and decomposing $X = K_{\eta(X)} + X^{\mathcal{D}}$, we have

$$d\eta^\lambda(V_i, Z) = d\eta^\lambda(X - X^{\mathcal{D}_i}, Z) = d\eta^\lambda(K_{\eta(X)}, Z) + \sum_{j\neq i} d\eta^\lambda(X^{\mathcal{D}_j}, Z) = 0,$$

where (in the last equality) the first term vanishes because η^λ is $\mathfrak{g}$-invariant, whereas the second term vanishes because for $j \neq i$, $\mathcal{D}_i$ is $d\eta^\lambda$-orthogonal to $\mathcal{D}_j$, see (43). □

We define a partial connection $\nabla^i\colon C^\infty(E^i) \times C^\infty(E^i) \to C^\infty(E^i)$ on the involutive subbundle $E^i \subseteq TN$, which pulls back by ι_w to the Tanaka connection of $\alpha_{w,i}$, as follows:

- ∇^i preserves $\mathcal{D}_i$;
- V_i, J_i and ω are ∇^i parallel;
- For any $X, Y \in C^\infty(\mathcal{D}_i)$, the torsion T^{∇^i} satisfies $T^{\nabla^i}(X, Y) = \omega(X, Y)V_i$ and $T^{\nabla^i}(V_i, J_iX) = -J_iT^{\nabla^i}(V_i, X)$.

In particular, ∇^i satisfies

$$g(\nabla^i_X Y, Z) = \check{g}(\check{\nabla}_{\check{X}}\check{Y}, \check{Z}) = \check{g}(\check{\nabla}^i_{\check{X}}\check{Y}, \check{Z})$$

We next show that ∇^i can be extended to a full connection ∇ on TN preserving $\mathcal{D}$, and such that:

- $\nabla|_{\mathcal{D}} = \pi^*\check{\nabla}$;
- $\iota_w^*(r_i \circ \nabla)$ is the Tanaka connection $\iota_w^*\nabla^i$ on $(N_i, \alpha_i, \mathcal{D}_i, J_i)$.

The first condition tells us that the torsion of $X, Y \in C^\infty(\mathcal{D})$ is the vertical part of $-[X, Y]$, that is

$$T^\nabla(X, Y) = -K_{\eta([X,Y])}. \tag{44}$$

Hence for $X, Y \in C^\infty(\mathcal{D}_i)$,

$$r_i(T^\nabla(X, Y)) = -r_i(K_{\eta([X,Y])}) = \omega(X, Y)V_i = T^{\nabla^i}(X, Y).$$

Let $\mathcal{R}^{\mathfrak{g}} := \bigoplus_{i=1}^{\ell} \mathcal{R}^{\mathfrak{g},i}$ be the rank ℓ subbundle of TN over N which is everywhere transverse to $\mathcal{D}$. We extend the endomorphism J of $\mathcal{D}$ by zero on $\mathcal{R}^{\mathfrak{g}}$, and define a linear connection ∇ on

$$TN = \bigoplus_{i \in \mathcal{I}} E^i = \bigoplus_{i \in \mathcal{I}} (\mathcal{D}_i \oplus \mathcal{R}^{\mathfrak{g},i}) = \mathcal{D} \oplus \mathcal{R}^{\mathfrak{g}}$$

such that

(i) ∇ agrees with the pullback connection $\pi^*\nabla^g$ on $\mathcal{D} \cong \pi^*TM$,

(ii) $\nabla V_i = 0$,

(iii) $\nabla_s X = [s, X] + \nabla_X s - \frac{1}{2}J(\mathcal{L}_s J)X$ for each section $s \in C^\infty(\mathcal{R}^{\mathfrak{g}})$ and $X \in C^\infty(\mathcal{D})$.

Lemma 10. *Let ∇ be a connection on N satisfying the conditions* (i)–(iii) *above. Then, $\iota_w^*(r_i \circ \nabla)$ is the Tanaka connection $\iota_w^*\nabla^i$ on $(N_i, \alpha_i, \mathcal{D}_i, J_i)$.*

Proof. We first show that $\nabla_s JX = J\nabla_s X$ for any $X \in C^\infty(TN)$ and $s \in C^\infty(\mathcal{R}^{\mathfrak{g}})$. It is clear that $\nabla_s Jt = J\nabla_s t = 0$ for $s, t \in C^\infty(\mathcal{R}^{\mathfrak{g}})$ and when $X \in C^\infty(\mathcal{D})$, we have $J\nabla_X s = 0$ as well as the decomposition

$$[s, X] = [s, X]^{\mathcal{D}} - \nabla_X s$$

with respect to the splitting $TN = \mathcal{D} \oplus \mathcal{R}^{\mathfrak{g}}$. Using these two facts and condition (iii), a straightforward calculation shows $\nabla_s JX = J\nabla_s X$. Together with this last identity, conditions (i)–(ii) ensure that $r_i \circ \nabla$ satisfies the first two defining properties of ∇^i on E^i. It remains to check the torsion properties. The first of these follows from (44). Moreover, using again condition (iii) we get

$$T^{\nabla}(s, JX) = -\tfrac{1}{2}J(\mathcal{L}_s J)JX = \tfrac{1}{2}J^2(\mathcal{L}_s J)X = -JT^{\nabla}(s, X)$$

for any $X \in C^\infty(TN)$ and $s \in C^\infty(\mathcal{R}^{\mathfrak{g}})$. This is the second torsion property. $\square$

Proof of Proposition 8. To compare the curvatures ∇^i and $\check{\nabla}^i$, we notice that

$$\begin{aligned} g(\nabla^i_{[X,Y]}Z, T) &= g(\nabla^i_{[X,Y]^{\mathcal{D}_i}}Z, T) - \omega_{\mathbf{L}}(\check{X}, \check{Y})g(\nabla^i_{V_i}Z, T) \\ &= g(\nabla_{[X,Y]^{\mathcal{D}}}Z, T) - \omega_{\mathbf{L}}(\check{X}, \check{Y})g(\nabla_{V_i^{\mathcal{K}}}Z, T) \\ &= \check{g}(\check{\nabla}^i_{[\check{X},\check{Y}]}\check{Z}, \check{T}) + \omega_{\mathbf{L}}(\check{X}, \check{Y})\check{A}^i(\check{Z}, \check{T}), \end{aligned}$$

where to go from first line to the third we have decomposed

$$\omega(X, Y)V_i = \omega(X, Y)V_i^{\mathcal{K}^{\mathfrak{g}}} - \sum_{j\neq i}[X, Y]^{\mathcal{D}_j}$$

(with $V_i^{\mathcal{K}^{\mathfrak{g}}}$ being the projection of V_i to $\mathcal{K}^{\mathfrak{g}}$), and we view $\check{A}^i(\check{Z}, \check{T}) = -g(\nabla_{V_i^{\mathcal{K}^{\mathfrak{g}}}}Z, T)$ as a $(0, 2)$-tensor on M, since pullbacks to N of smooth functions on M are $\mathcal{K}^{\mathfrak{g}}$-invariant. It then follows that

$$g(R^{\nabla^i}_{X,Y}Z, T) = \check{g}(R^{\check{\nabla}^i}_{\check{X},\check{Y}}\check{Z}, \check{T}) + \omega_{\mathbf{L}}(\check{X}, \check{Y})\check{A}^i(\check{Z}, \check{T}).$$

Applying the projection $\mathcal{B}^i$ to the both sides, and using that $\mathcal{B}^i(R^{\nabla^i})$ equals the Chern–Moser tensor of N_i (which is zero) as well as $\mathcal{B}^i(\omega_{\mathbf{L}} \otimes A^i) = 0$ (as $\mathcal{B}^i$ projects onto the space of $\omega_{\mathbf{L}}$-primitive Kähler curvature tensors), we obtain $B^i = \mathcal{B}^i(R^{\check{\nabla}^i}) = 0$. $\square$

4.2. Curvature for Levi–Kähler quotients of products of 3-spheres

A *labelled cuboid* $(\Delta, \mathbf{L})$ is a labelled Delzant polytope which has the combinatorics of an m-cube. By Corollary 2, any toric symplectic orbifold whose rational Delzant polytope $(\Delta, \mathbf{L})$ is a labelled cuboid admits a compatible toric Kahler metric $h_{\mathbf{L}}$ which is a Levi–Kähler quotient of an m-fold product of 3-spheres. We next use the explicit form (19) of the Kähler metric $h_{\mathbf{L}}$ in order to compute its scalar curvature. Up to a factor of $-1/2$, a Ricci potential is given by the log ratio of the symplectic volume form to a holomorphic volume form. Using $d\mathbf{t}^{(1,0)}$ to compute the latter, we readily obtain

$$\sum_{i\in\mathcal{I}} \log A_i(\xi_i) + \sum_{i\in\mathcal{I}} \log N_{i\infty}(\mu^\lambda) - 2\log \bigwedge_{i\in\mathcal{I}} (\xi_i dN_{i\infty} + dN_{i0}).$$

The derivatives of the first two terms are straightforward to compute, using that $d(N_{j\infty}(\mu^\lambda)) = -\sum_{i\in\mathcal{I}} N_{i\infty}(\mu^\lambda)\langle \mathbf{Q}_i, dN_{j\infty}\rangle d\xi_i$. For the third, observe by Cramer's rule that its exterior derivative is $\sum_{i\in\mathcal{I}} 2a_{ii}\, d\xi_i$ where the coefficients a_{ij} solve the linear system $\sum_{j\in\mathcal{I}}(\xi_j dN_{j\infty} + dN_{j0})a_{ij} = dN_{i\infty}$, i.e., $a_{ij} = \langle \mathbf{Q}_j, dN_{i\infty}\rangle$. Thus

$$\rho_{\mathbf{L}} = \frac{1}{2}\bigg(d\sum_{i\in\mathcal{I}} \alpha_i\theta_i\bigg), \tag{45}$$

where

$$\alpha_i = A_i'(\xi_i) - \sum_{j\in\mathcal{I}} \frac{\langle \mathbf{Q}_i, dN_{j\infty}\rangle N_{i\infty}(\mu^\lambda)}{N_{j\infty}(\mu^\lambda)} A_i(\xi_i) - 2\langle \mathbf{Q}_i, dN_{i\infty}\rangle A_i(\xi_i)$$

and $\theta_i = \langle \mathbf{Q}_i, d\mathbf{t}\rangle$. We are interested in the scalar curvature only, defined by

$$s_{\mathbf{L}} = \frac{2m\rho_{\mathbf{L}}\wedge\omega^{m-1}}{\omega^m}, \qquad \frac{\omega^{m-1}}{\omega^m} = -\frac{1}{m}\sum_{j\in\mathcal{I}} \frac{\prod_{k\neq j} d\xi_k\wedge\theta_k}{N_{j\infty}(\mu^\lambda)\, d\xi_1\wedge\theta_1\wedge\cdots\wedge d\xi_m\wedge\theta_m}.$$

Straightforward computation using (45) then yields

$$\begin{aligned} s_{\mathbf{L}} = &-\sum_{i\in\mathcal{I}} \frac{A_i''(\xi_i)}{N_{i\infty}(\mu^\lambda)} + \sum_{i\in\mathcal{I}} 2A_i'(\xi_i)\bigg(\mathbf{Q}_{ii} + \sum_{j\in\mathcal{I}} \mathbf{Q}_{ij}\bigg) \\ &- \sum_{i\in\mathcal{I}} A_i(\xi_i) N_{i\infty}(\mu^\lambda)\bigg(2\frac{\langle \mathbf{Q}_i, dN_{j\infty}\rangle^2}{N_{i\infty}(\mu^\lambda)^2} + \sum_{j\in\mathcal{I}}(4\mathbf{Q}_{ii}\mathbf{Q}_{ij} - \mathbf{Q}_{ij}^2) + \sum_{j,k} \mathbf{Q}_{ij}\mathbf{Q}_{ik}\bigg). \end{aligned} \tag{46}$$

where $\mathbf{Q}_{ij} = \langle \mathbf{Q}_i, dN_{j\infty}\rangle / N_{j\infty}(\mu^\lambda)$.

We now specialize to the case of projective cubes as in Section 3.3, where $N_{i\infty}(\mu^\lambda) = \mu_0 = 1/(b_0 + b_1\xi_1 + \cdots b_m\xi_m)$, independent of $1 \leqslant i \leqslant m$. Using

$$d\mu_0 = \langle dN_{j\infty}, d\mu\rangle = -\sum_{i=1}^m \mu_0 d\xi_i \langle \mathbf{Q}_i, dN_{j\infty}\rangle,$$

we obtain immediately that $\mathbf{Q}_{ij} = b_i$. The Ricci potential specializes to give

$$\mu_0^{m+2} \prod_{i=1}^{m} A_i(\xi_i), \tag{47}$$

as may be verified directly using (28) and (30), while the scalar curvature reduces to

$$\begin{aligned} s_{\mathbf{L}} = &- \sum_{i \in \mathcal{I}} \frac{A_i''(\xi_i)}{\mu_0} + \sum_{i \in \mathcal{I}} 2(m+1) b_i A_i'(\xi_i) \\ &- \sum_{i \in \mathcal{I}} (m+1)(m+2) \mu_0 b_i^2 A_i(\xi_i). \end{aligned} \tag{48}$$

4.3. Projective cubes and (w, p)-extremality

We now specialize to the case that the cuboid Δ is a projective cube, i.e., the intersections of pairs of opposite facets lie in a hyperplane. For any labelled projective cube $(\Delta, \mathbf{L})$, $h_{\mathbf{L}}$ is given by (30), and we provide here a characterization of this toric metric in terms of the toric geometry of $(\Delta, \mathbf{L})$, as developed in [2, 31, 37].

The starting point of our approach is based on a recent observation in [11], which in turn extends the formal GIT framework of [30, 33] (realizing the scalar curvature of a Kähler metric as a momentum map under the action of the group of hamiltonian transformations) to a larger family of related GIT problems. Let (M, ω) be a compact symplectic manifold (or orbifold) and $\mathrm{Ham}(M, \omega)$ the group of hamiltonian transformations. Fix a torus $\mathbb{T} \leqslant \mathrm{Ham}(M, \omega)$ and a positive hamiltonian $w > 0$ with $\mathrm{grad}_\omega w \in \mathfrak{t} := \mathrm{Lie}(\mathbb{T})$. Let $\mathcal{C}^{\mathbb{T}}(M, \omega)$ be the space of $\mathbb{T}$-invariant, ω-compatible complex structures on (M, ω) and $\mathrm{Ham}^{\mathbb{T}}(M, \omega)$ the subgroup of $\mathbb{T}$-equivariant hamiltonian transformation, acting naturally on $\mathcal{C}^{\mathbb{T}}(M, \omega)$. The Lie algebra of $\mathrm{Ham}^{\mathbb{T}}(M, \omega)$ is identified with the space $C_0^\infty(M)^{\mathbb{T}}$ of smooth, $\mathbb{T}$-invariant functions of integral zero, endowed with the $\mathrm{Ham}^{\mathbb{T}}(M, \omega)$ bi-invariant inner product

$$\langle h_1, h_2 \rangle_{w,p} := \int_M h_1 h_2 w^{-(p+1)} v_\omega, \tag{49}$$

where p is a real constant (which we call the *conformal dimension*) and $v_\omega = \omega^m / m!$ is the volume form of ω. The space $\mathcal{C}^{\mathbb{T}}(M, \omega)$ carries a formal

Fréchet Kähler structure, $(\mathbf{J}, \mathbf{\Omega}^{w,p})$, defined by

$$\mathbf{J}_J(\dot{J}) = J\dot{J}, \qquad \mathbf{\Omega}^{w,p}_J(\dot{J}_1, \dot{J}_2) = \frac{1}{2}\int_M \mathrm{tr}\Big(J\dot{J}_1\dot{J}_2\Big) w^{-(p-1)} v_\omega,$$

where the tangent space of $\mathcal{C}^{\mathbb{T}}(M,\omega)$ at J is identified to be the Fréchet space of smooth sections $\dot{J}$ of $\mathrm{End}(TM)$ satisfying

$$\dot{J}J + J\dot{J} = 0, \qquad \omega(\dot{J}\cdot, \cdot) + \omega(\cdot, \dot{J}\cdot) = 0.$$

The formal complex structure $\mathbf{J}$ is the same as the one in [30, 33] whereas the modified formal symplectic form $\mathbf{\Omega}^{w,p}$ stays closed (as can easily be checked).

In the following, we denote by g_J the Kähler metric corresponding to $J \in \mathcal{C}^{\mathbb{T}}(M,\omega)$ and by s_J and Δ_J the corresponding scalar curvature and Laplace operator. We then have a straightforward (mutatis mutandis) generalization of [11, Thm. 1] (which corresponds to $p = 2m$).

Lemma 11. *The action of* $\mathrm{Ham}^{\mathbb{T}}(M,\omega)$ *on* $(\mathcal{C}^{\mathbb{T}}(M,\omega), \mathbf{J}, \mathbf{\Omega}^{w,p})$ *is hamiltonian with a momentum map* $\mu\colon \mathcal{C}^{\mathbb{T}}(M,\omega) \to \big(C^\infty_0(M)^{\mathbb{T}}\big)^*$*, whose value at J is identified with the* $\langle\cdot,\cdot\rangle_{w,p}$*-dual of*

$$s_{J,w,p} := w^2 s_J - 2(p-1)w\Delta_J w - p(p-1)g_J^{-1}(dw, dw). \tag{50}$$

Note that $s_{J,w,p}$ is the trace of the conformal modification

$$\rho_{J,w,p} := w^2\rho_J + (p-1)w dd^c w - \tfrac{1}{2}p(p-1)dw \wedge d^c w \tag{51}$$

of the Ricci form.

Remark 10. As in [11], one can extend the definition of $(\mathbf{J}, \mathbf{\Omega}^{w,p})$ on $\mathcal{C}^{\mathbb{T}}(M,\omega)$ to the larger Frechét space $\mathcal{AK}^{\mathbb{T}}(M,\omega)$ of $\mathbb{T}$-invariant ω-compatible almost-Kähler structures J. Then, using the formulae in [35, Ch. 8] (see also [41, Lemma 2.1 & Prop. 3.1]), the momentum map for the action of $\mathrm{Ham}^{\mathbb{T}}(M,\omega)$ on $\mathcal{AK}^{\mathbb{T}}(M,\omega)$ is still given by Lemma 11, except that in (50) we must take s_J to be the *hermitian* scalar curvature of g_J (the trace of the Ricci form of the canonical hermitian connection, see e.g. [41]).

Using the contractibility of $\mathcal{AK}^G_\omega$, we obtain, as in [11], a generalized *Futaki invariant* $\mathfrak{F}^{\mathbb{T}}_{\omega,w,p}\colon \mathfrak{t} \to \mathbb{R}$ of $(M,\omega,\mathbb{T},w,p)$: for any vector field $H \in \mathfrak{t}$

with a hamiltonian h,

$$\mathfrak{F}^{\mathbb{T}}_{\omega,w,p}(H) := \int_M \mathring{s}_{J,w,p} h w^{-(p+1)} v_\omega \tag{52}$$

is independent of the choice of $J \in \mathcal{AK}^G_\omega$, where $\mathring{s}_{J,w,p}$ is the L^2-projection of $s_{J,w,p}$ onto functions with integral zero with respect to the volume form $w^{-(p+1)} v_\omega$.

Specializing to the case of a toric manifold (or orbifold) $(M, \omega, \mathbb{T})$, the above formalism allows one to extend the theory of extremal toric metrics from [31] to the (w,p)-extremal toric case (the case $p = 2m$ is developed in detail in [11]). In particular, we have the following result:

Proposition 9. *Let $(M, \omega, \mathbb{T})$ be a compact toric orbifold with labelled Delzant polytope $(\Delta, \mathbf{L})$ in $\mathbb{R}^m$ and w a positive affine-linear function on Δ. Then,*

(a) *There exists at most one (up to equivariant isometry) compatible toric metric g_J on $(M, \omega, \mathbb{T})$, for which $s_{J,w,p}$ is an affine-linear function.*

(b) *The affine-linear function in* (a) *is uniquely determined by $(\Delta, \mathbf{L}, w, p)$.*

Definition 12. The (unique) compatible toric metric satisfying the condition (a) of Proposition 9 is called the (w,p)-*extremal* metric of $(M, \omega, \mathbb{T})$.

Theorem 6. *Suppose $(\Delta, \mathbf{L})$ is a labelled projective cube in $\mathbb{R}^m$, corresponding to a compact toric orbifold $(M, \omega, \mathbb{T})$ and let $h_{\mathbf{L}}$ be the Levi–Kähler quotient metric defined by* (30). *Then $h_{\mathbf{L}}$ is the $(w, m+2)$-extremal metric of $(\Delta, \mathbf{L})$, where w is the unique up to scale positive affine-linear function on $\mathbb{R}^m$, vanishing on the hyperplane containing the intersections of opposite facets of Δ.*

Proof. We take $w = \mu_0$ and apply (51) with $p = m+2$, $\rho_J = \rho_{\mathbf{L}}$ to obtain

$$\begin{aligned} \rho_{\mathbf{L},\mu_0} &= \mu_0^2 \rho_J + (m+1)\big(\mu_0 dd^c \mu_0 - \tfrac{1}{2}(m+2) d\mu_0 \wedge d^c \mu_0\big) \\ &= -\tfrac{1}{2}\mu_0^2 dd^c \log\Big(\mu_0^{m+2} \prod_i A_i(\xi_i)\Big) \\ &\quad + (m+1)\big(\mu_0^2 dd^c \log \mu_0 - \tfrac{m}{2} d\mu_0 \wedge d^c \mu_0\big) \end{aligned} \tag{53}$$

$$= \tfrac{1}{2}\Big(-\mu_0^2 dd^c \log\prod_i A_i(\xi_i) + m\mu_0^2 dd^c\log\mu_0 - m(m+1)d\mu_0\wedge d^c\mu_0\Big).$$

We now compute

$$\begin{aligned} dd^c\log\prod_j A_i(\xi_i) &= \sum_i d\big(A_i'(\xi_i)\theta_i\big)\\ &= \sum_i A_i''(\xi_i)d\xi_i\wedge\theta_i - \sum_{i,j} b_i\mu_0 A_i'(\xi_i)d\xi_j\wedge\theta_j\\ dd^c\log\mu_0 &= -\sum_i d\big(b_i\mu_0 A_i(\xi_i)\theta_i\big)\\ &= -\sum_i b_i\mu_0 A_i'(\xi_i)d\xi_i\wedge\theta_i\\ &\quad + \sum_{i,j} A_i(\xi_i)\big(b_ib_j\mu_0^2 d\xi_j\wedge\theta_i + b_i^2\mu_0^2 d\xi_j\wedge\theta_j\big)d\mu_0\wedge d^c\mu_0\\ &= \sum_{i,j} b_i^2\mu_0^4 A_i(\xi_i)d\xi_i\wedge\theta_j. \end{aligned}$$

It follows that

$$s_{J,\mu_0} = \frac{2m\rho_{\mathbf{L},\mu_0}\wedge\omega^{m-1}}{\omega^m} = -\frac{\sum_{i=1}^m A_i''(\xi_i)}{b_0+b_1\xi_1+\cdots+b_m\xi_m},$$

which (as $\deg A_j(\xi)\leqslant 3$) is an affine-linear function in momenta. □

Remark 11. We notice that for $m=2$, $m+2=2m$ and $s_{J,w,4}$ computes the scalar curvature of the conformal oppositely oriented metric $\tilde{h}_{\mathbf{L}} = (1/w^2)h_{\mathbf{L}}$ of Proposition 6, i.e., $h_{\mathbf{L}}$ is w-extremal in the sense of [11].

4.4. Extremal Levi–Kähler quotients

Formula (48) shows that for $\ell = m > 2$, the Levi–Kähler metric $h_{\mathbf{L}}$ associated to a projective cube cannot be extremal unless $b_i = 0, i = 1,\dots,m$, i.e., M is the product of weighted projective lines. However, we show below that when $\ell = 2$, Levi–Kähler quotients of a product of two spheres $\mathbb{S}^{2m_1+1}\times\mathbb{S}^{2m_2+1}$ can provide new examples of extremal Kähler orbifolds.

4.4.1. Extremal Levi–Kähler quotients of $\mathbb{S}^3\times\mathbb{S}^3$. As any quadrilateral is a projective cube, $h_{\mathbf{L}}$ is $(w,4)$-extremal by Theorem 6. Furthermore, $s_{J,w,4}$ is the scalar curvature of the conformal metric $\tilde{h}_{\mathbf{L}} = (1/w^2)h_{\mathbf{L}}$, see [11]. By Proposition 6, $(h_{\mathbf{L}}, J)$ is either a product, of Calabi type, or a regular ambitoric Kähler metric of Segre type. We can then use [5, 39] to characterize the extremality of $h_{\mathbf{L}}$ as follows.

Proposition 10. *Let (M, ω) be a compact toric 4-orbifold whose rational Delzant polytope is a labelled quadrilateral $(\Delta, \mathbf{L})$ and $h_{\mathbf{L}}$ the corresponding Levi–Kähler metric. If $(\Delta, \mathbf{L})$ is a parallelogram, then $(M, \omega_{\mathbf{L}}, h_{\mathbf{L}})$ is an extremal toric orbifold which is the Kähler product of two extremal weighted projective lines; otherwise $h_{\mathbf{L}}$ is extremal if and only if the oppositely oriented ambitoric metric $\tilde{h}_{\mathbf{L}} = (1/w^2)h_{\mathbf{L}}$ has constant scalar curvature, or equivalently, $\tilde{h}_{\mathbf{L}}$ is a conformally Kähler, Einstein–Maxwell metric in the sense [11], where w is a positive affine linear function on Δ, determined up to positive scale by the property that it vanishes where the opposite sides of Δ intersect.*

Proof. The product case follows from Proposition 6 (see Section 3.4.1). When $h_{\mathbf{L}}$ is of Calabi-type, i.e. given by (36) for polynomials $A(x)$ and $B(y)$ of degree 2 or 3, the negative ambitoric metric $\tilde{h}_{\mathbf{L}} = x^2 h_{\mathbf{L}}$ is also of Calabi-type with respect to the variables (x, y) and functions $A(x)$ and $B(y)$. It follows from [4, 39] that $h_{\mathbf{L}}$ is extremal if and only if $\tilde{h}_{\mathbf{L}}$ is extremal if and only if $B(y)$ has degree 2 and $(xA(x))''(0) = -B''(0)$. As $\deg A \leqslant 3$, this is precisely the condition that $\tilde{h}_{\mathbf{L}}$ is of constant scalar curvature (see [4, Prop. 14]). The case when $h_{\mathbf{L}}$ is regular ambitoric (i.e., negative orthotoric) is treated similarly, using the local form (37) and [4, Prop. 11]. Finally, as $\tilde{h}_{\mathbf{L}} = (1/w^2)h_{\mathbf{L}}$ with w being a Killing potential with respect to $h_{\mathbf{L}}$, the scalar curvature $\tilde{h}_{\mathbf{L}}$ is constant if and only if $\tilde{h}_{\mathbf{L}}$ defines a conformally Kähler, Einstein–Maxwell metric, see [11]. □

Remark 12. The metric $h_{\mathbf{L}}$ is extremal (and hence $\tilde{h}_{\mathbf{L}}$ is Einstein–Maxwell) if and only if the affine function defined by $(\Delta, \mathbf{L})$ in Proposition 9(b) is constant. By an observation originating in [39], for a given quadrilateral Δ this places two linear constraints on the labels $\mathbf{L}$, see also [11]. Thus, the above characterization for the extremality of $h_{\mathbf{L}}$ lead to the following useful observation: given a compact convex quadrilateral Δ which is not a parallelogram, there is a two-parameter family of inward normals to the faces, such that the corresponding Levi–Kähler metric is extremal.

4.4.2. CSC Levi–Kähler quotients of $\mathbb{S}^5 \times \mathbb{S}^3$. We discuss here examples of Levi–Kähler quotients of constant scalar curvature (CSC), obtained from the generalized Calabi construction in §3.5, where the base is $S = \mathbb{C}P^1$ equipped with a Fubini–Study metric ω_S and the fibre V is a toric orbifold with Delzant image a simplex in $\mathbb{R}^2$. By Proposition 7, the resulting 6-dimensional orbifold (M, g, ω) is obtained as a Levi–Kähler quotient of $\mathbb{S}^5 \times \mathbb{S}^3$ as soon as the fibrewise metric is Bochner-flat (which is the condition to be Levi–Kähler in the case of one factor). As the extremality condition is difficult to characterize in general, we shall use the hamiltonian 2-form ansatz with $\ell = 2$ and $N = 1$ from [7, 9], which in turn is a special case of the generalized Calabi ansatz [9, §4]. We briefly recall the construction below and invite the Reader to consult [7, 9] for further details.

Let (S, g_S, ω_S) be a compact Riemann orbi-surface and η a real constant. We build a Kähler metric (g, ω) with a hamiltonian 2-form of order 2 and constant root η, defined on an orbifold fibration over $M \to S$, with fibres isomorphic to an orbifold quotient of a weighted projective plane. The Kähler metric (g, ω) is written on a dense subset $\mathring{M} \subseteq M$ as follows (see [9]):

$$
\begin{aligned}
g &= (\eta - \xi_1)(\eta - \xi_2) g_S + \frac{(\xi_1 - \xi_2) p_c(\xi_1)}{F(\xi_1)} d\xi_1^2 + \frac{(\xi_2 - \xi_1) p_c(\xi_2)}{F(\xi_2)} d\xi_2^2 \\
&\quad + \frac{F(\xi_1)}{(\xi_1 - \xi_2) p_c(\xi_1)} \big(\theta_1 + \xi_1 \theta_2\big)^2 + \frac{F(\xi_2)}{(\xi_2 - \xi_1) p_c(\xi_2)} \big(\theta_1 + \xi_2 \theta_2\big)^2 \\
\omega &= (\eta - \xi_1)(\eta - \xi_2)\omega_S + d\sigma_1 \wedge \theta_1 + d\sigma_2 \wedge \theta_2, \\
d\theta_1 &= -\eta \omega_S, \quad d\theta_2 = \omega_S \quad p_c(t) = (t - \eta), \ \sigma_1 = \xi_1 + \xi_2, \ \sigma_2 = \xi_1 \xi_2.
\end{aligned} \tag{54}
$$

Here $\xi_1 \in [-1, \beta]$, $\xi_2 \in [\beta, 1]$ (for some $|\beta| < 1$) are the *orthotoric* coordinates on the fibre, $|\eta| > 1$, and $F(x)$ is a smooth function which satisfies the positivity and boundary conditions

- $F(x)/p_c(x) > 0$ on $(\beta, 1)$; $F(x)/p_c(x) < 0$ on $(-1, \beta)$;
- $F(\pm 1) = F(\beta) = 0$.

It is easy to see that (54) is a special case of (38), with $N = 1$, $(x_1, x_2) = (\sigma_1, \sigma_2)$, $p_{11} = -\eta, p_{12} = 1, c_1 = \eta^2$, and a toric orbifold fibre whose Delzant polytope Δ is the the image of $[-1, \beta] \times [\beta, 1]$ under the map $(\sigma_1, \sigma_2) = (\xi_1, \xi_2, \xi_1 \xi_2)$ and labelling

$$
\begin{aligned}
L_{-1} &= -c_{-1}(\sigma_1 + \sigma_2 + 1), \quad L_{+1} = -c_1(-\sigma_1 + \sigma_2 + 1), \\
L_\beta &= -c_\beta(-\beta\sigma_1 + \sigma_2 + \beta^2),
\end{aligned}
$$

where $c_{\pm 1}\big(F/p_c\big)'(\pm 1) = 2 = c_\beta \big(F/p_c\big)'(\beta)$, see [9, Prop. 9]. Here we assume the usual rationality condition for the simplex $(\Delta, \mathbf{L})$, which certainly holds for $\beta, \eta, c_{\pm 1}, c_\beta \in \mathbb{Q}$.

We recall from [7] that the metric (54) is extremal if and only if the scalar curvature of g_S is a constant s and $F(x)$ is a polynomial of degree at most 5 satisfying

$$F''(\eta) = -s. \tag{55}$$

The metric g is of constant scalar curvature if, furthermore, the degree of $F(x)$ is at most 4. Likewise, by the same result, the fibrewise orthotoric metric

$$\begin{aligned} g^V = {} & \frac{(\xi_1 - \xi_2)p_c(\xi_1)}{F(\xi_1)} d\xi_1^2 + \frac{(\xi_2 - \xi_1)p_c(\xi_2)}{F(\xi_2)} d\xi_2^2 \\ & + \frac{F(\xi_1)}{(\xi_1 - \xi_2)p_c(\xi_1)} \big(dt_1 + \xi_1 dt_2\big)^2 + \frac{F(\xi_2)}{(\xi_2 - \xi_1)p_c(\xi_2)} \big(dt_1 + \xi_2 dt_2\big)^2 \end{aligned} \tag{56}$$

is Bochner-flat if and only if $F(x)/p_c(x)$ is a polynomial of degree at most 4. By Proposition 7, taking in (30) (S, ω_S) to be (an isometric orbifold quotient of) $\mathbb{C}P^1$ endowed with a Fubini–Study metric and $F(x) = -c(x^2 - 1)(x - \beta)(x - \eta)(x - \gamma)$ (resp. $F(x) = -c(x^2 - 1)(x - \beta)(x - \eta)$), one gets an ansatz for extremal (resp. constant scalar curvature) toric orbifolds, which are also Levi–Kähler quotients of $\mathbb{S}^5 \times \mathbb{S}^3$.

We further specialize to the constant scalar curvature case, i.e. $F(x) = -c(x^2 - 1)(x - \beta)(x - \eta)$ for a non-zero positive constant c. Then, (55) reduces to $2c(3\eta^2 - 2\beta\eta - 1) = s$, whereas the positivity conditions for $F(x)$ imply $\eta < -1$. Together with $Scal_S > 0$, these are the only constraints, subject to a rationality condition which is trivially solved by taking β, η, c rational. For instance, letting $\beta = 1/n, \eta = -n, c = 2/(3n^2 + 1), s = 4$ gives rise to a CSC Levi–Kähler quotient orbifold, which is not a product.

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