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Partial data inverse problems for semilinear elliptic equations with gradient nonlinearities

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We show that the linear span of the set of scalar products of gradients of harmonic functions on a bounded smooth domain $\Omega \subset \mathbb{R}^n$ which vanish on a closed proper subset of the boundary is dense in $L^1(\Omega)$. We apply this density result to solve some partial data inverse boundary problems for a class of semilinear elliptic PDE with quadratic gradient terms.

1. Introduction and statement of results

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a connected bounded open set with C^∞ boundary. In the paper [8] it is established that the linear span of the set of products of harmonic functions in $C^\infty(\overline{\Omega})$, which vanish on a closed proper subset of the boundary, is dense in $L^1(\Omega)$. This result is motivated by the Calderón inverse problem with partial data, see [24] and [40] for review, and it provides the solution of the linearized version of the partial data problem at the zero potential. The recent works [27] and [30] have exploited this density result to give a solution for the partial data inverse boundary problem for a class of semilinear elliptic PDE.

The purpose of this paper is twofold. First we shall give an extension of the density result of [8] where the set of products of harmonic functions which vanish on a closed proper subset of the boundary, is replaced by the set of scalar products of gradients of such functions. We shall then apply this density result to solve some partial data inverse problems for a class of semilinear elliptic PDE with quadratic gradient terms.

The first result of the paper, extending the corresponding result of [8], is as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a connected bounded open set with C^∞ boundary, let $\Gamma \subset \partial\Omega$ be an open nonempty subset of $\partial\Omega$, and let $\widetilde{\Gamma} =$*

$\partial\Omega \setminus \Gamma$. *Then the linear span of the set of scalar products of gradients of harmonic functions in* $C^\infty(\overline{\Omega})$, *which vanish on* $\widetilde{\Gamma}$, *is dense in* $L^1(\Omega)$.

Remark 1.2. Theorem 1.1 gives a solution to the linearized Calderón problem with partial data, given on an arbitrary open nonempty subset $\Gamma \subset \partial\Omega$, at a constant conductivity, see [8].

We shall next proceed to state our results concerning inverse boundary problems for a class of semilinear elliptic PDE with quadratic gradient terms. Specifically, we shall consider the following Dirichlet problem,

$$\begin{cases} -\Delta u + q(x)(\nabla u)^2 + V(x,u) = 0 & \text{in} \quad \Omega, \\ u = f \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

Here $q \in C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$, the Hölder space, and the function $V : \overline{\Omega} \times \mathbb{C} \to \mathbb{C}$ satisfies the following conditions:

(i) the map $\mathbb{C} \ni z \mapsto V(\cdot, z)$ is holomorphic with values in $C^\alpha(\overline{\Omega})$,

(ii) $V(x,0) = \partial_z V(x,0) = \partial_z^2 V(x,0) = 0$, for all $x \in \overline{\Omega}$.

We have also written $(\nabla u)^2 = \nabla u \cdot \nabla u$. It follows from (i) and (ii) that V can be expanded into a power series

$$V(x,z) = \sum_{k=3}^{\infty} V_k(x) \frac{z^k}{k!}, \quad V_k(x) := \partial_z^k V(x,0) \in C^\alpha(\overline{\Omega}), \tag{1.2}$$

converging in the $C^\alpha(\overline{\Omega})$ topology.

It is shown in Appendix A that there exist $\delta > 0$ and $C > 0$ such that when $f \in B_\delta(\partial\Omega) := \{f \in C^{2,\alpha}(\partial\Omega) : \|f\|_{C^{2,\alpha}(\partial\Omega)} < \delta\}$, the problem (1.1) has a unique solution $u = u_f \in C^{2,\alpha}(\overline{\Omega})$ satisfying $\|u\|_{C^{2,\alpha}(\overline{\Omega})} < C\delta$.

Let $\Gamma_1, \Gamma_2 \subset \partial\Omega$ be arbitrary non-empty open subsets of the boundary $\partial\Omega$. Associated to the problem (1.1), we define the partial Dirichlet–to–Neumann map $\Lambda_{q,V}^{\Gamma_1,\Gamma_2} f = \partial_\nu u_f|_{\Gamma_2}$, where $f \in B_\delta(\partial\Omega)$, supp $(f) \subset \Gamma_1$. Here ν is the unit outer normal to the boundary.

The second result of this paper is as follows.

Theorem 1.3. *Let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *be a connected bounded open set with* C^∞ *boundary, and let* $\Gamma_1, \Gamma_2 \subset \partial\Omega$ *be arbitrary open non-empty subsets of the boundary* $\partial\Omega$. *Let* $q_1, q_2 \in C^\alpha(\overline{\Omega})$ *and* $V^{(1)}, V^{(2)} : \overline{\Omega} \times \mathbb{C} \to \mathbb{C}$ *satisfy the assumptions (i) and (ii). Assume that* $\Lambda_{q_1,V^{(1)}}^{\Gamma_1,\Gamma_2} = \Lambda_{q_2,V^{(2)}}^{\Gamma_1,\Gamma_2}$. *Then* $q_1 = q_2$ *in* Ω *and* $V^{(1)} = V^{(2)}$ *in* $\Omega \times \mathbb{C}$.

Remark 1.4. To best of our knowledge, Theorem 1.3 is new even in the full data case $\Gamma_1 = \Gamma_2 = \partial\Omega$.

Remark 1.5. We would like to emphasize that in Theorem 1.3 the open non-empty sets $\Gamma_1, \Gamma_2 \subset \partial\Omega$ are completely arbitrary. It may be interesting to note that the corresponding partial data inverse problem is still open in dimensions $n \geq 3$ in the linear setting, even for the linear Schrödinger equation $-\Delta u + q(x)u = 0$ in Ω, say. In dimension $n = 2$ in the linear setting, the global identifiability in the partial data inverse problem is established in [14] when $\Gamma_1 = \Gamma_2$ is an arbitrary open non-empty portion of $\partial\Omega$, and in [15] when $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$, provided that some additional geometric assumptions are satisfied. We also refer to [7] for examples of non-uniqueness in the anisotropic Calderón problem when the Dirichlet and Neumann data are measured on disjoint subsets of the boundary in dimensions $n = 2, 3$.

Remark 1.6. To motivate the consideration of nonlinear elliptic PDE, discussed in this paper, let us mention that semilinear PDE with quadratic gradient terms occur naturally in the study of harmonic maps, harmonic heat flow maps, as well as Schrödinger maps, see [39], [3].

Following [30], we shall next discuss inverse boundary problems for semilinear elliptic equations with quadratic gradient terms, in the presence of an unknown obstacle. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a connected C^∞ boundary, and let $D \subset\subset \Omega$ be such that $\Omega \setminus \overline{D}$ is connected and $\partial D \in C^\infty$. Let us consider the following boundary problem,

$$\begin{cases} -\Delta u + q(x)(\nabla u)^2 + V(x,u) = 0 & \text{in} \quad \Omega \setminus \overline{D}, \\ u = 0 & \text{on} \quad \partial D, \\ u = f & \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

An application of Theorem A.1 of Appendix A, as before, gives that for all $f \in B_\delta(\partial\Omega)$, the problem (1.3) has a unique small solution $u \in C^{2,\alpha}(\overline{\Omega} \setminus D)$. Let $\Gamma_1, \Gamma_2 \subset \partial\Omega$ be arbitrary non-empty open subsets of the boundary $\partial\Omega$. We define the associated partial Dirichlet–to–Neumann map $\Lambda_{q,V}^{D,\Gamma_1,\Gamma_2}$ by

$$\Lambda_{q,V}^{D,\Gamma_1,\Gamma_2}(f) = \partial_\nu u|_{\Gamma_2}, \quad f \in B_\delta(\partial\Omega), \quad \text{supp } (f) \subset \Gamma_1.$$

We are interested in the inverse problem of determining the unknown obstacle D, the coefficient q, and the non-linear term V, all from the knowledge of the partial Dirichlet–to–Neumann map $\Lambda_{q,V}^{D,\Gamma_1,\Gamma_2}$.

The following result is analogous to [30, Theorem 1.2], with the novelty that we allow quadratic gradient terms in the nonlinearity, and that we can perform measurements on arbitrary open non-empty sets $\Gamma_1, \Gamma_2 \subset \partial\Omega$.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a bounded open set with connected C^∞ boundary, and let $\Gamma_1, \Gamma_2 \subset \partial\Omega$ be arbitrary open non-empty subsets of the boundary $\partial\Omega$. Let $D_1, D_2 \subset\subset \Omega$ be non-empty open subsets with C^∞ boundaries such that $\Omega \setminus \overline{D_j}$ are connected, $j = 1, 2$. Let $q_j \in C^\alpha(\overline{\Omega} \setminus D_j)$ and $V^{(j)} : (\overline{\Omega} \setminus D_j) \times \mathbb{C} \to \mathbb{C}$ satisfy the assumptions (i) and (ii), $j = 1, 2$. Assume that $\Lambda^{D_1,\Gamma_1,\Gamma_2}_{q_1,V^{(1)}} = \Lambda^{D_2,\Gamma_1,\Gamma_2}_{q_2,V^{(2)}}$. Then $D := D_1 = D_2$, $q_1 = q_2$ in $\Omega \setminus \overline{D}$ and $V^{(1)} = V^{(2)}$ in $(\Omega \setminus \overline{D}) \times \mathbb{C}$.*

Remark 1.8. It may be interesting to note that the simultaneous recovery of an obstacle and surrounding potentials in the linear setting, say in the case of the linear Schrödinger equation, constitutes an open problem, see [20], [30] for a discussion.

Let us remark that inverse boundary problems for nonlinear elliptic PDE have been studied extensively in the literature. To the best of our knowledge, the following main types of nonlinear scalar equations have been considered, under suitable assumptions on the nonlinearity:

(i) $-\Delta u + a(x, u) = 0$, see [22], [21], [43] for the full data problem in the Euclidean case, and [10], [29] for the manifold case, [16] for the partial data problem in the $n = 2$ case, and [27], [30] for the partial data problem when $n \ge 2$,

(ii) $-\Delta u + b(u, \nabla u) = 0$, see [19] for the partial data problem in the case $n = 3$,

(iii) $-\Delta u + q(x, \nabla u) = 0$, see [42] for the full data problem when $n = 2$,

(iv) $\nabla \cdot (\gamma(x, u)\nabla u) = 0$, see [41], [44] for the full data problem in the case $n \ge 2$,

(v) $\nabla \cdot (\overrightarrow{C}(x, \nabla u)) = 0$, see [5], [23], [12] for the full data problem,

(vi) $\nabla \cdot (c(u, \nabla u)\nabla u) = 0$, see [37] for the full data problem when $n \ge 2$.

A classical method for attacking inverse boundary problems for nonlinear elliptic PDE, going back to [18], consists of performing a first order linearization of the given nonlinear Dirichlet-to-Neumann map, allowing one to reduce the inverse problem to an inverse boundary problem for a linear elliptic equation, and to employ the available results in this case. A second order linearization of the nonlinear Dirichlet–to–Neumann map has also been successfully exploited in the works [2], [5], [23], [41], and [44]. The recent works [10], [29] have introduced a natural and powerful method of higher order linearizations of the nonlinear Dirichlet-to-Neumann map for inverse boundary problems for elliptic PDE, allowing one to solve such problems for nonlinear equations in situations where the corresponding inverse problems in the linear setting are open. This development of inverse boundary problems for nonlinear elliptic PDE was preceded by the pioneering work [28] for inverse problems for nonlinear hyperbolic PDE, see also [6], [33], and the references given there.

The problem of determining an unknown obstacle is of central significance in inverse scattering. The first uniqueness result for this problem goes back to Schiffer and Lax and Phillips [34, p. 173]. We refer to the works [17], [25], [26] for some other significant contributions, and to [20] for a review.

Let us now describe the main ideas of the proofs of Theorem 1.1, Theorem 1.3, and Theorem 1.7. First, the proof of Theorem 1.1 proceeds similarly to [8], with the only essential difference being that a certain Runge type approximation theorem needed here has to be established with respect to the H^1–topology, as opposed to an L^2–approximation result obtained in [8] .

The proof of Theorem 1.3 proceeds by the method of higher order linearizations, with Theorem 1.1 and the main result of [8] being the crucial ingredients.

As for Theorem 1.7, it is an immediate consequence of Theorem 1.3, once the obstacle has been recovered. Following [30], the determination of the obstacle is obtained by performing a first order linearization of the problem (1.3), and relying on a standard contradiction argument.

The paper is organized as follows. In Section 2 we establish Theorem 1.1. The proof of Theorem 1.3 occupies Section 3. Theorem 1.7 is proven in Section 4. In Appendix A we show the well-posedness of the Dirichlet problem for our semilinear elliptic equation with quadratic gradient terms, in the case of small boundary data.

2. Proof of Theorem 1.1

We shall follow the strategy of the work [8]. Let $f \in L^\infty(\Omega)$ be such that

$$\int_\Omega f \nabla u \cdot \nabla v dx = 0, \tag{2.1}$$

for any harmonic functions $u, v \in C^\infty(\overline{\Omega})$ satisfying $u|_{\widetilde{\Gamma}} = v|_{\widetilde{\Gamma}} = 0$. In view of the Hahn–Banach theorem, we have to show that $f = 0$ in Ω. This global statement will be obtained as a corollary of the following local result.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with C^∞ boundary, let $x_0 \in \partial\Omega$, and let $\widetilde{\Gamma} \subset \partial\Omega$ be the complement of an open boundary neighborhood of x_0. Then there exists $\delta > 0$ such that if we have* (2.1) *for any harmonic functions $u, v \in C^\infty(\overline{\Omega})$ satisfying $u|_{\widetilde{\Gamma}} = v|_{\widetilde{\Gamma}} = 0$, then $f = 0$ in $B(x_0, \delta) \cap \Omega$.*

Proposition 2.1 will be proved in Subsection 2.3. The passage from local to global results will be carried out in Subsection 2.2. Here an essential ingredient is a Runge type approximation theorem in the H^1–topology, established in Subsection 2.1 and extending [8, Lemma 2.2], where approximation in the L^2–sense was shown.

2.1. Runge type approximation

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with C^∞ boundary, and let us consider the L^2–dual of $H^1(\Omega)$, given by

$$\widetilde{H}^{-1}(\Omega) := \{v \in H^{-1}(\mathbb{R}^n) : \text{supp } (v) \subset \overline{\Omega}\},$$

see [9], [35]. Here the duality pairing is defined as follows: if $v \in \widetilde{H}^{-1}(\Omega)$ and $w \in H^1(\Omega)$, then we set

$$(v, w)_{\widetilde{H}^{-1}(\Omega), H^1(\Omega)} = (v, \text{Ext}(w))_{H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)}, \tag{2.2}$$

where $\text{Ext}(w) \in H^1(\mathbb{R}^n)$ is an arbitrary extension of w, and $(\cdot,\cdot)_{H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)}$ is the extension of L^2 scalar product $(\varphi, \psi)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \varphi(x)\overline{\psi(x)}dx$. Note that the definition (2.2) is independent of the choice of an extension $\text{Ext}(w)$, see [9, Lemma 22.7].

Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^n$, $n \geq 2$, be two bounded open sets with smooth boundaries such that $\Omega_2 \setminus \overline{\Omega_1} \neq \emptyset$. Assume that $\partial\Omega_1 \cap \partial\Omega_2 = \overline{U}$ where $U \subset \partial\Omega_1$ is open with C^∞ boundary. Associated to Ω_2, we let $\mathcal{G} : C^\infty(\overline{\Omega_2}) \to C^\infty(\overline{\Omega_2})$, $a \mapsto w$, be the solution operator to the Dirichlet problem,

$$\begin{cases} -\Delta w = a & \text{in} \quad \Omega_2, \\ w|_{\partial\Omega_2} = 0. \end{cases}$$

The following result is an extension of [8, Lemma 2.2].

Lemma 2.2. *The space*

$$W := \{\mathcal{G}a|_{\Omega_1} : a \in C^\infty(\overline{\Omega_2}),\ \text{supp}\ (a) \subset \Omega_2 \setminus \overline{\Omega_1}\}$$

is dense in the space

$$S := \{u \in C^\infty(\overline{\Omega_1}) : -\Delta u = 0 \ \text{in}\ \Omega_1,\ u|_{\partial\Omega_1 \cap \partial\Omega_2} = 0\},$$

with respect to the $H^1(\Omega_1)$–topology.

Proof. We shall use some ideas of [31], see also [4], [32]. Let $v \in \widetilde{H}^{-1}(\Omega_1)$ be such that

$$(v, \mathcal{G}a|_{\Omega_1})_{\widetilde{H}^{-1}(\Omega_1), H^1(\Omega_1)} = 0 \tag{2.3}$$

for any $a \in C^\infty(\overline{\Omega_2})$, supp $(a) \subset \Omega_2 \setminus \overline{\Omega_1}$. By the Hahn–Banach theorem, it suffices to show that $(v, u)_{\widetilde{H}^{-1}(\Omega_1), H^1(\Omega_1)} = 0$ for any $u \in S$.

We have $\mathcal{G}a \in H_0^1(\Omega_2)$ and let us view $\mathcal{G}a$ as an element of $H^1(\mathbb{R}^n)$ via an extension by 0 to $\mathbb{R}^n \setminus \Omega_2$. Then there exists a sequence $\varphi_j \in C_0^\infty(\Omega_2)$ such that $\varphi_j \to \mathcal{G}a$ in $H^1(\mathbb{R}^n)$. It follows from (2.3) that

$$\begin{aligned} 0 = (v, \mathcal{G}a)_{H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)} &= \lim_{j\to\infty} (v, \varphi_j)_{H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)} \\ &= \lim_{j\to\infty} (v, \varphi_j)_{H^{-1}(\Omega_2), H_0^1(\Omega_2)} = (v, \mathcal{G}a)_{H^{-1}(\Omega_2), H_0^1(\Omega_2)}. \end{aligned} \tag{2.4}$$

As $v \in \widetilde{H}^{-1}(\Omega_1)$, by [35, Theorem 3.29], there is a sequence $v_j \in C_0^\infty(\Omega_1)$ such that $v_j \to v$ in $H^{-1}(\mathbb{R}^n)$. Let $f \in H_0^1(\Omega_2)$ and $f_j \in C^\infty(\overline{\Omega_2}) \cap H_0^1(\Omega_2)$ be the unique solutions to the following Dirichlet problems,

$$\begin{cases} -\Delta f = v & \text{in} \quad \Omega_2, \\ f = 0 & \text{on} \quad \partial\Omega_2, \end{cases} \qquad \begin{cases} -\Delta f_j = v_j & \text{in} \quad \Omega_2, \\ f_j = 0 & \text{on} \quad \partial\Omega_2. \end{cases} \tag{2.5}$$

Now it follows from (2.4), (2.5) that

$$\begin{aligned} 0 = (v, \mathcal{G}a)_{H^{-1}(\Omega_2), H_0^1(\Omega_2)} &= \lim_{j\to\infty} (v_j, \mathcal{G}a)_{H^{-1}(\Omega_2), H_0^1(\Omega_2)} \\ &= \lim_{j\to\infty} (-\Delta f_j, \mathcal{G}a)_{H^{-1}(\Omega_2), H_0^1(\Omega_2)} = \lim_{j\to\infty} \int_{\Omega_2} (-\Delta f_j)\overline{\mathcal{G}a} dx \\ &= \lim_{j\to\infty} \int_{\Omega_2} f_j \overline{a} dx = \int_{\Omega_2} f \overline{a} dx. \end{aligned} \tag{2.6}$$

Here in the penultimate equality we use Green's formula, and the fact that $f_j|_{\partial\Omega_2} = \mathcal{G}a|_{\partial\Omega_2} = 0$. In the last equality in (2.6) we use that $f_j \to f$ in

$L^2(\Omega)$ by the well–posedness of the Dirichlet problem for $-\Delta$ in Ω_2, see [9, Theorem 23.4]. As $a \in C^\infty(\overline{\Omega_2})$, supp $(a) \subset \Omega_2 \setminus \overline{\Omega_1}$, is arbitrary, we get from (2.6) that $f = 0$ in $\Omega_2 \setminus \overline{\Omega_1}$. Since $f \in H_0^1(\Omega_2)$, we see that $f|_{\partial\Omega_1 \setminus \partial\Omega_2} = 0$, and therefore, $f|_{\partial\Omega_1} = 0$. Hence, $f \in H_0^1(\Omega_1)$, and we shall view f as an element of $H^1(\mathbb{R}^n)$ via an extension by 0 to $\mathbb{R}^n \setminus \Omega_1$.

Let $\widehat{f_j} \in C_0^\infty(\Omega_1)$ be such that $\widehat{f_j} \to f$ in $H^1(\mathbb{R}^n)$. Thus, $-\Delta \widehat{f_j} \to -\Delta f$ in $H^{-1}(\mathbb{R}^n)$. Let $u \in S$ and let $\mathrm{Ext}(u) \in H^1(\mathbb{R}^n)$ be an extension of u. Then integrating by parts, we have

$$
\begin{aligned}
(2.7)\quad (-\Delta f, \mathrm{Ext}(u))_{H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)} &= \lim_{j\to\infty} ((-\Delta \widehat{f_j}), \mathrm{Ext}(u))_{H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)} \\
&= \lim_{j\to\infty} \int_{\Omega_1} (-\Delta \widehat{f_j}) \overline{u} dx = 0.
\end{aligned}
$$

We shall consider the compactly supported distribution $g = -\Delta f - v \in H^{-1}(\mathbb{R}^n)$. As supp (v), supp $(f) \subset \overline{\Omega_1}$, we see that supp $(g) \subset \overline{\Omega_1}$, and it follows from (2.5) that supp $(g) \subset \partial\Omega_1$. As $\partial\Omega_1$ is a codimension 1 submanifold in $\mathbb{R}^n$, by [1, Theorem 5.1.13], [35, Lemma 3.39], we obtain that

$$
g = h \otimes \delta_{\partial\Omega_1}, \quad h \in H^{-1/2}(\partial\Omega_1).
$$

Furthermore, in view of (2.5), we have supp $(g) \subset \partial\Omega_1 \cap \partial\Omega_2 = \overline{U}$, and therefore, supp $(h) \subset \overline{U}$. Since $U \subset \partial\Omega_1$ is an open set with C^∞ boundary, by [35, Theorem 3.29], there exists a sequence $h_j \in C_0^\infty(U)$ such that $h_j \to h$ in $H^{-1/2}(\partial\Omega_1)$. Hence, we have

$$
\begin{aligned}
(2.8)\quad (g, \mathrm{Ext}(u))_{H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)} &= (h, u|_{\partial\Omega_1})_{H^{-1/2}(\partial\Omega_1), H^{1/2}(\partial\Omega_1)} \\
&= \lim_{j\to\infty} (h_j, u|_{\partial\Omega_1})_{H^{-1/2}(\partial\Omega_1), H^{1/2}(\partial\Omega_1)} \\
&= \lim_{j\to\infty} \int_{\partial\Omega_1} h_j \overline{u} dS = 0,
\end{aligned}
$$

where in the last equality we use that $u|_{\partial\Omega_1 \cap \partial\Omega_2} = 0$. It follows from (2.7) and (2.8) that

$$
\begin{aligned}
(v, u)_{\widetilde{H}^{-1}(\Omega_1), H^1(\Omega_1)} &= (-\Delta f, \mathrm{Ext}(u))_{H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)} \\
&\quad - (g, \mathrm{Ext}(u))_{H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)} = 0.
\end{aligned}
$$

This completes the proof of Lemma 2.2. □

2.2. From local to global results. Proof of Theorem 1.1

We shall follow [8]. We want to show that f vanishes inside Ω. Let us fix a point $x_1 \in \Omega$ and let $\theta : [0,1] \to \overline{\Omega}$ be a C^1 curve joining $x_0 \in \partial\Omega \setminus \widetilde{\Gamma}$ to x_1 such that $\theta(0) = x_0$, $\theta'(0)$ is the interior normal to $\partial\Omega$ at x_0 and $\theta(t) \in \Omega$, for all $t \in (0,1]$. Let us set

$$\Theta_\varepsilon(t) = \{x \in \overline{\Omega} : d(x, \theta([0,t])) \leq \varepsilon\}$$

so that $\Theta_\varepsilon(t)$ is a closed neighborhood of the curve ending at $\theta(t)$, $t \in [0,1]$. Let

$$I = \{t \in [0,1] : f \text{ vanishes a.e. on } \Theta_\varepsilon(t) \cap \Omega\}.$$

Note that by Proposition 2.1, $0 \in I$ if $\varepsilon > 0$ is small enough. One can easily see that I is a closed subset of $[0,1]$. If we show that I is open then as $[0,1]$ is connected, $I = [0,1]$. Hence, $x_1 \notin \text{supp }(f)$. Since x_1 is an arbitrary point of Ω, we have $f = 0$ on Ω, completing the proof of Theorem 1.1.

Thus, we only need to show that I is open. To this end, let $t \in I$ and $\varepsilon > 0$ be small enough so that $\partial\Theta_\varepsilon(t) \cap \partial\Omega \subset \partial\Omega \setminus \widetilde{\Gamma}$. For $\varepsilon > 0$ sufficiently small, the set $\partial\Theta_\varepsilon(t)$ intersects $\partial\Omega$ transversally, and in suitable local coordinates $y_1, \dots, y_n$ centered at x_0, $\partial\Omega$ is given by $y_n = 0$, and $\partial\Theta_\varepsilon(t)$ is given by $y_1 = 0$. It is then easy to see that the set $\Omega \setminus \Theta_\varepsilon(t)$ can be smoothed out into an open subset Ω_1 of Ω with smooth boundary so that

$$\Omega_1 \supset \Omega \setminus \Theta_\varepsilon(t), \quad \partial\Omega \cap \partial\Omega_1 \supset \widetilde{\Gamma},$$

and $\partial\Omega_1 \cap \partial\Omega = \overline{U}$ where $U \subset \partial\Omega_1$ is an open set with C^∞ boundary. Furthermore, let us augment the set Ω by smoothing out the set $\Omega \cup B(x_0, \varepsilon')$, with $0 < \varepsilon' \ll \varepsilon$ sufficiently small, into an open set Ω_2 with smooth boundary so that

$$\partial\Omega_2 \cap \partial\Omega \supset \partial\Omega_1 \cap \partial\Omega = \partial\Omega_1 \cap \partial\Omega_2 \supset \widetilde{\Gamma}.$$

Let G_{Ω_2} be the Green kernel associated to the open set Ω_2,

$$-\Delta_y G_{\Omega_2}(x,y) = \delta(x-y), \quad G_{\Omega_2}(x,\cdot)|_{\partial\Omega_2} = 0.$$

Consider the function

$$v(x,z) = \int_{\Omega_1} f(y) \nabla_y G_{\Omega_2}(x,y) \cdot \nabla_y G_{\Omega_2}(z,y) dy, \quad x,z \in \Omega_2 \setminus \overline{\Omega_1},$$

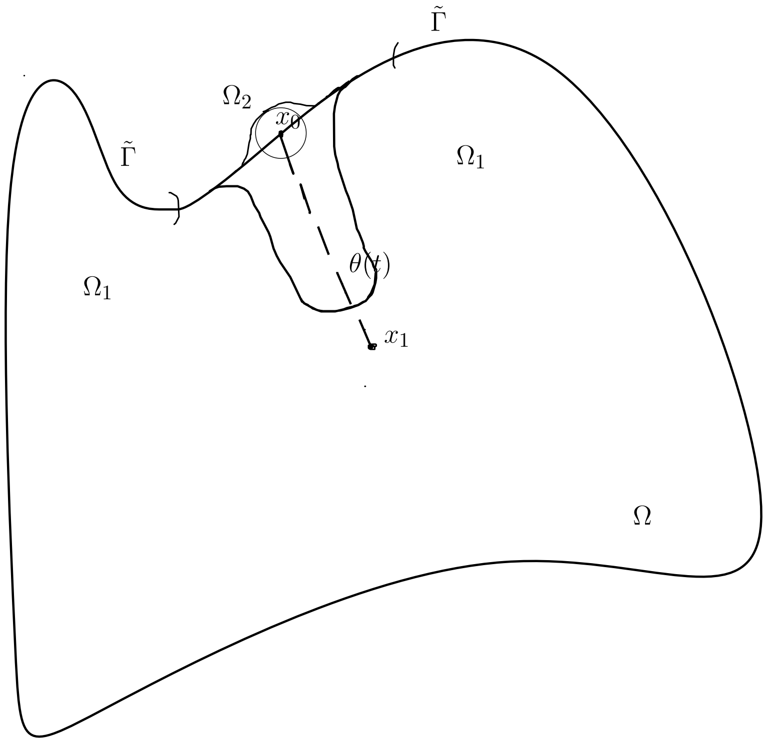

Figure 1: The open sets Ω_1 and Ω_2.

which is harmonic in both $x, z \in \Omega_2 \setminus \overline{\Omega_1}$. As $f = 0$ on $\Theta_\varepsilon(t) \cap \Omega$, we get

$$v(x,z) = \int_\Omega f(y)\nabla_y G_{\Omega_2}(x,y) \cdot \nabla_y G_{\Omega_2}(z,y)dy, \quad x, z \in \Omega_2 \setminus \overline{\Omega_1}.$$

When $x, z \in \Omega_2 \setminus \overline{\Omega}$, the Green functions $G_{\Omega_2}(x,\cdot), G_{\Omega_2}(z,\cdot) \in C^\infty(\overline{\Omega})$ are harmonic on Ω, and $G_{\Omega_2}(x,\cdot)|_{\tilde{\Gamma}} = G_{\Omega_2}(z,\cdot)|_{\tilde{\Gamma}} = 0$. By assumption (2.1), we have $v(x,z) = 0$ when $x, z \in \Omega_2 \setminus \overline{\Omega}$. Since $v(x,z)$ is harmonic when $x, z \in \Omega_2 \setminus \overline{\Omega_1}$ and $\Omega_2 \setminus \overline{\Omega_1}$ is connected, by unique continuation, $v(x,z) = 0$ when $x, z \in \Omega_2 \setminus \overline{\Omega_1}$, i.e.

$$\int_{\Omega_1} f(y)\nabla_y G_{\Omega_2}(x,y) \cdot \nabla_y G_{\Omega_2}(z,y)dy = 0, \quad x, z \in \Omega_2 \setminus \overline{\Omega_1}. \tag{2.9}$$

Letting $a \in C^\infty(\overline{\Omega_2})$, supp $(a) \subset \Omega_2 \setminus \overline{\Omega_1}$, $b \in C^\infty(\overline{\Omega_2})$, supp $(b) \subset \Omega_2 \setminus \overline{\Omega_1}$, multiplying (2.9) by $a(x)$, $b(z)$, and integrating, we obtain that

$$\int_{\Omega_1} f(y) \int_{\Omega_2} \nabla_y G_{\Omega_2}(x,y) a(x) dx \cdot \int_{\Omega_2} \nabla_y G_{\Omega_2}(z,y) b(z) dz dy = 0.$$

Hence, we have

$$\int_{\Omega_1} f \nabla u \cdot \nabla v dy = 0, \tag{2.10}$$

for all $u, v \in W$. By continuity of the bilinear form,

$$H^1(\Omega_1) \times H^1(\Omega_1) \to \mathbb{C}, \quad (u,v) \mapsto \int_{\Omega_1} f \nabla u \cdot \nabla v dy,$$

and by Lemma 2.2, we get (2.10) for any $u, v \in C^\infty(\overline{\Omega_1})$ harmonic in Ω_1 which vanish on $\partial\Omega_1 \cap \partial\Omega_2$. Now by Proposition 2.1, f vanishes on a neighborhood of $\partial\Omega \setminus (\partial\Omega_1 \cap \partial\Omega_2)$, and hence, I is an open set.

2.3. Proof of Proposition 2.1

We shall follow [8]. First using a conformal transformation of harmonic functions, we reduce to the following setting: $x_0 = 0$, the tangent plane to Ω at x_0 is given by $x_1 = 0$,

$$\Omega \subset \{x \in \mathbb{R}^n : |x + e_1| < 1\}, \quad \widetilde{\Gamma} = \{x \in \partial\Omega : x_1 \leq -2c\}$$

for some $c > 0$. Here $e_1 = (1, 0, \dots, 0)$ is the first coordinate vector.

Let $p(\xi) = \xi^2$, $\xi \in \mathbb{R}^n$, be the principal symbol of $-\Delta$, and let us denote by $p(\zeta) = \zeta^2$ its holomorphic extension to $\mathbb{C}^n$. We write

$$p^{-1}(0) = \{\zeta \in \mathbb{C}^n : \zeta^2 = 0\}.$$

Let $\zeta \in p^{-1}(0)$ and let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a cutoff function such that $\chi = 1$ on $\widetilde{\Gamma}$. Consider the following function

$$u(x,\zeta) = e^{-\frac{i}{h} x \cdot \zeta} + w(x,\zeta), \tag{2.11}$$

where w is the solution to the Dirichlet problem,

$$\begin{cases} -\Delta w = 0 \quad \text{in} \quad \Omega, \\ w|_{\partial\Omega} = -(e^{-\frac{i}{h} x\cdot\zeta} \chi)|_{\partial\Omega}. \end{cases}$$

Thus, $u \in C^\infty(\overline{\Omega})$, u is harmonic in Ω, and $u|_{\widetilde{\Gamma}} = 0$. We have

$$\begin{aligned}\|w\|_{H^1(\Omega)} &\le C\|e^{-\frac{i}{h}x\cdot\zeta}\chi\|_{H^{1/2}(\partial\Omega)} \\ &\le C\|e^{-\frac{i}{h}x\cdot\zeta}\chi\|^{1/2}_{H^1(\partial\Omega)}\|e^{-\frac{i}{h}x\cdot\zeta}\chi\|^{1/2}_{L^2(\partial\Omega)} \\ &\le C(1+h^{-1}|\zeta|)^{1/2}e^{\frac{1}{h}H_K(\mathrm{Im}\,\zeta)},\end{aligned} \tag{2.12}$$

where H_K is the supporting function of the compact subset $K = \mathrm{supp}\,\chi \cap \partial\Omega$ of the boundary,

$$H_K(\xi) = \sup_{x\in K} x\cdot\xi, \quad \xi \in \mathbb{R}^n.$$

Let us take $\chi \in C_0^\infty(\mathbb{R}^n)$ be such that supp $(\chi) \subset \{x \in \mathbb{R}^n : x_1 \le -c\}$ and $\chi = 1$ on $\{x \in \partial\Omega : x_1 \le -2c\}$. Then (2.12) implies that

$$\|w\|_{H^1(\Omega)} \le C(1+h^{-1}|\zeta|)^{1/2}e^{-\frac{c}{h}\mathrm{Im}\,\zeta_1}e^{\frac{1}{h}|\mathrm{Im}\,\zeta'|}, \tag{2.13}$$

when $\mathrm{Im}\,\zeta_1 \ge 0$.

The cancellation identity (2.1) gives that

$$\int_\Omega f(x)hDu(x,\zeta)\cdot hDu(x,\eta)dx = 0, \tag{2.14}$$

for all $\zeta, \eta \in p^{-1}(0)$, where $u(x,\zeta)$, $u(x,\eta)$ are harmonic functions of the form (2.11). It follows from (2.14) that

$$\begin{aligned}\int_\Omega f(x)\zeta\cdot\eta e^{-\frac{ix\cdot(\zeta+\eta)}{h}}dx = &\int_\Omega f(x)\zeta e^{-\frac{ix\cdot\zeta}{h}}\cdot hDw(x,\eta)dx \\ &+\int_\Omega f(x)\eta e^{-\frac{ix\cdot\eta}{h}}\cdot hDw(x,\zeta)dx \\ &-\int_\Omega f(x)hDw(x,\zeta)\cdot hDw(x,\eta)dx.\end{aligned}$$

Thus,

$$\begin{aligned}&\left|\int_\Omega f(x)\zeta\cdot\eta e^{-\frac{ix\cdot(\zeta+\eta)}{h}}dx\right| \\ &\le \|f\|_{L^\infty(\Omega)}\big(|\zeta|\|e^{-\frac{ix\cdot\zeta}{h}}\|_{L^2(\Omega)}\|hDw(x,\eta)\|_{L^2(\Omega)} \\ &\quad + |\eta|\|e^{-\frac{ix\cdot\eta}{h}}\|_{L^2(\Omega)}\|hDw(x,\zeta)\|_{L^2(\Omega)} \\ &\quad + \|hDw(x,\zeta)\|_{L^2(\Omega)}\|hDw(x,\eta)\|_{L^2(\Omega)}\big).\end{aligned} \tag{2.15}$$

Now when $\operatorname{Im}\zeta_1 \geq 0$, using the fact that $\Omega \subset \{x \in \mathbb{R}^n : |x+e_1| < 1\}$, we get

$$\|e^{-\frac{ix\cdot\zeta}{h}}\|_{L^2(\Omega)} \leq Ce^{\frac{|\operatorname{Im}\zeta'|}{h}}, \quad \zeta \in p^{-1}(0). \tag{2.16}$$

We obtain from (2.15) using (2.13) and (2.16) that

$$\begin{aligned}\left|\int_\Omega f(x)\zeta\cdot\eta e^{-\frac{ix\cdot(\zeta+\eta)}{h}}dx\right| &\leq C\|f\|_{L^\infty(\Omega)}e^{\frac{|\operatorname{Im}\zeta'|+|\operatorname{Im}\eta'|}{h}}e^{-\frac{c}{h}\min(\operatorname{Im}\zeta_1,\operatorname{Im}\eta_1)}\\ &\quad\times\big(|\zeta|(h^2+h|\eta|)^{1/2}+|\eta|(h^2+h|\zeta|)^{1/2}\\ &\quad+(h^2+h|\zeta|)^{1/2}(h^2+h|\eta|)^{1/2}\big),\end{aligned} \tag{2.17}$$

for all $\zeta,\eta \in p^{-1}(0)$, $\operatorname{Im}\zeta_1 \geq 0$, $\operatorname{Im}\eta_1 \geq 0$.

As in [8], consider the map

$$s : p^{-1}(0)\times p^{-1}(0) \to \mathbb{C}^n, \quad (\zeta,\eta)\mapsto \zeta+\eta.$$

Its differential at a point (ζ_0,η_0),

$$Ds(\zeta_0,\eta_0) : T_{\zeta_0}p^{-1}(0)\times T_{\eta_0}p^{-1}(0) \to \mathbb{C}^n, \quad (\zeta,\eta)\mapsto\zeta+\eta,$$

is surjective provided that $\mathbb{C}^n = T_{\zeta_0}p^{-1}(0)+T_{\eta_0}p^{-1}(0)$, i.e. ζ_0 and η_0 are linearly independent. In particular, the latter is true if $\zeta_0=\gamma$ and $\eta_0=-\overline{\gamma}$ with $\gamma=(i,1,0,\dots,0)\in\mathbb{C}^n$. Now $\zeta_0+\eta_0=2ie_1$, and therefore, the inverse function theorem implies that there exists $\varepsilon>0$ small such that any $z\in\mathbb{C}^n$, $|z-2ie_1|<2\varepsilon$, may be decomposed as $z=\zeta+\eta$ where $\zeta,\eta\in p^{-1}(0)$, $|\zeta-\gamma|<C_1\varepsilon$ and $|\eta+\overline{\gamma}|<C_1\varepsilon$ with some $C_1>0$. Furthermore, by rescaling, any $z\in\mathbb{C}^n$ such that $|z-2iae_1|<2\varepsilon a$ for some $a>0$, may be decomposed as

$$z=\zeta+\eta, \quad \zeta,\eta\in p^{-1}(0), \quad |\zeta-a\gamma|<C_1a\varepsilon, \quad |\eta+a\overline{\gamma}|<C_1a\varepsilon. \tag{2.18}$$

Now (2.18) implies that $|\operatorname{Im}\zeta'|<C_1a\varepsilon$, $|\operatorname{Im}\eta'|<C_1a\varepsilon$, $|\zeta|\leq Ca$, and $|\eta|\leq Ca$. If $\varepsilon>0$ is small enough, (2.18) gives that $\operatorname{Im}\zeta_1>a/2$, $\operatorname{Im}\eta_1>a/2$, and $|\zeta\cdot\eta|\geq a^2$, Hence, it follows from (2.17) and (2.18) that

$$\begin{aligned}\left|\int_\Omega f(x)e^{-\frac{ix\cdot z}{h}}dx\right| &\leq C\|f\|_{L^\infty(\Omega)}e^{-\frac{ca}{2h}}e^{\frac{2C_1a\varepsilon}{h}}\\ &\quad\times[a^{-1}(1+a)^{1/2}+a^{-2}(1+a)],\end{aligned} \tag{2.19}$$

for all $z\in\mathbb{C}^n$ such that $|z-2iae_1|<2\varepsilon a$ for some $a>0$ and $\varepsilon>0$ sufficiently small. Following [8] and choosing $a>1$ large, we see that the bound

(2.19) is completely analogous to the estimate (3.8) in [8]. We may therefore complete the proof of Proposition 2.1 by repeating the arguments of [8] exactly as they stand.

3. Proof of Theorem 1.3

We shall first establish that the knowledge of the partial Dirichlet–to–Neumann map $\Lambda_{q,V}^{\Gamma_1,\Gamma_2}$ allows us to recover the coefficient q in the quadratic gradient term in (1.1). To that end, let $\varepsilon=(\varepsilon_1,\varepsilon_2)\in\mathbb{C}^2$, and let $f_k\in C^\infty(\partial\Omega)$, supp $(f_k)\subset\Gamma_1$, $k=1,2$. An application of Theorem A.1 shows that for all $|\varepsilon|$ sufficiently small, the Dirichlet problem

$$
\begin{cases}-\Delta u_j+q_j(x)(\nabla u_j)^2+\sum_{k=3}^\infty V_k^{(j)}(x)\frac{u_j^k}{k!}=0 \quad \text{in} \quad \Omega,\\ u_j=\varepsilon_1 f_1+\varepsilon_2 f_2 \quad \text{on} \quad \partial\Omega,\end{cases} \tag{3.1}
$$

$j=1,2$, has a unique small solution $u_j=u_j(\cdot,\varepsilon)\in C^{2,\alpha}(\overline{\Omega})$, which depends holomorphically on $\varepsilon\in \text{neigh}(0,\mathbb{C}^2)$ with values in $C^{2,\alpha}(\overline{\Omega})$. We shall now carry out a second order linearization of the problem (3.1) and of the corresponding partial Dirichlet–to–Neumann maps. Note that the assumption (ii) on the nonlinearity V guarantees that V does not appear in the second order linearlization. Differentiating (3.1) with respect to ε_l, $l=1,2$, taking $\varepsilon=0$, and using that $u_j(x,0)=0$, we get

$$
\begin{cases}\Delta v_j^{(l)}=0 \quad \text{in} \quad \Omega,\\ v_j^{(l)}=f_l \quad \text{on} \quad \partial\Omega,\end{cases} \tag{3.2}
$$

where $v_j^{(l)}=\partial_{\varepsilon_l}u_j|_{\varepsilon=0}$. By the uniqueness and the elliptic regularity for the Dirichlet problem (3.2), we see that $v^{(l)}:=v_1^{(l)}=v_2^{(l)}\in C^\infty(\overline{\Omega})$, $l=1,2$.

Applying $\partial_{\varepsilon_1}\partial_{\varepsilon_2}|_{\varepsilon=0}$ to (3.1), we get

$$
\begin{cases}-\Delta(\partial_{\varepsilon_1}\partial_{\varepsilon_2}u_j|_{\varepsilon=0})+2q_j(x)\nabla\partial_{\varepsilon_1}u_j|_{\varepsilon=0}\cdot\nabla\partial_{\varepsilon_2}u_j|_{\varepsilon=0}=0 \quad \text{in} \quad \Omega,\\ \partial_{\varepsilon_1}\partial_{\varepsilon_2}u_j|_{\varepsilon=0}=0 \quad \text{on} \quad \partial\Omega,\end{cases} \tag{3.3}
$$

and letting $w_j=\partial_{\varepsilon_1}\partial_{\varepsilon_2}u_j|_{\varepsilon=0}$, (3.3) yields that

$$
\begin{cases}-\Delta w_j+2q_j(x)\nabla v^{(1)}\cdot\nabla v^{(2)}=0 \quad \text{in} \quad \Omega,\\ w_j=0 \quad \text{on} \quad \partial\Omega.\end{cases} \tag{3.4}
$$

The fact that $\Lambda_{q_1,V^{(1)}}^{\Gamma_1,\Gamma_2}(\varepsilon_1 f_1+\varepsilon_2 f_2)=\Lambda_{q_2,V^{(2)}}^{\Gamma_1,\Gamma_2}(\varepsilon_1 f_1+\varepsilon_2 f_2)$ for all small $\varepsilon_1,\varepsilon_2$ and all $f_1,f_2\in C^\infty(\partial\Omega)$ with supp (f_1), supp $(f_2)\subset\Gamma_1$ implies that

$\partial_\nu u_1|_{\Gamma_2} = \partial_\nu u_2|_{\Gamma_2}$. Hence, an application of $\partial_{\varepsilon_1}\partial_{\varepsilon_2}|_{\varepsilon=0}$ gives $\partial_\nu w_1|_{\Gamma_2} = \partial_\nu w_2|_{\Gamma_2}$. Multiplying (3.4) by $v^{(3)} \in C^\infty(\overline{\Omega})$ harmonic in Ω and applying Green's formula, we get

$$2\int_\Omega (q_1 - q_2)(\nabla v^{(1)} \cdot \nabla v^{(2)})v^{(3)}dx = \int_{\partial\Omega\setminus\Gamma_2} (\partial_\nu w_1 - \partial_\nu w_2)v^{(3)}dS = 0,$$

provided that supp $(v^{(3)}|_{\partial\Omega}) \subset \Gamma_2$. Hence, we obtain that

$$\int_\Omega (q_1 - q_2)(\nabla v^{(1)} \cdot \nabla v^{(2)})v^{(3)}dx = 0$$

for any $v^{(l)} \in C^\infty(\overline{\Omega})$ harmonic in Ω, $l = 1,2,3$, such that supp $(v^{(l)}|_{\partial\Omega}) \subset \Gamma_1$, $l = 1,2$, and supp $(v^{(3)}|_{\partial\Omega}) \subset \Gamma_2$. Taking $v^{(3)} \not\equiv 0$ and applying Theorem 1.1, we obtain that

$$(q_1 - q_2)v^{(3)} = 0 \quad \text{in} \quad \Omega.$$

Now $v^{(3)}$ is harmonic and therefore, the set $(v^{(3)})^{-1}(0)$ is of measure zero, see [36]. Hence $q_1 = q_2 =: q$ in Ω.

We now come to prove that $V^{(1)} = V^{(2)}$. To that end, it suffices to show that $V_m^{(1)} = V_m^{(2)}$ for all $m \geq 3$, see (3.1), which will be done inductively. Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \mathbb{C}^m$, $m \geq 3$, be small, and $f_k \in C^\infty(\partial\Omega)$, supp $(f_k) \subset \Gamma_1$, $k = 1, \dots, m$. Let $u_j = u_j(\cdot, \varepsilon) \in C^{2,\alpha}(\overline{\Omega})$, $j = 1,2$, be the unique small solution to the Dirichlet problem

$$\begin{cases} -\Delta u_j + q(x)(\nabla u_j)^2 + \sum_{k=3}^\infty V_k^{(j)}(x)\frac{u_j^k}{k!} = 0 & \text{in} \quad \Omega, \\ u_j = \varepsilon_1 f_1 + \dots + \varepsilon_m f_m & \text{on} \quad \partial\Omega. \end{cases} \tag{3.5}$$

We shall first discuss the case $m = 3$. The first linearization of (3.5) leads to the problem (3.2) with $l = 1,2,3$, and therefore, $\partial_{\varepsilon_l} u_1|_{\varepsilon=0} = \partial_{\varepsilon_l} u_2|_{\varepsilon=0} =: v^{(l)}$, $l = 1,2,3$. The second linearization of (3.5) gives rise to a problem of the form (3.3) with $q_j = q$, and therefore, $\partial_{\varepsilon_{l_1}}\partial_{\varepsilon_{l_2}} u_1|_{\varepsilon=0} = \partial_{\varepsilon_{l_1}}\partial_{\varepsilon_{l_2}} u_2|_{\varepsilon=0} =: w^{(l_1,l_2)}$, $l_1, l_2 \in \{1,2,3\}$. Applying $\partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3}|_{\varepsilon=0}$ to (3.5), we obtain that

$$\begin{cases} -\Delta w_j + V_3^{(j)} v^{(1)}v^{(2)}v^{(3)} = H & \text{in} \quad \Omega, \\ w_j = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $H(x) = -2q(x)[\nabla v^{(1)} \cdot \nabla w^{(2,3)} + \nabla v^{(2)} \cdot \nabla w^{(1,3)} + \nabla v^{(3)} \cdot \nabla w^{(1,2)}]$ is independent of j. It follows that

$$\int_\Omega (V_3^{(1)} - V_3^{(2)}) v^{(1)} v^{(2)} v^{(3)} v^4 dx = 0,$$

for any $v^{(l)} \in C^\infty(\overline{\Omega})$ harmonic in Ω, $l = 1, \dots, 4$, such that supp $(v^{(l)}|_{\partial\Omega}) \subset \Gamma_1$, $l = 1, 2, 3$, and supp $(v^{(4)}|_{\partial\Omega}) \subset \Gamma_2$. Arguing as in [27], using the density result of [8], we conclude that $V_3^{(1)} = V_3^{(2)}$. The general inductive argument can now be carried out exactly as in [27]. The proof of Theorem 1.3 is complete.

4. Proof of Theorem 1.7

Theorem 1.7 is an immediate consequence of Theorem 1.3, once the obstacle has been recovered. The proof of the fact that $D_1 = D_2$ is standard, see for instance [30], and is presented here for completeness and convenience of the reader.

Following [30], we proceed by performing a first order linearization of the problem (1.3). To that end, let $\varepsilon \in \mathbb{C}$, and let $f \in C^\infty(\partial\Omega)$, supp $(f) \subset \Gamma_1$. An application of Theorem A.1 shows that for all $|\varepsilon|$ sufficiently small, the Dirichlet problem

$$\begin{cases} -\Delta u_j + q_j(x)(\nabla u_j)^2 + V_j(x, u_j) = 0 & \text{in} \quad \Omega \setminus \overline{D_j}, \\ u_j = 0 & \text{on} \quad \partial D_j, \\ u_j = \varepsilon f & \text{on} \quad \partial\Omega, \end{cases} \tag{4.1}$$

$j = 1, 2$, has a unique small solution $u_j = u_j(\cdot, \varepsilon) \in C^{2,\alpha}(\overline{\Omega} \setminus D_j)$, which depends holomorphically on $\varepsilon \in \text{neigh}(0, \mathbb{C})$ with values in $C^{2,\alpha}(\overline{\Omega} \setminus D_j)$. Differentiating (4.1) with respect to ε, taking $\varepsilon = 0$, and writing $v_j = \partial_\varepsilon u_j|_{\varepsilon=0}$, we get

$$\begin{cases} -\Delta v_j = 0 & \text{in} \quad \Omega \setminus \overline{D_j}, \\ v_j = 0 & \text{on} \quad \partial D_j, \\ v_j = f & \text{on} \quad \partial\Omega. \end{cases} \tag{4.2}$$

$j = 1, 2$. The fact that $\Lambda^{D_1,\Gamma_1,\Gamma_2}_{q_1,V^{(1)}}(\varepsilon f) = \Lambda^{D_2,\Gamma_1,\Gamma_2}_{q_2,V^{(2)}}(\varepsilon f)$ for all small ε and all $f \in C^\infty(\partial\Omega)$ with supp $(f) \subset \Gamma_1$ implies that $\partial_\nu v_1|_{\Gamma_2} = \partial_\nu v_2|_{\Gamma_2}$.

Assume that $D_1 \neq D_2$, and assume for example that D_2 is not contained in D_1. Let G be the connected component of $\Omega \setminus (\overline{D_1} \cup \overline{D_2})$ whose boundary contains $\partial\Omega$. Then there exists a point $x_0 \in \partial D_2$ such that $x_0 \in \Omega \setminus \overline{D_1}$ and $x_0 \in \partial G$, see [17, p. 1579]. We reproduce the argument of [17] for completeness and convenience of the reader. Indeed, by our assumption and the fact that ∂D_1 is smooth, there is a point $x' \in D_2 \setminus \overline{D_1}$. Let $x'' \in G$ be arbitrary and since $\Omega \setminus \overline{D_1}$ is connected, there is a continuous path $s(t) \in \Omega \setminus \overline{D_1}$, for $t \in [0,1]$, such that $s(0) = x'$ and $s(1) = x''$. We let $x_0 = s(t_0)$ where $t_0 = \sup\{t : s(t) \in D_2\}$.

To complete the proof, we follow [30] and let $v = v_1 - v_2$. Then we have $-\Delta v = 0$ in G, $v|_{\partial\Omega} = 0$, and $\partial_\nu v|_{\Gamma_2} = 0$. By the unique continuation principle for harmonic functions and continuity of harmonic functions up to the boundary, we conclude that $v_1 = v_2$ in $\overline{G}$. In view of (4.2), we get $0 = v_2(x_0) = v_1(x_0)$. Let us fix some $f \in C^\infty(\partial\Omega)$, supp $(f) \subset \Gamma_1$, such that $f \geq 0$, $f \not\equiv 0$. As $x_0 \in \Omega \setminus \overline{D_1}$, the maximum principle yields that $v_1 \equiv 0$ in $\Omega \setminus \overline{D_1}$. Since v_1 is continuous up to the boundary of $\Omega \setminus \overline{D_1}$, we get a contradiction, and therefore, $D_1 = D_2$. The proof of Theorem 1.7 is complete.

Appendix A. Well-posedness of the Dirichlet problem for a class of semilinear elliptic equations with a quadratic gradient term

The purpose of this appendix is to show the well-posedness of the Dirichlet problem for a class of semilinear elliptic equations with small boundary data. The argument is standard and is given here for completeness and convenience of the reader.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with C^∞ boundary. Let $k \in \mathbb{N} \cup \{0\}$ and $0 < \alpha < 1$. The Hölder space $C^{k,\alpha}(\overline{\Omega})$ consists of all functions $u \in C^k(\overline{\Omega})$ such

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} := \sum_{|\alpha|=k} \sup_{x,y\in\Omega, x\neq y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\alpha} + \|u\|_{L^\infty(\Omega)} < \infty.$$

We shall write $C^\alpha(\overline{\Omega}) = C^{0,\alpha}(\overline{\Omega})$. For future reference, we remark that $C^{k,\alpha}(\overline{\Omega})$ is an algebra under pointwise multiplication, and

$$\text{(A.1)} \qquad \|uv\|_{C^{k,\alpha}(\overline{\Omega})} \leq C\big(\|u\|_{C^{k,\alpha}(\overline{\Omega})}\|v\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}\|v\|_{C^{k,\alpha}(\overline{\Omega})}\big), \quad u, v \in C^{k,\alpha}(\overline{\Omega}),$$

see [13, Theorem A.7]. We also have the corresponding spaces $C^{k,\alpha}(M)$, where M is a compact C^∞ manifold.

We shall be concerned with the following Dirichlet problem,

$$\begin{cases} -\Delta u + q(x)(\nabla u)^2 + V(x,u) = 0 \quad \text{in} \quad \Omega, \\ u = f \quad \text{on} \quad \partial\Omega. \end{cases} \tag{A.2}$$

Here $q \in C^\alpha(\overline{\Omega})$, for some $0 < \alpha < 1$, and the function $V : \overline{\Omega} \times \mathbb{C} \to \mathbb{C}$ satisfies the following conditions:

(a) the map $\mathbb{C} \ni z \mapsto V(\cdot, z)$ is holomorphic with values in the Hölder space $C^\alpha(\overline{\Omega})$,

(b) $V(x,0) = 0$, for all $x \in \overline{\Omega}$.

The condition (b) ensures that $u = 0$ is a solution to (A.2) when $f = 0$. It follows from (a) and (b) that V can be expanded into a power series

$$V(x,z) = \sum_{k=1}^{\infty} V_k(x) \frac{z^k}{k!}, \quad V_k(x) := \partial_z^k V(x,0) \in C^\alpha(\overline{\Omega}), \tag{A.3}$$

converging in the $C^\alpha(\overline{\Omega})$ topology. Assume for simplicity that $V_1 \in C^\infty(\overline{\Omega})$ and let us suppose furthermore that

(c) 0 is not a Dirichlet eigenvalue of $-\Delta + V_1$.

We have the following result.

Theorem A.1. *There exist $\delta > 0$, $C > 0$ such that for any $f \in B_\delta(\partial\Omega) := \{f \in C^{2,\alpha}(\partial\Omega) : \|f\|_{C^{2,\alpha}(\partial\Omega)} < \delta\}$, the problem* (A.2) *has a solution $u = u_f \in C^{2,\alpha}(\overline{\Omega})$ which satisfies*

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \le C\|f\|_{C^{2,\alpha}(\partial\Omega)}.$$

Furthermore, the solution u is unique within the class

$$\{u \in C^{2,\alpha}(\overline{\Omega}) : \|u\|_{C^{2,\alpha}(\overline{\Omega})} < C\delta\}$$

and it is depends holomorphically on $f \in B_\delta(\partial\Omega)$.

Proof. We shall follow [29] and in order to prove this result we shall rely on the implicit function theorem for holomorphic maps between complex

Banach spaces, see [38, p. 144]. To that end, let us set

$$B_1 = C^{2,\alpha}(\partial\Omega), \quad B_2 = C^{2,\alpha}(\overline{\Omega}), \quad B_3 = C^{\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\partial\Omega),$$

and consider the map,

$$\begin{aligned} &F : B_1 \times B_2 \to B_3, \\ &F(f,u) = (-\Delta u + q(x)(\nabla u)^2 + V(x,u), u|_{\partial\Omega} - f). \end{aligned} \tag{A.4}$$

Let us first show that the map F has indeed the mapping property given in (A.4). We have $-\Delta u \in C^{\alpha}(\overline{\Omega})$ and an application of (A.1) gives $q(x)(\nabla u)^2 \in C^{\alpha}(\overline{\Omega})$. We only need to check that $V(x, u(x)) \in C^{\alpha}(\overline{\Omega})$. To this end, let us first observe that by Cauchy's estimates, the coefficients $V_k(x)$ in (A.3) satisfy

$$\|V_k\|_{C^{\alpha}(\overline{\Omega})} \le \frac{k!}{R^k} \sup_{|z|=R} \|V(\cdot, z)\|_{C^{\alpha}(\overline{\Omega})}, \quad R > 0. \tag{A.5}$$

Using (A.1) and (A.5), we get for all $k = 1, 2, \dots,$

$$\left\| \frac{V_k}{k!} u^k \right\|_{C^{\alpha}(\overline{\Omega})} \le \frac{C^k}{R^k} \|u\|^k_{C^{\alpha}(\overline{\Omega})} \sup_{|z|=R} \|V(\cdot, z)\|_{C^{\alpha}(\overline{\Omega})}. \tag{A.6}$$

Choosing $R = 2C\|u\|_{C^{\alpha}(\overline{\Omega})}$, we see that the series $\sum_{k=1}^{\infty} V_k(x) \frac{z^k}{k!}$ converges in $C^{\alpha}(\overline{\Omega})$ and therefore, $V(x, u(x)) \in C^{\alpha}(\overline{\Omega})$. Furthermore,

$$\|V(\cdot, u(\cdot))\|_{C^{\alpha}(\overline{\Omega})} \le \sup_{|z|=2C\|u\|_{C^{\alpha}(\overline{\Omega})}} \|V(\cdot, z)\|_{C^{\alpha}(\overline{\Omega})}.$$

We next claim that the map F in (A.4) is holomorphic. To this end, let us observe that since F is clearly locally bounded, it suffices verify the weak holomorphy, see [38, p. 133]. In doing so, let $(f_0, u_0), (f, u) \in B_1 \times B_2$, and let us show that the map

$$\lambda \mapsto F((f_0, u_0) + \lambda(f, u))$$

is holomorphic in $\mathbb{C}$ with values in B_3. Clearly, we only have to check that the map $\lambda \mapsto V(x, u_0(x) + \lambda u_1(x))$ is holomorphic in $\mathbb{C}$ with values in $C^{\alpha}(\overline{\Omega})$.

This follows from the fact that the series

$$\sum_{k=1}^{\infty} \frac{V_k}{k!}(u_0 + \lambda u_1)^k$$

converges in $C^\alpha(\overline{\Omega})$, locally uniformly in $\lambda \in \mathbb{C}$, see (A.6).

We have $F(0,0) = 0$ and the partial differential $\partial_u F(0,0) : B_2 \to B_3$ is given by

$$\partial_u F(0,0)v = (-\Delta v + V_1 v, v|_{\partial\Omega}).$$

In view of (c), an application of [11, Theorem 6.15] allows us to conclude that the map $\partial_u F(0,0) : B_2 \to B_3$ is a linear isomorphism.

By the implicit function theorem, see [38, p. 144], we get that there exists $\delta > 0$ and a unique holomorphic map $S : B_\delta(\partial\Omega) \to C^{2,\alpha}(\overline{\Omega})$ such that $S(0) = 0$ and $F(f, S(f)) = 0$ for all $f \in B_\delta(\partial\Omega)$. Setting $u = S(f)$ and noting that S is Lipschitz continuous and $S(0) = 0$, we see that

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \le C\|f\|_{C^{2,\alpha}(\partial\Omega)}.$$

The proof is complete. □

Corollary A.2. *The map*

$$B_\delta(\partial\Omega) \to C^{1,\alpha}(\overline{\Omega}), \quad f \mapsto \partial_\nu u_f|_{\partial\Omega}$$

is holomorphic.

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References

[1] M. Agranovich, Sobolev Spaces, Their Generalizations and Elliptic Problems in Smooth and Lipschitz Domains, Springer Monographs in Mathematics. Springer, Cham, (2015).

[2] Y. Assylbekov and T. Zhou, *Direct and inverse problems for the nonlinear time-harmonic Maxwell equations in Kerr-type media*, preprint, `arXiv:1709.07767`.

[3] I. Bejenaru, *Quadratic nonlinear derivative Schrödinger equations. I*, IMRP Int. Math. Res. Pap. (2006), Art. ID 70630, 84 pp.

[4] F. Browder, *Approximation by solutions of partial differential equations*, Amer. J. Math. **84** (1962), 134–160.

[5] C. Cârstea, G. Nakamura, and M. Vashisth, *Reconstruction for the coefficients of a quasilinear elliptic partial differential equation*, Appl. Math. Lett. **98** (2019), 121–127.

[6] X. Chen, M. Lassas, L. Oksanen, and G. Paternain, *Detection of Hermitian connections in wave equations with cubic non-linearity*, preprint, `arXiv:1902.05711`.

[7] T. Daudé, N. Kamran, and F. Nicoleau, *Non-uniqueness results for the anisotropic Calderón problem with data measured on disjoint sets*, Ann. Inst. Fourier (Grenoble) **69** (2019), no. 1, 119–170.

[8] D. Dos Santos Ferreira, C. Kenig, J. Sjöstrand, and G. Uhlmann, *On the linearized local Calderón problem*, Math. Res. Lett. **16** (2009), no. 6, 955–970.

[9] G. Eskin, *Lectures on linear partial differential equations*, Graduate Studies in Mathematics **123**, American Mathematical Society, Providence, RI, (2011).

[10] A. Feizmohammadi and L. Oksanen, *An inverse problem for a semilinear elliptic equation in Riemannian geometries*, preprint, `arXiv:1904.00608`.

[11] D. Gilbarg and N. Trudinger, Elliptic Partial Differential Equations of Second Order, reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, (2001).

[12] D. Hervas and Z. Sun, *An inverse boundary value problem for quasilinear elliptic equations*, Comm. Partial Differential Equations **27** (2002), no. 11–12, 2449–2490.

[13] L. Hörmander, *The boundary problems of physical geodesy*, Arch. Rational Mech. Anal. **62** (1976), no. 1, 1–52.

[14] O. Imanuvilov, G. Uhlmann, and M. Yamamoto, *The Calderón problem with partial data in two dimensions*, J. Amer. Math. Soc. **23** (2010), no. 3, 655–691.

[15] O. Imanuvilov, G. Uhlmann, and M. Yamamoto, *Inverse boundary value problem by measuring Dirichlet data and Neumann data on disjoint sets*, Inverse Problems **27** (2011), no. 8, 085007, 26 pp.

[16] O. Imanuvilov and M. Yamamoto, *Unique determination of potentials and semilinear terms of semilinear elliptic equations from partial Cauchy data*, J. Inverse Ill-Posed Probl. **21** (2013), no. 1, 85–108.

[17] V. Isakov, *On uniqueness in the inverse transmission scattering problem*, Comm. Partial Differential Equations **15** (1990), no. 11, 1565–1587.

[18] V. Isakov, *On uniqueness in inverse problems for semilinear parabolic equations*, Arch. Rational Mech. Anal. **124** (1993), no. 1, 1–12.

[19] V. Isakov, *Uniqueness of recovery of some quasilinear partial differential equations*, Comm. Partial Differential Equations **26** (2001), no. 11–12, 1947–1973.

[20] V. Isakov, *Inverse obstacle problems*, Inverse Problems **25** (2009), no. 12, 123002, 18 pp.

[21] V. Isakov and A. Nachman, *Global uniqueness for a two-dimensional semilinear elliptic inverse problem*, Trans. Amer. Math. Soc. **347** (1995), no. 9, 3375–3390.

[22] V. Isakov and J. Sylvester, *Global uniqueness for a semilinear elliptic inverse problem*, Comm. Pure Appl. Math. **47** (1994), no. 10, 1403–1410.

[23] K. Kang and G. Nakamura, *Identification of nonlinearity in a conductivity equation via the Dirichlet–to–Neumann map*, Inverse Problems **18** (2002), no. 4, 1079–1088.

[24] C. Kenig and M. Salo, *Recent progress in the Calderón problem with partial data*, Inverse Problems and Applications, 193–222, Contemp. Math. **615**, Amer. Math. Soc., Providence, RI, (2014).

[25] A. Kirsch and R. Kress, *Uniqueness in inverse obstacle scattering*, Inverse Problems **9** (1993), no. 2, 285–299.

[26] A. Kirsch and L. Päivärinta, *On recovering obstacles inside inhomogeneities*, Math. Methods Appl. Sci. **21** (1998), no. 7, 619–651.

[27] K. Krupchyk and G. Uhlmann, *A remark on partial data inverse problems for semilinear elliptic equations*, Proceedings of the AMS, to appear.

[28] Y. Kurylev, M. Lassas, and G. Uhlmann, *Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations*, Invent. Math. **212** (2018), no. 3, 781–857.

[29] M. Lassas, T. Liimatainen, Y-H. Lin, and M. Salo, *Inverse problems for elliptic equations with power type nonlinearities*, preprint, `arXiv:1903.12562`.

[30] M. Lassas, T. Liimatainen, Y-H. Lin, and M. Salo, *Partial data inverse problems and simultaneous recovery of boundary and coefficients for semilinear elliptic equations*, preprint, `arXiv:1905.02764`.

[31] M. Lassas, T. Liimatainen, and M. Salo, *The Poisson embedding approach to the Calderón problem*, Math. Annalen, to appear.

[32] M. Lassas, T. Liimatainen, and M. Salo, *The Calderón problem for the conformal Laplacian*, preprint `arXiv:1612.07939`.

[33] M. Lassas, G. Uhlmann, and Y. Wang, *Inverse problems for semilinear wave equations on Lorentzian manifolds*, Comm. Math. Phys. **360** (2018), no. 2, 555–609.

[34] P. Lax and R. Phillips, Scattering Theory. Pure and Applied Mathematics, Vol. 26 Academic Press, New York–London, (1967).

[35] W. McLean, Strongly Elliptic Systems and Boundary Integral Equations, Cambridge University Press, Cambridge, (2000).

[36] B. Mityagin, *The zero set of a real analytic function*, preprint, `arXiv:1512.07276v1`.

[37] C. Muñoz and G. Uhlmann, *The Calderon problem for quasilinear elliptic equations*, preprint, `arXiv:1806.09586`.

[38] J. Pöschel and E. Trubowitz, Inverse Spectral Theory, Pure and Applied Mathematics **130**, Academic Press, Inc., Boston, MA, (1987).

[39] D. Tataru, *Nonlinear geometric dispersive equations: an overview*, Mathematical Seminar, pp. 216–235, AIP Conf. Proc., 1329, Amer. Inst. Phys., Melville, NY, (2011).

[40] G. Uhlmann, *Inverse problems: seeing the unseen*, Bull. Math. Sci. **4** (2014), no. 2, 209–279.

[41] Z. Sun, *On a quasilinear inverse boundary value problem*, Math. Z. **221** (1996), no. 2, 293–305.

[42] Z. Sun, *Inverse boundary value problems for a class of semilinear elliptic equations*, Adv. in Appl. Math. **32** (2004), no. 4, 791–800.

[43] Z. Sun, *An inverse boundary-value problem for semilinear elliptic equations*, Electron. J. Differential Equations (2010), No. 37, 5 pp.

[44] Z. Sun and G. Uhlmann, *Inverse problems in quasilinear anisotropic media*, Amer. J. Math. **119** (1997), no. 4, 771–797.

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