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Slope filtrations of F-isocrystals and logarithmic decay

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Let k be a perfect field of positive characteristic and let X be a smooth irreducible quasi-compact scheme over k. The Drinfeld-Kedlaya theorem states that for an irreducible F-isocrystal on X, the gap between consecutive generic slopes is bounded by one. In this note we provide a new proof of this theorem. Our proof utilizes the theory of F-isocrystals with r-log decay. We first show that a rank one F-isocrystal with r-log decay is overconvergent if $r < 1$. Next, we establish a connection between slope gaps and the rate of log-decay of the slope filtration. The Drinfeld-Kedlaya theorem then follows from a patching argument.

1. Introduction

1.1. Motivation

Let k be a perfect field of positive characteristic and let X be a smooth irreducible quasi-compact scheme over k. When studying motives over X, one typically studies their ℓ-adic realization for some $\ell \neq p$. These are lisse ℓ-adic sheaves on X, which correspond to continuous ℓ-adic representations of $\pi_1^{et}(X)$. While lisse ℓ-adic sheaves are sufficient for studying the ℓ-adic and archimedean properties of a motive, thus far they have been insufficient for studying p-adic questions. For example, for a smooth proper fibration $f : Y \to X$, we know that the Frobenius eigenvalues of $R_i^{et} f_*(\mathbb{Q}_\ell)$ at a closed point $x \in X$ has ℓ-adic valuation zero, but there does not exist such a sweeping general statement about the p-adic valuations. In general, the p-adic valuations will change as x varies. It is therefore natural to ask how the p-adic valuations behave as one varies x (i.e. how does the p-adic Newton polygon of the characteristic polynomial of the Frobenius vary). By considering the p-adic realization of a motive, which are F-isocrystals, there are several beautiful statements about the variation of these Newton polygons. The first general result along these lines is due to Grothendeick, and says that the Newton polygon of a generic point lies below the Newton polygon of any specialization. Another significant result is the de Jong-Oort purity theorem, which tells us these Newton polygons are constant on an open subscheme and jump on a closed subscheme of codimension one. More recently, we have the Drinfeld-Kedlaya theorem (see [5]). This theorem states that for an irreducible F-isocrystal, the gaps between slopes of the generic Newton polygon are bounded by one. The purpose of this article is to provide a new proof of this theorem using F-isocrystals with logarithmic decay. Along the way, we prove two results about the logarithmic decay property, which are of intrinsic interest. The first result, Proposition 4.2, suggests that isocrystals with small rates of logarithmic decay are overconvergent. The second result, Theorem 5.1, describes an interesting connection between the slope filtration and the log-decay condition.

1.2. Statement of the main result and proof outline

Let M be either a convergent F-isocrystal or an overconvergent F-isocrystal on X whose rank is n (see §3). For any point $x \in X$, we may associate to $M|_x$ rational numbers $a_x^1(M), \dots, a_x^n(M) \in \mathbb{Q}$, with $a_x^1(M) \leq \cdots \leq a_x^n(M)$, which we call the slopes of M_x (see [8]). Informally, one may think of the slopes as the p-adic valuations of the "eigenvalues" of the Frobenius acting on x^*M. The Drinfeld-Kedlaya theorem is the following:

Theorem 1.1. *Assume that M is irreducible. Let $\eta \in X$ be the generic point. Then*

$$|a_\eta^{i+1}(M) - a_\eta^i(M)| \leq 1,$$

for each i.

In the first step of the proof of Theorem 1.1, we study a rank one isocrystals M on $\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m$ with r-log decay. In Theorem 4.2, we prove that if $r < 1$ then a tensor power $M^{\otimes p^k}$ has a log-connection. When M has a compatible Frobenius structure, we find that a higher tensor power extends to $\mathbb{A}_k^{n+m}$. This implies that the representation of $\pi_1^{et}(\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m)$ corresponding to M is potentially unramified at the coordinate planes, and thus M is overconvergent by a result of Kedlaya (see [13, Theorem 2.3.7]). It would be interesting to know if all isocrystals on $\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m$ with r-log-decay are overconvergent when $r < 1$.

For the second step of the proof, we consider an overconvergent F-isocrystal M on $\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m$ whose Newton polygon remains constant for each point $x \in \mathbb{G}^n_{m,k} \times \mathbb{A}^m_k$. Thus M obtains a slope filtration in the category of convergent F-isocrystals. That is, if $a^i_x(M) < a^{i+1}_x(M)$, there exists a convergent sub-F-isocrystal M_i of M, such that $a^j_x(M_i) = a^j_x(M)$ for all $j \leq i$ and all $x \in \mathbb{G}^n_{m,k} \times \mathbb{A}^m_k$. In Proposition 5.6 we prove that M_i has r_i-log-decay, where $r_i = \frac{1}{a^{i+1}_x(M) - a^i_x(M)}$. Combining this with the results of the previous paragraph, we see that M_i is overconvergent if $a^{i+1}_x(M) - a^i_x(M) > 1$.

The final step involves a geometric patching argument. We first consider a generically étale alteration $Y \to X$ with compactification $\overline{Y}$ such that $\overline{Y} - Y$ is a normal crossing divisor. Using the results of [11] we find finite étale maps locally on Y onto $\mathbb{G}^n_{m,k} \times \mathbb{A}^m_k$. This allows us to use the ideas of the previous paragraphs.

1.3. Relationship with previous approaches

The proof of Drinfeld and Kedlaya in [5] can be summarized as follows: first they prove that if $U \subset X$ is a dense open subscheme, the restriction functor from convergent F-isocrystals on X to F-isocrystals on U is fully faithful. This builds upon several other difficult fully faithful results due to Kedlaya and Shiho (see [10], [12, Theorem 5.1], and [17]). Let M be an F-isocrystal on X. Let $U \subset X$ be the locus of points where $a^i_\eta(M) = a^i_x(M)$. When we restrict M to U, we obtain a slope filtration. In particular, $M|_U$ corresponds to an element of $\mathrm{Ext}^1(M_1, M_2)$, where the slopes of M_1 are less than those of M_2. In [5], they show that $\mathrm{Ext}^1(M_1, M_2)$ is trivial when the smallest slope of M_2 is greater than one plus the largest slope of M_1. Therefore a gap between slopes larger than one implies a decomposition $M|_U = M_1 \oplus M_2$. This decomposition provides idempotent morphisms from $M|_U \to M|_U$, which extend to M by the aforementioned fully faithfulness result.

Our proof can be viewed as orthogonal to the Drinfeld-Kedlaya approach in two facets. First, instead of restricting to the constant locus of the Newton polygon, we prove that the Newton polygon remains constant along all of X. This allows us to completely bypass any sort of fully faithfulness result when M is convergent and only use the fully faithfulness of the overconvergent F-isocrystals to convergent F-isocrystals functor when M is overconvergent (see [10]). Second is the proof that $\mathrm{Ext}^1(M_1, M_2)$ is trivial, which can be traced back to Kedlaya's thesis (see [9, Theorem 5.2.1]). The underlying idea is that when the gaps between the slopes are larger than one, the connection is preserved at the corresponding step of the descending slope filtration due to de Jong (see [3]). The descending slope filtration only exists over some purely inseparable pro-cover, but using the connection it is possible to descend part of the filtration to the original base. This is in contrast to our proof, where we show that the pertinent step of the ascending slope filtration descends using the notion of r-logarithmic decay.

It is also worth mentioning that Drinfeld and Kedlaya assume that M is indecomposible. This is decidedly stronger than Theorem 1.1, where we assume M is irreducible. However, for the applications in [5] and the other applications we are aware of (e.g. [16]), Theorem 1.1 is sufficient. Of course, one could apply the Ext^1 result in [5] together with Theorem 1.1 to obtain this more general result.

Finally, let us mention previous work of the author, where we proved Theorem 1.1 for curves over a finite field (see [15, Corollary 1.8]). In this work, Theorem 1.1 was a corollary of a difficult monodromy theorem for rank one convergent F-isocrystals and an analysis of the slope filtration. In particular, using a monodromy theorem ([15, Proposition 6.15]) and class field theory, we showed that for $r < 1$, a rank one F-isocrystal with r-log-decay is overconvergent. This is the same as Theorem 4.4. However, we maintain that the present approach is preferable and necessary. First, the proof of Theorem 4.4 in this article relies on a study of the underlying differential equation. This elementary approach completely bypasses the technical monodromy theorem used in [15]. It is also amenable to the higher dimensional situation, where ramification theory is much more technical. Second, in this article we may take our ground field to be any perfect field k. Lastly, in this paper we deal with varieties of arbitrary dimension. Although one could use the Lefschetz theorem for F-isocrystals, due to Abe and Esnault (see [1]), to obtain results on higher dimensional varieties, this only works for finite fields. It also has the downside of being less direct of than the proof presented here.

1.4. A remark on logarithmic decay F-isocrystals

The notion of F-isocrystals with a log-decay Frobenius structure was introduced by Dwork-Sperber and plays a prominent role in Wan's work on unit-root L-functions (see [6], [18], and [19]). The log-decay condition for Frobenius structures arise naturally in the study of unit-root F-isocrystals. However, they only study F-isocrystals over $\mathbb{G}_{m,k}^n$. In [15], the author studied F-isocrystals with log-decay in both the Frobenius structure and the differential equation over curves. We studied the rate of log-decay of the slope filtration and the monodromy properties of F-isocrystals with log-decay. In the present article, we utilize the log-decay notion for a general variety X

in a somewhat ad-hoc manner. We find an alteration $Y \to X$, whose compactification $\overline{Y}$ is smooth and $\overline{Y} - Y$ is a normal crossing divisor D. Then we cover $\overline{Y}$ with open subschemes that admit an etale map onto $\mathbb{A}^n$, taking D to coordinate planes. This lets us use an explicit definition of r-log-decay for F-isocrystals on $\mathbb{G}^n_{m,k}$. Although sufficient for our applications, it is not clear if the property of having r-log-decay is intrinsic to an F-isocrystal on X. What if we choose a different alteration Y or find different étale maps onto $\mathbb{A}^n$? It would be interesting to either find an intrinsic definition of logarithmic decay or to prove that the ad-hoc notion used in this article is intrinsic.

2. Rings of functions on polyannuli

Let K be $W(k)[\frac{1}{p}]$ and let σ the lift of the Frobenius morphism on k. Let $n > 0$ and let $m \geq 0$. Consider indeterminates $T_1, \ldots, T_{n+m}$. We define the following rings:

$$\mathcal{A} = K\langle T_1^{\pm}, \ldots T_n^{\pm}, T_{n+1}, \ldots, T_{n+m}\rangle$$
$$\mathcal{A}_{(i)} = K\langle T_1^{\pm}, \ldots T_{i-1}^{\pm}, T_{i+1}^{\pm}, \ldots T_n^{\pm}, T_{n+1}, \ldots, T_{n+m}\rangle$$

We may extend σ to $\mathcal{A}$ by having σ send T_i to T_i^p. We let $v_p(.)$ be the Gauss valuation on $\mathcal{A}$ and $\mathcal{A}_{(i)}$, normalized so that $v_p(p) = 1$ (more generally $v_{p^f}(.)$ is the normalization with $v_{p^f}(p) = \frac{1}{f}$). Let $x(T) \in \mathcal{A}$. For each $i = 1, \ldots, n$ we may write

$$x(T) = \sum_{d \in \mathbb{Z}} a_d T_i^d,$$

with $a_d \in \mathcal{A}_{(i)}$ and $v_p(a_d) \to \infty$ as $|d| \to \infty$. We refer to this as the T_i-adic expansion of $x(T)$. Using this expansion, we define some truncations of $x(T)$:

$$W_<^{(i)}(x(T)) = \sum_{d<-1} a_d T_i^d \qquad W_\geq^{(i)}(x(T)) = \sum_{d \geq 0} a_d T_i^d.$$

Note that $W_<^{(i)}(W_\geq^{(j)}(x(T))) = W_\geq^{(j)}(W_<^{(i)}(x(T)))$. Furthermore, we define the T_i-adic residue to be

$$\mathrm{Res}_i(x(T)) = a_{-1}.$$

Next, we define the T_i-adic j-th partial valuation as

$$w_j^{(i)}(x(T)) = \min_{v_p(a_d) \leq j} \{d\}.$$

Using these partial valuations, we define the ring of overconvergent functions and the ring of r-log-decay functions:

$$\mathcal{A}^\dagger = \left\{ x(T) \in \mathcal{A} \;\middle|\; \begin{array}{l} \text{there exists } c > 0 \text{ such that for } i = 1, \dots, n \\ \text{and } j \gg 0, \text{ we have } w_j^{(i)}(x(T)) \geq -cj \end{array} \right\}$$
$$\mathcal{A}^r = \left\{ x(T) \in \mathcal{A} \;\middle|\; \begin{array}{l} \text{there exists } c > 0 \text{ such that for } i = 1, \dots, n \\ \text{and } j \gg 0, \text{ we have } w_j^{(i)}(x(T)) \geq -cp^{rj} \end{array} \right\}.$$

Note that σ restricts to endomorphisms of $\mathcal{A}^\dagger$ and $\mathcal{A}^r$. For $r < 1$ and $x \in \mathcal{A}^r$, the T_i-adic primitive of $W_<^{(i)}(x)$ converges to an element of $\mathcal{A}^{r+1}$:

$$\int W_<^{(i)}(x(T)) dT_i = \sum_{d < -1} a_d \frac{T_i^{d+1}}{d+1}.$$

3. *F*-isocrystals

Let X be a smooth irreducible quasi-compact scheme over k. We will freely use the notion of convergent and overconvergent F-isocrystals. For a high level overview, we recommend [14]. We let $\mathbf{F} - \mathbf{Isoc}^\dagger(X)$ denote the category of overconvergent F-isocrystals on X (see [14] for precise definitions) and we let $\mathbf{F} - \mathbf{Isoc}(X)$ denote the category of convergent F-isocrystals on X. For a dense open immersion $U \subset X$ we let $\mathbf{F} - \mathbf{Isoc}(U, X)$ denote the category of F-isocrystals on U overconvergent along $X - U$. For any finite extension E of $\mathbb{Q}_p$, we let $\mathbf{F} - \mathbf{Isoc}(X) \otimes E$ (resp $\mathbf{F} - \mathbf{Isoc}^\dagger(X) \otimes E$, $\mathbf{F} - \mathbf{Isoc}(U, X) \otimes E$) denote the category whose objects are objects in $\mathbf{F} - \mathbf{Isoc}(X)$ (resp $\mathbf{F} - \mathbf{Isoc}^\dagger(X)$, $\mathbf{F} - \mathbf{Isoc}(U, X)$) with a $\mathbb{Q}_p$-linear action of E. Given an open subscheme $V \subset U$ and $W \subset X$, such that $V \subset W$ is an open immersion, there is a natural restriction functor $\mathbf{F} - \mathbf{Isoc}(U, X) \to \mathbf{F} - \mathbf{Isoc}(V, W)$. If M is an object of $\mathbf{F} - \mathbf{Isoc}(U, X)$, we refer to the image of M in $\mathbf{F} - \mathbf{Isoc}(V, W)$ as the restriction of M to the pair (V, W).

Now let M be an object of $\mathbf{F} - \mathbf{Isoc}(X)$. For any $x \in X$ we define $b_x^i(M) = a_x^1(M) + \cdots + a_x^i(M)$ and let $NP_x(M)$ be the lower convex hull of the vertices $(i, b_x^i(M))$. If (i, y) is a vertex of $NP_x(M)$ for all $x \in X$, then by a theorem of Katz (see [8, Theorem 2.4.2] or [14, Theorem 4.1]) there exists a rank i subobject M_i of M in $\mathbf{F} - \mathbf{Isoc}(X)$ such that $a_x^j(M_i) = a_x^j(M)$

for $j \leq i$. When the y-coordinate of this vertex is zero, we refer to this subobject as the unit-root subcrystal, denoted by M^{u-r}.

3.1. F-isocrystals on $\mathbb{G}_{m,k}$ as (σ^f, ∇)-modules over $\mathcal{A}, \mathcal{A}^r$, and $\mathcal{A}^\dagger$

When $U = \mathbb{G}_{m,k}^n \times \mathbb{A}_k^m$ and $X = \mathbb{A}_k^{n+m}$, we may view objects of $\mathbf{F} - \mathbf{Isoc}(U)$ and $\mathbf{F} - \mathbf{Isoc}(U, X)$ as differential equations over the rings introduced in §2 with a compatible Frobenius structure. Let R be either $\mathcal{A}^\dagger$, $\mathcal{A}^r$, or $\mathcal{A}$. Let S be either $\mathcal{A}^r$ or $\mathcal{A}$ with $R \subset S$.

Definition 3.1. A σ^f-module is a locally free R-module M equipped with a σ^f-semilinear endomorphism $\varphi : M \to M$ whose linearization is an isomorphism. More precisely, we have $\varphi(am) = \sigma^f(a)\varphi(m)$ for $a \in R$ and $\varphi : R \otimes_{\sigma^f} M \to M$ is an isomorphism.

Definition 3.2. Let Ω_R be the module of differentials of R over K. Let $\delta_T : R \to \Omega_R$ to be the exterior derivative. A ∇-module over R is a locally free R-module M equipped with a connection. That is, M comes with a K-linear map $\nabla : M \to M \otimes_R \Omega_R$ satisfying the Liebnitz rule: $\nabla(am) = \delta_T(a)m + a\nabla(m)$. Note that $M \otimes_R S$ has the structure of a ∇-module over S. Given a ∇-module over R, we obtain an induced map $\nabla : M \otimes_R \Omega_R \to M \otimes_R \wedge^2 \Omega_R$. We say that M is integrable if $\nabla \circ \nabla = 0$.

Definition 3.3. Let M be a ∇-module over R with rank d. We say that M has a regular singularity if there exists a basis of M such that the connection is defined by a matrix of 1-forms

$$f_1(T)\frac{dT_1}{T_1} + \cdots + f_n(T)\frac{dT_n}{T_n} + f_{n+1}(T)dT_{n+1} + \cdots + f_{n+m}(T)dT_{n+m},$$

where $f_i(T)$ is a $d \times d$ matrix with entries in $K\langle T_1, \ldots, T_{n+m}\rangle$.

Definition 3.4. By abuse of notation, define $\sigma^f : \Omega_R \to \Omega_R$ be the map induced by pulling back the differential along σ^f. We define a (σ^f, ∇)-module to be an R-module M that is both a σ^f-module and an integrable ∇-module with the following compatibility condition:

$$\begin{array}{ccc} M & \xrightarrow{\nabla} & M \otimes \Omega_R \\ \downarrow{\scriptstyle \sigma^f} & & \downarrow{\scriptstyle \sigma^f \otimes \sigma^f} \\ M & \xrightarrow{\nabla} & M \otimes \Omega_R. \end{array}$$

We denote the category of (σ^f, ∇)-modules over R by $\mathbf{M\Phi}^{f\nabla}_R$ and we denote the category of (σ, ∇)-modules over R by $\mathbf{M\Phi}^{\nabla}_R$. There are functors $\mathbf{M\Phi}^{f\nabla}_R \to \mathbf{M\Phi}^{f\nabla}_S$ obtained by base changing to S. We say that an object M of $\mathbf{M\Phi}^{f\nabla}_{\mathcal{A}}$ is overconvergent (resp. has r-log-decay) if it lies in the essential image of $\mathbf{M\Phi}^{f\nabla}_{\mathcal{A}^\dagger} \to \mathbf{M\Phi}^{f\nabla}_{\mathcal{A}}$ (resp. $\mathbf{M\Phi}^{f\nabla}_{\mathcal{A}^r} \to \mathbf{M\Phi}^{f\nabla}_{\mathcal{A}}$). For an object M of $\mathbf{M\Phi}^{\nabla}_R$, we write $M \otimes_R S$ for its image in $\mathbf{M\Phi}^{\nabla}_S$. There are equivalences of categories

$$\begin{aligned} \mathbf{M\Phi}^{f\nabla}_{\mathcal{A}} &\longleftrightarrow \mathbf{F}-\mathbf{Isoc}(\mathbb{G}^n_{m,k} \times \mathbb{A}^m_k) \otimes \mathbb{Q}_{p^f} \\ \mathbf{M\Phi}^{f\nabla}_{\mathcal{A}^\dagger} &\longleftrightarrow \mathbf{F}-\mathbf{Isoc}^\dagger(\mathbb{G}^n_{m,k} \times \mathbb{A}^m_k, \mathbb{A}^{n+m}_k) \otimes \mathbb{Q}_{p^f}. \end{aligned}$$

There are natural functors $\epsilon_f : \mathbf{M\Phi}^{\nabla}_{\mathcal{A}} \to \mathbf{M\Phi}^{f\nabla}_{\mathcal{A}}$ and $\epsilon_f^\dagger : \mathbf{M\Phi}^{\nabla}_{\mathcal{A}^\dagger} \to \mathbf{M\Phi}^{f\nabla}_{\mathcal{A}^\dagger}$, which are obtained by iterating the Frobenius map f times.

4. Connections on polyannuli with r-log decay for $r < 1$

In this section we study rank one ∇-modules with small rates of logarithmic decay. We modify an argument of Crew (see the proof of [2, Proposition 4.11]) to show that a free rank one object of $\mathbf{M\Phi}^{f\nabla}_{\mathcal{A}^r}$ for $r < 1$ are overconvergent in $\mathbf{M\Phi}^{f\nabla}_{\mathcal{A}}$.

Lemma 4.1. *Let M be an integrable ∇-module over $\mathcal{A}$ or $\mathcal{A}^r$ that is free with rank one. Let e be a basis of M and write*

$$\nabla(e) = (f_1(T)dT_1 + \cdots + f_{n+m}(T)dT_{n+m}) \otimes e.$$

We have $Res_i(f_i) \in K$ for $i = 1, \ldots, n$. Furthermore, if $W^{(i)}_<(f_i) = 0$ then for each $j \neq i$ we have $W^{(i)}_\geq(f_j) = f_j$.

Proof. Since ∇ is integrable we know $\partial_i f_j = \partial_j f_i$. The lemma follows immediately. □

Proposition 4.2. *Let M be a free rank one integrable ∇-module over $\mathcal{A}^r$ for some $r < 1$. Then there exists m such that $M^{\otimes p^m} \otimes_{\mathcal{A}^r} \mathcal{A}$ has a regular singularity.*

Proof. Let e be a basis of M and let $c_{1,1}, \dots, c_{1,n+m} \in \mathcal{A}^r$ with $\nabla(e) = \sum c_{1,i} dT_i \otimes e$. Since $r < 1$ we know that

$$h_1 = \int W_<^{(1)}(c_{1,1}) dT_1$$

converges in $\mathcal{A}$. Thus, for τ sufficiently large we may consider the basis $e_1 = \exp(p^\tau h_1) e^{\otimes p^\tau}$ of $M^{\otimes p^\tau} \otimes_{\mathcal{A}^r} \mathcal{A}$. We have

$$\begin{aligned} \nabla^{\otimes p^\tau}(e_1) &= \sum_{i=1}^{d} c_{2,i} dT_i \otimes e_1 \\ c_{2,i} &= p^\tau \partial_i h_1 + p^\tau c_{1,i}. \end{aligned}$$

By our definition of h_1 we know that $W_<^{(1)}(c_{2,1}) = 0$, so by Lemma 4.1 we have $W_\geq^{(1)}(c_{2,i}) = c_{2,i}$ $i > 1$. As $W_\geq^{(1)}(\partial_i h_1) = 0$ this gives

$$\begin{aligned} W_<^{(2)}(c_{2,2}) &= W_<^{(2)}(W_\geq^{(1)}(c_{2,2})) \\ &= p^\tau W_<^{(2)}(W_\geq^{(1)}(c_{1,2})), \end{aligned}$$

which is contained in $\mathcal{A}^r$. In particular

$$h_2 = \int W_<^{(2)}(c_{2,2}) dT_2$$

converges in $\mathcal{A}$. After increasing τ, we may consider the basis $e_2 = \exp(h_2) e_1$ of $M^{\otimes p^\tau} \otimes_{\mathcal{A}^r} \mathcal{A}$. We have

$$\begin{aligned} \nabla^{\otimes p^\tau}(e_2) &= \sum_{i=1}^{d} c_{3,i} dT_i \otimes e_2 \\ c_{3,i} &= p^\tau \partial_i (h_1 + h_2) + p^\tau c_{1,i}. \end{aligned}$$

By our definition of h_2, we know that $w_<^{(j)}(c_{3,j}) = 0$ for $j = 1, 2$. Then by Lemma 4.1 we have $W_\geq^{(1)}(W_\geq^{(2)}(c_{3,3})) = c_{3,3}$. Since the truncation operators commute with each other and $W_\geq^{(j)}(\partial_i h_j) = 0$ for $j = 1, 2$, we find as above that

$$W_<^{(3)}(c_{3,3}) = p^\tau W_<^{(3)}(W_\geq^{(1)}(W_\geq^{(2)}(c_{1,3}))),$$

which is contained in $\mathcal{A}^r$. This allows us to define h_3. The proposition follows from repeating this process. □

Proposition 4.3. *Let M be a free rank one (σ^f, ∇)-module over $\mathcal{A}^r$ for some $r < 1$. Then a tensor power of $M \otimes_{\mathcal{A}^r} \mathcal{A}$ extends to $K\langle T_1, \dots, T_{n+m}\rangle$. More precisely, for some $k > 0$, there exists a basis of $M^{\otimes k} \otimes_{\mathcal{A}^r} \mathcal{A}$ such that the Frobenius and connection matrices have entries in $K\langle T_1, \dots, T_{n+m}\rangle$.*

Proof. By Proposition 4.2 we may assume that the connection of $M \otimes_{\mathcal{A}^r} \mathcal{A}$ has regular singularities. This means $M \otimes_{\mathcal{A}^r} \mathcal{A} = e\mathcal{A}$ and

$$\nabla(e) = (f_1 dT_1 + \cdots + f_{n+m} dT_{n+m}) \otimes e,$$

where $f_i = \frac{c_i}{T_i} + g_i$ with $c_i \in K$ and $g_i \in K\langle T_1, \dots, T_{n+m}\rangle$. Let $a \in \mathcal{A}$ satisfy $\varphi(e) = ae$. The compatibility between φ and ∇ imply

$$\frac{\partial_i a}{a} = p^f T_i^{p^f-1} f_i^{\sigma^f} - f_i. \tag{1}$$

This gives $\mathrm{Res}_i(\frac{\partial_i a}{a}) = p^f c_i^{\sigma^f} - c_i$. This residue is an integer n_i, so we have $c_i = \frac{n_i}{p^f-1}$.

It follows that the connection of $M^{\otimes(p^f-1)} \otimes_{\mathcal{A}^r} \mathcal{A}$ has integer exponents. In particular we have

$$\nabla(e^{\otimes p^f-1}) = (h_1 dT_1 + \cdots + h_{n+m} dT_{n+m}) \otimes e^{\otimes p^f-1},$$

where $h_i = \frac{n_i}{T_i} + (p^f - 1) g_i$. Now consider the basis of $M^{\otimes(p^f-1)} \otimes_{\mathcal{A}^r} \mathcal{A}$ defined by

$$e_0 = \prod T_i^{-n_i} e^{\otimes p^f-1}.$$

The connection in terms of this basis is

$$\nabla^{\otimes p^f-1}(e_0) = (b_1 dT_1 + \cdots + b_{n+m} T_{n+m}) \otimes e_0,$$

where $b_i \in K\langle T_1, \dots, T_{n+m}\rangle$. If $\varphi^{\otimes p^f-1}(e_0) = a_0 e_0$, we have

$$\partial_i a_0 = (p^f T_i^{p^f-1} b_i^{\sigma^f} - b_i) a_0,$$

from which we deduce that the T_i-adic expansion of a_0 contains no negative powers of T_i. □

Theorem 4.4. *Let M be a free rank one object of $\mathbf{F} - \mathbf{Isoc}(\mathbb{G}_m^n \times \mathbb{A}_k^m) \otimes \mathbb{Q}_{p^f}$, so that we may regard M as an (σ^f, ∇)-module over $\mathcal{A}$. If the connection descends to $\mathcal{A}^r$ for some $r < 1$, then M is overconvergent along the divisor $T_1 \dots T_n = 0$.*

Proof. We may assume that M is unit-root, and therefore corresponds to a p-adic character $\rho : \pi_1(\mathbb{G}_m^n \times \mathbb{A}_k^m) \to \mathbb{Q}_{p^f}^\times$. By Proposition 4.3 there exists $k > 0$ such that $M^{\otimes k}$ extends to an F-isocrystal on $\mathbb{A}_k^{n+m}$, meaning that $\rho^{\otimes k}$ extends to a representation of $\pi_1(\mathbb{A}_k^{n+m})$. This implies ρ is potentially unramified along $T_1 \dots T_n = 0$ as in [13, Definition 2.3.8]. By [13, Theorem 2.3.9], we know that M is overconvergent along the divisor $T_1 \dots T_n = 0$. $\square$

Question 4.5. Let $r < 1$. When M is a rank one integrable ∇-module over $\mathcal{A}^r$, the proof of Proposition 4.2 can be modified to show that M is overconvergent along the divisor $T_1 \dots T_n = 0$. One may ask if this is true for ∇-modules defined over $\mathcal{A}^r$ of higher rank. Similarly, are all (σ^f, ∇)-modules defined over $\mathcal{A}^r$ overconvergent along the divisor $T_1 \dots T_n = 0$?

5. Slope filtrations and rate of logarithmic decay

Let M be a free (σ, ∇)-module over $\mathcal{A}^\dagger$ of rank d. We will assume that the Newton polygon of x^*M remains constant as we vary over all points $x : \mathrm{Spec}(k_0) \to \mathbb{G}_{m,k}^n \times \mathbb{A}^m$, that the slopes are non-negative, and that the slope zero occurs exactly once. In particular, the object $M \otimes_{\mathcal{A}^\dagger} \mathcal{A}$ of $\mathbf{M\Phi}_{\mathcal{A}}^{\nabla}$ contains a rank one subobject M^{u-r}. We will further assume that each subobject in the slope filtration is free over $\mathcal{A}$. The main result of this section is:

Theorem 5.1. *Let s be the smallest nonzero slope of M and let $r = \frac{1}{s}$. There exists f such that $\epsilon_f(M^{u-r})$ has r-log-decay.*

We first introduce some auxiliary subrings of $\mathcal{A}^r$ and $\mathcal{A}^\dagger$:

$$\mathcal{A}^{r,c} = \left\{ x(T) \in \mathcal{A} \;\middle|\; \begin{array}{l} w_k^{(i)}(x(T)) \geq -cp^{rk} \text{ for } k > 0 \\ \text{and } w_0^{(i)}(x(T)) \geq 0 \text{ for } i = 1, \dots, m+n \end{array} \right\},$$
$$\mathcal{A}^{\dagger,r,c} = \mathcal{A}^{r,c} \cap \mathcal{A}^\dagger.$$

Note that $\mathcal{A}^{r,c}$ is p-adically complete, unlike $\mathcal{A}^r$. When R is any of these rings, we let $\mathcal{O}_R$ denote the ring of elements in R whose Gauss valuation is greater than or equal to zero. For f large enough, there exists $\omega \in K$ with $v_{p^f}(\omega) = s$. The following lemma follows from the definition of $\mathcal{A}^{r,c}$.

Lemma 5.2. *We have the following:*

1) *For $x \in \mathcal{A}^{r,c}$ we have $x^{\sigma^f} \in \mathcal{A}^{r,p^f c}$ and $\omega x \in \mathcal{A}^{r,p^{-f}c}$.*

2) Let $x \in \mathcal{A}$ and assume that $w_0^{(i)}(x) \geq 0$ for $i = 1, \ldots, m+n$. If $\omega x \in \mathcal{A}^{r,c}$ then $x \in \mathcal{A}^{r,p^f c}$.

Lemma 5.3. *Let $x, y \in \mathcal{O}_{\mathcal{A}^\dagger}$. Assume that $x^{-1}, \omega y \in \mathcal{O}_{\mathcal{A}^{\dagger,r,c}}$ and $w_0^{(i)}(y) \geq 0$ for each $i = 1, \ldots, m+n$. Then*

1) We have $x^{-\sigma^f} \omega y \in \mathcal{O}_{\mathcal{A}^{\dagger,r,c}}$.

2) If $\omega | y$ then $(x+y)^{-1} \in \mathcal{O}_{\mathcal{A}^{\dagger,r,c}}$.

Proof. We know that $y \in \mathcal{O}_{\mathcal{A}^{\dagger,r,p^f c}}$, which means $x^{-\sigma^f} y \in \mathcal{O}_{\mathcal{A}^{\dagger,r,p^f c}}$. Thus $x^{-\sigma^f} \omega y \in \mathcal{O}_{\mathcal{A}^{\dagger,r,c}}$. Since $\mathcal{O}_{\mathcal{A}^{r,c}}$ is p-adically complete, we know the geometric series $(1 + x^{-1}y)^{-1}$ is contained in $\mathcal{O}_{\mathcal{A}^{r,c}}$. Furthermore, since $1 + x^{-1}y \in \mathcal{O}_{\mathcal{A}^\dagger}^\times$, we know that $(1 + x^{-1}y)^{-1} \in \mathcal{O}_{\mathcal{A}^{\dagger,r,c}}$. Thus $(x+y)^{-1} = x^{-1}(1 + x^{-1}y)^{-1}$ is contained in $\mathcal{O}_{\mathcal{A}^{\dagger,r,c}}$. □

In the next two lemmas, we show that after a change of basis, the Frobenius matrix of M satisfies certain properties.

Lemma 5.4. *There exists a basis of M whose Frobenius is given by a matrix*

$$A = \begin{pmatrix} A_{1,1} & \omega A_{1,2} \\ \omega A_{2,1} & \omega A_{2,2} \end{pmatrix},$$

where $A_{i,j}$ are matrices with entries in $\mathcal{O}_{\mathcal{A}^\dagger}$ and $A_{1,1} \in \mathcal{O}_{\mathcal{A}^\dagger}^\times$.

Proof. Choose a basis of M and let $A_0 \in GL_d(\mathcal{A}^\dagger)$ be the matrix of the Frobenius structure in terms of this basis. By [8, Theorem 2.4.2], there is a slope filtration $M_0 \subset M_1 \subset \cdots \subset M_s = M \otimes_{\mathcal{A}^\dagger} \mathcal{A}$, such that M_{i+1}/M_i is isoclinic of slope α_i. Furthermore, our assumptions on the slopes of M imply that $\alpha_0 = 0$ and $\alpha_i \geq s$ for $i \geq 1$. By taking f sufficiently large, we may assume that there exists $\omega_i \in K$ such that $v_{p^f}(\omega_i) = \alpha_i$ for each i. There exists $N \in GL_d(\mathcal{A})$ such that $NA_0N^{-\sigma^f}$ is of the form

$$\begin{pmatrix} A_{1,1} & * & \ldots & * \\ 0 & A_{2,2} & * \ldots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \ldots & A_{s,s} \end{pmatrix},$$

where $A_{i,i}$ gives the Frobenius structure of M_{i+1}/M_i. We let $K(\omega_i)$ denote the rank one object of $\mathbf{M\Phi}_{\mathcal{A}}^\nabla$ whose Frobenius structure is multiplication by ω_i^{-1} and whose connection is the zero map. Then $M_{i+1}/M_i \otimes K(\omega_i)$ is a unit-root F-isocrystal, and therefore has a basis whose Frobenius matrix has entries in $\mathcal{O}_{\mathcal{A}}$. This follows from the construction of the Frobenius structure from the corresponding Galois representation (see [7, Theorem 4.1.1]). Thus, for $i \geq 1$ we may assume that $A_{i,i}$ has entries in $\omega_i \mathcal{O}_{\mathcal{A}} \subseteq \omega \mathcal{O}_{\mathcal{A}}$. Furthermore, after skew conjugation by a diagonal matrix whose entries are highly divisible by p, we may assume that the $*$'s in the top right of this matrix are elements of $\omega \mathcal{O}_{\mathcal{A}}$. Therefore we may assume that

$$NA_0 N^{-\sigma^f} \equiv \begin{pmatrix} A_{1,1} & 0 \\ 0 & 0 \end{pmatrix} \mod \omega \mathcal{O}_{\mathcal{A}}.$$

The lemma follows by taking $N_0 \in GL_d(\mathcal{A}^\dagger)$ whose entries are sufficiently close to those of N. □

Lemma 5.5. *After a change of variables, we may assume the following holds for all i:*

1) $w_0^{(i)}(A_{1,1}^{-1}) \geq 0$

2) $w_0^{(i)}(A_{1,2}) \geq 0$ *and* $w_0^{(i)}(A_{2,2}) \geq 0$.

Proof. To prove the first claim, it is enough to prove $w_0^{(i)}(A_{1,1}) \leq 0$. Let $C_i = \begin{pmatrix} T_i^k & 0 \\ 0 & 1_{d-1} \end{pmatrix}$. For k large enough, we see that $C_i A C_i^{-\sigma^f}$ satisfies the desired property. For the second claim we set $D_i = \begin{pmatrix} 1 & 0 \\ 0 & T_i^{-k} 1_{d-1} \end{pmatrix}$. When k is large enough, we find that $D_i A D_c^{-\sigma^f}$ satisfies the second property without changing the $(1,1)$-entry. □

By Lemma 5.5 we know that for c sufficiently large the entries of $A_{1,1}^{-1}$, $\omega A_{2,1}$, $\omega A_{1,2}$ and $\omega A_{2,2}$ are contained in $\mathcal{O}_{\mathcal{A}^{\dagger,r,c}}$. The connection is given by the differential matrices

$$C_1 dT_1 + \cdots + C_{n+m} dT_{n+m},$$

where the C_i are $d \times d$ matrices with entries in $\mathcal{O}_{\mathcal{A}^\dagger}$.

Proposition 5.6. *There exists* $N = \begin{pmatrix} 1 & 0 \\ N_{2,1} & 1_{d-1} \end{pmatrix}$ *where* $N_{2,1}$ *has entries in* $\mathcal{O}_{\mathcal{A}^{r,c}}$ *and* $NAN^{-\sigma^f}$ *is of the form* $A' = \begin{pmatrix} A'_{1,1} & \omega A_{1,2} \\ 0 & A'_{2,2} \end{pmatrix}$.

Proof. We will show inductively that there exists $N_k = \begin{pmatrix} 1 & 0 \\ N_{2,1,k} & 1_{d-1} \end{pmatrix}$ such that :

1) $A_k = N_k A N_k^{-\sigma^f}$ is of the form $\begin{pmatrix} A_{1,1,k} & \omega A_{1,2} \\ \omega^k A_{2,1,k} & \omega A_{2,2,k} \end{pmatrix}$
2) The entries of $A_{1,1,k}^{-1}, \omega A_{1,2,k}, \omega A_{2,2,k}$, and $\omega^k A_{2,1,k}$ are contained in $\mathcal{O}_{\mathcal{A}^{\dagger,r,c}}$
3) We have $w_0^{(i)}(A_{1,1,k}^{-1}) \geq 0$, $w_0^{(i)}(A_{1,2,k}) \geq 0$ and $w_0^{(i)}(A_{2,2,k}) \geq 0$.
4) For all k we have $N_k \equiv N_{k-1} \mod \omega^k$.

The result will follow by taking the limit of the N_k as $k \to \infty$. When $k = 1$ this follows from Lemma 5.5. Now let $k > 1$ and assume N_k exists. We define

$$N_k = \begin{pmatrix} 1 & 0 \\ -A_{1,1,k}^{-1}\omega^k A_{2,1,k} & 0 \end{pmatrix},$$

and set $A_{k+1} = N_k A_k N_k^{-\sigma^f}$. It is immediate that (1), (3), and (4) are satisfied. We can verify (2) using Lemma 5.3. □

Proof of Theorem 5.1. Let N and A' be as in Proposition 5.6. After changing basis by N, the connection is given by the matrix of 1-forms:

$$D_1 dT_1 + \cdots + D_{n+m} dT_{n+m},$$

where $D_i = (\partial_i N)N^{-1} + NC_i N^{-1}$. In particular, the D_i has entries in $\mathcal{O}_{\mathcal{A}^r}$. Compatibility between the connection and Frobenius give the relation:

$$\partial_i A' + D_i A' = p^f A' D_i^{\sigma^f}.$$

Write $D_i = \begin{pmatrix} R_i & S_i \\ U_i & V_i \end{pmatrix}$, where V is a $(d-1) \times (d-1)$-matrix. Continuing with the notation from Proposition 5.6 and considering the lower left corner, we obtain

$$U_i A'_{1,1} = p^f A'_{2,2} U_i^{\sigma^f}.$$

From here it is clear that $U_i = 0$. It follows that the R_i and $A'_{1,1}$ describe the unit-root sub-F-isocrystal M^{u-r} of M. As the connection and Frobenius structure are defined over $\mathcal{O}_{\mathcal{A}^r}$, we see that M^{u-r} has r-log-decay. □

Remark 5.7. In previous work of the author, we introduced a concept of strict r-log-decay when considering F-isocrystals over a curve. We say that a (σ^f, ∇)-module has strict r-log-decay if it has r-log-decay, but does not have r'-log-decay for any $r' < r$. Conjecture 4.11 in [15] states that the differences between consecutive slope determine the strict rates of log-decay of the slope filtration. More precisely, assume that $a_\eta^{i+1}(M) - a_\eta^i(M) > 0$ and let M_i be the convergent sub-F-isocrystal of M with $a_\eta^j(M_i) = a_\eta^j(M)$ for $j \leq i$. Then [15, Conjecture 4.11] states that M_i should have $\frac{1}{a_\eta^{i+1}(M) - a_\eta^i(M)}$-log-decay when M is irreducible. It is natural to formulate this conjecture for higher dimensional varieties as well. Using Theorem 5.1 and Theorem 4.4 we may prove this conjecture for generically ordinary objects of $\mathbf{F} - \mathbf{Isoc}(\mathbb{G}_m^n \times \mathbb{A}_k^m)$. It would be interesting to explore other higher dimensional cases of this conjecture.

6. Proof of the main result

Proposition 6.1. *Let M be a free rank d object of $\mathbf{F} - \mathbf{Isoc}(\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m, \mathbb{A}_k^{n+m})$. Assume that the Newton polygon of M remains constant on $x \in \mathbb{G}_{m,k}^n \times \mathbb{A}_k^m$. We also assume that each subobject in the slope filtration is also free. Let η be the generic point of $\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m$. We assume that $a_\eta^{i+1}(M) - a_\eta^i(M) > 0$ and let M_i denote the subobject of M in $\mathbf{F} - \mathbf{Isoc}(\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m)$ with $a_\eta^j(M_i) = a_\eta^j(M)$ for $j \leq i$. If $a_\eta^{i+1}(M) - a_\eta^i(M) > 1$, then M_i is overconvergent along $T_1 \dots T_n = 0$ (i.e. M_i is in the essential image of $\mathbf{F} - \mathbf{Isoc}(\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m, \mathbb{A}_k^{n+m}) \to \mathbf{F} - \mathbf{Isoc}(\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m)$).*

Proof. Assume $a_\eta^{i+1}(M) - a_\eta^i(M) > 1$. Let $r = \frac{1}{a_\eta^{i+1}(M) - a_\eta^i(M)}$. Note that M_i is overconvergent along $T_1 \dots T_n = 0$ if and only if $\det(M_i)$ is overconvergent along $T_1 \dots T_n = 0$. To see this, we adopt an common exterior product argument (see the proof of [9, Theorem 5.2.1] or the proof of [8, Theorem 2.4.2]). We realize M as a (σ^f, ∇)-module over $\mathcal{A}^\dagger$. Let r be the rank of M_i, which is a (σ^f, ∇)-module over $\mathcal{A}$. If $\det(M_i)$ is overconvergent on $\mathcal{A}^\dagger$, then by [10] we know that $\det(M_i)$ is a subobject of $\bigwedge^r M$ in $\mathbf{M\Phi}^{f}{}^{\nabla}_{\mathcal{A}^\dagger}$. Let M' be the subobject of M in $\mathbf{M\Phi}^{f}{}^{\nabla}_{\mathcal{A}^\dagger}$ consisting of elements $v \in M$ satisfying $v \wedge w = 0$ for all $w \in \det(M_i)$. Then $M' \otimes \mathcal{A}$ is isomorphic to M_i in $\mathbf{M\Phi}^{f}{}^{\nabla}_{\mathcal{A}}$, which means M_i is overconvergent along $T_1 \dots T_n = 0$. Thus, we may replace M with $\wedge^i M$ twisted so that the smallest slope is 0 and prove that the unit-root sub-F-isocrystal M^{u-r} is overconvergent along $T_1 \dots T_n = 0$. By Theorem 5.1 we know that $\epsilon_f(M^{u-r})$ has r-log-decay for some $f > 0$. As $r < 1$, we know from

Theorem 4.4 that $\epsilon_f(M^{u-r})$ is overconvergent along $T_1 \dots T_n = 0$. The functor ϵ_f for the corresponding Galois representations corresponds to the composition $\rho : \pi_1(\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m) \to GL_n(\mathbb{Q}_p) \to GL_n(\mathbb{Q}_{p^f})$. It follows that M^{u-r} is potentially unramified along $T_1 \dots T_n = 0$ and is therefore overconvergent along $T_1 \dots T_n = 0$ by [13, Theorem 2.3.9]. □

Lemma 6.2. *Let A_0 and B_0 be smooth k-algebras. Let $f_0 : A_0 \to B_0$ be a finite étale morphism. Let B (resp A) be a p-adically complete $W(k)$-algebra with $B \otimes_{W(k)} k = B_0$ (resp $A \otimes_{W(k)} k = A_0$) and let $f : A \to B$ be a lifting of f_0. If A and B are flat then f is finite étale.*

Proof. Since f_0 is finite étale, there exists $g_0(x_0) \in A_0[x_0]$ of degree d such that $B_0 = A_0[x_0]/g_0(x_0)$ and $g_0'(x_0)$ is a unit in B_0. Let $x \in B$ be a lift of x_0. We claim that B is isomorphic to $M = A \oplus xA \oplus \cdots \oplus x^{d-1}A$ as an A-module via the natural map $\theta : M \to B$. It suffices to show $\theta_n : M \otimes \mathbb{Z}/p^n\mathbb{Z} \to B \otimes \mathbb{Z}/p^n\mathbb{Z}$ is an isomorphism for all n. When $n = 1$ this is true because f_0 is finite étale. Assume that θ_n is an isomorphism. Let $y \in B \otimes \mathbb{Z}/p^{n+1}\mathbb{Z}$. We can find $x \in M \otimes \mathbb{Z}/p^n\mathbb{Z}$ such that $\theta_n(x) - y \in p^n(B \otimes \mathbb{Z}/p^{n+1})$. Since θ_0 is an isomorphism we can find $z \in M$ such that $p^n\theta_n(z) = \theta_n(x) - y$. This proves surjectivity. The injectivity of θ_{n+1} follows from the flatness assumption. This shows that B is a finite A-algebra. Furthermore, there exists $g(T) \in A[T]$ of degree d such that $B = A[T]/g(T)$ and x corresponds to T. Clearly g reduces to g_0 modulo p, so we know $g'(x) \neq 0$ is a unit in B. □

Theorem 6.3. *(Drinfeld-Kedlaya) Let k be perfect field of characteristic p. Let X be a smooth irreducible quasi-compact scheme over k. Let M be an irreducible object of $\mathbf{F} - \mathbf{Isoc}(X)$ or $\mathbf{F} - \mathbf{Isoc}^\dagger(X)$. Then for each i*

$$|a_\eta^{i+1}(M) - a_\eta^i(M)| \leq 1.$$

Proof. We first take M to be an object of $\mathbf{F} - \mathbf{Isoc}(X)$. Let $\eta \in X$ be a generic point. Assume that $a_\eta^{i+1}(M) - a_\eta^i(M) > 1$. We will show that for every closed point $x \in X$, we have $b_x^i(M) = b_\eta^i(M)$, which will imply M is not irreducible. By replacing M with a twist of $\wedge^i M$, we may assume that $b_\eta^1(M) = 0$ and $b_\eta^2(M) > 1$. The de Jong-Oort purity theorem (see [4]) tells us that the locus in X where $NP_x(M)$ lies above $NP_\eta(M)$ is a closed subscheme $D \subset X$ of. Let $x_0 \in D$ be a closed point. Let $i : C \hookrightarrow X$ be a smooth curve containing x_0. We further assume that the set theoretic intersection of D and C is equal to $\{x_0\}$ and let $U = C - \{x_0\}$. As $U \cap D$ is empty, there exists a rank one unit-root convergent sub-F-isocrystal M^{u-r} contained in $M|_U$. After shrinking C, we may find a morphism $f : C \to \mathbb{A}^1$ such that f

restricted to $C - \{x_0\}$ is finite étale over $\mathbb{G}_m$ of degree d (see [11, Theorem 2]). Consider $N = f_* M|_U$, which is overconvergent and the subcrystal $N^{u-r} = f_* M^{u-r}$. By Proposition 6.1 we know that N^{u-r} is overconvergent along 0, which implies M^{u-r} is overconvergent along x_0. Using [13, Theorem 2.3.9], we see that $(M^{u-r})^{\otimes n}$ extends to all of C for some large n. It follows that $b^1_{x_0}(M) = 0$.

Next, let M be an object of $\mathbf{F}-\mathbf{Isoc}^\dagger(X)$ and assume that $a^{i+1}_\eta(M) - a^i_\eta(M) > 1$. By the previous paragraph, we know that there is a convergent sub-F-isocrystal $M_i \subset M$ on X. We claim that M_i is overconvergent. As in the previous paragraph, we may assume $b^2_\eta(M) - b^1_\eta(M) > 1$ and that $b^1_\eta(M) = 0$. We will prove that M^{u-r} is overconvergent. Let $f : Y \to X$ be a generically etale morphism such that Y has a smooth compatification $\overline{Y}$ and $E = \overline{Y} - Y$ is a normal crossing divisor. Let $N = f^* M$ and $N^{u-r} = f^* M^{u-r}$. For $x \in E$, we let $U \subset \overline{Y}$ be an affine neighborhood of x and let $V = U - (U \cap E)$. Let $\mathbb{U}$ be a smooth lifting of U over $W(k)$ and let $\mathcal{U}$ be the rigid fiber of $\mathbb{U}$. Then we may regard $N|_V$ as a locally free sheaf $\mathcal{N}$ on a strict neighborhood $\mathcal{V}_\epsilon$ of the tube $]V[$ in $\mathcal{U}$ with a connection and a compatible Frobenius structure. After shrinking U, we may assume that $\mathcal{N}$ is a free $\mathcal{O}_{\mathcal{V}_\epsilon}$-module. Shrinking U further, may also assume each step of the slope filtration is a free $\mathcal{O}_{\mathcal{V}}$-module. By [11, Theorem 2] there exists a finite étale morphism $\overline{\pi} : U \to \mathbb{A}^{m+n}_k$ such that $\overline{\pi}(U \cap E)$ is the union of n coordinate hyperplanes. Now consider $\overline{\pi}_*(N|_V)$, which is an object of $\mathbf{F}-\mathbf{Isoc}(\mathbb{G}^n_{m,k} \times \mathbb{A}^m_k, \mathbb{A}^{n+m}_k)$ and $\pi_*(N^{u-r}|_V)$, which is the unit-root sub-F-isocrystal of $\pi_*(N|_V)$. We may lift $\overline{\pi}$ to a map $\pi : \mathbb{U} \to \mathrm{Spec}(W(k)\langle T_1, \ldots, T_{n+m}\rangle)$ and by Lemma 6.2 we see that π is finite étale. In particular, the rigid fiber of this map π^{rig} is finite étale and thus $(\pi^{rig}|_{\mathcal{V}_\epsilon})_*(\mathcal{N})$ is free. In particular, we may regard $\pi_*(N|_V)$ as a free (σ, ∇)-module over $\mathcal{A}^\dagger$. Similarly, each step of the slope filtration of $\pi_*(N|_V)$ is a free (σ, ∇)-module over $\mathcal{A}$. By Proposition 6.1 we know that $\pi_*(N^{u-r}|_V)$ is overconvergent along $T_1 \ldots T_n = 0$, which in turn means $N^{u-r}|_V$ is overconvergent along $U - V$. As being overconvergent can be checked Zariski locally, we see that N^{u-r} is overconvergent.

Now let W be a dense open subset of X and $Z \subset Y$ such that $f : Z \to W$ is a finite étale morphism of degree d. Note that $f_*(N^{u-r}|_Z)$ is isomorphic to $(M^{u-r})^d|_W$ and $f_* N^{u-r}$ is overconvergent, which means that $M^{u-r}|_W$ is overconvergent. It follows that M^{u-r} is overconvergent. □

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