

Two-solvable and two-bipolar knots with large four-genera

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For every integer g , we construct a 2-solvable and 2-bipolar knot whose topological 4-genus is greater than g . Note that 2-solvable knots are in particular algebraically slice and have vanishing Casson-Gordon obstructions. Similarly all known smooth 4-genus bounds from gauge theory and Floer homology vanish for 2-bipolar knots. Moreover, our knots bound smoothly embedded height four gropes in D^4 , an a priori stronger condition than being 2-solvable. We use new lower bounds for the 4-genus arising from $L^{(2)}$ -signature defects associated to meta-metabelian representations of the fundamental group.

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1. Introduction

A knot K in S^3 is *slice* if there exists a locally flat proper embedding $D^2 \hookrightarrow D^4$ such that the boundary of D^2 is the knot K . This idea of ‘4-dimensional triviality’ can be generalized in a number of ways, perhaps most easily by approximating a disc by a small genus surface. The 4-genus $g_4(K)$ of a knot K in S^3 is the minimal possible genus $g(\Sigma)$ of an orientable surface Σ with a locally flat proper embedding $\Sigma \hookrightarrow D^4$ in the 4-ball, where Σ has a single boundary component whose image coincides with K . From this point of view, a knot is approximately slice if it has small 4-genus. However, this perspective does not give successively closer approximations to sliceness; there also exist many knots of 4-genus one, such as the trefoil, which intuitively seem far from slice.

An alternate approach is to approximate the slice disc exterior $X_D := D^4 \setminus \nu(D^2)$, a compact 4-manifold with the three key properties that (i) $\partial X_D = M_K$, the 0-surgery of S^3 along K ; (ii) the inclusion induces an isomorphism $H_1(M_K) \cong H_1(X_D)$; and (iii) $H_2(X_D) = 0$. We therefore think of a compact 4-manifold W with $\partial W = M_K$ such that $i_*: H_1(M_K) \rightarrow H_1(W)$ is an isomorphism and some condition on $H_2(W)$ is satisfied as an approximation to a slice disc exterior. One might ask that $H_2(X_D)$ is of small rank, but a little thought shows that this essentially recovers the 4-genus condition, besides again not yielding arbitrarily refined approximations.

In [16], Cochran, Orr, and Teichner introduced a new perspective, motivated by surgery theory, in which one allows $H_2(W)$ to be arbitrarily large but requires that it is generated by almost disjointly embedded surfaces with a condition on the image of their fundamental groups in $\pi_1(W)$. See Section 2 for the precise definition. In fact, they give an infinite family of increasingly strict conditions, indexed by $h \in \frac{1}{2}\mathbb{N}$: a knot is said to be *h-solvable* if its 0-surgery bounds a slice disc exterior approximation satisfying the h th such condition. It is an open question whether any knot which is *h-solvable* for all h must be slice, and in general knots which are *h-solvable* for large h are hard to distinguish from slice knots.

The idea of solvability is closely related to the more geometric notion of bounding a *grope* of large height. A grope of height 1 is defined to be an orientable surface of arbitrary genus and a single boundary component, and a grope of height n is obtained by attaching boundaries of gropes of height $n - 1$ to an orientable surface along standard basis curves. We refer to [16, 21], or our Section 10 for the precise definition. A grope of larger height is a better approximation to a disc. Gropes are ingredients of fundamental importance for the topological disc embedding technology of Freedman and

Quinn [20, 21] on 4-manifolds, and also in the work of Cochran, Orr and Teichner [16] discussed above, where it was shown that if a knot K bounds an embedded framed grope of height h in D^4 then K is $(h - 2)$ -solvable. The converse remains an open question.

It is natural to ask whether the 4-genus and grope/ n -solvability approximations to sliceness have any relationship.

Question 1.1 ([3, Remark 5.6]). For a fixed h , do there exist h -solvable knots, i.e. knots which are close to slice in the sense of [16], which have arbitrarily large 4-genera, and hence are far from slice in the first sense?

This question seem to be difficult, one reason for which is that existing methods for extracting lower bounds for the topological 4-genus are not effective for h -solvable knots with $h \geq 2$. The simplest lower bounds are the Tristram-Levine signature function and Taylor's bound [38], the best possible bound for the 4-genus coming from the Seifert form. For algebraically slice knots these lower bounds vanish. In [25], Gilmer showed that there are algebraically slice knots with arbitrarily large 4-genus using Casson-Gordon signatures [1, 2]. In [3], Cha showed that there exist knots with arbitrarily large 4-genus which are algebraically slice and have vanishing Casson-Gordon signatures, using Cheeger-Gromov Von Neumann $L^{(2)}$ ρ -invariants corresponding to metabelian fundamental group representations. The above abelian and metabelian lower bounds can be used to give affirmative answers to Question 1.1 for the initial cases $h = 0, 1$, but these lower bounds vanish for h -solvable knots with $h \geq 2$. Extending Cha's ρ -invariant approach beyond the metabelian level to give further lower bounds for the 4-genus was left open, essentially because of difficulties arising from non-commutative algebra.

In this paper, we present a new method that avoids the non-commutative algebra problem. It enables us to go one step further than Gilmer and Cha, by combining a Casson-Gordon type approach and $L^{(2)}$ -signatures associated with representations to 3-solvable groups i.e. solvable groups with length 3 derived series. Here is our main result.

Theorem 1.2. *For each $g \in \mathbb{N}$, there exists a 2-solvable knot K with $g_4(K) > g$. Moreover, K bounds an embedded framed grope of height 4 in D^4 .*

Moreover, the knots of Theorem 1.2 are *2-bipolar* in the sense of Cochran, Harvey and Horn [14]. We give the definition in Section 2, noting for now that the notion of bipolarity is an approximation to being smoothly slice, which combines the idea of Donaldson's diagonalization theorem with fundamental

group information related to gropes and derived series. Also, for a 2-bipolar knot, the invariants τ , Υ , ε , ν^+ from Heegaard-Floer homology, as well as the d -invariants of p/q surgery, all cannot prove that the knot is not smoothly slice, and consequently cannot bound the smooth 4-genus [14]. This also holds for gauge theoretic obstructions such as those arising from Donaldson's theorem and the 10/8 theorem.

Theorem 1.2 answers the $h = 2$ case of Question 1.1, and prompts us to conjecture that the answer is 'yes' in general. In fact, we make a bolder conjecture.

Conjecture 1.3. *Let K be an h -solvable knot which is not torsion in \mathcal{C} . Then $\{\#^n K\}$ is a collection of h -solvable knots containing knots with arbitrarily large 4-genera.*

A knot which did not satisfy the second sentence would be an example of a non-torsion knot with stable 4-genus zero i.e. $\lim_{n \rightarrow \infty} g_4(nK)/n = 0$, and it is unknown whether any such knots exist [34]. Thus a counterexample to this conjecture would also be very interesting.

One might also wonder whether there exist highly bipolar knots with large smooth 4-genus, especially with the additional requirement that they be topologically slice. The following question seems to be unknown even in the case $h = 0$.

Question 1.4. Do there exist topologically slice h -bipolar knots with large smooth 4-genus?

As above, there are many reasonable candidate knots with which one might hope to answer 'yes.' The main result of [7] gave many examples of topologically slice, h -bipolar knots K which are of infinite order, even modulo the subgroup of $(h + 1)$ -bipolar knots, and a smooth/bipolar analogue of Conjecture 1.3 suggests we should expect $\#^n K$ to have arbitrarily large smooth 4-genus as $n \rightarrow \infty$.

Summary of the construction and proof. In order to construct 2-bipolar knots bounding height four gropes, we take connected sums of sufficiently many copies of the seed ribbon knot $R := 11_{n74}$, and perform satellite operations on a collection of judiciously chosen infection curves $\{\alpha_i^+, \alpha_i^-\}$, with α_i^\pm lying in the second derived subgroup $\pi_1(S^3 \setminus R)^{(2)}$ of the knot group of the i th copy of R . Our choice of (R, α^+, α^-) is depicted on the right side of Figure 3. We use knots $\{J_i^+, J_i^-\}$ with Arf invariant zero for

the companions of the satellite operations, chosen so that the $\{J_i^+\}$ have increasingly large negative Tristram-Levine signature functions and the $\{J_i^-\}$ have increasingly large positive signature functions.

Let K be the result of these satellite operations. In Proposition 2.3, we show that K is 2-solvable; in Proposition 2.6, we show that K is 2-bipolar; and in Proposition 10.7, we show that K bounds a grope of height 4 in D^4 . Writing K_i for the knot resulting from the satellite construction on $(R, \alpha_i^\pm, J_i^\pm)$, we have $K = \#_{i=1}^N K_i$. Let M_{K_i} be the zero-surgery manifold of K_i and write $Y := \bigsqcup_{i=1}^N M_{K_i}$.

The main idea of our proof is as follows. If there were a surface Σ of genus g embedded in D^4 with boundary K , then there would be an associated 4-manifold Z with boundary Y and a quotient Γ of $\pi_1(Z)$ such that the $L^{(2)}$ ρ -invariant

$$\rho^{(2)}(Y, \Gamma) := \rho^{(2)}(Y, \phi: \pi_1(Y) \rightarrow \pi_1(Z) \rightarrow \Gamma)$$

would be bounded above by a constant depending only on g and the base knot R . However, by choosing the infection knots $\{J_i^\pm\}$ to have suitably large Tristram-Levine signature functions, $L^{(2)}$ -induction will imply that $\rho^{(2)}(Y, \Gamma)$ must be very large so long some curve α_i^\pm represents an element of $\pi_1(Y)$ mapping nontrivially to Γ . The key difficulty is to show that this must always be the case, recalling that Γ depends on the hypothesized surface Σ .

In Example 6.1 we present a slightly simpler construction of a family of 2-solvable knots with arbitrary 4-genera, starting with connected sums of the ribbon knot 8_8 and performing a single satellite construction on each copy of 8_8 as indicated in Figure 1.

Coefficient systems: comparison with earlier methods. To show the nontriviality of some α_i^\pm in Γ , we use twisted homology over a metabelian representation to define the coefficient system. Although the representation is non-abelian, we use the ideas of Casson and Gordon [2] to define finitely generated twisted homology modules over a *commutative* principal ideal domain. The commutativity enables us to consider the “size” of the twisted homology modules in terms of the minimal number of generators, generalizing the abelian representation case in e.g. [3]. Supposing that the 4-genus is small compared to the size of the twisted first homology, we show that there is a *meta-metabelian* quotient Γ of $\pi_1(Z)$, i.e. a quotient whose third derived subgroup vanishes, in which one of the α_i^\pm is nontrivial in order to eventually obtain a contradiction. In previous approaches to slice obstructions using $L^{(2)}$ -signature defects corresponding to representations to groups with nontrivial n th derived subgroups for $n \geq 2$, the homology modules associated to non-abelian representations were over non-commutative rings,

for which it is still unknown how to implement an analogous generating rank argument.

In our method, it is also crucial to use $L^{(2)}$ -signatures over amenable groups that are *not torsion-free*, which were developed in [4, 9] and deployed in a similar context in [35].

The smooth slice genus. We remark that concordance obstructions predicated on a smooth embedding cannot be used to draw conclusions about locally flat surfaces, and hence cannot be used to prove our result. On the other hand our knots have arbitrarily large smooth 4-genus, since a smooth embedding of a surface in D^4 is in particular a locally flat embedding.

If we were interested in the smooth 4-genus version of Theorem 1.2, currently known techniques using Heegaard Floer homology or gauge theory would not apply. It is unknown whether the Rasmussen s -invariant, which does provide a lower bound on the smooth 4-genus of a knot, must vanish for 2-bipolar knots. Our knots are even the first examples in the literature of 1-bipolar knots with large 4-genus, though for that result one could use a simpler Casson-Gordon signature argument analogous to [25].

Horn's results. The fact that $g_4(K)$ is large implies that the base surface of any embedded grope in D^4 with boundary K must have large genus. The main theorem of Horn [28] gives examples, for each g and each n , of knots bounding height n gropes such that the base surface of any height n grope must have genus at least g . However, Horn's example knots are not known to have large 4-genera: he was only able to provide lower bounds on the genera of surfaces that extend to an embedding of a height n grope.

Organization of the paper. The next four sections are concerned with background theory. Section 2 recalls the definitions of the derived series of a group, a useful variation called the local derived series, and what it means for a knot to be h -solvable or h -bipolar. We also explain here how to construct h -solvable and h -bipolar knots using the satellite construction. Section 3 introduces some conventions for dealing with disconnected manifolds, in particular as relates to representations of their fundamental groupoids and associated twisted homology groups. Section 4 recalls the Cheeger-Gromov von Neumann $L^{(2)}$ ρ -invariant $\rho^{(2)}(Y, \phi)$ of a closed 3-manifold Y together with a homomorphism of its fundamental group $\pi_1(Y) \rightarrow \Gamma$ to a group Γ , and gives the facts about this invariant that we will need. Section 5 describes homology twisted with metabelian representations. In particular we consider coefficient systems inspired by Casson-Gordon invariants [2].

Section 6 begins the proof of Theorem 1.2, by precisely stating the criteria that will imply certain knots have large topological 4-genus, giving a brief outline of the proof, and providing examples meeting those criteria. Section 7 proves some technical lemmas that are vital in arranging that the representation used for our ρ -invariant computation is suitably nontrivial. For this, we control the size of the homology groups of certain covering spaces. In Section 8 we review a standard cobordism used in the proof of Theorem 1.2, and carefully investigate the way metabelian representations extend over this cobordism. Section 9 proves the main theorem by bounding the ρ -invariant in two different ways as described above. Section 10 proves that our knots bound height four embedded gropes.

2. The solvable and bipolar filtrations

In this section we recall the definitions of the solvable and bipolar filtrations, and how to construct highly solvable or bipolar knots. We will also need, later in the article, not just the standard derived series of a group but also the local derived series [4, 12, 13].

Definition 2.1. Let G be a group. The h th derived subgroup $G^{(h)}$ of G is defined recursively via $G^{(0)} := G$ and $G^{(h)} = [G^{(h-1)}, G^{(h-1)}]$ for $h \geq 1$. Moreover, for any sequence $\mathcal{S} = (S_i)_{i \in \mathbb{N}}$ of abelian groups, define the h th \mathcal{S} -local derived subgroup of G recursively by $G_{\mathcal{S}}^{(0)} := 0$ and, for $h \geq 1$,

$$G_{\mathcal{S}}^{(h)} := \ker \left\{ G_{\mathcal{S}}^{(h-1)} \rightarrow G_{\mathcal{S}}^{(h-1)} / [G_{\mathcal{S}}^{(h-1)}, G_{\mathcal{S}}^{(h-1)}] \right. \\ \left. \rightarrow (G_{\mathcal{S}}^{(h-1)} / [G_{\mathcal{S}}^{(h-1)}, G_{\mathcal{S}}^{(h-1)}]) \otimes_{\mathbb{Z}} S_h \right\}.$$

We remark that a group G is called *metabelian* if $G^{(1)} \neq 0$ but $G^{(2)} = 0$ and analogously *meta-metabelian* if $G^{(2)} \neq 0$ but $G^{(3)} = 0$. This explains some language from the introduction.

For any sequence \mathcal{S} and any $h \in \mathbb{N}$ we have that $G^{(h)} \subseteq G_{\mathcal{S}}^{(h)}$. Note that since for fixed $h \in \mathbb{N}$ the subgroup $G_{\mathcal{S}}^{(h)}$ only depends on the first h terms of \mathcal{S} , we will often take $\mathcal{S} = (S_1, \dots, S_h)$ to be a partial sequence. We will be particularly interested in $\mathcal{S} = (\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q})$ for a prime p .

For $h \in \mathbb{N}_{\geq 0}$, we now define h -solvability of a knot. As indicated in the introduction, there is an extension of this definition to $h \in \frac{1}{2}\mathbb{N}_{\geq 0}$. We do not require this more general definition, and refer the reader to [16, Definition 1.2] for details.

Definition 2.2. A knot K is *h-solvable* if there exists a compact spin 4-manifold W such that $\partial W = M_K$, the inclusion induced map $H_1(M_K) \rightarrow H_1(W)$ is an isomorphism, and there exist embedded surfaces with trivial normal bundle D_1, \dots, D_k and L_1, \dots, L_k in W such that

- 1) The surfaces are pairwise disjoint except for D_j and L_j , which for each $j = 1, \dots, k$ intersect transversely in a single point.
- 2) The second homology classes represented by $D_1, \dots, D_k, L_1, \dots, L_k$ generate $H_2(W)$.
- 3) The inclusion induced maps $\pi_1(D_i) \rightarrow \pi_1(W)$ and $\pi_1(L_i) \rightarrow \pi_1(W)$ have image contained in $\pi_1(W)^{(h)}$.

This gives a filtration of the knot concordance group by subgroups \mathcal{F}_h consisting of the concordance classes of *h-solvable* knots, explored in [15–18], among others. Every 1-solvable knot is algebraically slice and every 2-solvable knot has vanishing Casson-Gordon invariant sliceness obstruction. In particular, as mentioned in the introduction, the traditional 4-genus lower bounds of Tristram-Levine and Casson-Gordon signatures cannot be usefully employed with 2-solvable knots.

The satellite operation interacts particularly nicely with the solvable filtration. We remind the reader that given a knot R , *infection curves* $\alpha_1, \dots, \alpha_k$ in $S^3 \setminus \nu(R)$ that form an unlink in S^3 , and infection knots J_1, \dots, J_k , the *satellite of R by $\{J_i\}$ along $\{\alpha_i\}$* is defined to be the image of R in

$$\left(S^3 \setminus \bigsqcup_{i=1}^k \nu(\alpha_i) \right) \cup \bigsqcup_{i=1}^k E_{J_i} \cong S^3,$$

where E_{J_i} is the exterior of J_i and the identification is made so that a 0-framed longitude of α_i , denoted by $\lambda(\alpha_i)$, is identified with a meridian of J_i and vice versa. We denote this knot by $R_\alpha(J)$. The next proposition comes from [17, Proposition 3.1]. We will apply it with $h = 2$ to see that the knots we construct are 2-solvable.

Proposition 2.3. *Let R be a slice knot and $\{\alpha_i\}_{i=1}^k$ be a collection of unknotted, unlinked curves in $S^3 \setminus R$ such that $[\alpha_i] \in \pi_1(M_R)^{(h)}$ for all $i = 1, \dots, k$. If for each $i = 1, \dots, k$ the knot J_i has $\text{Arf}(J_i) = 0$, then $R_\alpha(J)$ is *h-solvable*.*

While our discussions have been thus far focused on the topological category, there are analogous notions of smooth sliceness, concordance, and

4-genera of knots. There is considerable interest in understanding the structure of \mathcal{T} , the collection of topologically slice knots modulo smooth concordance. Here the h -solvable filtration is of no use, since every topologically slice knot lies in $\bigcap_{h=0}^{\infty} \mathcal{F}_h$. This prompted Cochran-Harvey-Horn to define the *bipolar filtration* as follows.

Definition 2.4. A knot K is h -positive (respectively h -negative) if there exists a smoothly embedded disc D in a smooth simply connected 4-manifold V such that $\partial(V, D) = (S^3, K)$ and such that there exist disjointly embedded surfaces S_1, \dots, S_k in $V \setminus \nu(D)$ which form a basis for $H_2(V)$ such that for each $i = 1, \dots, k$,

- 1) The surface S_i has $S_i \cdot S_i = +1$ (respectively $S_i \cdot S_i = -1$).
- 2) The inclusion induced map $\pi_1(S_i) \rightarrow \pi_1(V \setminus D)$ has image contained in the subgroup $\pi_1(V \setminus D)^{(h)}$.

Note that smoothly slice knots are h -positive for all $h \in \mathbb{N}$, that the connected sum of two h -positive knots is h -positive, and that any knot that can be unknotted by changing crossings from positive to negative (negative to positive) is 0-positive (0-negative) [14].

Definition 2.5. We say that a knot K is h -bipolar if it is both h -positive and h -negative.

The following proposition, inspired by [7, Lemma 2.3], gives us a way to construct h -bipolar knots; we will apply it when $h = 2$.

Proposition 2.6. Let R be a smoothly slice knot and let η^+ and η^- be curves in the complement of R that form an unlink in S^3 . Suppose that each η^{\pm} represents a class in $\pi_1(S^3 \setminus R)^{(h)}$, and that for any knot J we have that $R_{\eta^+}(J)$ and $R_{\eta^-}(J)$ are both smoothly slice. Then for any 0-positive knot J^+ and 0-negative knot J^- , the satellite knot $R_{\eta^+, \eta^-}(J^+, J^-)$ is h -bipolar.

Proof. Since $R_{\eta^+}(J^+)$ is slice and J^- is 0-negative, the knot

$$R_{\eta^+, \eta^-}(J^+, J^-) = (R_{\eta^+}(J^+))_{\eta^-}(J^-)$$

is h -negative by [14, Proposition 3.3]. We see that

$$R_{\eta^+, \eta^-}(J^+, J^-) = (R_{\eta^-}(J^-))_{\eta^+}(J^+)$$

is h -positive by a symmetric argument. □

3. Disconnected manifolds, fundamental groups and twisted homology

We will need to understand the twisted homology of a connected 4-manifold X with disconnected boundary Y . In this section, we establish some technical details in this setting, for example by defining inclusion maps from the twisted homology of Y to that of X and showing that there is a long exact sequence of the homology of the pair (X, Y) . On a first reading we encourage the reader to skim this section, focusing on the paragraph leading into Definition 3.1 and the statement of Proposition 3.3. A similar discussion can be found in [23, Section 2.1].

We note once and for all that manifolds are oriented and either compact or arising as an infinite cover of a compact manifold. For a manifold V , we write $p: \tilde{V} \rightarrow V$ for the universal cover.

Let $Y = \bigsqcup_{i=1}^N Y_i$ be a compact ℓ -dimensional manifold with N connected components. Let $y_i \in Y_i$ be a basepoint for each connected component. Let S be a ring with unity and let A be a left S -module. A representation Φ of the fundamental groupoid of Y into $\text{Aut}(A)$ is equivalent to a homomorphism

$$\Phi = \prod_{i=1}^N \Phi_i: \prod_{i=1}^N \pi_1(Y_i, y_i) \rightarrow \text{Aut}(A)$$

from the free product of the fundamental groups of the connected components to $\text{Aut}(A)$. We will use the following examples.

- (a) Let Γ be a group. Then we will take $A = S = \mathbb{Z}\Gamma$, with $\Phi_i: \pi_1(Y_i, y_i) \rightarrow \Gamma \subseteq \text{Aut}(A)$, where $g \in \Gamma$ acts on A by left multiplication. We will also take $A = \mathcal{N}\Gamma$, the group Von Neumann algebra of Γ , discussed in Section 4.
- (b) The ring S is a commutative PID and $A = S^r$, together with a homomorphism

$$\Phi_i: \pi_1(Y_i, y_i) \rightarrow GL_r(S) = \text{Aut}(S^r).$$

For each i we use the representation $\Phi_i: \pi_1(Y_i, y_i) \rightarrow \text{Aut}(A)$ to give A a right $\mathbb{Z}[\pi_1(Y_i, y_i)]$ -module structure. Then we let $C_*(\tilde{Y}_i)$ be a cellular chain complex obtained by lifting some CW decomposition of Y_i (or a CW complex homotopy equivalent to Y_i in the case that Y_i is a topological 4-manifold),

and define the homology of Y twisted by Φ to be

$$H_*(Y; A) := \bigoplus_{i=1}^N H_*(A \otimes_{\mathbb{Z}[\pi_1(Y_i, y_i)]} C_*(\tilde{Y}_i)).$$

Now suppose that $Y = \partial X$, where X is a compact connected $(\ell + 1)$ -dimensional manifold with $\partial X = Y = \bigsqcup_{i=1}^N Y_i$. A schematic of a similar situation is shown in Figure 6. Let $x \in X$ be a basepoint and let $\tau_i: [0, 1] \rightarrow X$ be a path from x to y_i . The paths τ_i induce homomorphisms $\iota_i: \pi_1(Y_i, y_i) \rightarrow \pi_1(X, x)$, by $\gamma \mapsto \tau_i \gamma \bar{\tau}_i$.

Definition 3.1. We say that $\Phi: \prod_{i=1}^N \pi_1(Y_i, y_i) \rightarrow \text{Aut}(A)$ extends over X if there is a homomorphism $\Psi: \pi_1(X, x) \rightarrow \text{Aut}(A)$ such that $\Psi \circ \iota_i = \Phi_i$ for each $i = 1, \dots, N$.

Use the inclusion $j_i: Y_i \rightarrow X$ to define the pullback cover of Y_i in terms of the universal cover of X via the diagram:

$$\begin{array}{ccc} \tilde{Y}_i^X & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow p \\ Y_i & \xrightarrow{j_i} & X \end{array}$$

The pullback \tilde{Y}_i^X is given by pairs $\{(y, \tilde{x}) \in Y_i \times \tilde{X} \mid j_i(y) = p(\tilde{x})\}$. Apply the action of the group $\pi_1(X, x)$ on \tilde{X} to the second factor to obtain an action of $\pi_1(X, x)$ on \tilde{Y}_i^X . This is defined since the action on \tilde{X} is equivariant with respect to p . The action of $\pi_1(X, x)$ on \tilde{Y}_i^X induces an action of $\mathbb{Z}[\pi_1(X, x)]$ on the chain complex $C_*(\tilde{Y}_i^X)$.

Lemma 3.2. We have a homeomorphism

$$\pi_1(X, x) \times_{\pi_1(Y, y_i)} \tilde{Y}_i \cong \tilde{Y}_i^X,$$

where by definition the left hand side means: $\pi_1(X, x) \times \tilde{Y}_i / \sim$, where $(\gamma, \tilde{y}) \sim (\gamma', \tilde{y}')$ if there is $g \in \pi_1(Y, y_i)$ such that $\gamma \iota_i(g) = \gamma'$ and $g \cdot \tilde{y} = \tilde{y}'$.

Proof. Start with the covering space $\pi_1(Y_i, y_i) \rightarrow \tilde{Y}_i \rightarrow Y_i$ with fibre $\pi_1(Y_i, y_i)$, and then apply the ‘product over $\pi_1(Y_i, y_i)$ ’ construction to obtain a covering space $\pi_1(X, x) \times_{\pi_1(Y_i, y_i)} \pi_1(Y_i, y_i) \rightarrow \pi_1(X, x) \times_{\pi_1(Y_i, y_i)} \tilde{Y}_i \rightarrow Y_i$. Since

$\pi_1(X, x) \times_{\pi_1(Y_i, y_i)} \pi_1(Y_i, y_i) \cong \pi_1(X, x)$ as discrete spaces and affine sets over $\pi_1(X, x)$, this fibre bundle is homeomorphic to

$$\pi_1(X, x) \rightarrow \pi_1(X, x) \times_{\pi_1(Y_i, y_i)} \tilde{Y}_i \rightarrow Y_i.$$

Since both $\pi_1(X, x) \times_{\pi_1(Y_i, y_i)} \tilde{Y}_i$ and \tilde{Y}^X are covering spaces of Y_i corresponding to the homeomorphism $\iota_i: \pi_1(Y_i, y_i) \rightarrow \pi_1(X, x)$, they are homeomorphic by the classification of covering spaces. \square

It follows from Lemma 3.2 that we have an chain isomorphism of $C_*(\tilde{Y}_i^X)$ with $\mathbb{Z}[\pi_1(X, x)] \otimes_{\mathbb{Z}[\pi_1(Y_i, y_i)]} C_*(\tilde{Y}_i)$. Consider the sequence of chain maps:

$$\begin{aligned} A \otimes_{\mathbb{Z}[\pi_1(Y_i, y_i)]} C_*(\tilde{Y}_i) &\xrightarrow{\cong} A \otimes_{\mathbb{Z}[\pi_1(X, x)]} \mathbb{Z}[\pi_1(X, x)] \otimes_{\mathbb{Z}[\pi_1(Y_i, y_i)]} C_*(\tilde{Y}_i) \\ &\xrightarrow{\cong} A \otimes_{\mathbb{Z}[\pi_1(X, x)]} C_*(\tilde{Y}_i^X) \\ &\rightarrow A \otimes_{\mathbb{Z}[\pi_1(X, x)]} C_*(\tilde{X}). \end{aligned}$$

The first map sends $a \otimes c \mapsto a \otimes 1 \otimes c$. The second map uses the isomorphism discussed above, and the third map is induced by j_i .

This chain level map induces a map on homology $(j_i)_*: H_*(Y_i; A) \rightarrow H_*(X; A)$, which in turn induces

$$\bigoplus_{i=1}^N (j_i)_*: \bigoplus_{i=1}^N H_*(Y_i; A) \cong H_*(Y; A) \rightarrow H_*(X; A).$$

Let $\tilde{Y}^X := \bigsqcup_{i=1}^N \tilde{Y}_i^X$. Then by identifying \tilde{Y}^X with its image in \tilde{X} we also have relative twisted homology groups

$$H_*(X, Y; A) := H_*(A \otimes_{\mathbb{Z}[\pi_1(X, x)]} C_*(\tilde{X}, \tilde{Y}^X)).$$

The chain maps above fit into a short exact sequence of chain complexes

$$\begin{aligned} 0 \rightarrow \bigoplus_{i=1}^N A \otimes_{\mathbb{Z}[\pi_1(Y_i, y_i)]} C_*(\tilde{Y}_i) &\rightarrow A \otimes_{\mathbb{Z}[\pi_1(X, x)]} C_*(\tilde{X}) \\ &\rightarrow A \otimes_{\mathbb{Z}[\pi_1(X, x)]} C_*(\tilde{X}, \tilde{Y}^X) \rightarrow 0. \end{aligned}$$

That this is exact follows from the chain isomorphism

$$A \otimes_{\mathbb{Z}[\pi_1(Y_i, y_i)]} C_*(\tilde{Y}_i) \cong A \otimes_{\mathbb{Z}[\pi_1(X, x)]} C_*(\tilde{Y}_i^X).$$

The short exact sequence of chain complexes gives rise to a long exact sequence in homology, which we record in the next proposition.

Proposition 3.3. *With a fixed choice of paths $\{\tau_i\}$ and a choice of representation $\Phi: \prod_{i=1}^n \pi_1(Y_i, y_i) \rightarrow \text{Aut}(A)$ that extends over X , there is a long exact sequence in twisted homology*

$$\cdots \rightarrow H_k(Y; A) \rightarrow H_k(X; A) \rightarrow H_k(X, Y; A) \rightarrow H_{k-1}(Y; A) \rightarrow \cdots$$

with $H_k(Y; A) \rightarrow H_k(X; A)$ the inclusion induced map discussed above.

In later sections we work with many different representations of a given fundamental group(oid), and so we emphasize the representation Φ rather than the module A by writing $H_k^\Phi(Y)$ for $H_k(Y; A)$.

4. $L^{(2)}$ -signature invariants

In this section we introduce the Von Neumann $L^{(2)}$ ρ -invariant of a closed (not necessarily connected) 3-manifold equipped with a representation of its fundamental group or groupoid, and we recall the key properties of this invariant required for the proof of Theorem 1.2. In particular, we review the Cheeger-Gromov bound, a satellite formula, and an upper bound in terms of the second Betti number of a bounding 4-manifold.

Definition 4.1. Let Y be a closed oriented 3-manifold, let Γ be a discrete group, and let $\phi: \pi_1(Y) \rightarrow \Gamma$ be a representation. Note that Y might be disconnected, in which case we use the conventions of Section 3. Suppose that ϕ extends to $\Phi: \pi_1(W) \rightarrow \Gamma$ where W is a compact oriented 4-manifold with $\partial W = Y$. The *von Neumann $L^{(2)}$ ρ -invariant* of (Y, ϕ) is the signature defect

$$\rho^{(2)}(Y, \phi) = \sigma_\Gamma^{(2)}(W, \Phi) - \sigma(W),$$

where $\sigma_\Gamma^{(2)}(W, \Phi)$ is the $L^{(2)}$ -signature of the intersection form $\lambda_\Gamma: H_2(W, \mathcal{N}\Gamma) \times H_2(W, \mathcal{N}\Gamma) \rightarrow \mathcal{N}\Gamma$ and $\sigma(W)$ is the ordinary signature of the intersection form on $H_2(W; \mathbb{Q})$. Here the $L^{(2)}$ -signature is defined via the completion $\mathbb{Z}\Gamma \rightarrow \mathbb{C}\Gamma \rightarrow \mathcal{N}\Gamma$ to the Von Neumann algebra, and the spectral theory of operators on $\mathcal{N}\Gamma$ -modules. We refer to [16, Section 5] and [4, Section 3.1] for more details. In particular, $\rho^{(2)}(Y, \phi)$ only depends on the pair (Y, ϕ) since both the $L^{(2)}$ signature and the ordinary signature satisfy Novikov additivity and also $\sigma_\Gamma^{(2)}(V, \Phi) = \sigma(V)$ for a *closed* 4-manifold V . (See [10, p. 323] and [16, Lemma 5.9].)

This invariant was originally defined by Cheeger and Gromov via Riemannian geometry and η -invariants, independently of any bounding 4-manifold, so the above definition could be taken as a proposition that the two definitions coincide. For our purposes, as is common in the knot concordance literature, it is simpler to take the above as the definition; for a discussion, see [16, Section 5] and [18].

Example 4.2. Let M_J be the zero-framed surgery manifold of a knot $J \subset S^3$ and let $\phi: \pi_1(M_J) \rightarrow \mathbb{Z}$ be the abelianization map. Then

$$\rho_0(J) := \rho^{(2)}(M_J, \mathbb{Z}) = \int_{\omega \in S^1} \sigma_\omega(J) d\omega,$$

where $\sigma_\omega(J)$ is the Tristram-Levine signature of J at $\omega \in S^1$, that is the signature of $(1 - \omega)V + \overline{(1 - \omega)}V^T$ for V a Seifert matrix of J . See [16, Lemma 5.4] for the proof.

We will need the following theorem of Cheeger and Gromov, establishing a universal bound for the ρ -invariants of a fixed closed 3-manifold Y .

Theorem 4.3 ([11]). *Let Y be a closed oriented 3-manifold. Then there exists a constant C such that $|\rho^{(2)}(Y, \phi)| \leq C$ for any discrete group Γ and any representation $\phi: \pi_1(Y) \rightarrow \Gamma$.*

We will refer to the infimum of all such constants C as the Cheeger-Gromov constant of Y , denoted $C(Y)$. We note that [6] has given a proof of Theorem 4.3 using the signature defect definition of $\rho^{(2)}(Y, \phi)$ given above, and has given explicit bounds for $C(Y)$ in terms of the triangulation complexity of Y .

The following proposition comes from [15].

Proposition 4.4. *Let $K = R_\alpha(J)$ be the result of a satellite operation on a knot R by infection knots $\{J_k\}$ along infection curves $\{\alpha_k\}$. Let $\phi: \pi_1(M_K) \rightarrow \Gamma$, and suppose that for some $h \in \mathbb{N}$ we have $\alpha_k \in \pi_1(M_R)^{(h)}$ for all k and $\Gamma^{(h+1)} = 1$. Suppose that for all k , either $\phi(\alpha_k) = 1$ or $\phi(\alpha_k)$ is infinite order in Γ . Then the restriction induced maps $\pi_1(M_R \setminus \bigsqcup \nu(\alpha_k)) \rightarrow \Gamma$ and $\pi_1(E_{J_k}) \rightarrow \Gamma$ extend uniquely to $\phi_0: \pi_1(M_R) \rightarrow \Gamma$ and $\phi_k: \pi_1(M_{J_k}) \rightarrow \Gamma$ and we have*

$$\rho^{(2)}(M_K, \phi) = \rho^{(2)}(M_R, \phi_0) + \sum_k \rho^{(2)}(M_{J_k}, \phi_k).$$

Proof. The proof of [15, Lemma 2.3] applies, with the following modification. The original statement of this proposition assumes the additional hypothesis that Γ is a poly-torsion-free-abelian (PTFA) group. However, an inspection of the proof shows that we need only assume that for each k either $\phi(\alpha_k) = 1 \in \Gamma$ or $\phi(\alpha_k)$ is infinite order. In the case that $\phi(\alpha_k) \neq 1$ in Γ , they need in the proof of [15, Lemma 2.3] that $H_1(\alpha_k; \mathbb{Z}\Gamma) = 0$. But since $\phi(\alpha_k)$ is infinite order, $H_1(\alpha_k; \mathbb{Z}\Gamma)$ is the first homology of $\mathbb{R} \times \Gamma / \langle \alpha_k \rangle$, which vanishes. \square

We will apply Proposition 4.4 when $\Gamma = G/G_{(\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q})}^{(3)}$ for some group G and $h = 2$. For such Γ , any curves $\alpha_k \in \pi_1(M_R)^{(2)}$ satisfy the hypothesis of the proposition, since then $\phi(\alpha_k) \in \Gamma^{(2)}$ and $\Gamma^{(2)}/\Gamma^{(3)}$ is torsion-free.

Under the assumptions of Proposition 4.4, we have that the map $\phi_k : \pi_1(M_{J_k}) \rightarrow \Gamma$ factors through the abelianization map. To see this, note that each meridian of J_k is identified with a longitude of α_k , which lies in $\pi_1(M_R)^{(h)}$ and hence is sent to $\Gamma^{(h)}$. So the image of $\phi_k : \pi_1(M_{J_k}) \rightarrow \Gamma$ is contained in $\Gamma^{(h)}$, which is an abelian group since $\Gamma^{(h+1)} = 1$. When $\Gamma^{(h)}/\Gamma^{(h+1)}$ is torsion-free, as occurs when $\Gamma = G/G_{(\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q})}^{(3)}$ and $h = 2$, we therefore have that ϕ_k is either the zero map or maps onto a copy of \mathbb{Z} in Γ . By the principle of $L^{(2)}$ -induction [16, Proposition 5.13] and Example 4.2, we have that

$$\rho^{(2)}(M_{J_k}, \phi_k) = \begin{cases} \rho_0(J_k) & \text{if } \phi_k \neq 0 \\ 0 & \text{if } \phi_k = 0. \end{cases}$$

Finally, note that since a meridian of J_k is identified with a longitude $\lambda(\alpha_k)$ of α_k in M_K , we have that ϕ_k is the zero map if and only if $\phi(\lambda(\alpha_k)) = 0$. We summarize the results of the above discussion for later use.

Proposition 4.5. *Let $K = R_\alpha(J)$ be the result of a satellite operation on R by infection knots $\{J_k\}$ along infection curves $\{\alpha_k\}$ lying in $\pi_1(M_R)^{(2)}$. Let $\Gamma = G/G_{(\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q})}^{(3)}$ for some group G and prime p , and let $\phi : \pi_1(M_K) \rightarrow \Gamma$. Then the restriction induced maps $\pi_1(M_R \setminus \bigsqcup \nu(\alpha_k)) \rightarrow \Gamma$ and $\pi_1(E_{J_k}) \rightarrow \Gamma$ extend uniquely to maps $\phi_0 : \pi_1(M_R) \rightarrow \Gamma$ and $\phi_k : \pi_1(M_{J_k}) \rightarrow \Gamma$. Moreover,*

$$\begin{aligned} \rho^{(2)}(M_K, \phi) &= \rho^{(2)}(M_R, \psi_0) + \sum_k \rho^{(2)}(M_{J_k}, \phi_k) \\ &= \rho^{(2)}(M_R, \phi_0) + \sum_k \delta_k(\psi) \rho_0(J_k), \end{aligned}$$

where

$$\delta_k(\psi) = \begin{cases} 0 & \text{if } \psi(\lambda(\alpha_k)) = 0 \\ 1 & \text{if } \psi(\lambda(\alpha_k)) \neq 0. \end{cases}$$

The following straightforward consequence of [4, Theorem 3.11] will provide our key upper bound on $L^{(2)}$ -signatures. Strebel’s class of groups $D(\mathbb{Z}_p)$ was defined in [37]; we will not recall the definition. We will use the fact that for any group G and any $h \in \mathbb{N}$, we have that $\Lambda = G/G_S^{(h)}$ is amenable and lies in $D(\mathbb{Z}_p)$ provided S_i is either \mathbb{Q} or $\mathbb{Z}_{p^{a_i}}$ for every $i \in \mathbb{N}$ [9, Lemma 6.8].

Theorem 4.6. *Let Z be a 4-manifold with boundary $\partial Z = Y$ and let $\phi : \pi_1(Y) \rightarrow \pi_1(Z) \rightarrow \Lambda$ be a homomorphism, where Λ is amenable and in Strebel’s class $D(\mathbb{Z}_p)$. Then $|\rho^{(2)}(Y, \phi)| \leq 2 \dim_{\mathbb{Z}_p} H_2(Z, \mathbb{Z}_p)$.*

Proof of Theorem 4.6. Let \tilde{Z} be the cover of Z corresponding to the kernel of the map $\pi_1(Z) \rightarrow \Lambda$. Since Z is a compact 4-manifold with boundary, it has the homotopy type of a finite 3-dimensional CW complex. This follows from [30, §1(III)] to get a finite CW complex, combined with [39, Corollary 5.1] to restrict the dimension of the CW complex to three. Let C_* be the corresponding chain complex, and let $\tilde{C}_* := C_*(\tilde{Z})$ denote the chain complex of \tilde{Z} . Since Λ is amenable and in Strebel’s [37] class $D(\mathbb{Z}_p)$, [4, Theorem 3.11] tells us that

$$\begin{aligned} \dim^{(2)} H_2(Z; \mathcal{N}\Lambda) &= \dim_{\mathbb{Z}_p} H_2(\mathbb{Z}_p \otimes_{\mathbb{Z}\Lambda} \tilde{C}_*) \\ &= \dim_{\mathbb{Z}_p} H_2(\mathbb{Z}_p \otimes_{\mathbb{Z}} C_*) = \dim_{\mathbb{Z}_p} H_2(Z; \mathbb{Z}_p). \end{aligned}$$

It follows that

$$\begin{aligned} |\rho^{(2)}(Y, \phi)| &= |\sigma_{\Lambda}^{(2)}(Z) - \sigma(Z)| \leq \dim^{(2)} H_2(Z; \mathcal{N}\Lambda) + \dim_{\mathbb{Q}} H_2(Z; \mathbb{Q}) \\ &\leq 2 \dim_{\mathbb{Z}_p} H_2(Z; \mathbb{Z}_p). \end{aligned}$$

For the final inequality, we use the universal coefficient theorem to deduce that $\dim_{\mathbb{Q}} H_2(Z; \mathbb{Q}) \leq \dim_{\mathbb{Z}_p} H_2(Z; \mathbb{Z}_p)$. □

5. Metabelian twisted homology

In this section we review Casson-Gordon type metabelian representations of knot groups, and the resulting twisted homology. The behavior of infection curves in this twisted homology will be key to our proof of Theorem 1.2.

We now let S denote a commutative PID and let Q denote its quotient field. We will often take $S = \mathbb{F}[t^{\pm 1}]$ and $Q = \mathbb{F}(t)$ for some field \mathbb{F} , as well as $S = \mathbb{Z}$ and $Q = \mathbb{Q}$.

Let X be a space homotopy equivalent to a finite CW-complex and let A be a left S -module given the structure of a right $\mathbb{Z}[\pi_1(X)]$ -module by a homomorphism $\phi: \pi_1(X) \rightarrow \text{Aut}(A)$. Note that $\pi_1(X)$ naturally acts on \widetilde{C}_* , the chain complex of the universal cover \widetilde{X} of X , on the left. Then as in Section 3, the twisted homology $H_*(X; A)$ is defined to be

$$H_*^\phi(X) := H_*(A \otimes_{\mathbb{Z}[\pi_1(X)]} \widetilde{C}_*).$$

We will be particularly interested in the following metabelian representations. Suppose that we have a preferred surjection $\varepsilon: H_1(X) \rightarrow \mathbb{Z}$. For every $r \in \mathbb{N}$, we let $p_r: \mathbb{Z} \rightarrow \mathbb{Z}_r$ be the usual projection map and let X^r denote the r -fold cyclic cover of X corresponding to $\ker(p_r \circ \varepsilon)$. Note that covering transformations give $H_1(X^r)$ the structure of a $\mathbb{Z}[\mathbb{Z}_r]$ -module. Choosing a preferred element $\gamma_0 \in \pi_1(X)$ with $\varepsilon(\gamma_0) = +1$ then gives us a map

$$\rho_{\gamma_0}: \pi_1(X) \rightarrow \mathbb{Z} \ltimes H_1(X^r) \quad \text{by } \gamma \mapsto (t^{\varepsilon(\gamma)}, [\gamma_0^{-\varepsilon(\gamma)} \gamma]),$$

where $\gamma_0^{-\varepsilon(\gamma)} \gamma \in \pi_1(X^r) \leq \pi_1(X)$ and $[\gamma_0^{-\varepsilon(\gamma)} \gamma]$ denotes the image of $\gamma_0^{-\varepsilon(\gamma)} \gamma$ under the Hurewicz map.

Given any choice of a homomorphism $\chi: H_1(X^r) \rightarrow \mathbb{Z}_m$, we let $\xi_m = e^{2\pi i/m}$ and obtain a map $\theta_\chi: \mathbb{Z} \ltimes H_1(X^r) \rightarrow \text{GL}_r(\mathbb{Q}(\xi_m)[t^{\pm 1}])$ by

$$(t^j, a) \mapsto \begin{bmatrix} 0 & \dots & 0 & t \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix}^j \begin{bmatrix} \xi_m^{\chi(a)} & 0 & \dots & 0 \\ 0 & \xi_m^{\chi(t \cdot a)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \xi_m^{\chi(t^{k-1} \cdot a)} \end{bmatrix}.$$

We then let $S = \mathbb{Q}(\xi_m)[t^{\pm 1}]$ and $A = \mathbb{Q}(\xi_m)[t^{\pm 1}]^r$, noting that $\theta_\chi \circ \rho_{\gamma_0}$ gives A a right $\mathbb{Z}[\pi_1(X)]$ -module structure. These representations appear in [22, 24, 26, 31, 32], modelled on the covering spaces used in the definition of Casson-Gordon invariants [2]. We refer to such representations as *Casson-Gordon type representations*.

In particular, given an oriented knot K and a preferred meridian $\mu \in \pi_1(X_K)$, the canonical abelianization map $\varepsilon: \pi_1(X_K) \rightarrow \mathbb{Z}$ has $\varepsilon(\mu) = +1$. Note that since the zero-framed longitude λ_K of K is an element of $\pi_1(X_K)^{(2)}$, for every $r \in \mathbb{N}$ the map $\rho_\mu: \pi_1(X_K) \rightarrow \mathbb{Z} \ltimes H_1(X_K^r)$ extends uniquely over $\pi_1(M_K)$. The homology $H_1(X^r)$ splits canonically as $H_1(\Sigma_r(K)) \oplus \mathbb{Z}$, where

$\Sigma_r(K)$ is the r th cyclic branched cover of S^3 along K . Our map $\chi: H_1(X^r) \rightarrow \mathbb{Z}_m$ will always be chosen to factor through the projection map to $H_1(\Sigma_r(K))$.

In the case $r = 2$ we have that t must act by -1 on $H_1(\Sigma_2(K))$, as discussed in the first paragraphs of [19], and so we can conveniently decompose $\theta_\chi \circ \rho_\mu$ differently as $\theta \circ f_\chi$, where

$$f_\chi: \pi_1(M_K) \rightarrow \mathbb{Z} \times \mathbb{Z}_m$$

$$\gamma \mapsto (t^{\varepsilon(\gamma)}, \chi([\mu^{-\varepsilon(\gamma)}\gamma]))$$

and

$$\theta: \mathbb{Z} \times \mathbb{Z}_m \rightarrow \text{GL}_2(\mathbb{Q}(\xi_m)[t^{\pm 1}])$$

$$(t^j, a) \mapsto \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix}^j \begin{bmatrix} \xi_m^a & 0 \\ 0 & \xi_m^{-a} \end{bmatrix}.$$

The following proposition is a slight modification of a result of [35, Prop. 7.1], and gives the key connection between a certain derived series and metabelian homology, when $m = q^s$ is a prime power.

Proposition 5.1. *Let W be a 4-manifold with boundary $\partial W = Y$. Let $\Phi: \pi_1(W) \rightarrow \text{Aut}(A)$ be a representation that factors through $\mathbb{Z} \times \mathbb{Z}_{q^s}$ for some prime q , and let $\phi: \pi_1(Y) \rightarrow \text{Aut}(A)$ be the composition of Φ with the inclusion map $\pi_1(Y) \rightarrow \pi_1(W)$. Let $\eta \in \pi_1(Y)^{(2)}$ and suppose η is sent to the identity in $\pi_1(W)/\pi_1(W)_{(\mathbb{Q}, \mathbb{Z}_{q^s}, \mathbb{Q})}^{(3)}$. Then for any $v \in A$ and any $\tilde{\eta}$, a lift of η to the cover of W induced by Φ , we have that the class $[v \otimes \tilde{\eta}]$ in $H_1^\phi(Y)$ maps to 0 in $H_1^\Phi(W)$.*

Proof. The proof of [35, Prop. 7.1] (with its first and last sentences deleted) applies verbatim. □

Finally, we recall the twisted Blanchfield form. In analogy to the linking form on the torsion part of the ordinary first homology of a closed oriented 3-manifold, if $\phi: \pi_1(M_K) \rightarrow \text{GL}_r(\mathbb{Q}(\xi_m)[t^{\pm 1}])$ arises as above then there is a metabelian twisted Blanchfield form [35]

$$\text{Bl}^\phi: H_1^\phi(M_K) \times H_1^\phi(M_K) \rightarrow \mathbb{Q}(\xi_m)(t)/\mathbb{Q}(\xi_m)[t^{\pm 1}].$$

Note that in the above circumstance $H_1^\phi(M_K)$ is a torsion $\mathbb{Q}(\xi_m)[t^{\pm 1}]$ -module, by the corollary to [2, Lemma 4]; see also [24]. In Section 6.1, we will need

to know that this form is *sesquilinear* [36]. That is, letting $\bar{}$ denote the involution of $\mathbb{Q}(\xi_m)[t^{\pm 1}]$ induced by sending $t \rightarrow t^{-1}$ and $a + bi \mapsto a - bi$, we have

$$\text{Bl}^\phi(px, qy) = p\bar{q}\text{Bl}^\phi(x, y) \quad \text{for every } p, q \in \mathbb{Q}(\xi_m)[t^{\pm 1}] \text{ and } x, y \in H_1^\phi(M_K).$$

6. Main theorem and examples

Here is the result that we use to show that certain satellite knots have large 4-genus.

Theorem 6.1. *Let R be a ribbon knot and let η^1, \dots, η^r be curves in $S^3 \setminus \nu(R)$ that form an unlink in S^3 such that each η^j represents an element of $\pi_1(M_R)^{(2)}$. Suppose that there is a prime p such that for every nontrivial character $\chi: H_1(\Sigma_2(R)) \rightarrow \mathbb{Z}_p$ we have*

- (1) *The module $H_1^{\theta \circ f_\chi}(M_R) := H_1\left(\mathbb{Q}(\xi_p)[t^{\pm 1}]^2 \otimes_{\mathbb{Z}[\pi_1(M_R)]} C_*(\widetilde{M_R})\right)$ is nontrivial and generated by the collection $\{[1, 0] \otimes [\eta^j]\}_{j=1}^r$.*
- (2) *The order of $H_1^{\theta \circ f_\chi}(M_R)$ is relatively prime to $\Delta_R(t)$ over $\mathbb{Q}(\xi_{p^a})(t)$ for all $a > 0$.*

Let $m_R > 0$ denote the generating rank of the p -primary part of $H_1(\Sigma_2(R))$ and let d_R denote the number of distinct orders of $H_1^{\theta \circ f_\chi}(M_R)$ as χ ranges over all nontrivial characters from $H_1(\Sigma_2(R))$ to \mathbb{Z}_p .

Now fix $g > 0$ and suppose that $N \geq \frac{4g(d_R+1)+2}{m_R}$ and that the collection of knots $\{J_i^j \mid 1 \leq i \leq N, 1 \leq j \leq r\}$ satisfy

$$|\rho_0(J_i^j)| > 2(2g + N - 1) + NC_R + \sum_{k=1}^{i-1} \sum_{\ell=1}^r |\rho_0(J_k^\ell)| + \sum_{\ell=1}^{j-1} |\rho_0(J_i^\ell)|$$

for each $1 \leq i \leq N, 1 \leq j \leq r$. Then the knot $K = \#_{i=1}^N R_{\eta^1, \dots, \eta^r}(J_i^1, \dots, J_i^r)$ has 4-genus at least $g + 1$.

We remark that both m_R and d_R depend not only on the ribbon knot R but also on the choice of prime p , though for convenience we suppress this from the notation. We remark also that Theorem 6.1 can be generalized to consider higher prime power branched covers by appropriately changing the constants; we leave the details of that to the interested reader.

For convenience, we write η_i^j for the curve η^j in the i th copy of R in $\#_{i=1}^N R$. Note that we can also write $K = \#_{i=1}^N K_i$, where

$$K_i := R_{\eta^1, \dots, \eta^j}(J_i^1, \dots, J_i^r).$$

We will prove Theorem 6.1 in Section 9 by assuming that $g_4(K) \leq g$ and obtaining a contradiction as follows.

Under the assumption that $g_4(K) \leq g$, we construct a manifold Z with $\partial Z = Y := \bigsqcup_{i=1}^N M_{K_i}$, $\chi(Z) = 2g$, and a few other nice properties. We then let

$$\psi: \pi_1(Y) \rightarrow \pi_1(Z) \rightarrow \Lambda := \pi_1(Z)/\pi_1(Z)_{(\mathbb{Q}, \mathbb{Z}_{p^a}, \mathbb{Q})}^{(3)}$$

be the map induced by inclusion. Since Λ is amenable and in $D(\mathbb{Z}_p)$ [4, Lemma 4.3], Proposition 4.6 gives an upper bound on $|\rho^{(2)}(Y, \psi)|$ in terms of g . Our result follows from obtaining a contradictory lower bound on $|\rho^{(2)}(Y, \psi)|$. By Proposition 4.5 and our choices of the J_i^j knots, we will obtain a contradiction if for some i and j we have $\lambda(\eta_i^j) \notin \pi_1(Z)_{(\mathbb{Q}, \mathbb{Z}_{p^a}, \mathbb{Q})}^{(3)}$. By Proposition 5.1, this will be implied if we can construct some representation $\phi: \pi_1(Y) \rightarrow \mathbb{Z} \times \mathbb{Z}_{p^a}$ which extends over $\pi_1(Z)$ to Φ such that for some i and j , the element $[1, 0] \otimes [\lambda(\eta_i^j)] \in H_1^\phi(Y)$ is not in the kernel of the inclusion induced map $H_1^\phi(Y) \rightarrow H_1^\Phi(Z)$. The technical work of the proof consists of showing that such a map ϕ must exist under the assumption that $g_4(K) \leq g$ together with our construction of K .

We will first give some examples of knots satisfying the hypotheses of the theorem and then prove some technical lemmas in the next two sections. In particular we will need to gain some control over the size of certain homology groups, in order to show that some curve η_i^j always survives into a suitable 3-solvable quotient of the fundamental group of the complement of a hypothesized locally flat embedded surface of genus g . Of course Theorem 1.2 follows immediately from Theorem 6.1 together with the examples exhibited in Section 6.2 below and (for the grope bounding result) Proposition 10.7.

It is relatively easy to find examples of seed ribbon knots satisfying the hypotheses of Theorem 6.1, at least with the help of a computer program to compute twisted metabelian homology, as developed in [35]. In Section 6.1 we give one such example of a pair (R, α) , and describe the appropriate infection by knots with Arf invariant zero and large signature in order to obtain 2-solvable large 4-genus knots.

It is a little harder to find suitable seed knots R that also satisfy Proposition 2.6, and therefore produce 2-solvable and 2-bipolar large 4-genus examples for the proof of Theorem 1.2.. Nonetheless, we exhibit such a seed knot R with suitable infection curves in Section 6.2, and describe the appropriate infection by 0-bipolar knots with large signature in order to obtain 2-bipolar, 2-solvable large 4-genus knots.

6.1. Example 1: a 2-solvable knot with large 4-genus

Let R denote the ribbon knot 8_8 , with the unknotted curve α in $S^3 \setminus \nu(R)$ illustrated in Figure 1.

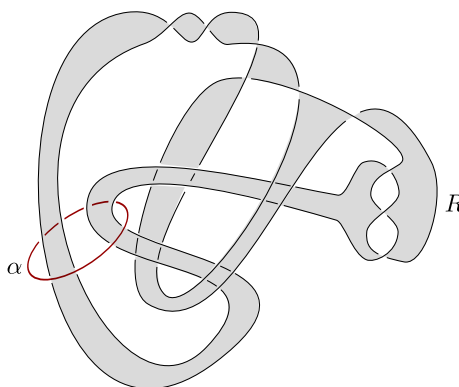


Figure 1: The knot $R = 8_8$ with a grey Seifert surface and a red infection curve α .

This is the same knot-curve pair (R, α) as in [35, Example 8.1], with a slight isotopy to make it more obvious that the curve α bounds a surface in the complement of a Seifert surface for R , and hence lies in $\pi_1(M_R)^{(2)}$. We will need a few computations from that paper. First, $H_1(\Sigma_2(\mathbb{R})) \cong \mathbb{Z}_{25}$. Note that rescaling a character $\chi : H_1(\Sigma_2(\mathbb{R})) \rightarrow \mathbb{Z}_5$ by a nonzero element of \mathbb{Z}_5 does not change the underlying $(\mathbb{Z} \times \mathbb{Z}_5)$ -covering space of M_R , and hence preserves the isomorphism type of the twisted homology. It is therefore not hard to check that given any nontrivial character $\chi : H_1(\Sigma_2(R)) \rightarrow \mathbb{Z}_5$, the corresponding twisted homology $H_1^{\theta \circ f_\chi}(M_R)$ is isomorphic to $\mathbb{Q}(\xi_5)[t^{\pm 1}] / \langle t^2 - 3t + 1 \rangle$. Since α is in $\pi_1(M_R)^{(2)}$, it lifts to a curve $\tilde{\alpha}$ in the f_χ covering space of M_R , and hence $x := [1, 0] \otimes \tilde{\alpha}$ is an element of $H_1^{\theta \circ f_\chi}(M_R)$. Finally, the metabelian twisted Blanchfield pairing $Bl^{\theta \circ f_\chi}(x, x)$ is non-zero in $\mathbb{Q}(\xi_5)(t) / \mathbb{Q}(\xi_5)[t^{\pm 1}]$, as was computed in [35, Example 8.1].

Lemma 6.2. *The element x generates $H_1^{\theta \circ f_\chi}(M_R)$.*

Proof. Supposing for a contradiction that x does not generate. Let y be some generator for $H_1^{\theta \circ f_x}(M_R)$. Note that $\text{Bl}^{\theta \circ f_x}(y, y)$ is of the form $\frac{q(t)}{t^2 - 3t + 1}$ for some $q(t) \in \mathbb{Q}(\xi_5)[t^{\pm 1}]$. The polynomial $t^2 - 3t + 1$ factors as $(t - w_+)(t - w_-)$ for $w_{\pm} = \frac{3 \pm \sqrt{5}}{2} \in \mathbb{Q}(\xi_5)$. Therefore, if x does not generate then it must be homologous to $c(t - w_*)y$ for some $c \in \mathbb{Q}(\xi_5)[t^{\pm 1}]$ and $* \in \{\pm\}$; without loss of generality, say $* = +$. But then we can obtain a contradiction, since

$$\begin{aligned} \text{Bl}^{\theta \circ f_x}(x, x) &= \text{Bl}^{\theta \circ f_x}(c(t - w_+)y, c(t - w_+)y) \\ &= (t - w_+) \overline{(t - w_+)} c \bar{c} \text{Bl}^{\theta \circ f_x}(y, y) \\ &= (t - w_+)(t^{-1} - w_+) c \bar{c} \frac{q(t)}{t^2 - 3t + 1} \\ &= -t^{-1} w_+ (t - w_+)(t - w_-) c \bar{c} \frac{q(t)}{t^2 - 3t + 1} \\ &= -t^{-1} w_+ c \bar{c} q(t) = 0 \in \mathbb{Q}(\xi)(t) / \mathbb{Q}(\xi_5)[t^{\pm 1}]. \end{aligned}$$

□

Now, fix some $g \geq 0$ and let $N = 8g + 2$, noting that m_R , the generating rank of the 5-primary part of $H_1(\Sigma_2(R))$, is 1 and that there is only one isomorphism class of $H_1^{\theta \circ f_x}(M_R)$ and so $d_R = 1$ as well. Note that $t^2 - 3t + 1$ is relatively prime to $\Delta_R(t) = 2 - 6t + 9t^2 - 6t^3 + 2t^4$, even considered over \mathbb{C} . By [6, Theorem 1.9], $C_R = 10^9$ is an upper bound for the Cheeger-Gromov constant $C(M_R)$ of the 0-surgery on R . For $k = 1, \dots, N$, let J_k be a knot with $\text{Arf}(J_k) = 0$ and

$$(1) \quad \int_{\omega \in S^1} \sigma_{\omega}(J_k) d\omega > 2(2g + N - 1) + C_R N + \sum_{i=1}^{k-1} \left(\int_{\omega \in S^1} \sigma_{\omega}(J_i) d\omega \right).$$

We can achieve this by taking J_k to be a sufficiently large even connected sum of negative trefoils, since for the negative trefoil

$$\int_{\omega \in S^1} \sigma_{\omega}(J_i) d\omega = 4/3 > 0.$$

In fact, the numerically minded reader can easily verify that Equation (1) is satisfied if we define J_k to be the connected sum of $10^{k+10}g$ negative trefoils.

We note that $K = \#_{i=1}^N R_{\alpha}(J_i)$ is a 2-solvable knot (by Proposition 2.3) which satisfies the hypotheses of Theorem 6.1, and hence has topological 4-genus at least $g + 1$.

6.2. Example 2: a 2-solvable, 2-bipolar knot with large 4-genus

Let R be the knot depicted on the left of Figure 2. On the right of Figure 2 we see a genus 2 Seifert surface F for R along with two sets of *derivative curves* for R : each of the two component links $D = D_1 \cup D_2$ (blue) and $L = L_1 \cup L_2$ (red) generates a half-rank summand of $H_1(F)$, forms a slice link (in fact, an unlink) in S^3 , and is 0-framed by F .

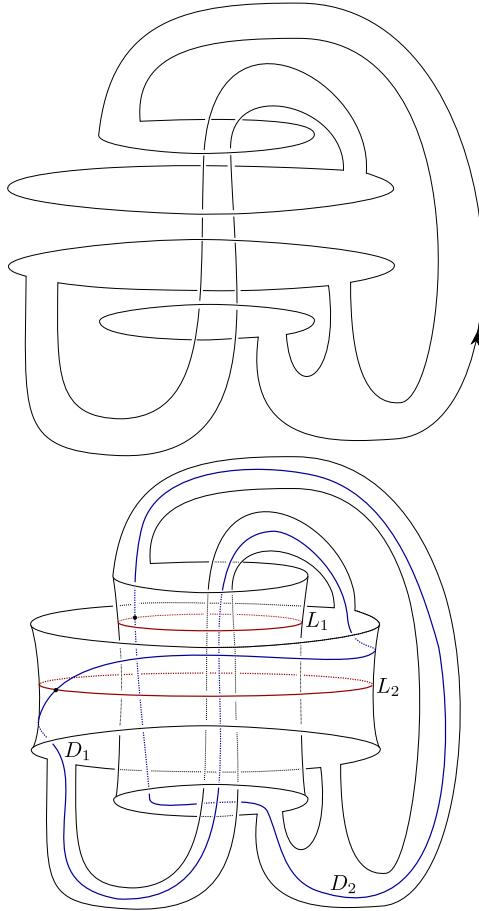


Figure 2: The knot $R = 11_n74 = P(3, -3, 3, -2)$ (left) and a Seifert surface for R along with two derivative links $L = L_1 \cup L_2$ and $D = D_1 \cup D_2$ (right).

Now let $\beta_1, \beta_2, \gamma_1, \gamma_2$ be the curves indicated on the left of Figure 3. These curves are disjoint from F and hence lie in $\pi_1(M_R)^{(1)}$. Note that the indicated basepoints should be thought of as living in a plane ‘far below’

the plane of the diagram; in that plane they can be connected, uniquely up to homotopy, to a single preferred basepoint for $\pi_1(M_R)$.

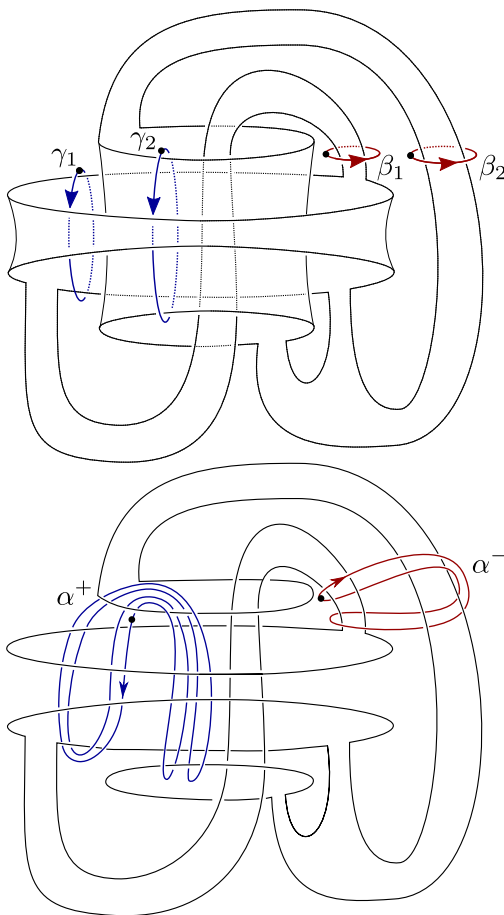


Figure 3: Unknotted curves $\beta_1, \beta_2, \gamma_1, \gamma_2$ in $\pi_1(M_R)^{(1)}$ (left) and α^\pm in $\pi_1(M_R)^{(2)}$ (right).

Let $\alpha^- = [\beta_1, \beta_2]$ and $\alpha^+ = [\gamma_1, \gamma_2]$, where $[v, w] = vwv^{-1}w^{-1}$; unknotted representatives for α^\pm are shown on the right of Figure 3. Note that $\alpha^\pm \in \pi_1(M_R)^{(2)}$. Since α^+ has no geometric linking with either component of the link D , for any knot J the satellite knot $R_{\alpha^+}(J)$ still has a smoothly slice derivative, and hence is itself smoothly slice. Similarly, since α^- has no geometric linking with either component of the link L , the satellite knot $R_{\alpha^-}(J)$ is slice for every knot J . Therefore, by Proposition 2.6, for

any 0-positive knot J^+ and 0-negative knot J^- , the knot $R_{\alpha^+, \alpha^-}(J^+, J^-)$ is 2-bipolar.

We now proceed to verify the conditions of Theorem 6.1, so that for appropriate choices of $N \in \mathbb{N}$ and of J_i^\pm , $i = 1, \dots, N$, the knot

$$\#_{i=1}^N R_{\alpha^+, \alpha^-}(J_i^+, J_i^-)$$

will have large topological 4-genus.

Note that $H_1(\Sigma_2(R)) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$, and so (up to rescaling by a nonzero constant), there are four nontrivial characters $\chi: H_1(\Sigma_2(R)) \rightarrow \mathbb{Z}_3$. We compute that for three of these characters, which we call χ_1, χ_2 , and χ_3 , the resulting twisted homology is

$$H_1^{\theta \circ f_{\chi_i}}(M_R) \cong \mathbb{Q}(\xi_3)[t^{\pm 1}] / \langle (t-1)^2 \rangle =: A_1.$$

For the fourth character, denoted by χ_4 , we compute that the twisted homology is

$$H_1^{\theta \circ f_{\chi_4}}(M_R) \cong \mathbb{Q}(\xi_3)[t^{\pm 1}] / \langle t^2 - 14t + 1 \rangle =: A_2.$$

Crucially, for any nontrivial χ_i the lifts of α^+ and α^- to the cover of M_R induced by f_{χ_i} generate $H_1^{\theta \circ f_{\chi_i}}(M_R)$. (More precisely, $[1, 0] \otimes [\alpha^+]$ and $[1, 0] \otimes [\alpha^-]$ generate.)

As in [35], we computed the twisted homology using a Maple program, available for download on the authors' websites. The program obtains a presentation for the twisted homology using the Wirtinger presentation, taking the Fox derivatives, and then applying the representation. It then simplifies the presentation by row and column operations to obtain a diagonal matrix. Keeping track of how the original generators, which can be identified in the knot diagram, are modified under the sequence of row and column operations, we not only compute the twisted homology $H_1^{\theta \circ f_{\chi_i}}(M_R)$ but also can identify which elements the curves $[1, 0] \otimes [\alpha^\pm]$ represent in $H_1^{\theta \circ f_{\chi_i}}(M_R)$. Note also that the orders of A_1 and A_2 are both relatively prime to $\Delta_R(t) = (t^2 - t + 1)^2$ even over \mathbb{C} .

Now let $g \in \mathbb{N}$ be given. By our discussion above, we have $m_R = d_R = 2$, and so we let

$$N := \frac{4g(2+1) + 2}{2} = 6g + 1.$$

Let C_R denote an upper bound for the Cheeger-Gromov constant $C(M_R)$. For each $i = 1, \dots, N$, successively pick m_i to be even and large enough that

$J_i^+ := \#^{m_i}T_{2,3}$ satisfies

$$\rho_0(J_i^+) < -2(2g + N - 1) - NC_R + \sum_{j=1}^{i-1} \rho_0(J_j^+) - \sum_{j=1}^{i-1} \rho_0(J_j^-),$$

and then pick m'_i to be even and large enough that $J_i^- := -\#^{m'_i}T_{2,3}$ satisfies

$$\rho_0(J_i^-) > 2(2g + N - 1) + NC_R - \sum_{j=1}^i \rho_0(J_j^+) + \sum_{j=1}^{i-1} \rho_0(J_j^-).$$

Note that in particular $\rho_0(J_i^+) < 0 < \rho_0(J_i^-)$ for all i .

We now let $K_i = R_{\alpha^+, \alpha^-}(J_i^+, J_i^-)$ and $K = \#_{i=1}^N K_i$. Observe that K is 2-solvable by Proposition 2.3 and 2-bipolar by Proposition 2.6; K also satisfies the hypotheses of Theorem 6.1, so $g_4(K) \geq g + 1$. In Section 10, we will show that K bounds an embedded grope of height four in the 4-ball. The knot K therefore gives the example claimed in Theorem 1.2.

It now remains to prove Theorem 6.1.

7. Controlling the size of some homology groups

This section contains some technical results needed for the proof of Theorem 6.1, with the theme that we need to control the size of certain homology groups of some covering spaces.

We start this section with an elementary algebraic lemma. This lemma and the one after it are very similar to, and are inspired by, results of Levine in [33], in particular Lemma 4.3 of Part I and Proposition 3.2 of Part II. To avoid citing lemmas that were written for a different situation, and for the edification of the reader, we provide short self-contained proofs.

Lemma 7.1. *Let $F: M \rightarrow M$ be an endomorphism of a finitely generated free $\mathbb{Z}[\mathbb{Z}_2]$ -module M such that*

$$\text{Id} \otimes F: \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} M \rightarrow \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} M$$

is an isomorphism, where \mathbb{Z} is a $\mathbb{Z}[\mathbb{Z}_2]$ -module via the trivial action of \mathbb{Z}_2 on \mathbb{Z} . Then

$$\text{Id} \otimes F: \mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} M \rightarrow \mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} M$$

is also an isomorphism.

Proof. Let $A + BT \in \mathbb{Z}[\mathbb{Z}_2]$ be the determinant of F , where $A, B \in \mathbb{Z}$ and $T \in \mathbb{Z}_2$ denotes the generator. Then $A + B = \pm 1$ by hypothesis. Thus $A^2 - B^2 = (A + B)(A - B) = \pm(A - B) \neq 0$ since $A + B \equiv A - B$ modulo 2. Now

$$(A + BT) \cdot \frac{1}{A^2 - B^2}(A - BT) = 1,$$

so over $\mathbb{Q}[\mathbb{Z}_2]$ the determinant of F is invertible, and hence $\text{Id}_{\mathbb{Q}[\mathbb{Z}_2]} \otimes F$ is an isomorphism as desired. \square

Next we apply this lemma to obtain some control on the size of the homology of double covering spaces.

Lemma 7.2. *Let $f: X \rightarrow Y$ be a map of finite CW complexes such that*

$$f_*: H_i(X; \mathbb{Z}) \rightarrow H_i(Y; \mathbb{Z})$$

is an isomorphism for $i = 0$ and a surjection for $i = 1$. Let $\varphi: \pi_1(Y) \rightarrow \mathbb{Z}_2$ be a surjective homomorphism and let X^2, Y^2 be the induced 2-fold covers. Then

$$f_*: H_i(X^2; \mathbb{Q}) \cong H_i(X; \mathbb{Q}[\mathbb{Z}_2]) \rightarrow H_i(Y^2; \mathbb{Q}) \cong H_i(Y; \mathbb{Q}[\mathbb{Z}_2])$$

is also an isomorphism for $i = 0$ and a surjection for $i = 1$.

Proof. The zeroth and first relative homology groups vanish, that is

$$H_i(Y, X; \mathbb{Z}) = 0 \quad \text{for } i = 0, 1.$$

Thus the cellular chain complex $(C_*(Y, X; \mathbb{Z}), \partial_*)$ admits a partial chain contraction: writing C_* to abbreviate $C_*(Y, X; \mathbb{Z})$, the partial chain homotopy comprises maps $s_0: C_0 \rightarrow C_1$ and $s_1: C_1 \rightarrow C_2$ such that $\partial \circ s_0 = \text{Id}: C_0 \rightarrow C_0$ and $\partial \circ s_1 + s_0 \circ \partial = \text{Id}: C_1 \rightarrow C_1$.

To see this, follow the proof of the fundamental lemma of homological algebra: for each basis element $x_i \in C_0$, choose a lift $y_i \in C_1$ with $\partial y_i = x_i$, and define $s_0(x_i) = y_i$, and then extend linearly. Such a y_i exists since $\partial: C_1 \rightarrow C_0$ is surjective. Then for each generator $z_i \in C_1$, consider $\text{Id}(z_i) - s_0 \circ \partial(z_i)$. Since

$$\partial(\text{Id}(z_i) - s_0 \circ \partial(z_i)) = \partial(z_i) - \partial \circ s_0 \circ \partial(z_i) = \partial(z_i) - \text{Id} \circ \partial(z_i) = 0,$$

we have that $\text{Id}(z_i) - s_0 \circ \partial(z_i)$ is a cycle. Hence there is a $v_i \in C_2$ such that $\partial v_i = \text{Id}(z_i) - s_0 \circ \partial(z_i)$. Define $s_1(z_i) := v_i$, and extend linearly to define s_1

on all of C_1 . Then $\partial \circ s_1(z_i) = \partial v_i = \text{Id}(z_i) - s_0 \circ \partial(z_i)$ for every generator z_i of C_1 . This completes the construction of a partial chain homotopy.

Now consider the chain complex $D_* := C_*(Y, X; \mathbb{Z}[\mathbb{Z}_2]) \cong C_*(Y^2, X^2)$, the relative chain complex of the 2-fold covering spaces. Since the cellular chain groups are finitely generated free modules, the partial chain contraction s_0, s_1 lifts to maps $\tilde{s}_0: D_0 \rightarrow D_1$ and $\tilde{s}_1: D_1 \rightarrow D_2$.

We claim that these maps induce a partial chain contraction after tensoring over $\mathbb{Q}[\mathbb{Z}_2]$. To see the claim, the maps

$$F := \tilde{s}_0 \circ \partial: D_0 \rightarrow D_0 \quad \text{and} \quad G := \partial \circ \tilde{s}_1 + \tilde{s}_0 \circ \partial: D_1 \rightarrow D_1$$

are endomorphisms of the free modules D_0 and D_1 respectively, that become automorphisms when tensored over \mathbb{Z} . That is,

$$\begin{aligned} \text{Id} \otimes F: \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_0 &\rightarrow \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_0, \\ \text{Id} \otimes G: \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_1 &\rightarrow \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_1 \end{aligned}$$

are isomorphisms. By Lemma 7.1,

$$\begin{aligned} \text{Id} \otimes F: \mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_0 &\rightarrow \mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_0, \\ \text{Id} \otimes G: \mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_1 &\rightarrow \mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_1 \end{aligned}$$

are also isomorphisms. Thus

$$\text{Id} \otimes \tilde{s}_i: \mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_i \rightarrow \mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_{i+1}$$

is a partial chain contraction and

$$H_i(\mathbb{Q}[\mathbb{Z}_2] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D_*) = H_i(Y^2, X^2; \mathbb{Q}) \cong H_i(Y, X; \mathbb{Q}[\mathbb{Z}_2]) = 0$$

for $i = 0, 1$. The lemma follows from the long exact sequence of the pair (Y, X) (Proposition 3.3). □

Our next lemma requires some facts about finitely generated modules over commutative PIDs, which we remind the reader of in order to establish notation.

Definition 7.3. Let S be a commutative PID with quotient field Q , and let A be a finitely generated module over S .

- 1) $TA := \{a \in A \text{ such that } sa = 0 \text{ for some } s \in S\}$, the S -torsion submodule of A .

- 2) $A^\wedge := \text{Hom}_S(A, Q/S)$. If A is torsion (i.e. $A = TA$), then A and A^\wedge are non-canonically isomorphic.
- 3) Given a map of S modules $f: A \rightarrow B$, we abbreviate $f|_{TA}: TA \rightarrow TB$ by $f|_T$. We emphasize that $\text{coker}(f|_T)$ is therefore isomorphic to $TB/\text{Im}(f|_{TA})$, not $B/\text{Im}(f|_{TA})$.
- 4) We say that A has *generating rank k over S* if A is generated as an S -module by k elements but not by $k - 1$ elements, and write $\text{g-rk}_S A = k$. It follows immediately from the definition that if A surjects onto B then $\text{g-rk}_S B \leq \text{g-rk}_S A$. It is also true and easy to check that if $B \leq A$ then $\text{g-rk}_S B \leq \text{g-rk}_S A$, though this is less obvious and uses that S is a commutative PID.
- 5) By the fundamental theorem of finitely generated modules over PIDs, there exist $j, k \in \mathbb{N}$ and elements $s_1, \dots, s_k \in S$ such that there is a (non-canonical) isomorphism

$$A \cong S^j \oplus TA \cong S^j \oplus \bigoplus_{i=1}^k S/\langle s_i \rangle.$$

When $j > 0$ we say that the *order* of A is $|A| = 0$ and when $j = 0$ we say that the order of A is $|A| = \prod_{i=1}^k s_i$. This is well-defined up to multiplication by units in S . The key property of order we use is that if $f: A \rightarrow B$ is a map of S -modules with $\ker(f)$ torsion, then $|\text{Im}(f)| = |A|/|\ker(f)|$.

We will need the following basic lemma in the proof of Theorem 6.1, noting for future use that $\mathbb{F}[t^{\pm 1}]$ is a Euclidean domain whenever \mathbb{F} is a field.

Lemma 7.4. *Let A be a finitely generated module over a Euclidean domain S , hence non-canonically isomorphic to $S^m \oplus TA$ for some $m \geq 0$. Suppose that B is a submodule of A such there exists $C \subseteq A/B$ of generating rank $\ell > m$. Then there exists a module $C' \subseteq TA/(B \cap TA)$ of generating rank $\ell - m$ such that the order of C' divides the order of C .*

Proof. Let a_1, \dots, a_ℓ be elements of A such that $[a_1], \dots, [a_\ell]$ generate $C \subseteq A/B$. Pick a decomposition of $A \cong S^m \oplus TA$ and use it to write each $a_i = (s_i^j)_{j=1}^m \oplus \alpha_i$ for $(s_i^j)_{j=1}^m \in S^m$ and $\alpha_i \in TA$. Since S is a Euclidean domain, row-reduction of the $\ell \times m$ matrix M with $M_{i,j} := s_i^j$ yields a matrix M' in Hermite normal form. Since $\ell > m$, we have that M' contains at least

$\ell - m$ rows of zeros. By taking the corresponding linear combinations of the a_i , we obtain a new collection of elements $a'_i = (t_i^j)_{j=1}^m \oplus \alpha'_i$ such that the collection of $[a'_i]$ generate C . Moreover, for $i > m$ we have that $t_i^j = 0$ for all $j = 1, \dots, m$ and so $a'_i = \alpha'_i \in TA$. Note that for $i > m$ we have that $\alpha'_i \neq 0$, since if $\alpha_i = a'_i = 0$ then the generating rank of C would be strictly less than ℓ . For similar reasons, we see that the generating rank of the submodule of $TA/(B \cap TA)$ generated by $\alpha'_{m+1}, \dots, \alpha'_\ell$ is exactly $\ell - m$. So let C' be this submodule. Finally, the order of C' divides the order of C because C' is a submodule of C . \square

We will use the next lemma twice, once in the proof of Proposition 7.6, and then again in Step 3 of the proof of Theorem 6.1 in Section 9.

Lemma 7.5. *Let X be a 4-manifold with boundary $\partial X = Y$. Let S be a commutative PID with quotient field Q and let $A := S^n$ for some $n \in \mathbb{N}$. Suppose there is a representation Φ of the fundamental groupoid of Y into $\text{Aut}(A)$ that extends over X , as in Section 3. Consider the long exact sequence (Proposition 3.3) of S -modules of the pair, with coefficients taken in A :*

$$\dots \rightarrow H_2(X) \xrightarrow{j_2} H_2(X, Y) \xrightarrow{\partial} H_1(Y) \xrightarrow{i_1} H_1(X) \xrightarrow{j_1} H_1(X, Y) \rightarrow \dots$$

Suppose that $H_1(X, Y)$ is torsion. Then $\ker(j_1|_T) \cong \text{coker}(j_2|_T)$.

Proof. Unless otherwise specified, all homology groups are taken with coefficients in A . In this proof, we sometimes abbreviate $H_1(X, Y)$ by H to make certain long exact sequences more concise.

First, we argue that the Bockstein homomorphism β is an isomorphism. This Bockstein arises in the long exact sequence of Ext groups [27, IV, Prop. 7.5] associated to the short exact sequence $0 \rightarrow S \rightarrow Q \rightarrow Q/S \rightarrow 0$, as follows:

$$\text{Ext}_S^0(H, Q) \rightarrow \text{Ext}_S^0(H, Q/S) \xrightarrow{\beta} \text{Ext}_S^1(H, S) \rightarrow \text{Ext}_S^1(H, Q).$$

Since Q is an injective S -module, $\text{Ext}_S^1(H, Q)$ vanishes, and since $H = H_1(X, Y)$ is torsion we have $\text{Ext}_S^0(H, Q) = 0$. Thus β is an isomorphism.

Therefore Poincaré duality, universal coefficients, and the (inverse of the) Bockstein homomorphism together induce natural isomorphisms fitting into

a commutative diagram:

$$\begin{array}{ccccccccc}
 TH_2(X) & \xrightarrow[\cong]{P.D.} & TH^2(X, Y) & \xrightarrow[\cong]{U.C.} & \text{Ext}_S^1(H, S) & \xrightarrow[\cong]{\beta^{-1}} & \text{Ext}_S^0(H, Q/S) & \xrightarrow[\cong]{} & TH^\wedge \\
 \downarrow j_2|_T & & \downarrow & & \downarrow & & \downarrow & & \downarrow (j_1|_T)^\wedge \\
 TH_2(X, Y) & \xrightarrow[\cong]{P.D.} & TH^2(X) & \xrightarrow[\cong]{U.C.} & \text{Ext}_S^1(H_1(X), S) & \xrightarrow[\cong]{\beta^{-1}} & \text{Ext}_S^0(H_1(X), Q/S) & \xrightarrow[\cong]{} & TH_1(X)^\wedge
 \end{array}$$

By the naturality of the above sequence of maps, the following square commutes and so $\text{coker}(j_2|_T) \cong \text{coker}((j_1|_T)^\wedge)$.

$$(2) \quad \begin{array}{ccc}
 TH_2(X) & \xrightarrow{j_2|_T} & TH_2(X, Y) \\
 \cong \downarrow & & \downarrow \cong \\
 TH_1(X, Y)^\wedge & \xrightarrow{(j_1|_T)^\wedge} & TH_1(X)^\wedge
 \end{array}$$

Now let $P := \ker(j_1|_T) \leq TH_1(X)$ and define $\Phi: \text{coker}((j_1|_T)^\wedge) \rightarrow P^\wedge$ by $\Phi([f]) = f|_P$. Observe that Φ is well-defined, since for any $g \in TH_1(X, Y)^\wedge$ we have

$$(j_1|_T)^\wedge(g)(x) = g(j_1(x)) = g(0) = 0 \quad \text{for all } x \in P = \ker(j_1|_T).$$

Also, Φ is onto: given any $f \in P^\wedge$ (i.e. a map $f: P \rightarrow Q/S$), since $P \leq TH_1(X)$ and Q/S is an injective S -module we can extend f to a map $g: TH_1(X) \rightarrow Q/S$, and will have that $\Phi([g]) = f$. Therefore, in order to show that Φ is an isomorphism it is enough to show that $|\text{coker}((j_1|_T)^\wedge)| = |P^\wedge|$. Note that

$$(3) \quad |\text{coker}((j_1|_T)^\wedge)| = \frac{|TH_1(X)^\wedge|}{|\text{Im}((j_1|_T)^\wedge)|} = \frac{|TH_1(X)^\wedge| |\ker((j_1|_T)^\wedge)|}{|TH_1(X, Y)^\wedge|}.$$

Also, $(j_1|_T)^\wedge(f) = 0$ if and only if $f(j_1|_T(x)) = 0$ for all $x \in TH_1(X)$ if and only if f vanishes on $\text{Im}(j_1|_T)$, so

$$|\ker((j_1|_T)^\wedge)| = \frac{|TH_1(X, Y)^\wedge|}{|\text{Im}(j_1|_T)|}.$$

Therefore we can rewrite Equation (3) as

$$|\text{coker}((j_1|_T)^\wedge)| = \frac{|TH_1(X)^\wedge|}{|\text{Im}(j_1|_T)|} = \frac{|TH_1(X)|}{|\text{Im}(j_1|_T)|} = |P| = |P^\wedge|.$$

So $\Phi: \text{coker}((j_1|_T)^\wedge) \rightarrow \ker(j_1|_T)^\wedge$ is an isomorphism. Since $\ker(j_1|_T)$ is a torsion S -module, there is an (albeit non-canonical) identification $\ker(j_1|_T) \cong (\ker(j_1|_T))^\wedge$, and so we have as desired that

$$\text{coker}(j_2|_T) \cong \text{coker}((j_1|_T)^\wedge) \cong_{\Phi} \ker(j_1|_T)^\wedge \cong \ker(j_1|_T). \quad \square$$

Note that in particular this implies that $\ker(j_1|_T)_p \cong \text{coker}(j_2|_T)_p$ for any prime p , where for a \mathbb{Z} -module G we write G_p for the p -primary part.

Next we apply the control on homology of double covers gained in Lemma 7.2 along with the homological algebra of Lemma 7.5 to manifolds M^3 and V^4 .

Proposition 7.6. *Let M be a homology $S^1 \times S^2$, and let V be a connected 4-manifold with boundary M such that the inclusion induced map $H_1(M) \xrightarrow{i_*} H_1(V)$ is an isomorphism. Suppose that $H_1(M^2) \cong \mathbb{Z} \oplus G$, where G is torsion. Then for any prime p the p -primary part of $TH_1(M^2)/\ker(TH_1(M^2) \rightarrow TH_1(V^2))$ has generating rank at least $n := \frac{m-2\chi(V)}{2}$, where m denotes the generating rank of the p -primary part of $TH_1(M^2)$.*

Proof. Observe that $H_i(V, M; \mathbb{Z}) = 0$ for $i = 0, 1$. It follows from Lemma 7.2 that

$$H_i(V^2, M^2; \mathbb{Q}) \cong H_i(V, M; \mathbb{Q}[\mathbb{Z}_2]) = 0 \quad \text{for } i = 0, 1.$$

Therefore $\dim H_1(V^2, \mathbb{Q}) \leq \dim H_1(M^2, \mathbb{Q}) = 1$ and

$$\dim H_3(V^2; \mathbb{Q}) = \dim H^3(V^2; \mathbb{Q}) = \dim H_1(V^2, M^2; \mathbb{Q}) = 0.$$

Also note that

$$2\chi(V) = \chi(V^2) = 1 - \dim H_1(V^2, \mathbb{Q}) + \dim H_2(V^2, \mathbb{Q}).$$

We therefore have that $\dim H_2(V^2, \mathbb{Q}) = 2\chi(V) + \dim H_1(V^2, \mathbb{Q}) - 1$ is at most $2\chi(V)$. Now consider the following long exact sequence:

$$\dots \rightarrow H_2(V^2) \xrightarrow{j_2} H_2(V^2, M^2) \xrightarrow{\partial} H_1(M^2) \xrightarrow{i_1} H_1(V^2) \xrightarrow{j_1} H_1(V^2, M^2) \dots$$

Since $H_1(V^2, M^2; \mathbb{Q}) = 0$, Lemma 7.5 tells us that

$$\text{g-rk } \ker(j_1|_T)_p = \text{g-rk } \text{coker}(j_2|_T)_p.$$

Moreover, for any finitely generated abelian group A , we have that $A \cong \mathbb{Z}^a \oplus TA$ for some $a \in \mathbb{N}_{\geq 0}$ and hence that $\text{g-rk}(A \otimes \mathbb{Z}_p) = a + \text{g-rk}(A_p)$. In

particular

$$\text{g-rk}(\ker(j_1|_T) \otimes \mathbb{Z}_p) = \text{g-rk}(\text{coker}(j_2|_T) \otimes \mathbb{Z}_p).$$

Combining $\text{g-rk}(A \otimes \mathbb{Z}_p) = a + \text{g-rk}(A_p)$ with $H_1(M^2) \cong \mathbb{Z} \oplus TH_1(M^2)$, we have that in order to show as desired that the generating rank of the p -primary part of $(TH_1(M^2)/\ker(i_1|_T))$ is at least n , it suffices to show

$$\text{g-rk}((H_1(M^2)/\ker(i_1)) \otimes \mathbb{Z}_p) \geq n + 1.$$

Note that $H_1(M^2)/\ker(i_1) \cong \text{Im}(i_1) \cong \ker(j_1)$, so if $\text{g-rk}(\ker(j_1) \otimes \mathbb{Z}_p) \geq n + 1$ then we are done. Similarly, since $\text{g-rk}(H_1(M^2) \otimes \mathbb{Z}_p) = m + 1$ and

$$\ker(i_1) = \text{Im}(\partial) \cong H_2(V^2, M^2)/\ker(\partial) = H_2(V^2, M^2)/\text{Im}(j_2) = \text{coker}(j_2),$$

if $\text{g-rk}(\text{coker}(j_2) \otimes \mathbb{Z}_p) \leq m - n$, then we are also done.

Now suppose for a contradiction that both $\text{g-rk}(\text{coker}(j_2) \otimes \mathbb{Z}_p) > m - n$ and $\text{g-rk}(\ker(j_1) \otimes \mathbb{Z}_p) \leq n$. Note that since $H_3(V^2, M^2) \cong H^1(V^2) \cong H^1(M^2) \cong H_2(M^2)$, the ranks of $H_2(V^2, M^2)$ and $H_2(V^2)$ coincide and so $H_2(V^2, M^2)$ splits (albeit non-canonically) as $\mathbb{Z}^{b_2(V^2)} \oplus TH_2(V^2, M^2)$. Thus we obtain our desired contradiction as follows:

$$\begin{aligned} 2\chi(V) + n = m - n &< \text{g-rk}(\text{coker}(j_2) \otimes \mathbb{Z}_p) \\ &\leq \text{g-rk}((H_2(V^2, M^2)/\text{Im}(j_2|_T)) \otimes \mathbb{Z}_p) \\ &= \text{g-rk}(\mathbb{Z}^{b_2(V^2)} \otimes \mathbb{Z}_p) \\ &\quad + \text{g-rk}((TH_2(V^2, M^2)/\text{Im}(j_2|_T)) \otimes \mathbb{Z}_p) \\ &= b_2(V^2) + \text{g-rk}(\text{coker}(j_2|_T) \otimes \mathbb{Z}_p) \\ &= b_2(V^2) + \text{g-rk}(\ker(j_1|_T) \otimes \mathbb{Z}_p) \\ &\leq b_2(V^2) + \text{g-rk}(\ker(j_1) \otimes \mathbb{Z}_p) \leq 2\chi(V) + n. \quad \square \end{aligned}$$

8. A standard cobordism

In this section we study a standard cobordism U between the zero-framed surgery manifold of a connected sum of knots $M_K = M_{\#_{i=1}^N K_i}$ and the disjoint union $Y := \bigsqcup_{i=1}^N M_{K_i}$ of the zero-framed surgery manifolds of the summands K_i . In particular we need to understand the behavior of certain representations of the fundamental groups. We will also explicitly choose the basepoints $\{x_i\}$ and paths $\{\tau_i\}$ necessary to define twisted homology for disconnected manifolds, as discussed in Section 3.

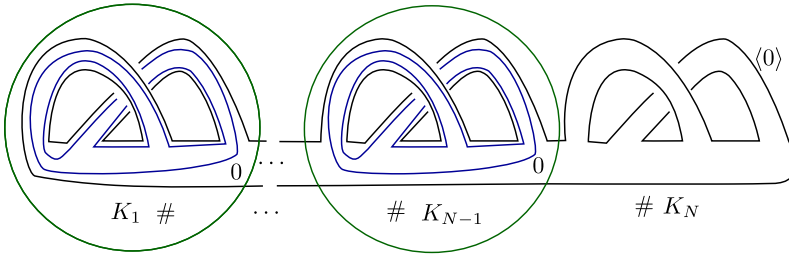


Figure 4: A Kirby diagram for U .

Let U' be $M_K \times [0, 1]$ with $(N - 1)$ 0-framed 2-handles attached along ‘longitudes of K_i .’ A schematic of a relative Kirby diagram for U' is given by the black and blue curves of Figure 4. Note that we depict each K_i as the boundary of a Seifert surface G_i , and hence $K = \#_{i=1}^N K_i$ as the boundary of $\natural_{i=1}^N G_i$. Since repeatedly sliding the black 0-framed curve over the blue curves gives the standard surgery diagram for Y' , we have $\partial_+(U') = Y' := \#_{i=1}^N M_{K_i}$. Now let U'' be $Y' \times [1, 2]$ together with $(N - 1)$ 3-handles attached along 2-spheres (whose outline is indicated in green in Figure 4) so that $\partial_+ U'' = Y = \bigsqcup_{i=1}^N M_{K_i}$. Let $U = U' \cup_{Y'} U''$.

We now consider the points, arcs, and closed curves shown in Figure 5.

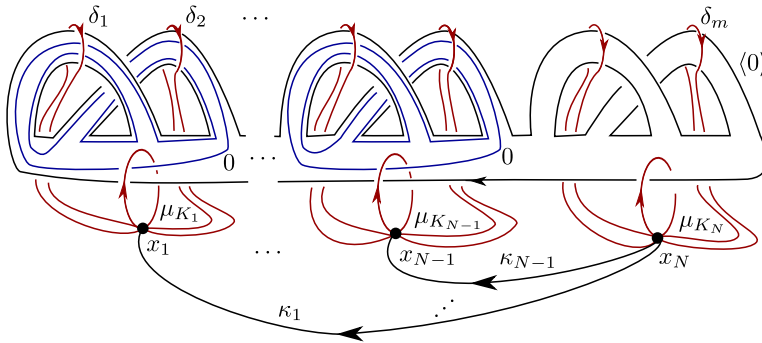


Figure 5: Basepoints x_i , arcs κ_i , and meridians μ_i for $i = 1, \dots, N$ and closed curves δ_j for $j = 1, \dots, m$ in M_K .

Note that the curves δ_j for $j = 1, \dots, m$ form a normal generating set for the first commutator subgroup of $\pi_1(M_K, x_N)$, when suitably based using the arcs κ_i .

The attaching regions for the 2-handles of U' avoid

$$\bigcup_{i=1}^N \mu_{K_i} \cup \bigcup_{i=1}^{N-1} \kappa_i \cup \bigcup_{j=1}^m \delta_j \subset M_K,$$

and so the points $x'_i := x_i \times \{1\}$, arcs $\kappa'_i := \kappa_i \times \{1\}$, and loops $\mu'_{K_i} := \mu_{K_i} \times \{1\}$ and $\delta'_j := \delta_j \times \{1\}$ lie in $\partial_+ U' = Y'$ for all $i = 1, \dots, N$ and $j = 1, \dots, m$. Similarly, the attaching regions for the 3-handles of U'' avoid

$$\left(\bigcup_{i=1}^N \mu'_{K_i} \cup \bigcup_{j=1}^m \delta'_j \right) \subset Y',$$

and so the loops $\mu''_{K_i} := \mu'_{K_i} \times \{2\}$ and $\delta''_j := \delta'_j \times \{2\}$ lie in $\partial_+ U'' = Y$ for all $i = 1, \dots, N$ and $j = 1, \dots, m$. For each $i = 1, \dots, N$ we have an inclusion-induced map

$$\begin{aligned} \iota_i &: \pi_1(M_{K_i}, x'_i \times \{2\}) \rightarrow \pi_1(U'', x'_N) \\ &\text{by } \beta \mapsto \kappa'_i \cdot (x'_i \times [1, 2]) \cdot \beta \cdot \overline{(x'_i \times [1, 2])} \cdot \overline{\kappa'_i}. \end{aligned}$$

Let $U = U' \cup_{Y'} U''$, and note that we also have an inclusion-induced map

$$\iota: \pi_1(U'', x'_N) \rightarrow \pi_1(U, x_N) \quad \text{by } \gamma \mapsto (x_N \times [0, 1]) \cdot \gamma \cdot \overline{(x_N \times [0, 1])}.$$

In the language of Section 3, $\iota \circ \iota_i$ is induced by the path from x_N to x''_i given by

$$\tau_i = (x_N \times [0, 1]) \cdot \kappa'_i \cdot (x'_i \times [1, 2]).$$

We return to using the notation from Section 5 in order to state and prove the following.

Proposition 8.1. *Let $K = \#_{i=1}^N K_i$ and U be the standard cobordism from M_K to $\bigsqcup_{i=1}^N M_{K_i}$ as above. Let $p \in \mathbb{N}$ and choose maps $\chi_i: H_1(\Sigma_2(K_i)) \rightarrow \mathbb{Z}_p$ for $i = 1, \dots, N$, so $(\chi_i)_{i=1}^N: H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_p$. Let μ_{K_N} be the preferred meridian for K and for $i = 1, \dots, N$ let μ''_{K_i} be the preferred meridian for K_i . Then the map*

$$f_{(\chi_i)_{i=1}^N}^K: \pi_1(M_K, x_N) \rightarrow \mathbb{Z} \ltimes \mathbb{Z}_p$$

extends uniquely to a map $F: \pi_1(U, x_N) \rightarrow \mathbb{Z} \ltimes \mathbb{Z}_p$. Also, the composition

$$f_i: \pi_1(M_{K_i}, x'_i \times \{2\}) \xrightarrow{\iota_i} \pi_1(U'', x'_N) \xrightarrow{\iota} \pi_1(U, x_N) \xrightarrow{F} \mathbb{Z} \ltimes \mathbb{Z}_p$$

satisfies $f_i = f_{\chi_i}^{K_i}$.

Proof. Notice that $\pi_1(U) = \pi_1(M_K)/\langle \lambda_{K_1}, \dots, \lambda_{K_{N-1}} \rangle$. Therefore, since each λ_{K_i} bounds a subsurface of $\bigsqcup_{i=1}^N G_i$ and hence lies in $\pi_1(M_K)^{(2)}$, the map $f_{(\chi_i)_{i=1}^N}^K$ extends uniquely as desired.

Observe that for all $i = 1, \dots, N$ we have

$$\begin{aligned} \iota(\iota_i(\mu''_{K_i})) &= \iota(\kappa'_i \cdot (x'_i \times [1, 2]) \cdot \mu''_{K_i} \cdot \overline{(x'_i \times [1, 2])} \cdot \overline{\kappa'_i}) \\ &= \iota(\kappa'_i \cdot \mu'_{K_i} \cdot \overline{\kappa'_i}) \\ &= (x_N \times [0, 1]) \cdot \kappa'_i \cdot \mu'_{K_i} \cdot \overline{\kappa'_i} \cdot \overline{(x_N \times [0, 1])} = \mu_{K_N} \in \pi_1(U, x_N). \end{aligned}$$

Therefore

$$f_i(\mu''_{K_i}) = F(\mu_{K_N}) = f_{(\chi_i)_{i=1}^N}^K(\mu_{K_N}) = (t, 0) = f_{\chi_i}^{K_i}(\mu''_{K_i}) \in \mathbb{Z} \times \mathbb{Z}_p.$$

For each $i = 1, \dots, N$, every element $\gamma \in \pi_1(M_{K_i}, x''_i)$ can be written as $\gamma = (\mu''_{K_i})^{\varepsilon(\gamma_i)} a$ for some element $a \in \pi_1(M_{K_i}, x''_i)^{(1)}$. Moreover, the collection of δ''_j corresponding to K_i in Figure 5 normally generate $\pi_1(M_{K_i}, x''_i)^{(1)}$. It therefore suffices to check that f_i agrees with $f_{\chi_i}^{K_i}$ on μ''_{K_i} (as done above) and on the collection of δ''_j corresponding to K_i .

Supposing that δ''_j corresponds to K_i , we have that

$$\begin{aligned} \iota(\iota_i(\delta''_j)) &= \iota(\kappa'_i \cdot (x'_i \times [1, 2]) \cdot \delta''_j \cdot \overline{(x'_i \times [1, 2])} \cdot \overline{\kappa'_i}) \\ &= \iota(\kappa'_i \cdot \delta'_j \cdot \overline{\kappa'_i}) \\ &= (x_N \times [0, 1]) \cdot \kappa'_i \cdot \delta'_j \cdot \overline{\kappa'_i} \cdot \overline{(x_N \times [0, 1])} \\ &= \kappa_i \delta_j \overline{\kappa_i} \in \pi_1(U, x_N). \end{aligned}$$

Now fix a lift \widetilde{x}_N of x_N to M_K^2 , the double cover of M_K . Since x_N does not lie in a tubular neighborhood of K , we can think of \widetilde{x}_N as lying in $E_K^2 \subseteq \Sigma_2(K)$ as well. The inclusion induced maps $M_K \rightarrow U$ and $M_{K_i} \rightarrow U$ induce isomorphisms on first homology, and so the double cover U^2 is a cobordism from M_K^2 to $\bigsqcup_{i=1}^N M_{K_i}^2$. For each $i = 1, \dots, N$, lifting the arc

$$\kappa_i \cdot (x_i \times [0, 1]) \cdot (x'_i \times [1, 2])$$

to U^2 starting at \widetilde{x}_N gives a preferred basepoint $\widetilde{x''_i}$ in $M_{K_i}^2$. As before, we also think of this basepoint as lying in $E_{K_i}^2 \subseteq \Sigma_2(K_i)$. We can therefore speak of the lift $\widetilde{\gamma}$ of a curve γ based at x_N (respectively, x''_i) to $\Sigma_2(K)$ (respectively, $\Sigma_2(K_i)$) by choosing the lift with basepoint \widetilde{x}_N (respectively, $\widetilde{x''_i}$).

Remark 8.2. A choice of basepoint is technically always necessary to define f_{χ}^K , though this was suppressed in Section 5 in our discussion of the connected case, where it was less important.

Therefore

$$f_i(\delta_j'') = F(\kappa_i \delta_j \overline{\kappa_i}) = f_{(\chi_k)_{k=1}^N}^K(\kappa_i \delta_j \overline{\kappa_i}) = (0, (\chi_k)_{k=1}^N(\widetilde{\kappa_i \delta_j \overline{\kappa_i}})).$$

Similarly,

$$f_{\chi_i}^{K_i}(\delta_j'') = (0, \chi_i(\widetilde{\delta_j''})).$$

It therefore only remains to show that

$$(\chi_k)_{k=1}^N(\widetilde{\kappa_i \delta_j \overline{\kappa_i}}) = \chi_i(\widetilde{\delta_j''}).$$

First, note that $\chi_k(\widetilde{\kappa_i \delta_j \overline{\kappa_i}}) = 0$ unless $k = i$. Also, the homology class of $\widetilde{\kappa_i \delta_j \overline{\kappa_i}}$ in

$$H_1(\Sigma_2(K_i)) \subset \bigoplus_{i=1}^N H_1(\Sigma_2(K_i)) \cong H_1(\Sigma_2(K))$$

is exactly the same as that of $\widetilde{\delta_j''}$ in $H_1(\Sigma_2(K_i))$, and so we have that

$$\chi_i(\widetilde{\kappa_i \delta_j \overline{\kappa_i}}) = \chi_i(\widetilde{\delta_j''}). \quad \square$$

We note for later use that the inclusion induced maps $M_K \rightarrow U$ and $M_{K_i} \rightarrow U$ give isomorphisms on first homology, and that $H_2(U) \cong \mathbb{Z}^N$ and $H_3(U) \cong \mathbb{Z}^N$.

9. Proof of Theorem 6.1

Since the proof of Theorem 6.1 is rather long, for the reader's convenience we outline the main steps of the argument, with references to key results from elsewhere in the paper.

- 1) (Proposition 9.1.) Construct a 4-manifold V with boundary $\partial V = M_K$ such that the inclusion induced map $H_1(M_K) \rightarrow H_1(V)$ is an isomorphism and $H_2(V) \cong \mathbb{Z}^{2g}$. Let U denote the standard cobordism between M_K and $Y := \bigsqcup_{i=1}^N M_{K_i}$ discussed in Section 8 and let $Z := V \cup_{M_K} U$. Note for later use that $H_2(Z) = \mathbb{Z}^{2g+N-1}$ and $\chi(Z) = 2g$.

- 2) (Propositions 8.1 and 9.2.) Show that we can choose maps $\chi_i : H_1(M_{K_i}) \rightarrow \mathbb{Z}_p$ such that the corresponding map $\phi : \prod_{i=1}^N \pi_1(M_{K_i}) \rightarrow \mathbb{Z} \times \mathbb{Z}_p$ extends to a map $\Phi : \pi_1(Z) \rightarrow \mathbb{Z} \times \mathbb{Z}_{p^a}$ for some $a \geq 1$ and such that at least $n := \frac{m_R N - 4g}{2}$ of the χ_i are nonzero.
- 3) (Claim 9.4.) Show that for some $1 \leq i \leq N$ and $1 \leq j \leq r$, the element $[1, 0] \otimes [\lambda(\eta_i^j)]$ in $H_1^\phi(Y) = H_1(\mathbb{Q}(\xi_p)[t^{\pm 1}]^2 \otimes_{\mathbb{Z}[\pi_1(Y)]} C_*(\tilde{Y}))$ does not map to 0 in $H_1^\Phi(Z)$. (Recall that $\lambda(\eta_i^j)$ is a longitude of the infection curve η_i^j and lies in $M_{K_i} \subset Y$.) This step, which contains much of the technical work of the theorem, crucially relies on our assumption that for every nontrivial $\chi : H_1(\Sigma_2(R)) \rightarrow \mathbb{Z}_p$ we know that the collection $\{[1, 0] \otimes [\eta^j]\}$ generates $H_1^{\theta \circ f_\chi}(R)$ and that the order of $H_1^{\theta \circ f_\chi}(R)$ is relatively prime to $\Delta_R(t)$.
- 4) (Last two paragraphs of Section 9.) Construct a local coefficient derived series representation $\pi_1(Y) \rightarrow \Lambda$ extending over $\pi_1(Z)$ and bound the $L^{(2)}$ ρ -invariant $\rho^{(2)}(Y, \Lambda)$ in two different ways to get a contradiction. Essentially, since $\chi(Z) = 2g$ and our representation extends over $\pi_1(Z)$, Theorem 4.6 implies that $|\rho^{(2)}(Y, \Lambda)|$ is small, while our assumptions on $|\rho_0(J_i^j)|$ together with Step 3, Proposition 4.4, and Proposition 5.1 will imply that $|\rho^{(2)}(Y, \Lambda)|$ is very large.

We now prove the two propositions crucial to Steps 1 and 2, respectively.

Proposition 9.1. *Let $N \in \mathbb{N}$ be arbitrary and $K = \#_{i=1}^N K_i$ be a knot with $g_4(K) \leq g$. Then there exists a compact connected 4-manifold V such that, letting U denote the standard cobordism between M_K and $Y := \sqcup_{i=1}^N M_{K_i}$ from Section 8, $Z := U \cup_{M_K} V$ satisfies:*

- (i) $\partial Z = Y$,
- (ii) $H_2(Z) = \mathbb{Z}^{2g+N-1}$, and
- (iii) $\chi(Z) = 2g$.

Proof. Let F' be a locally flat surface embedded in D^4 with $\partial F' = K$ and $g(F') = g$. Following [3, Prop. 5.1], we construct a topological 4-manifold V with boundary $\partial V = M_K$, $H_1(M_K) \rightarrow H_1(V)$ an isomorphism, and $H_2(V) \cong \mathbb{Z}^{2g}$, as follows. Let $X = X_0(K)$ denote the 0-trace of K , the 4-manifold obtained from D^4 by attaching a 0-framed 2-handle along a neighborhood of K . Let F be the closed surface in X obtained by taking the union of F' with a core of the 2-handle. Note that since F is locally flat it has a normal bundle by [21, Section 9.3]. Observe that $F \cdot F = 0$, and so $\nu(F) \cong F \times D^2$.

Now, let $V = (X \setminus \nu(F)) \cup_{\partial\nu(F)} H \times S^1$, where H is any handlebody with $\partial H = F$. A Mayer-Vietoris argument shows that $H_1(V) \cong \mathbb{Z}$, with generator a meridian to F , and that $H_2(V) \cong \mathbb{Z}^{2g}$. Note that by Poincaré duality and universal coefficients, we have

$$H_3(V) \cong H^1(V, M_K) \cong \text{Hom}(H_1(V, M_K), \mathbb{Z}) \cong \text{Hom}(0, \mathbb{Z}) \cong 0.$$

So in particular the Euler characteristic of V is $\chi(V) = 1 - 1 + 2g - 0 + 0 = 2g$.

Let U be the standard cobordism between M_K and $\bigsqcup_{i=1}^N M_{K_i}$ discussed rather extensively in Section 8. Now let $Z = V \cup_{M_K} U$, as illustrated schematically in Figure 6.

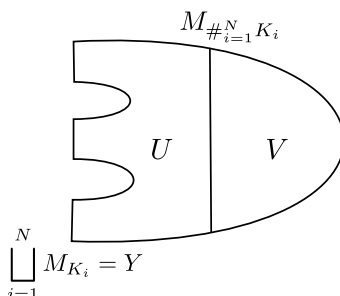


Figure 6: A schematic diagram of $Z = U \cup_{M_K} V$.

Note that $H_2(Z) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}^{N-1}$, and the inclusion induced map $H_1(M_K) \rightarrow H_1(Z)$ is an isomorphism, as are each of the maps $H_1(M_{K_i}) \rightarrow H_1(Y) \rightarrow H_1(Z)$ for $i = 1, \dots, N$. Also, $H_3(Z) \cong \mathbb{Z}^{N-1}$ and $H_4(Z) = 0$. So the Euler characteristic of Z is

$$\chi(Z) = 1 - 1 + (2g + N - 1) - (N - 1) = 2g. \quad \square$$

Proposition 9.2. *Let R be a ribbon knot and p be a prime, and let m_R denote the generating rank of the p -primary part of $H_1(\Sigma_2(R))$. Fix $N \in \mathbb{N}$, and for each $i = 1, \dots, N$ let K_i be a knot obtained by infection along an unlink $\{\eta^j\}_{j=1}^r$ in the complement of R such that each η^j represents an element of $\pi_1(M_R)^{(1)}$. Let $K = \#_{i=1}^N K_i$, and suppose that M_K bounds a compact connected 4-manifold V such that $H_1(M_K) \rightarrow H_1(V)$ is an isomorphism.*

Then there exist $\chi_i: H_1(\Sigma_2(R)) \rightarrow \mathbb{Z}_p$, for $i = 1, \dots, N$ such that:

- (a) *at least $\frac{m_R N - 2\chi(V)}{2}$ of the χ_i are nonzero, and*

(b) for some $a > 0$, there exists a map $\pi_1(V) \rightarrow \mathbb{Z} \times \mathbb{Z}_{p^a}$ such that the composition $\pi_1(M_K) \rightarrow \pi_1(V) \rightarrow \mathbb{Z} \times \mathbb{Z}_{p^a}$ is given by the post-composition of $f_{\oplus_{i=1}^N \chi_i^{K_i}}$ with the inclusion $\mathbb{Z} \times \mathbb{Z}_p \hookrightarrow \mathbb{Z} \times \mathbb{Z}_{p^a}$.

Proof. For convenience, let $n = \frac{m_R N - 2\chi(V)}{2}$. There is a canonical identification $H_1(M_K^2) \cong \mathbb{Z} \oplus H_1(\Sigma_2(K))$, and so given any $(\chi_i^{K_i})_{i=1}^N$ we obtain not just a map $\chi: H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_p$ but also a map $\bar{\chi}: H_1(M_K^2) \rightarrow \mathbb{Z}_p$ by sending the \mathbb{Z} coordinate to zero. Since the inclusion $H_1(M_K) \rightarrow H_1(V)$ is an isomorphism, it therefore suffices to show that there are homomorphisms $(\chi_i^R)_{i=1}^N: H_1(\Sigma_2(R)) \rightarrow \mathbb{Z}_p$, at least n of which are nonzero, such that the map

$$\bar{\chi} := \overline{(\chi_i^{K_i})_{i=1}^N}: H_1(M_K^2) \rightarrow \mathbb{Z}_p$$

extends over $H_1(V^2)$, perhaps after expanding its codomain to \mathbb{Z}_{p^a} for some $a > 0$. Note that $\bar{\chi}$ extends over $H_1(V^2)$ up to enlarging its codomain if and only if $\bar{\chi}$ vanishes on

$$H := \ker(H_1(M_K^2) \xrightarrow{i_1} H_1(V^2)).$$

The group of characters $TH_1(M_K^2) \rightarrow \mathbb{Z}_p$ is isomorphic to $H_1(\Sigma_2(K), \mathbb{Z}_p)$, which is in turn congruent to $(\mathbb{Z}_p^{m_R})^N$, where we recall that m_R denotes the generating rank of the p -primary part of $H_1(\Sigma_2(R))$. The subgroup of characters vanishing on H is in bijective correspondence with $(TH_1(M_K^2)/H) \otimes \mathbb{Z}_p$.

Note that M_K is a homology $S^1 \times S^2$ with $\text{g-rk}(TH_1(M^2) \otimes \mathbb{Z}_p) = m_R N$. Therefore, by Proposition 7.6, the p -primary part of $TH_1(M_K^2)/H$ has generating rank at least $\frac{m_R N - 2\chi(V)}{2} = n$. Therefore $TH_1(M_K^2)/H$ has a subgroup isomorphic to \mathbb{Z}_p^n . Our desired result now follows from a linear algebra argument (see the proof of [29, Theorem 6.1]): every subgroup of \mathbb{Z}_p^M isomorphic to \mathbb{Z}_p^ℓ ($0 \leq \ell \leq M$) has an element at least ℓ of whose coordinates are nonzero. □

Now we prove Theorem 6.1.

Proof of Theorem 6.1. Suppose for the sake of contradiction that there is some locally flat surface F' embedded in D^4 with $\partial F' = K$ and $g(F') = g$. Let U, V , and Z be as in Proposition 9.1. Note that as discussed in Section 8 we have a standard choice of basepoints and paths inducing inclusion maps; for the rest of the proof, these choices will remain fixed though not explicitly discussed.

We pause to establish notation. For a knot J in S^3 , we denote its exterior by E_J . For a manifold X with $H_1(X) \cong \mathbb{Z}$, we denote its canonical double cover by X^2 . The choice of a meridian μ_J determines a splitting $\pi_1(M_J) \cong \mathbb{Z} \ltimes \mathcal{A}(J)$, where $\mathcal{A}(J)$ denotes the Alexander module of J . Note that $H_1(\Sigma_2(J))$ is naturally identified with $\mathcal{A}(J)/\langle t + 1 \rangle$, and so a map $\chi: H_1(\Sigma_2(J)) \rightarrow \mathbb{Z}_p$ induces a map

$$f_\chi: \pi_1(M_J) \xrightarrow{\cong} \mathbb{Z} \ltimes \mathcal{A}(J) \rightarrow \mathbb{Z} \ltimes H_1(\Sigma_2(J)) \xrightarrow{Id \ltimes \chi} \mathbb{Z} \ltimes \mathbb{Z}_p.$$

Note that in the setting of Proposition 9.1, since $H_1(M_K) \rightarrow H_1(Z) \cong \mathbb{Z}$ is an isomorphism, Z also has a canonical double cover Z^2 . It is easy to check that $Z^2 = V^2 \cup_{M_K^2} U^2$ and that $\partial Z^2 = \bigsqcup_{i=1}^N M_{K_i}^2$.

For each $i = 1, \dots, N$, we have a canonical, linking form-preserving identification of $H_1(\Sigma_2(K_i))$ with $H_1(\Sigma_2(R))$ coming from the degree one maps $E_{J_i^j} \rightarrow E_{\text{unknot}}$. Given a map $\chi^R: H_1(\Sigma_2(R)) \rightarrow \mathbb{Z}_p$ we will use χ^{K_i} to denote the corresponding map from $H_1(\Sigma_2(K_i)) \rightarrow \mathbb{Z}_p$, and vice versa. We will also always identify $H_1(\Sigma_2(K))$ with $\bigoplus_{i=1}^N H_1(\Sigma_2(K_i))$ in the canonical, linking form-preserving way.

Define $n := \frac{m_R N - 4g}{2}$. (Note that with $\chi(V) = 2g$ this agrees with the definition of n used above.) We wish to show that there exist $\chi_i^R: H_1(\Sigma_2(R)) \rightarrow \mathbb{Z}_p$, for $i = 1, \dots, N$, such that at least n of the χ_i^R are nonzero and for some $a > 0$, there exists a map $\pi_1(Z) \rightarrow \mathbb{Z} \ltimes \mathbb{Z}_{p^a}$ such that the composition $\pi_1(M_{K_i}) \xrightarrow{\iota \circ \iota_i} \pi_1(Z) \rightarrow \mathbb{Z} \ltimes \mathbb{Z}_{p^a}$ is given by the postcomposition of $f_{\chi_i^{K_i}}$ with the inclusion $\mathbb{Z} \ltimes \mathbb{Z}_p \hookrightarrow \mathbb{Z} \ltimes \mathbb{Z}_{p^a}$. Henceforth, we will implicitly take the usual inclusion of \mathbb{Z}_p in \mathbb{Z}_{p^a} without further comment.

We will accomplish this in a somewhat indirect fashion, by focusing on constructing an appropriate map on $\pi_1(M_K)$ which extends over $\pi_1(U)$ and $\pi_1(V)$ separately. By Proposition 8.1, given any choice of $\chi_1^R, \dots, \chi_N^R: H_1(\Sigma_2(R)) \rightarrow \mathbb{Z}_p$, the map $f_{(\chi_i^{K_i})_{i=1}^N}: \pi_1(M_K) \rightarrow \mathbb{Z} \ltimes \mathbb{Z}_p$ extends uniquely to a map $F: \pi_1(U) \rightarrow \mathbb{Z} \ltimes \mathbb{Z}_p$ such that when we consider the composition

$$F \circ \iota \circ \iota_i: \pi_1(M_{K_i}) \rightarrow \mathbb{Z} \ltimes \mathbb{Z}_p,$$

we have $F \circ \iota \circ \iota_i = f_{\chi_i^{K_i}}$. By applying Proposition 9.2 to our K and V and extending over U as discussed above, we obtain $\chi = (\chi_1^R, \dots, \chi_N^R)$ with at least $\frac{m_R N - 4g}{2} =: n$ of the χ_i nonzero together with a map $F: \pi_1(Z) \rightarrow \mathbb{Z} \ltimes \mathbb{Z}_{p^a}$ such that the composition

$$\pi_1(M_{K_i}) \xrightarrow{\iota \circ \iota_i} \pi_1(Z) \rightarrow \mathbb{Z} \ltimes \mathbb{Z}_{p^a}$$

is given by $f_{\chi_i^{\kappa_i}}$.

As described in Section 5, we have a fixed map $\theta: \mathbb{Z} \ltimes \pi_1(\Sigma_2(K)) \rightarrow \mathrm{GL}_2(\mathbb{Q}(\xi_{p^a})[t^{\pm 1}])$. By post-composing F and each $f_{\chi_i^{\kappa_i}}$ with this map, we obtain

$$\begin{aligned} \Phi &= \theta \circ F: \pi_1(Z) \rightarrow \mathrm{GL}_2(\mathbb{Q}(\xi_{p^a})[t^{\pm 1}]), \\ \phi_i &= \theta \circ f_{\chi_i^{\kappa_i}}: \pi_1(M_{K_i}) \rightarrow \mathrm{GL}_2(\mathbb{Q}(\xi_{p^a})[t^{\pm 1}]). \end{aligned}$$

We let

$$\phi = \prod_{i=1}^N \phi_i: \prod_{i=1}^N \pi_1(M_{K_i}) \rightarrow \mathrm{GL}_2(\mathbb{Q}(\xi_{p^a})[t^{\pm 1}]).$$

For convenience, let $\mathbb{F} = \mathbb{Q}(\xi_{p^a})$, $S = \mathbb{F}[t^{\pm 1}]$, $Q = \mathbb{F}(t)$, and S/p be shorthand for $S/\langle p(t) \rangle$ for any polynomial $p(t) \in S$. Since our infection curves α_i live in the second derived subgroup of M_{R_0} , the degree one maps $f_i: E_{J_i^j} \rightarrow E_{\mathrm{unknot}}$ give us an identification

$$f_*: H_1^\phi(Y, S) = \bigoplus_{i=1}^N H_1^{\phi_i}(M_{K_i}, S) \xrightarrow{\cong} \bigoplus_{i=1}^N H_1^{\theta \circ f_{\chi_i^{\kappa_i}}, S}(M_R)$$

where the maps $\chi_i^R: H_1(\Sigma_2(R)) \rightarrow \mathbb{Z}_p \hookrightarrow \mathbb{Z}_{p^a}$ are as above. We now work towards proving the following claim.

Claim 9.3.

$$\begin{aligned} H := \bigoplus_{\{i|\chi_i^R \neq 0\}} H_1^{\phi_i}(M_{K_i}) \text{ is not contained} \\ \text{in } \ker(H_1^\phi(Y; \mathbb{F}[t^{\pm 1}]) \rightarrow H_1^\Phi(Z; \mathbb{F}[t^{\pm 1}])). \end{aligned}$$

Proof of Claim 9.3. First, note that H has generating rank at least $\lceil k/d_R \rceil$, since for some nontrivial $\chi_0: H_1(\Sigma_2(R)) \rightarrow \mathbb{Z}_p$ there is a submodule of H isomorphic to $\left(H_1^{\theta \circ f_{\chi_0}}(M_R)\right)^{\lceil k/d_R \rceil}$. Note that if $\chi_i^R = 0$ then

$$H_1^{\theta \circ f_{\chi_i^R}}(M_R) \cong \mathcal{A}_{\mathbb{Q}}(R) \otimes_{\mathbb{Q}[t^{\pm 1}]} \mathbb{F}[t^{\pm 1}].$$

We therefore have that

$$H_1^\phi(Y, S) \cong (\mathcal{A}_{\mathbb{Q}}(R) \otimes_{\mathbb{Q}[t^{\pm 1}]} \mathbb{F}[t^{\pm 1}])^{N-k} \oplus H$$

where $k \geq \frac{m_R N - 4g}{2}$ is the number of nonzero χ_i^R .

We now compute the rank of $H_2^\Phi(Z; Q)$. We can immediately see that $H_0^\Phi(Z; Q) \cong H_0^\phi(Y; Q) \cong 0$, since $H_0^\Phi(Z)$ and $H_0^\phi(Y)$ are annihilated by $t - 1$. Note that for each $i = 1, \dots, N$ the inclusion map $Y_i \rightarrow Z$ induces an isomorphism on $H_0(-; \mathbb{Z})$ and $H_1(-; \mathbb{Z})$. By the proof of [24, Proposition 4.1], modified to use only a partial chain contraction for $C_*^\Phi(Z, Y_i; Q)$ in degrees 0, 1, as in [16, Proposition 2.10], this implies that the map $H_1^{\phi_i}(Y_i; Q) \rightarrow H_1^\Phi(Z; Q)$ is onto. We have already observed that $H_1^{\phi_i}(Y_i)$ is a torsion S -module and so $H_1^{\phi_i}(Y_i; Q) = 0$; it follows that $H_1^\Phi(Z; Q) = 0$ as well. Consideration of the long exact sequence of the pair (Z, Y) then allows us to conclude that $H_1^\Phi(Z, Y; Q) = 0$. By Poincaré-Lefschetz duality, universal coefficients, and the long exact sequence of (Z, Y) with Q -coefficients we have that

$$H_3^\Phi(Z; Q) \cong H_\Phi^1(Z, Y; Q) \cong \text{Hom}(H_1^\Phi(Z, Y; Q), Q) \cong 0.$$

Finally, since Z is a topological 4-manifold and hence homotopy equivalent to a finite CW complex with cells of dimension at most 3 (see the proof of Theorem 4.6 for references for this fact), we have that $H_j^\Phi(Z; \mathbb{F}(t)) = 0$ for all $j \geq 4$. Re-computing $\chi(Z)$ with Q -coefficients, we obtain

$$2g = \chi(Z) = 0 - 0 + \dim_Q H_2^\Phi(Z; Q) - 0 + 0 = \dim_Q H_2^\Phi(Z; Q).$$

We now return to working with $S = \mathbb{F}[t^{\pm 1}]$ -coefficients and consider the long exact sequence of Proposition 3.3

$$\begin{aligned} \dots \rightarrow H_2^\phi(Y) &\xrightarrow{i_2} H_2^\Phi(Z) \xrightarrow{j_2} H_2^\Phi(Z, Y) \\ &\xrightarrow{\partial} H_1^\phi(Y) \xrightarrow{i_1} H_1^\Phi(Z) \xrightarrow{j_1} H_1^\Phi(Z, Y) \rightarrow \dots \end{aligned}$$

Suppose now for a contradiction that $H \leq \ker(i_1)$. Since

$$\begin{aligned} \ker(i_1) &= \text{Im}(\partial) \cong H_2^\Phi(Z, Y) / \ker(\partial) = H_2^\Phi(Z, Y) / \text{Im}(j_2) \\ &\cong (S^{2g} \oplus TH_2^\Phi(Z, Y)) / \text{Im}(j_2), \end{aligned}$$

it follows that $(S^{2g} \oplus TH_2^\Phi(Z, Y)) / \text{Im}(j_2)$ has a submodule H' isomorphic to H .

By applying Lemma 7.4 with $A = H_2^\Phi(Z, Y)$, $B = \text{Im}(j_2)$, and $C = H'$ we obtain that $TH_2^\Phi(Z, Y) / (\text{Im}(j_2) \cap TH_2^\Phi(Z, Y))$ contains a submodule H'' of generating rank at least $\lceil k/d_R \rceil - 2g$ and of order which divides the order of H' and so is relatively prime to Δ_R . Since $\text{Im}(j_2|_T) \subseteq \text{Im}(j_2) \cap TH_2^\Phi(Z, Y)$, it follows immediately that $\text{coker}(j_2|_T) = TH_2^\Phi(Z, Y) / \text{Im}(j_2|_T)$

contains a submodule of generating rank at least $\lceil k/d_R \rceil - 2g$ and of order relatively prime to Δ_R .

As argued above, we have that $H_1^\Phi(Z, Y; Q) = 0$, i.e. that $H_1^\Phi(Z, Y)$ is torsion, and so we can apply Lemma 7.5 to conclude that

$$\text{coker}(j_2|_T) \cong \ker(j_1|_T) = \ker(j_1) = \text{Im}(i_1).$$

Therefore $\text{Im}(i_1)$ has a submodule of generating rank at least $\lceil k/d_R \rceil - 2g$ and of order that is relatively prime to Δ_R . Since $k \geq n = \frac{m_R N - 4g}{2}$ and $N \geq \frac{4g(d_R + 1) + 2}{m_R}$, we obtain that $\lceil k/d_R \rceil - 2g > 0$ and so there is a submodule of $\text{Im}(i_1)$ isomorphic to S/s for some nontrivial polynomial s relatively prime to Δ_R . This is our desired contradiction, since $H \leq \ker(i_1)$ also implies that $\text{Im}(i_1)$ is a quotient of $(\mathcal{A}_\mathbb{Q}(R) \otimes_{\mathbb{Q}[t^{\pm 1}]} \mathbb{F}[t^{\pm 1}])^{N-k}$, which has order Δ_R^{N-k} and therefore cannot contain a submodule isomorphic to S/s . This completes the proof of the claim. \square

Claim 9.4. *For some $1 \leq i \leq N$ and $1 \leq j \leq r$, the element $[1, 0] \otimes [\lambda(\eta_i^j)]$ does not map to 0 in $H_1^\Phi(Z)$.*

Proof of Claim 9.4. Observe that since the longitude $\lambda(\eta_i^j)$ of η_i^j is in the second derived subgroup of $\pi_1(M_R)$ it must lift to a curve l_i^j in the cover \widetilde{M}_R of M_R determined by ϕ_i . (In fact, it lifts to $\mathbb{Z} \times \mathbb{Z}_{p^a}$ copies – pick one.) Since whenever $\chi_i^R \neq 0$ we have that the collection $\{[1, 0] \otimes [l_i^j]\}_{j=1}^r$ generates $H_1^{\phi_i}(M_R)$, our argument that $H \not\leq \ker(i_1)$ in fact implies that for at least one i and j with $1 \leq i \leq N$ and $1 \leq j \leq r$, we have $i_1([1, 0] \otimes [l_i^j]) \neq 0$ in $H_1^\Phi(Z)$. This completes the proof of Claim 9.4 and of Step 3. \square

We are now ready to complete the proof of Theorem 6.1, as described in Step 4, by constructing a new representation of $\pi_1(Y)$ and bounding $\rho^{(2)}(Y, \psi)$ in two different ways to derive a contradiction. Let

$$\psi: \pi_1(Y) \rightarrow \pi_1(Z) \rightarrow \Lambda := \pi_1(Z)/\pi_1(Z)_{(\mathbb{Q}, \mathbb{Z}_{p^a}, \mathbb{Q})}^{(3)}$$

be the map induced by inclusion. Since Λ is amenable and in $D(\mathbb{Z}_p)$ [4, Lemma 4.3] and ψ evidently extends over $\pi_1(Z)$, Theorem 4.6 and the fact from Step 1 that $H_2(Z) \cong \mathbb{Z}^{2g+N-1}$ tells us that

$$(4) \quad |\rho^{(2)}(Y, \psi)| \leq 2 \dim_{\mathbb{Z}_p} H_2(Z, \mathbb{Z}_p) = 2(2g + N - 1).$$

Let (i_0, j_0) be the maximal tuple (with respect to the lexicographic ordering) such that $[1, 0] \otimes [l_i^j]$ does not map to 0 in $H_1^\Phi(Z)$. Proposition 5.1

implies that $\lambda(\eta_{i_0}^{j_0}) \notin \pi_1(Z)_{(\mathbb{Q}, \mathbb{Z}_{p^a}, \mathbb{Q})}^{(3)}$. Moreover, Proposition 4.5 tells us that, letting $\delta_i^j(\psi) = \begin{cases} 1, & \psi(\lambda(\eta_i^j)) \neq 0 \\ 0, & \psi(\lambda(\eta_i^j)) = 0 \end{cases}$, we have

$$(5) \quad \begin{aligned} \rho^{(2)}(Y, \psi) &= \sum_{i=1}^N \rho^{(2)}(M_{R_\alpha(J_i)}, \psi|_{M_{R_\alpha(J_i)}}) \\ &= \sum_{i=1}^N \left(\rho^{(2)}(M_R, \psi_i^0) + \sum_{j=1}^r \delta_i^j(\psi) \rho_0(J_i^j) \right). \end{aligned}$$

Since $|\rho^{(2)}(M_R, \psi_i^0)| \leq C_R$ for all i , the tuple (i_0, j_0) is maximal such that $\delta_i^j(\psi) \neq 0$, and $J_{i_0}^{j_0}$ satisfies

$$|\rho_0(J_{i_0}^{j_0})| > 2(2g + N - 1) + NC_R + \sum_{k=1}^{i_0-1} \sum_{\ell=1}^r |\rho_0(J_k^\ell)| + \sum_{\ell=1}^{j_0-1} |\rho_0(J_{i_0}^\ell)|,$$

Equation 5 gives the desired contradiction with Equation 4, which completes the proof of Theorem 6.1. □

10. Height four gropes

In Proposition 10.7 below, we will show the following: the knot K in Section 6.2 bounds a framed grope of height 4 embedded in D^4 . For the reader's convenience, we begin by recalling the definition of a (capped) grope, a certain type of 2-complex.

Definition 10.1 (Grove of height h [16, 21]). A *capped surface*, or a *capped grope of height 1*, is an oriented surface of genus $g > 0$ with nonempty connected boundary, together with discs attached along the $2g$ curves of a standard symplectic basis for the surface. The discs are called *caps*. If G is a capped grope of height $h - 1$, then a 2-complex obtained by replacing each cap of G with a capped surface is called a *capped grope of height h* . A *grope of height h* is obtained by removing caps from a capped grope of height h . It is also called the *body* of the capped grope. The initial surface that the inductive construction starts with is called the *base surface*, and the *boundary of a grope*, ∂G , is the boundary of its base surface.

A (capped) grope defined above is often called *disc-like*. An *annulus-like* (capped) grope is defined in the same way, starting from a base surface with two boundary components.

Remark 10.2. It is not a priori obvious that a 2-complex G known to be a grope has a well-defined height, but it is true. For the reader's convenience, we give a quick argument. Let $\tau \subset G$ be the singular set of the grope union its boundary, i.e. the 1-complex consisting of the points where G is not locally homeomorphic to an open disc. Then $G \setminus \tau$ consists of a collection of open surfaces, many of which are planar. Removing the subset of G corresponding to the non-planar surfaces (the interior of the 'top stage' of G) gives a new grope with a strictly smaller singular set; we can then repeat the above procedure. In this perspective, the height of a grope is exactly the number of such steps needed to reduce the grope to a circle. We leave to the reader the analogous argument that the height of a capped grope is well-defined, as well as the intrinsic definition of the i th stage of a grope, $1 \leq i \leq h$.

A (capped) grope admits a standard embedding in the upper half 3-space $\mathbb{R}_+^3 = \{z \geq 0\}$ which takes the boundary to \mathbb{R}^2 . Compose it with $\mathbb{R}_+^3 \hookrightarrow \mathbb{R}_+^4$, take a regular neighborhood in \mathbb{R}_+^4 , and possibly perform finitely many plumbings. An embedding of the result in a 4-manifold is called an *immersed framed (capped) grope*. If no plumbing is performed, then we say that it is *embedded*. Often we will regard an immersed/embedded (capped) grope as a 2-complex, but it is always assumed to be framed in this sense. In addition, we assume that each intersection in an immersed capped grope is always between a cap and a surface in the body, following the convention of [8]. Note that in a simply connected 4-manifold, an embedded grope without caps can be promoted to an immersed capped grope.

Returning to our case, recall that the knot K in Section 6.2 is the connected sum of satellite knots. We will use the following terminology and results from [5, 8], which also consider link versions.

Definition 10.3 (Satellite capped grope [5, Definition 4.2], [8, Definition 4.2]). Suppose K is a knot in S^3 and α is an unknotted circle in S^3 disjoint from K . Let E_α be the exterior of α , and let λ_α be a zero linking longitude on ∂E_α . A *satellite capped grope* for (K, α) is a disc-like capped grope G immersed in $E_\alpha \times I$ such that the boundary of G is $\lambda_\alpha \times 0$, the body of G is disjoint from $K \times I$, and the caps are transverse to $K \times I$.

Definition 10.4 (Capped grope concordance [8, Definition 4.3]). A *capped grope concordance* between two knots J and J' is an annulus-like

capped grope immersed in $S^3 \times I$ such that the base surface is bounded by $J \times 0 \cup -J' \times 1$.

Proposition 10.5 ([8, Section 4.1]). *Suppose that there is a satellite capped grope of height h for (K, α) and a capped grope concordance of height ℓ between two knots J and J' . Then there is a capped grope concordance of height $h + \ell$ between the satellite knots $K_\alpha(J)$ and $K_\alpha(J')$.*

The height $h + \ell$ capped grope concordance in Proposition 10.5 is obtained by a “product” construction described in [8, Definition 4.4]. The last ingredient we need is the following result from [16].

Proposition 10.6 ([16, Remark 8.14]). *A knot in S^3 with trivial Arf invariant bounds a capped grope of height two immersed in D^4 .*

We can now prove the following.

Proposition 10.7. *Let $K = \#_{i=1}^n R_{\alpha^+, \alpha^-}(J_i^+, J_i^-)$ be a connected sum of satellite knots, where (R, α^+, α^-) is as in the right of Figure 3 and $\{J_i^+, J_i^-\}_{i=1}^n$ a collection of knots with vanishing Arf invariant. Then K bounds an embedded grope of height 4 in D^4 .*

Proof. First, note that it suffices to show that each $R_{\alpha^+, \alpha^-}(J_i^+, J_i^-)$ bounds an embedded grope of height 4, since we can then take the boundary connected sum of such gropes to obtain one with boundary K . We therefore show that under the hypothesis that $\text{Arf}(J^+) = \text{Arf}(J^-) = 0$ the knot $R_{\alpha^+, \alpha^-}(J^+, J^-)$ bounds a grope of height 4.

Observe that the curve α^- in Figure 3 bounds a disjoint capped grope of height two embedded in S^3 , where the body surfaces are disjoint from the knot R but the caps are allowed to intersect R . This is a geometric analogue of the commutator relation $\alpha^- = [\beta_1, \beta_2]$ where the curves β_1 and β_2 shown in the left of Figure 3 are again commutators in the fundamental group.

Indeed, in the planar diagram in the right of Figure 3, the bounded region enclosed by α^- is the projection of an obviously seen embedded disc which intersects R in four points, and by tubing on this disc, one obtains a genus one surface, shown in red in Figure 7, which is disjoint from R . This surface is the base surface of the promised height two grope bounded by α^- . The curves β_1 and β_2 are parallel to standard basis curves of the base surface, and they bound disjoint genus one surfaces obtained by tubing the obviously seen discs along the knot R , as illustrated in Figure 7. Attach them to the base stage surface to obtain a height two grope.

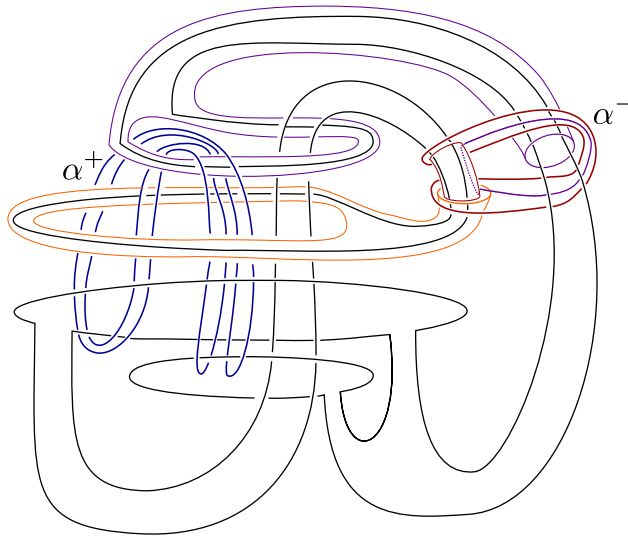


Figure 7: An embedded height 2 grope with boundary α^- in $S^3 \setminus (R \sqcup \alpha^+)$.

Note that all the surfaces used above are disjoint from the other curve α^+ , so by performing the satellite construction, we obtain a height two grope in $S^3 \setminus R_{\alpha^+}(J^+)$ bounded by α^- . Identify S^3 with $S^3 \times 0 \subset S^3 \times I$, push the interior of the grope into the interior of $S^3 \times I$, and add caps using the simple connectedness of $S^3 \times I$ as noted above. Apply general position to make the caps transverse to $R_{\alpha^+}(J^+) \times I$, to obtain a satellite capped grope for $(R_{\alpha^+}(J^+), \alpha^-)$.

Since the knot J^- has trivial Arf invariant, J^- bounds a capped grope of height two in D^4 , by Proposition 10.6. Remove, from D^4 , a small open 4-ball which intersects the capped grope in an unknotted 2-disc lying in the interior of the base surface, to obtain a capped grope concordance of height two between J^- and the trivial knot. By Proposition 10.5 and the above paragraph, the satellite knot $R_{\alpha^+, \alpha^-}(J^+, J^-) = (R_{\alpha^+}(J^+))_{\alpha^-}(J^-)$ is height 4 capped grope concordant to the knot $R_{\alpha^+}(J^+)$. Forget the caps of this capped grope concordance, and attach a slicing disc for the knot $R_{\alpha^+}(J^+)$, to obtain a grope of height 4 bounded by $R_{\alpha^+, \alpha^-}(J^+, J^-)$. \square

Remark 10.8. A similar argument shows the existence of a bounding grope of height 4 for the simpler example in Section 6.1. In this case, the height two surfaces constructed by “tubing along the knot R ” in the 3-space are

not disjoint, but the intersection can be removed by pushing the surfaces into 4-space. We omit the details.

Acknowledgements

The first and third authors thank the Max Planck Institute for Mathematics in Bonn, where they were visiting when part of the work on this paper occurred. The second author thanks Shelly Harvey for stimulating conversations. The first author was partly supported by NRF grant 2019R1A3B2067839. Finally, we thank the anonymous referee for a careful reading and useful suggestions which improved the paper.

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RECEIVED APRIL 16, 2019

ACCEPTED MARCH 20, 2020