Math. Res. Lett.
Volume 28, Number 2, 471–509, 2021

Construction of G_2-instantons via twisted connected sums

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We propose a method to construct G_2–instantons over a compact twisted connected sum G_2–manifold, applying a gluing result of Sá Earp and Walpuski to instantons over a pair of 7–manifolds with a tubular end. In our example, the moduli spaces of the ingredient instantons are non-trivial, and their images in the moduli space over the asymptotic cross-section K3 surface intersect transversely. Such a pair of asymptotically stable holomorphic bundles is obtained using a twisted version of the Hartshorne-Serre construction, which can be adapted to produce other examples. Moreover, their deformation theory and asymptotic behaviour are explicitly understood, results which may be of independent interest.

1. Introduction

We address the existence problem of G_2–instantons over twisted connected sums as formulated by the third author and Walpuski in [16], and we produce the first examples to date of solutions obtained by a nontrivially *transversal* gluing process.

Recall that a G_2*–manifold* (X, g_ϕ) is a Riemannian 7–manifold together with a torsion-free G_2*–structure*, that is, a non-degenerate closed 3–form ϕ satisfying a certain non-linear partial differential equation; in particular, ϕ induces a Riemannian metric g_ϕ with $\mathrm{Hol}(g_\phi) \subset G_2$. A G_2*–instanton* is a connection A on some G–bundle $E \to X$ such that $F_A \wedge *\phi = 0$. Such solutions have a well-understood elliptic deformation theory of index 0 [15], and some form of 'instanton count' of their moduli space is expected to yield new invariants of 7–manifolds, much in the same vein as the Casson invariant and instanton Floer homology from flat connections on 3–manifolds [6, 7]. While some important analytical groundwork has been established towards that goal [17], major compactification issues remain and this suggests that a thorough understanding of the general theory might currently have to be postponed in favour of exploring a good number of functioning examples. The present paper proposes a method to construct a potentially large number of such instances.

Readers interested in a more detailed account of instanton theory on G_2–manifolds are kindly referred to the introductory sections of [14, 16] and works cited therein.

1.1. G_2-instantons over twisted connected sums

An important method to produce examples of compact 7–manifolds with holonomy exactly G_2 is the *twisted connected sum* (TCS) construction [2, 3, 11], outlined in Section 2.1. It consists of gluing a pair of asymptotically cylindrical (ACyl) Calabi–Yau 3–folds obtained from certain smooth projective 3–folds called *building blocks*. A building block (Z, S) is given by a projective morphism $f : Z \to \mathbb{P}^1$ such that $S := f^{-1}(\infty)$ is a smooth anticanonical K3 surface, under some mild topological assumptions (see Definition 2.1); in particular, S has trivial normal bundle. Choosing a convenient Kähler structure on Z, one can make $V := Z \setminus S$ into an ACyl Calabi–Yau 3–fold, that is, a non-compact Calabi–Yau manifold with a tubular end modelled on $\mathbb{R}_+ \times \mathbb{S}^1 \times S$ [3, Theorem 3.4]. Then $\mathbb{S}^1 \times V$ is an ACyl G_2–manifold with a tubular end modelled on $\mathbb{R}_+ \times \mathbb{T}^2 \times S$.

When a pair $(Z_\pm, S_\pm)$ of building blocks admits a *matching* $\mathfrak{r} : S_+ \to S_-$ (see Definition 2.2), there exists a so-called hyper-Kähler rotation between the K3 surfaces 'at infinity'. In this case, the corresponding pair $\mathbb{S}^1 \times V_\pm$ of ACyl G_2–manifolds is truncated at a large 'neck length' T and, intertwining the circle components in the tori $\mathbb{T}^2_\pm$ along the tubular end, glued to form a compact 7-manifold

$$X = Z_+ \#_{\mathfrak{r}} Z_- := S^1 \times V_+ \cup_{\mathfrak{r}} S^1 \times V_-.$$

For large enough T_0, this twisted connected sum X carries a family of G_2-structures $\{\phi_T\}_{T \geq T_0}$ with $\mathrm{Hol}(\phi_T) = G_2$ [3, Theorem 3.12]. The construction is summarised in the following statement.

Theorem 1.1 ([3, Corollary 6.4]). *Given a matching $\mathfrak{r} : S_+ \to S_-$ between a pair of building blocks $(Z_\pm, S_\pm)$ with Kähler classes $\mathsf{k}_\pm \in H^{1,1}(Z_\pm)$ such that $(\mathsf{k}_{+|S_+})^2 = (\mathsf{k}_{-|S_-})^2$, there exists a family of torsion-free G_2-structures $\{\phi_T : T \gg 1\}$ on the closed 7-manifold $X = Z_+ \#_{\mathfrak{r}} Z_-$ with $\mathrm{Hol}(\phi_T) = G_2$.*

Theorem 1.1 raises a natural programme in gauge theory, aimed at constructing G_2-instantons over compact manifolds obtained as a TCS, originally outlined in [15]. Starting from holomorphic bundles over $Z_\pm$ with a suitable stability property, corresponding to Hermitian Yang-Mills metrics over the ACyl Calabi-Yau components [14, Theorem 58], it is possible to glue a hypothetical pair of such solutions into a G_2-instanton, provided a number of technical conditions are met (see below). In the present paper we develop a constructive method to obtain explicit examples of such instanton gluing in many interesting cases, so it is important to recall in detail the assumptions of this gluing theorem.

Let A be an ASD instanton on a $\mathbb{P}U(n)$-bundle $\mathscr{F}$ over a Kähler surface S. The linearisation of the instanton moduli space $\mathcal{M}_S$ near A is modelled on the kernel of the deformation operator

$$\mathbb{D}_{\mathbb{A}} := \mathrm{d}_A^* \oplus \mathrm{d}_A^+ : \Omega^1(S, \mathfrak{g}_{\mathscr{F}}) \to (\Omega^0 \oplus \Omega^+)(S, \mathfrak{g}_{\mathscr{F}}),$$

where $\mathfrak{g}_{\mathscr{F}}$ denotes the adjoint bundle associated to $\mathscr{F}$. Let F be the corresponding holomorphic vector bundle (*cf.* Donaldson-Kronheimer [5]), and denote by f the Hitchin-Kobayashi isomorphism:

$$f : H^1(S, \mathscr{E}nd_0(F)) \xrightarrow{\sim} H^1_A := \ker \mathbb{D}_{\mathbb{A}}. \tag{1}$$

Theorem 1.2 ([16, Theorem 1.2]). *Let $Z_\pm$,$S_\pm$, $\mathsf{k}_\pm$, $\mathfrak{r}$, X and ϕ_T be as in Theorem 1.1. Let $F_\pm \to Z_\pm$ be a pair of holomorphic vector bundles such that the following hold:*

Asymptotic stability: *$F_\pm|_{S_\pm}$ is μ-stable with respect to $\mathsf{k}_\pm|_{S_\pm}$. Denote the corresponding ASD instanton by $A_{\infty,\pm}$.*

Compatibility: *There exists a bundle isomorphism $\bar{\mathfrak{r}} : F_+|_{S_+} \to F_-|_{S_-}$ covering the hyper-Kähler rotation $\mathfrak{r}$ such that $\bar{\mathfrak{r}}^* A_{\infty,-} = A_{\infty,+}$.*

Inelasticity: *There are no infinitesimal deformations of $F_\pm$ fixing the restriction to $S_\pm$:*

$$H^1(Z_\pm, \mathscr{E}nd_0(F_\pm)(-S_\pm)) = 0.$$

Transversality: *If $\lambda_\pm := f_\pm \circ \mathrm{res}_\pm : H^1(Z_\pm, \mathscr{E}nd_0(F_\pm)) \to H^1_{A_{\infty,\pm}}$ denotes the composition of restrictions to $S_\pm$ with the isomorphism (1), then the image of λ_+ and $\bar{\mathfrak{r}}^* \circ \lambda_-$ intersect trivially in the linear space $H^1_{A_{\infty,+}}$:*

$$\mathrm{im}(\lambda_+) \cap \mathrm{im}(\bar{\mathfrak{r}}^* \circ \lambda_-) = \{0\}.$$

Then there exists a $U(r)$-bundle $\mathscr{F}$ over X and a family of connections $\{A_T : T \gg 1\}$ on the associated $\mathbb{P}U(r)$-bundle, such that each A_T is an irreducible unobstructed G_2-instanton over (X, ϕ_T).

Geometrically, the maps λ_+ and $\bar{\mathfrak{r}}^* \circ \lambda_-$ can be seen as linearisations of the natural inclusions of the moduli of asymptotically stable bundles $\mathcal{M}_{Z_\pm}$ into the moduli of ASD instantons $\mathcal{M}_{S_+}$ over the K3 surface 'at infinity', and we think of $H^1_{A_{\infty,+}}$ as a tangent model of $\mathcal{M}_{S_+}$ near the ASD instanton $A_{\infty,+}$. Then the transversality condition asks that the actual inclusions intersect transversally at $A_{\infty,+} \in \mathcal{M}_{S_+}$. That the intersection points are isolated reflects that the resulting G_2-instanton is rigid, since it is unobstructed and the deformation problem has index 0.

1.2. Gluing Hartshorne-Serre instanton bundles

In [2, 3, 11], building blocks Z are produced by blowing up Fano or semi-Fano 3-folds along the base curve $\mathscr{C}$ of an anticanonical pencil (see Proposition 2.3). By understanding the deformation theory of pairs (Y, S) of semi-Fanos Y and anticanonical K3 divisors $S \subset Y$, one can produce hundreds of thousands of pairs with the required matching (see Section 2.2).

In order to apply Theorem 1.2 to produce G_2-instantons over the resulting twisted connected sums, one first requires some supply of asymptotically stable, inelastic vector bundles $F \to Y$. Moreover, to satisfy the hypotheses of compatibility and transversality, one would in general need some understanding of the deformation theory of triples (Y, S, F).

It is important to observe that in the so-called *rigid* case, when $H^1_{A_{\infty,+}} = \{0\}$, transversality is automatic, since the instantons that are glued are isolated points in their moduli spaces. Using rigid bundles adds further constraints to the matching problem for the building blocks, but during the preparation of this article Walpuski [18] was able to exhibit such examples.

In this paper we study the non-rigid case, where the moduli spaces involved are non-trivial, and transversality is a genuine condition. Such examples have not previously appeared in the literature, but are relevant because they open the possibility of obtaining a conjectural instanton number on the G_2-manifold X as a genuine Lagrangian intersection within the moduli space $\mathcal{M}_{S_+}$ over the K3 cross-section along the neck, which can be addressed by enumerative methods in the future.

Our method is to use the Hartshorne-Serre construction to obtain families of bundles over the building blocks, for which the deformation theory can be sufficiently explicitly understood to solve the matching problem and prove transversality. As a proof of concept, we focus on a special case where the moduli space of asymptotically stable bundles over each building block is parametrised by the blow-up curve $\mathscr{C}$ itself. This simplifies the problem by separating the deformation theory of the bundles from the deformation theory of the pair (Y, S). We can therefore *first* find matchings between two semi-Fano families using the techniques from [3], and *then* exploit the high degree of freedom in the choice of the blow-up curve $\mathscr{C}$ (see Lemma 2.5) to satisfy the compatibility and transversality hypotheses.

We carry out all the computations for one particular pair of building blocks, which is detailed in Examples 2.7 and 2.8.

Theorem 1.3. *There exists a matching pair of building blocks $(Z_\pm, S_\pm)$, obtained as $Z_\pm = \mathrm{Bl}_{\mathscr{C}_\pm} Y_\pm$ for $Y_+ = \mathbb{P}^1 \times \mathbb{P}^2$ and the double cover $Y_- \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^2$ branched over a $(2,2)$ divisor, with rank 2 holomorphic bundles $F_\pm \to Z_\pm$ satisfying the hypotheses of Theorem 1.2.*

1.3. Survey of the proof of Theorem 1.3

- We construct holomorphic bundles on building blocks from certain complete intersection subschemes, via the Hartshorne-Serre correspondence [Theorem 3.1]. In Section 3.1, we establish conditions on the parameters of the Hartshorne-Serre construction that are conducive to application of Theorem 1.2. In Sections 3.2 and 3.3, we construct families of bundles $\{E_\pm\}$, over the particular blocks $Y_\pm$ of Theorem 1.3, satisfying these constraints.
- In Section 4.1, we recall sufficient conditions for the stability of $E_{\pm|S_\pm}$. Then, in Section 4.2, we focus on the moduli space $\mathcal{M}^s_{S_+,\mathcal{A}_+}(v_{S_+})$ of stable bundles on S_+, where the problems of compatibility and transversality therefore "take place". Here $Y_+ = \mathbb{P}^1 \times \mathbb{P}^2$, $S_+ \subset Y_+$ is the anticanonical K3 divisor and, for a smooth curve $\mathscr{C}_+ \in |-K_{Y_+|S_+}|$, the block $Z_+ := \mathrm{Bl}_{\mathscr{C}_+} Y_+$ is in the family obtained from Example 2.7.

  We show that $\mathcal{M}^s_{S_+,\mathcal{A}_+}(v_{S_+})$ is isomorphic to S_+ itself, and that the restrictions of the family of bundles E_+ correspond precisely to the blow-up curve $\mathscr{C}_+$. Now, given a rank 2 bundle $E_+ \to Z_+$ such that $\mathcal{G} := E_{+|S_+} \in \mathcal{M}^s_{S_+,\mathcal{A}_+}(v_{S_+})$, the restriction map

$$\text{res} : H^1(Z_+, \mathscr{E}nd_0(E_+)) \to H^1(S_+, \mathscr{E}nd_0(\mathcal{G})) \tag{2}$$

  corresponds to the derivative at E_+ of the map between instanton moduli spaces. Combining with Lemma 2.5, which guarantees the freedom to choose $\mathscr{C}_+$ when constructing the block Z_+ from S_+, we arrive at the following key step.

  Theorem 1.4. *For every stable bundle $\mathcal{G} \in \mathcal{M}^s_{S_+,\mathcal{A}_+}(v_{S_+})$ and every line $V \subset H^1(S_+, \mathscr{E}nd_0(\mathcal{G}))$, there is a smooth base locus curve $\mathscr{C}_+ \in |-K_{Y_+|S_+}|$ and an exceptional fibre $\ell_+ \subset \widetilde{\mathscr{C}}_+$ corresponding by Hartshorne-Serre to an inelastic vector bundle $E_+ \to Z_+$, such that $E_{+|S_+} = \mathcal{G}$ and the restriction map* (2) *has image V.*

- In Section 5 we give the rather technical proof that the bundles $E_\pm$ are inelastic, together with some auxiliary topological properties.
- Finally, in Section 6 we explain how to deduce Theorem 1.3 from Theorem 1.4. More precisely, let $\mathfrak{r} : S_+ \to S_-$ be a matching between $Y_+ = \mathbb{P}^1 \times \mathbb{P}^2$ and $Y_- \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^2$. Then Theorem 6.3 argues that for

any $E_- \to Z_-$ as above we can (up to a twist by holomorphic line bundles $\mathcal{R}_\pm \to Z_\pm$) choose the smooth curve $\mathscr{C}_+ \in |-K_{Y_+|S_+}|$ in the construction of Z_+ so that there is a Hartshorne-Serre bundle $E_+ \to Z_+$ that matches E_- transversely. Then the bundles $F_\pm := E_\pm \otimes \mathcal{R}_\pm$ satisfy all the gluing hypotheses of Theorem 1.2.

While we made the expository choice of unfolding the construction of an example progressively along the paper, an alternative read focused on the general theory could follow through Sections 2.1, 2.2, 3.1, 4.1, 5.1 and 5.2.

Remark 1.5. Theorem 1.4 is stronger than required by our argument. Indeed should the claim hold merely for *generic* $\mathcal{G}$ and V, then we could argue that every $E_- \to Z_-$ has a perturbation that can be matched transversely by some $E_+ \to Z_+$, which is good enough to construct examples.

2. G$_2$-manifolds via semi-Fano 3-folds

The goal of this section is to present a concrete example of a matching of building blocks, constructed from a certain pair of Fano 3-folds. We begin by reviewing some background about the construction and the matching problem.

2.1. Building blocks from semi-Fano 3-folds and twisted connected sums

Definition 2.1. A *building block* is a nonsingular algebraic 3-fold Z together with a projective morphism $f : Z \to \mathbb{P}^1$ satisfying the following assumptions:

(i) The anti-canonical class $-K_Z \in H^2(Z, \mathbb{Z})$ is primitive.

(ii) $S = f^{-1}(\infty)$ is a non-singular K3 surface and $S \sim -K_Z$.
Identify $H^2(S, \mathbb{Z})$ with the K3 lattice L (*i.e.* choose a marking for S), and let N denote the image of $H^2(Z, \mathbb{Z}) \to H^2(S, \mathbb{Z})$.

(iii) The inclusion $N \hookrightarrow L$ is primitive.

(iv) The groups $H^3(Z, \mathbb{Z})$ and $H^4(Z, \mathbb{Z})$ are torsion-free.

In particular, building blocks are simply-connected [2, §5.1]. Theorem 1.1 states that one can construct closed G_2-manifolds from pairs of building blocks that match in the following sense.

Definition 2.2. Let $Z_\pm$ be complex 3-folds, $S_\pm \subset Z_\pm$ smooth anticanonical K3 divisors and $\mathsf{k}_\pm \in H^2(Z_\pm)$ Kähler classes. We call a *matching* of (Z_+, S_+, k_+) and (Z_-, S_-, k_-) a diffeomorphism $\mathsf{r}\colon S_+ \to S_-$ such that $\mathsf{r}^*\mathsf{k}_- \in H^2(S_+)$ and $(\mathsf{r}^{-1})^*\mathsf{k}_+ \in H^2(S_-)$ have type $(2,0)+(0,2)$.

We also say that $\mathsf{r} : S_+ \to S_-$ is a matching of Z_+ and Z_- if there are Kähler classes $\mathsf{k}_\pm$ so that the above holds.

Let us briefly summarise the construction in Theorem 1.1. For any building block (Z, S), the noncompact 3–fold $V := Z \setminus S$ admits ACyl Ricci-flat Kähler metrics [8, Theorem D], hence an ACyl Calabi-Yau structure. This Calabi-Yau structure can be specified by choosing a Kähler class $\mathsf{k} \in H^{1,1}(Z)$ and a meromorphic $(3,0)$-form with a simple pole along S. The asymptotic limit of the Calabi-Yau structure defines a hyper-Kähler structure on S.

Given a pair of such Calabi-Yau manifolds $V_\pm$ and a so-called *hyper-Kähler rotation* $\mathsf{r} : S_+ \to S_-$ (see [3, Definition 3.9]), one can apply [11, Theorem 5.34] to glue $S^1 \times V_\pm$ into a closed manifold X with a 1-parameter family of torsion-free G_2-structures (see [3, Theorem 3.12]). Given a matching r between a pair of building blocks $(Z_\pm, S_\pm, \mathsf{k}_\pm)$, one can make the choices in the definition of the ACyl Calabi-Yau structure so that r becomes a hyper-Kähler rotation of the induced hyper-Kähler structures (*cf.* [3, Theorem 3.4 and Proposition 6.2]). Combining these steps proves Theorem 1.1.

For all but 2 of the 105 families of Fano 3-folds, the base locus of a generic anti-canonical pencil is smooth. This also holds for most families in the wider class of 'semi-Fano 3-folds' in the terminology of [2], *i.e.* smooth projective 3-folds where $-K_Y$ defines a morphism that does not contract any divisors. We can then obtain building blocks using [3, Proposition 3.15]:

Proposition 2.3. *Let Y be a semi-Fano 3-fold with $H^3(Y,\mathbb{Z})$ torsion-free, $|S_0, S_\infty| \subset |-K_Y|$ a generic pencil with (smooth) base locus $\mathscr{C}$, $S \in |S_0, S_\infty|$ generic, and Z the blow-up of Y at $\mathscr{C}$. Then S is a smooth K3 surface, its proper transform in Z is isomorphic to S, and (Z, S) is a building block. Furthermore*

(i) *the image N of $H^2(Z,\mathbb{Z}) \to H^2(S,\mathbb{Z})$ equals that of $H^2(Y,\mathbb{Z}) \to H^2(S,\mathbb{Z})$;*

(ii) *$H^2(Y,\mathbb{Z}) \to H^2(S,\mathbb{Z})$ is injective and the image N is primitive in $H^2(S,\mathbb{Z})$.*

Remark 2.4. Alternatively we could say that $S \in |-K_Y|$ and $\mathscr{C} \in |-K_{Y|S}|$ are generic and smooth. For $H^0(Y, -K_Y) \to H^0(S, -K_{Y|S})$ is surjective, so there really is an $S_\infty \in |-K_Y|$ that intersects S in $\mathscr{C}$, and then $|S_\infty : S|$ is a pencil with base locus $\mathscr{C}$.

Note that if $Y_\pm$ is a pair of semi-Fanos and $\mathfrak{r} : S_+ \to S_-$ is a matching in the sense of Definition 2.2, then $\mathfrak{r}$ also defines a matching of building blocks constructed from $Y_\pm$ using Proposition 2.3. Thus given a pair of matching semi-Fanos we can apply Theorem 1.1 to construct closed G_2-manifolds, *but* this still involves choosing the blow-up curves $\mathscr{C}_\pm$. For later use we make an observation concerning these blow-up curves, which will play an especially important role in our transversality argument in Section 4.2.

Lemma 2.5. *Let Y be a semi-Fano, $S \in |-K_Y|$ a smooth K3 divisor, and suppose that the restriction of $-K_Y$ to S is very ample. Then given any point $x \in S$ and any (complex) line $V \subset T_xS$, there exists an anticanonical pencil containing S whose base locus $\mathscr{C}$ is smooth, contains x, and $T_x\mathscr{C} = V$.*

Proof. The sections of $-K_{Y|S}$ define an embedding $S \hookrightarrow \mathbb{P}^g$, for some $g \geq 3$. The image of V defines a line in $\mathbb{P}^g$, intersecting S in a finite number of points (generically just in x if $g > 3$). Consider the sections of S by hyperplanes $H \subset \mathbb{P}^g$ that contain V. These form a $(g-2)$–dimensional family, with base locus $S \cap V$. By Bertini's theorem, a generic section $H \cap S$ in this family is smooth away from $S \cap V$. On the other hand, for each point $y \in S \cap V$, certainly a generic section is smooth at y—indeed, $H \cap S$ is smooth at y as long as T_yH does not contain T_yS. Hence there is a smooth section $\mathscr{C} := H \cap S$ with $T_x\mathscr{C} = V$. □

2.2. The matching problem

We now explain in more detail the argument of [3, §6] for finding matching building blocks $(Z_\pm, S_\pm)$. The blocks will be obtained by applying Proposition 2.3 to a pair of semi-Fanos $Y_\pm$, from some given pair of deformation types $\mathcal{Y}_\pm$.

A key deformation invariant of a semi-Fano Y is its Picard lattice $\mathrm{Pic}(Y) \cong H^2(Y; \mathbb{Z})$. For any anticanonical K3 divisor $S \subset Y$, the injection $\mathrm{Pic}(Y) \hookrightarrow H^2(S; \mathbb{Z})$ is primitive. The intersection form on $H^2(S; \mathbb{Z})$ of any K3 surface is isometric to $L_{K3} := 3U \oplus 2E_8$, the unique even unimodular lattice of signature $(3, 19)$. We can therefore identify $\mathrm{Pic}(Y)$ with a primitive sublattice $N \subset L_{K3}$ of the K3 lattice, uniquely up to the action of the isometry group $O(L_{K3})$ (this is usually uniquely determined by the isometry class of N as an abstract lattice).

Given a matching $\mathtt{r}\colon S_+ \to S_-$ between anticanonical divisors in a pair of semi-Fanos, we can choose the isomorphisms $H^2(S_\pm;\mathbb{Z}) \cong L_{K3}$ compatible with $\mathtt{r}^*$, hence identify $\mathrm{Pic}(Y_+)$ and $\mathrm{Pic}(Y_-)$ with a *pair* of primitive sublattices $N_+, N_- \subset L_{K3}$. While the $O(L_{K3})$ class of $N_\pm$ individually depends only on $Y_\pm$, the $O(L_{K3})$ class of the pair (N_+, N_-) depends on $\mathtt{r}$, and we call (N_+, N_-) the *configuration* of $\mathtt{r}$.

Many important properties of the resulting twisted connected sum only depend on the hyper-Kähler rotation in terms of the configuration. Given a pair $\mathcal{Y}_\pm$ of deformation types of semi-Fanos it is therefore interesting to know which configurations of their Picard lattices are realised by some hyper-Kähler rotation. Let us use the following terminology. Given a primitive sublattice $N \subset L_{K3}$ and $A \in N$ such that $A^2 > 0$, recall that an *N-polarised* K3 surface is a K3 surface S together with a marking $h\colon H^2(S;\mathbb{Z}) \cong L_{K3}$ such that $N \subseteq h(\mathrm{Pic}(S))$, and A corresponds to an ample class on S. For an open subcone $\mathrm{Amp}_{\mathcal{Y}} \subset N \otimes \mathbb{R}$, let us call a set $\mathcal{Y}$ of semi-Fanos $(N, \mathrm{Amp}_{\mathcal{Y}})$-*generic* if a generic N-polarised K3-surface (S,h) has an embedding $i\colon S \hookrightarrow Y$ as an anticanonical divisor in some $Y \in \mathcal{Y}$, such that $h \circ i^* : \mathrm{Pic}(Y) \to L_{K3}$ is an isomorphism onto N and the image of the ample cone of Y contains $\mathrm{Amp}_{\mathcal{Y}}$.

Given a configuration $N_+, N_- \subset L_{K3}$, let

$$N_0 := N_+ \cap N_-, \quad \text{and} \quad R_\pm := N_\pm \cap N_\mp^\perp.$$

We say that the configuration is *orthogonal* if $N_\pm$ are rationally spanned by N_0 and $R_\pm$ (geometrically, this means that the reflections in N_+ and N_- commute). Then there are sufficient conditions for a given orthogonal configuration to be realised by some matching [3, Proposition 6.17]:

Proposition 2.6. *Let $N_\pm \subset L_{K3}$ be a configuration of two primitive sublattices of signatures $(1, r_\pm - 1)$. Let $\mathcal{Y}_\pm$ be $(N_\pm, \mathrm{Amp}_{\mathcal{Y}_\pm})$-generic sets of semi-Fano 3–folds, and assume that*

- *the configuration is orthogonal, and*
- *$R_\pm \cap \mathrm{Amp}_{\mathcal{Y}_\mp} \neq \emptyset$.*

Then there exist $Y_\pm \in \mathcal{Y}_\pm$, $S_\pm \in |-K_{Y_\pm}|$, and a matching $\mathtt{r} : S_+ \to S_-$ with the given configuration. Moreover, the Kähler classes $\mathtt{k}_\pm$ on $Y_\pm$ in Definition 2.2 can be chosen so that $\mathtt{k}_{\pm|S_\pm}$ is arbitrarily close to any given element $R_\pm \cap \mathrm{Amp}_{\mathcal{Y}_\mp}$.

Fixing henceforth a primitive sublattice $N \subset L_{K3}$, every nonempty deformation type $\mathcal{Y}$ of semi-Fano 3-folds is $(N, \mathrm{Amp}_{\mathcal{Y}})$-generic for some $\mathrm{Amp}_{\mathcal{Y}}$ [2, Proposition 6.9] . For most pairs of deformation types $\mathcal{Y}_\pm$, one can apply results of Nikulin [13] to embed the perpendicular direct sum $N_+ \perp N_-$ primitively in L_{K3}. Thus one obtains a configuration satisfying the hypotheses of Proposition 2.6. This is used in [3] and [4] to produce many examples of twisted connected sum G_2-manifolds.

Now consider the problem of finding matching bundles $F_\pm \to Z_\pm$ in order to construct G$_2$-instantons by application of Theorem 1.2. For the compatibility hypothesis it is necessary that

$$c_1(F_+|_{S_+}) = \mathfrak{r}^* c_1(F_-|_{S_-}) \in H^2(S_+).$$

Identifying $H^2(S_+;\mathbb{Z}) \cong L_{K3} \cong H^2(S_-;\mathbb{Z})$ compatibly with $\mathfrak{r}^*$, this means we need

$$c_1(F_{+|S_+}) \;=\; c_1(F_{-|S_-}) \;\in\; N_+ \cap N_- \;=\; N_0.$$

Hence, if N_0 is trivial, both $c_1(F_{\pm|S_\pm})$ must also be trivial, which is a very restrictive condition on the first Chern classes of our bundles. To allow more possibilities, we want matchings $\mathfrak{r}$ whose configuration $N_+, N_- \subset L_{K3}$ has non-trivial intersection N_0.

Table 4 of [4] lists all 19 possible matchings of Fano 3-folds with Picard rank 2 such that N_0 is non-trivial. In this paper, the pair of building blocks we will consider comes from that table. The relevant Fano 3-folds are $Y_+ = \mathbb{P}^1 \times \mathbb{P}^2$ and the double cover $Y_- \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^2$ branched over a $(2,2)$ divisor. We explain the reasons why these Fano 3-folds were chosen in Remark 3.4. Moreover, we suggest other Fano 3-folds that are relevant to produce examples of G$_2$-instantons.

2.3. The Fano 3-folds $Y_+ = \mathbb{P}^1 \times \mathbb{P}^2$ and $Y_- \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^2$

Example 2.7. The product $Y_+ = \mathbb{P}^1 \times \mathbb{P}^2$ is a Fano 3-fold. Let $|S_0, S_\infty| \subset \left|-K_{Y_+}\right|$ be a generic pencil with (smooth) base locus $\mathscr{C}_+$ and $S_+ \in |S_0, S_\infty|$ generic. Denote by $r_+ : Z_+ \to Y_+$ the blow-up of Y_+ in $\mathscr{C}_+$, by $\widetilde{\mathscr{C}_+}$ the exceptional divisor and by ℓ_+ a fibre of $p_1 : \widetilde{\mathscr{C}_+} \to \mathscr{C}_+$. The proper transform of S_+ in Z_+ is also denoted by S_+, and (Z_+, S_+) is a building block by Proposition 2.3. We fix classes

$$H_+ := r_+^*\big(\big[\mathbb{P}^1 \times \mathbb{P}^1\big]\big) \quad \text{and} \quad G_+ = r_+^*\big(\big[\{x\} \times \mathbb{P}^2\big]\big) \in H^2(Z_+),$$

where x is a point, and also

$$h_+ := r_+^*\big([\{x\} \times \mathbb{P}^1]\big) \quad \text{and} \quad g_+ := r_+^*\big([\mathbb{P}^1 \times \{x\}]\big) \in H^4(Z_+).$$

The Picard group of S_+ has rank at least 2, containing

$$A_+ := G_{+|S_+} \quad \text{and} \quad B_+ := H_{+|S_+}.$$

Moreover, the sublattice N_+ spanned by A_+ and B_+ has intersection form represented by the matrix

$$M_+ := \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}.$$

NB.: Clearly $-K_{Y_+}$ is very ample, thus also $-K_{Y_+|S_+}$, so Y_+ lends itself to application of Lemma 2.5.

Example 2.8. A double cover $\pi : Y_- \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^2$ branched over a smooth $(2,2)$ divisor D is a Fano 3-fold. Let $|S_0, S_\infty| \subset \big|-K_{Y_-}\big|$ be a generic pencil with (smooth) base locus $\mathscr{C}_-$ and $S_- \in |S_0, S_\infty|$ generic. Denote by $r_- : Z_- \to Y_-$ the blow-up of Y_- in $\mathscr{C}_-$, and by $\widetilde{\mathscr{C}_-}$ the exceptional divisor. The proper transform of S_- in Z_- is also denoted by S_-, and (Z_-, S_-) is a building block by Proposition 2.3. We fix classes

$$H_- := (r_- \circ \pi)^*\big([\mathbb{P}^1 \times \mathbb{P}^1]\big) \quad \text{and}$$
$$G_- = (r_- \circ \pi)^*\big([\{x\} \times \mathbb{P}^2]\big) \in H^2(Z_-),$$

where x is a point, and also

$$h_- := \frac{1}{2}(r_- \circ \pi)^*\big([\{x\} \times \mathbb{P}^1]\big) \quad \text{and}$$
$$g_- := \frac{1}{2}(r_- \circ \pi)^*\big([\mathbb{P}^1 \times \{x\}]\big) \in H^4(Z_-).$$

For a generic $x \in \mathbb{P}^2$, the curve $\{x\} \times \mathbb{P}^1$ meets the branching divisor D transversely in two points, and the pre-image of $\{x\} \times \mathbb{P}^1$ in Y_- is an irreducible rational curve, whose Poincaré dual is mapped to $2h_-$ by r_-^*. Note, however, that there is a quartic curve $Q \subset \mathbb{P}^2$ (defined by the discriminant of the quadric polynomial corresponding to restriction of D to $\{x\} \times \mathbb{P}^1$) such that for generic $x \in Q$, the curve $\{x\} \times \mathbb{P}^1$ is tangent to D. For such x, the pre-image of $\{x\} \times \mathbb{P}^1$ in Y_- is a union of two lines. Such lines are parametrised by the pre-image $\widetilde{Q}$ of Q in S_-. The proper transform W in Z_- of such a line is Poincaré dual to h_-.

The Picard group of S_- has rank at least 2, containing

$$A_- := G_{-|S_-} \quad \text{and} \quad B_- := H_{-|S_-}.$$

The sublattice N_- generated by these vectors has intersection form represented by $M_- := \begin{pmatrix} 0 & 4 \\ 4 & 2 \end{pmatrix}$.

According to Table 5 of [4], we can find a matching between Y_+ and Y_- choosing the ample classes

$$\mathcal{A}_+ := A_+ + B_+ \quad \text{and} \quad \mathcal{A}_- := 2A_- + B_-.$$

Moreover $N_0 \subset N_+$ is generated by $5A_+ - 3B_+$ and $N_0 \subset N_-$ is generated by $5A_- - 2B_-$ (both have square -72).

Given a matching, we can take any smooth $\mathscr{C}_\pm \in |-K_{Y_\pm|S_\pm}|$ and apply Proposition 2.3 to construct building blocks $(Z_\pm, S_\pm)$, then apply Theorem 1.1 to obtain a twisted connected sum.

3. Twisted Hartshorne-Serre bundles over building blocks

The Hartshorne-Serre construction generalises the correspondence between divisors and line bundles, under certain conditions, in the sense that bundles of higher rank are associated to subschemes of higher codimension. We recall the rank 2 version, as an instance of Arrondo's formulation [1, Theorem 1]:

Theorem 3.1. *Let $W \subset Z$ be a local complete intersection subscheme of codimension 2 in a smooth algebraic variety. If there exists a line bundle $\mathcal{L} \to Z$ such that*

- $H^2(Z, \mathcal{L}^*) = 0$,
- $\wedge^2 \mathcal{N}_{W/Z} = \mathcal{L}|_W$,

then there exists a rank 2 vector bundle such that

(i) $\wedge^2 E = \mathcal{L}$,

(ii) *E has a global section whose vanishing locus is W.*

We will refer to such E as the *Hartshorne-Serre bundle obtained from W (and $\mathcal{L}$).*

3.1. A technique to construct matching bundles

Let Y be a semi-Fano 3–fold and (Z, S) be the block constructed as a blow-up of Y along the base locus $\mathscr{C}$ of a generic anti-canonical pencil [Proposition 2.3]. We now describe a general approach for making the choices of $\mathcal{L}$ and W in Theorem 3.1, in order to construct a Hartshorne-Serre bundle $E \to Z$ which, up to a twist, yields the bundle F meeting the requirements of Theorem 1.2.

(i) As explained in Section 2.2, for compatibility we need $c_1(F_{|S}) \in N_0$. However, this is too restrictive for producing bundles with suitable asymptotics under the Hartshorne-Serre construction. Instead, we obtain a rank 2 vector bundle E and a line bundle $\mathcal{R}$ such that:

$$c_1(E \otimes \mathcal{R}_{|S}) \in N_0 \tag{3}$$

and set $F := E \otimes \mathcal{R}$. The properties of asymptotic stability and inelasticity will be equivalent for E and for F, hence we can work directly with E. Moreover, since

$$c_1(E \otimes \mathcal{R}) = c_1(E) + 2c_1(\mathcal{R}),$$

the existence of a line bundle $\mathcal{R}$ such that (3) holds is equivalent to

$$c_1(E_{|S}) \in N_0 \bmod 2\,\mathrm{Pic}(S).$$

(ii) By the Hoppe stability criterion [Proposition 4.3], if our E is asymptotically stable with respect to a polarisation $\mathcal{A} \perp N_0$ in the Kähler cone $\mathcal{K}_Z$, then necessarily $\mu_{\mathcal{A}}(E_{|S}) > 0$, so one must also arrange

$$c_1(E_{|S}) \cdot \mathcal{A} > 0.$$

(iii) If we choose a genus 0 curve W by identifying the first Chern classes, the condition $\wedge^2 \mathcal{N}_{W/Z} = \mathcal{L}|_W$ of Theorem 3.1 is equivalent to:

$$(S - c_1(\mathcal{L})) \cdot W = 2. \tag{4}$$

On the block (Z_+, S_+), we choose a fibre $W_+ := \ell_+$ of the map $p_1 : \widetilde{\mathscr{C}}_+ \to \mathscr{C}_+$, where $\widetilde{\mathscr{C}}_+$ is the exceptional divisor of $Z_+ \to Y_+$, to obtain in fact a *family* of bundles $\{E_+ \to Z_+\}$ parametrised by $\mathscr{C}_+$. The large freedom to move the curve $\mathscr{C}_+$ without changing S_+, as stated in Lemma 2.5, will be essential to the proof of Theorem 1.4. In this case, (4) becomes

$$c_1(\mathcal{L}_+) \cdot \ell_+ = -1.$$

A little more generally, one could chose W_+ as the disjoint union of k fibres $\ell_1, ..., \ell_k$ of p_1, provided $\wedge^2 \mathcal{N}_{\ell_i/Z_+} = \mathcal{L}|_{\ell_i}$ for each fibre. In any case, condition (4) remains as above.

(iv) We denote by $\mathcal{M}^s_{S,\mathcal{A}}(v)$ the moduli space of $\mathcal{A}$-μ-stable bundles on S with Mukai vector $v := v(E_{|S})$ [*cf.* Section 4.2]. According to Theorem 3.1, we have $c_2(E) = [W]$, hence

$$\begin{aligned} \dim \mathcal{M}^s_{S,\mathcal{A}}(v) &= 4 \cdot c_2(E_{|S}) - c_1(E_{|S})^2 - 6 \\ &= 4 \cdot S \cdot W - c_1(E_{|S})^2 - 6. \end{aligned} \tag{5}$$

In general, we may achieve $\dim \mathcal{M}^s_{S,\mathcal{A}}(v) = 2k$, for $k \in \mathbb{N}^*$, by adopting $W_+ = \sqcup \ell_i$ the disjoint union of exceptional fibres as above in (iii), so (5) reads

$$c_1(E_{+|S_+})^2 = 2k - 6.$$

However, to prove transversality, it will be simplest to impose $\dim \mathcal{M}^s_{S,\mathcal{A}}(v) = 2$, so (5) becomes $2 = S \cdot W - \frac{1}{4} c_1(E_{|S})^2$, and, choosing $W_+ = \ell_+$,

$$c_1(E_{+|S_+})^2 = -4.$$

(v) Our most restrictive constraint is the vanishing of the Hartshorne-Serre obstruction [*cf.* Theorem 3.1]:

$$H^2(Z, \mathcal{L}^*_+) = 0, \tag{6}$$

Moreover, condition (iii) imposes $\mathcal{L}_+ = -S_+ + r^*_+(D)$, with $D \in \mathrm{Pic}(Y_+)$, fitting in the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{Z_+}(-r^*_+(D)) \longrightarrow \mathcal{L}^*_+ \longrightarrow \mathcal{O}_{Z_+}(-r^*_+(D))_{|S_+} \longrightarrow 0. \tag{7}$$

Applying Serre duality to the associated long exact sequence, we have:

$$H^2(Z, \mathcal{L}^*_+) \longrightarrow H^0(S_+, \mathcal{O}_{S_+}(D_{|S_+})) \longrightarrow H^0(Z, \mathcal{O}_{Z_+}(r^*_+(D) - S_+)).$$

Theoretically, one could set out to prove that

$$H^0(S_+, \mathcal{O}_{S_+}(D_{|S_+})) \hookrightarrow H^0(Z, \mathcal{O}_{Z_+}(r^*_+(D) - S_+)),$$

but this is very unlikely, because in non-trivial instances $r_+^*(D) - S_+$ is seldom effective. So in practice we show that

$$H^0(S_+, \mathcal{O}_{S_+}(D_{|S_+})) = 0. \tag{8}$$

Hence in general our technique leads to $D_{|S_+}$ which is not effective. One the other hand, by (ii), $\mu_{\mathcal{A}_+}(D_{|S_+}) > 0$. If we choose Y_+ a rank 2 semi-Fano and assume (i), in practice, it has led to $D = H_+ - G_+$, for some H_+ and G_+ generating $\operatorname{Pic} S_+$ (see *e.g.* Section 2.3) such that $\mu_{\mathcal{A}_+}(H_{+|S_+}) > \mu_{\mathcal{A}_+}(G_{+|S_+})$.

Suppose that indeed $H^2(Z, \mathcal{L}^*) = 0$ and $D = H_+ - G_+$ as above. Our technique to obtain transversality imposes in addition $H^1(Z_+, \mathcal{L}_+^*) = 0$ (see Section 4.2), in order to identify the bundles $E_{+|S_+}$ with points of S_+. Then the exact sequence (7) yields:

$$H^1(S_+, \mathcal{O}_{S_+}(D_{|S_+})) = H^2(Z_+, \mathcal{O}_{Z_+}(-r_+^*(D))).$$

This condition is not so easy to check. However, when H_+ and G_+ are the classes of simply connected divisors, we can prove that the right-hand side vanishes using Lemma 3.6 below; together with (8) this gives $\chi(D_{|S_+}) = 0$, which in turn implies $\chi(\mathcal{L}_+) = 0$. Furthermore, by Riemann–Roch, $D_{|S_+}^2 = -4$. This imposes $k = 1$ in (iv).

In conclusion, the conditions $H^2(Z, \mathcal{L}_+^*) = H^1(Z, \mathcal{L}_+^*) = 0$ are quite restrictive and in trying to achieve them we may impose, by excess,

$$D_{|S_+}^2 = -4 \quad \text{and} \quad \chi(\mathcal{L}_+) = 0.$$

(vi) As to inelasticity, in the case $\dim \mathcal{M}_{S,\mathcal{A}}^s(v) = 2$, Corollary 5.8 gives us the necessary and sufficient condition on the dimension of the family of curves of class W:

$$\dim H^0(W, \mathcal{N}_{W/Z}) = \dim H^0(Z, E) + H^1(Z, \mathcal{L}^*), \tag{9}$$

which further constrains the coupled choice of $W_\pm$ and $\mathcal{L}_\pm$. Following (v), we must actually check that

$$\dim H^0(W, \mathcal{N}_{W/Z}) = \dim H^0(Z, E) = 1 + H^0(Z, \mathcal{L} \otimes \mathcal{I}_W),$$

which can be calculated easily.

Summary 3.2. Let $(Z_\pm, S_\pm)$ be the building blocks constructed by blowing-up $N_\pm$-polarised semi-Fano 3-folds $Y_\pm$ along the base locus $\mathscr{C}_\pm$ of a generic anti-canonical pencil [*cf.* Proposition 2.3]. Let $N_0 \subset N_\pm$ be the sublattice of orthogonal matching, as in Section 2.2. Let $\mathcal{A}_\pm$ be the restriction of an ample class of $Y_\pm$ to $S_\pm$ which is orthogonal to N_0. We look for the Hartshorne-Serre parameters $W_\pm$ and $\mathcal{L}_\pm$ of Theorem 3.1, where $W_+ = \ell_+$ is an exceptional fibre in Z_+, W_- is a genus 0 curve in Z_- and $\mathcal{L}_\pm \to Z_\pm$ are line bundles such that:

(i) $c_1(\mathcal{L}_{\pm|S_\pm}) \in N_0 \mod 2\,\mathrm{Pic}(S_\pm)$;

(ii) $c_1(\mathcal{L}_{\pm|S_\pm}) \cdot \mathcal{A}_\pm > 0$;

(iii) $c_1(\mathcal{L}_+) \cdot \ell_+ = -1$ and $(S_- - c_1(\mathcal{L}_-)) \cdot W_- = 2$;

(iv) $c_1(\mathcal{L}_{+|S_+})^2 = -4$ and $S_- \cdot W_- - \frac{1}{4} c_1(\mathcal{L}_{-|S_-})^2 = 2$;

(v) $\chi(\mathcal{L}_+^*) = 0$;

(vi) $\dim H^0(W, \mathcal{N}_{W/Z}) = 1 + H^0(Z, \mathcal{L} \otimes \mathcal{I}_W)$.

Remark 3.3. Suppose $F_\pm$ satisfy the hypotheses of Theorem 1.2. Then the restrictions $F_\pm|_{S_\pm}$ have degree $c_1(F_\pm|_{S_\pm}) \cdot \mathcal{A}_\pm = 0$, because $c_1(F_\pm|_{S_\pm}) \in N_0$ and $\mathcal{A}_\pm \perp N_0$. Moreover, $F_\pm|_{S_\pm}$ are μ-$\mathcal{A}_\pm$-stable, hence also their duals, so $H^0(F_\pm|_{S_\pm}) = H^0(F_\pm^*|_{S_\pm}) = 0$. By Serre duality, this ensures that $H^2(F_\pm|_{S_\pm}) = 0$, thus

$$\chi(F_\pm|_{S_\pm}) \le 0.$$

Furthermore, in order to get 2-dimensional moduli spaces

$$\mathcal{M}_\pm := \mathcal{M}^s_{S_\pm, \mathcal{A}_\pm}(v(F_\pm|_{S_\pm}))$$

from the formula (see Theorem 4.1)

$$\dim \mathcal{M}_\pm = 10 - 4\chi(F_\pm|_{S_\pm}) + c_1(F_\pm|_{S_\pm})^2,$$

we need:

$$c_1(F_\pm|_{S_\pm})^2 \le -8 \text{ and } c_1(F_\pm|_{S_\pm})^2 \equiv 0 \mod 4.$$

Twisting $F_\pm$ by a line bundle, we can always assume that $c_1(F_\pm|_{S_\pm})$ is primitive in N_0. Therefore, if the lattice N_0 has rank 1, it must be generated by an element of square at most -8 and divisible by 4.

Remark 3.4. From condition (v) of Summary 3.2, we see that it is convenient to have an element in the lattice N_+ of square -4. Together with the conditions of Remark 3.3, this is why we consider $\mathbb{P}^1 \times \mathbb{P}^2$: its Picard lattice contains elements of square -4, and it matches its double cover branched over a $(2,2)$ divisor with $N_0 \simeq (-72)$. Looking at Table 2 of [4], another possibility would be the pair of matching Fano 3-folds numbered 25 and 14, given by the blow-up of $\mathbb{P}^3$ on an elliptic curve that is the intersection of two quadrics and the blow-up of V_5 (section of the Plücker-embedded Grassmannian $Gr(2,5) \subset \mathbb{P}^9$ by a subspace of codimension 3) on an elliptic curve that is the intersection of two hyperplane sections. However, we did not manage to find examples with these blocks because of the restrictive condition (v). On the other hand, we found 6 other possible matchings with the suitable conditions, considering semi-Fano 3-folds of rank 2.

In conclusion, while our approach does produce an original solution to the transversal gluing problem, it does not lend itself to the immediate mass-production of examples. In order to achieve that, the main constraint (v) should be suitably relaxed, possibly by a finer Hartshorne–Serre theorem or by an improved technique to get transversality. The reader who might wish to join in the effort can follow this recipe:

Step 1: Find two matching $N_\pm$-polarized semi-Fano 3-folds $Y_\pm$ such that:
- (i) there exists $x \in N_+$ such $x^2 = -4$
- (ii) there exists a primitive element $y \in N_0$ such that $y^2 \leq -8$ and 4 divides y^2.

Step 2: Find $\mathcal{L}_\pm$ and W_- which verify the conditions of Summary 3.2 (perhaps with a computer).

Step 3: The following must be checked by *ad-hoc* methods:
- (i) $H^2(\mathcal{L}_\pm^*) = 0$, for the Hartshorne-Serre construction [Theorem 3.1];
- (ii) $H^1(\mathcal{L}_+^*) = 0$ for our transversality method;
- (iii) that divisors with small slope do not contain W, for asymptotic stability [Proposition 4.3 (ii)];
- (iv) $H^1(E) = 0$ for inelasticity [Corollary 5.8].

Step 4: Conclude with similar arguments as in the proofs of Theorems 1.4 and 6.3.

3.2. Construction of E_+ over $Y_+ = \mathbb{P}^1 \times \mathbb{P}^2$

In this section all the objects considered are related to the building block (Z_+, S_+) obtained by blowing up $Y_+ = \mathbb{P}^1 \times \mathbb{P}^2$ from Example 2.7. We omit for simplicity the $+$ subscript.

In view of the constraints in Summary 3.2, we apply Theorem 3.1 to $Z = \mathrm{Bl}_{\mathscr{C}} Y$ as above, with parameters

$$W = \ell \quad \text{and} \quad \mathcal{L} = \mathcal{O}_Z(-S - G + H).$$

Proposition 3.5. *Let (Z, S) be a building block as in Example 2.7, $\mathscr{C}$ a pencil base locus and $\ell \subset Z$ an exceptional fibre of $\widetilde{\mathscr{C}} \to \mathscr{C}$. There exists a rank 2 Hartshorne-Serre bundle $E \to Z$ obtained from ℓ such that:*

(i) $c_1(E) = -S - G + H$,

(ii) *E has a global section with vanishing locus ℓ.*

We start the proof of Proposition 3.5 with a basic lemma that will be invoked several times later on.

Lemma 3.6. *Let X be a complex manifold and D be an effective prime divisor.*

(i) *If X is simply-connected, then $H^1(X, \mathcal{O}_X(-D)) = 0$.*

(ii) *If D is simply-connected and X has no global holomorphic 2-form, then $H^2(X, \mathcal{O}_X(-D)) = 0$.*

(iii) *If X is a K3 surface, then $D^2 \geq -2$.*

Proof. Items (i) and (ii) follow immediately from the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(-D) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

Item (iii) is straightforward from Riemann-Roch:

$$\frac{1}{2}D^2 + 2 = \chi(\mathcal{O}_X(D)) = h^0 - h^1 + h^2,$$

where $h^0 \geq 1$ because D is effective, $h^1 = 0$ by (i) and $h^2 = 0$ by Serre duality. □

To conclude the proof of Proposition 3.5, we apply Theorem 3.1 using the following:

Lemma 3.7. *In the hypotheses of Proposition 3.5,*

(i) $H^i(\mathcal{O}_Z(S + G - H)) = 0$, *for $i = 1, 2$.*

(ii) $\mathcal{L}_{|\ell} = \wedge^2 \mathcal{N}_{\ell/Z} = \mathcal{O}_\ell(-1)$.

Proof.

(i) In view of the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(G-H) \longrightarrow \mathcal{O}_Z(S+G-H) \longrightarrow \mathcal{O}_S(A-B) \longrightarrow 0,$$

and Serre duality, it suffices to check that $H^i(Z, \mathcal{O}_Z(G-H)) = 0$ and $H^i(S, \mathcal{O}_S(B-A)) = 0$, for $i \in \{1,2\}$. The latter is trivial, because neither $B-A$ nor $A-B$ are effective divisors. Moreover by Riemann–Roch, we also have $H^1(S, \mathcal{O}_S(B-A)) = 0$. As to the former, the divisor G is the class of a blow-up of $\mathbb{P}^2$ on 4 points. Denoting by h the pull back in G of the class of a line in $\mathbb{P}^2$, as before, we have:

$$0 \longrightarrow \mathcal{O}_Z(-H) \longrightarrow \mathcal{O}_Z(G-H) \longrightarrow \mathcal{O}_G(-h) \longrightarrow 0,$$

and, by Lemma 3.6, $H^i(Z, \mathcal{O}_Z(-H)) = 0$ and $H^i(G, \mathcal{O}_G(-h)) = 0$ for $i \in \{1,2\}$.

(ii) Clearly $c_1(\mathcal{O}_Z(-S-G+H)_{|\ell}) = (-S-G+H)\cdot \ell = -1$. Now, since ℓ is a line, we have $c_1(\mathcal{T}_\ell) = 2$; moreover, line bundles on ℓ are classified by their first Chern class, so it suffices to check that $c_1(\wedge^2 \mathcal{N}_{\ell/Z}) = -1$. Indeed, using $S \cdot \ell = 1$, this follows by adjunction:

$$c_1(\wedge^2 \mathcal{N}_{\ell/Z}) = c_1(\mathcal{N}_{\ell/Z}) = c_1((\mathcal{T}_Z)_{|\ell}) - c_1(\mathcal{T}_\ell) = S\cdot\ell - 2 = -1. \qquad \square$$

We now compute some topological facts about the Hartshorne-Serre bundle E we just constructed in Proposition 3.5. These will be essential for the inelasticity results in Section 5 but not elsewhere, so one may wish to skim through the proof on a first read.

Recall that, by Theorem 3.1, there is a global section $s \in H^0(E)$ such that $s^{-1}(0) = \ell$, where ℓ is a fibre of the map $p_1 : \widetilde{\mathscr{C}} \to \mathscr{C}$. Hence, we have the following exact sequence:

$$(10) \qquad 0 \longrightarrow \mathcal{O}_Z \xrightarrow{\ s\ } E \longrightarrow \mathcal{I}_\ell \otimes \mathcal{O}_Z(-S-G+H) \longrightarrow 0,$$

where $\mathcal{I}_\ell$ is the ideal sheaf of ℓ in Z.

Lemma 3.8. *We have $H^0(E) = \mathbb{C}$ and $H^1(E) = 0$.*

Proof. That $H^0(E) = \mathbb{C}$ follows directly from (10), since $-S-G+H$ is not an effective divisor and so $H^0(\mathcal{O}_Z(-S-G+H)) = 0$.

Similarly, since building blocks are simply-connected, the vanishing of $H^1(E)$ reduces to that of $H^1(\mathcal{I}_\ell \otimes \mathcal{O}_Z(-S-G+H))$. Twisting by $\mathcal{O}_Z(-S-G+H)$ the structural exact sequence of ℓ in Z, we have

$$0 \longrightarrow \mathcal{I}_\ell \otimes \mathcal{O}_Z(-S-G+H) \longrightarrow \mathcal{O}_Z(-S-G+H) \longrightarrow \mathcal{O}_\ell(-1) \longrightarrow 0,$$

so we only have to establish that $H^1(\mathcal{O}_Z(-S-G+H)) = 0$. In the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(-S-G+H) \longrightarrow \mathcal{O}_Z(-G+H) \longrightarrow \mathcal{O}_S(-A+B) \longrightarrow 0$$

the divisor $-A+B$ is not effective, so $H^0(\mathcal{O}_S(-A+B)) = 0$. On the other hand, the divisor H is the class of a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ on 12 points. Denoting by h and g the classes of the pull-back to H of the lines in $\mathbb{P}^1 \times \mathbb{P}^1$, respectively, the group $H^1(\mathcal{O}_Z(-G+H))$ must vanish by Lemma 3.6 and the following exact sequences:

$$0 \longrightarrow \mathcal{O}_Z(-G) \longrightarrow \mathcal{O}_Z(-G+H) \longrightarrow \mathcal{O}_H(g-h) \longrightarrow 0;$$

$$0 \longrightarrow \mathcal{O}_H(-h) \longrightarrow \mathcal{O}_H(g-h) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0. \qquad \square$$

3.3. Construction of E_- over $Y_- \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^2$

Similarly, in this section all the objects considered are related to building block Z_- obtained by blowing up $Y_- \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^2$ from Example 2.8. We also omit the $-$ subscript.

We apply Theorem 3.1 to Z as above, with

$$[W] = h \quad \text{and} \quad \mathcal{L} = \mathcal{O}_Z(G).$$

See Example 2.8 for the notation. (As described there, the possible choices of the line W are parametrised by an open subset of a surface $\widetilde{Q} \subset S$.)

Proposition 3.9. *Let (Z, S) be a building block provided in Example 2.8 and W a line of class h. There exists a rank 2 Hartshorne-Serre bundle $E \to Z$ obtained from W such that:*

(i) $c_1(E) = G$,

(ii) *E has a global section with vanishing locus W.*

As before, Proposition 3.9 is a direct application of Theorem 3.1, using:

Lemma 3.10. *In the hypotheses of Proposition 3.9,*

(i) $H^1(\mathcal{O}_Z(-G)) = H^2(\mathcal{O}_Z(-G)) = 0$.

(ii) $\mathcal{L}_{|W} = \wedge^2\mathcal{N}_{W/Z} = \mathcal{O}_W$.

Proof.

(i) This is immediate from Lemma 3.6.

(ii) We proceed as in the proof of Lemma 3.7 (ii). We have $c_1(\mathcal{O}_Z(G)_{|W}) = 0$, because $G \cdot W = 0$. On the other hand, since $c_1(\mathcal{T}_Z) = 2H + G - \widetilde{\mathscr{C}}$ we have

$$c_1((\mathcal{T}_Z)_{|W}) = (2H + G - \widetilde{\mathscr{C}}) \cdot W = 2.$$

Since W is a line, $c_1(T_W) = 2$. It follows that

$$c_1(\wedge^2\mathcal{N}_{W/Z}) = c_1((\mathcal{T}_Z)_{|W}) - c_1(\mathcal{T}_W) = 0. \qquad \square$$

Again, the following topological facts about the Hartshorne-Serre bundle E_- from Proposition 3.9 will be used in Section 5.

By Theorem 3.1, there is a global section $s \in H^0(E)$ such that $(s)_0 = W$ is a line of class h. Hence, we have the following exact sequence:

$$0 \longrightarrow \mathcal{O}_Z \xrightarrow{s} E \longrightarrow \mathcal{I}_W \otimes \mathcal{O}_Z(G) \longrightarrow 0, \tag{11}$$

where $\mathcal{I}_W$ is the ideal sheaf of W in Z.

Lemma 3.11. *We have $H^0(E) = \mathbb{C}^2$ and $H^1(E) = 0$.*

Proof. We follow the same approach as in the proof of Lemma 3.8. That $H^0(E) = \mathbb{C}^2$ reduces, by (11), to the fact that $H^0(\mathcal{I}_W \otimes \mathcal{O}_Z(G)) = \mathbb{C}$, since $H^0(\mathcal{O}_Z) = \mathbb{C}$ and $H^1(\mathcal{O}_Z) = 0$. Indeed, there is only one global section of $\mathcal{O}_Z(G)$ that vanishes on the line W.

Similarly, for the vanishing of $H^1(E)$, it suffices to check that $H^1(\mathcal{I}_W \otimes \mathcal{O}_Z(G)) = 0$. Twisting by $\mathcal{O}_Z(G)$ the structural exact sequence of W in Z, we have

$$0 \longrightarrow \mathcal{I}_W \otimes \mathcal{O}_Z(G) \longrightarrow \mathcal{O}_Z(G) \longrightarrow \mathcal{O}_W \longrightarrow 0.$$

Since $H^0(\mathcal{I}_W \otimes \mathcal{O}_Z(G)) = \mathbb{C}$, $H^0(\mathcal{O}_Z(G)) = \mathbb{C}^2$ and $H^0(\mathcal{O}_W) = \mathbb{C}$, the second map $H^0(\mathcal{O}_Z(G)) \to H^0(\mathcal{O}_W)$ is necessarily surjective. So, we only have to prove that $H^1(\mathcal{O}_Z(G)) = 0$, which is clear from the exact sequence

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_Z(G) \longrightarrow \mathcal{O}_G \longrightarrow 0. \qquad \square$$

4. The moduli space of stable bundles on S

In Section 4.1, we deduce the asymptotic stability of $E_\pm$ (Propositions 4.4 and 4.5), as well as the dimension of the corresponding moduli space at infinity (Proposition 4.6). In Section 4.2, we establish the freedom to choose the base locus curve $\mathscr{C}_+$ in order to match any given asymptotic incidence condition (Theorem 1.4).

We begin by recalling some known facts on moduli spaces of semi-stable sheaves on a K3 surface S (see [9]). A *Mukai vector* is a triple

$$v = (r, l, s) \in \left(H^0 \oplus H^2 \oplus H^4\right)(S, \mathbb{Z}).$$

We define a pairing between Mukai vectors (r, l, s) and (r', l', s') as follows:

$$(r, l, s) \cdot (r', l', s') := l \cdot l' - rs' - r's.$$

The Mukai vector of a vector bundle $E \to S$ is defined as

$$v(E) := (\operatorname{rk} E,\, c_1(E),\, \chi(E) - \operatorname{rk} E)\,,$$

with $\chi(E) = \frac{c_1(E)^2}{2} + 2\operatorname{rk} E - c_2(E)$.

The local structure of the moduli space of stable bundles over a K3 surface S can be computed in several ways, which trace back to the work of Maruyama (see [12, Proposition 6.9]):

Theorem 4.1 (Maruyama). *Let $L \to S$ be a polarised K3 surface and denote by $\mathcal{M}^s_{S,L}(v)$ the moduli space of isomorphism classes of L-slope-stable vector bundles on S with Mukai vector v. If $\mathcal{M}^s_{S,L}(v)$ is not empty, then it is a quasi-projective complex manifold of dimension $v^2 + 2$ and its Zariski tangent space at a point E admits the following isomorphisms:*

$$T_E\mathcal{M}^s_{S,L}(v) = \operatorname{Ext}^1(E, E) = H^1(\mathscr{E}nd(E)).$$

Furthermore,

$$\begin{aligned}\dim \mathcal{M}^s_{S,L}(v) &= -\chi(\mathscr{E}nd_0(E)) = 2(\operatorname{rk} E)^2 - 2\chi(E)\operatorname{rk} E + c_1(E)^2 \\ &= (1 - \operatorname{rk}(E))c_1(E)^2 + 2(\operatorname{rk} E)c_2(E) - 2(\operatorname{rk} E)^2 + 2.\end{aligned}$$

NB.: We recall that a polarisation over a nonsingular projective variety Y is determined by an ample line bundle $L \to Y$. The $L-$*degree* and the L–*slope* of a coherent sheaf $E \to Y$ are, respectively,

$$\deg_L E := c_1(E) \cdot L^{\dim Y - 1} \quad \text{and} \quad \mu_L(E) := \frac{\deg_L E}{\operatorname{rk}(E)}. \tag{12}$$

Then E is *(semi-)slope-stable* if, for every proper coherent subsheaf $F \subset E$ such that E/F is torsion-free, one has

$$\mu_L(F) \underset{(\leq)}{<} \mu_L(E).$$

If E is locally free, it suffices to check stability for *reflexive* subsheaves.

4.1. Asymptotic stability of the Hartshorne-Serre bundles $E_\pm$

We need suitable stability criteria for bundles over $S_\pm$. Following [10], a variety Y is called *polycyclic* if its Picard group is free Abelian. Given a polarisation $L \to Y$, the L-*degree* of a divisor $D \subset \operatorname{Pic}(Y)$ is [cf. (12)]

$$\delta_L(D) := \deg_L \mathcal{O}_Y(D).$$

Corollary 4.2 ([10, Corollary 4]). *Let $\mathcal{G} \to Y$ be a holomorphic vector bundle of rank 2 over a polycyclic variety with* $\operatorname{Pic}(Y) \simeq \mathbb{Z}^{l+1}$ *and polarisation L.*

The bundle $\mathcal{G}$ is (semi)-stable if and only if

$$H^0(\mathcal{G} \otimes \mathcal{O}_Y(D)) = 0$$

for all $D \in \operatorname{Pic}(Y)$ such that

$$\delta_L(D) \underset{(<)}{\leq} -\mu_L(\mathcal{G}).$$

Proposition 4.3 ([10, Proposition 10]). *Let Y be a smooth polycyclic variety endowed with a polarization L. Let $E \to Y$ be a rank 2 Hartshorne–Serre bundle obtained from some $W \subset Y$ as in Theorem 3.1. Then E is stable (resp. semi-stable) if*

(i) $\mu_L(E) > 0$ *(resp. $\mu_L(E) \geq 0$), and*

(ii) *for all hyper-surfaces $\mathcal{S}$ with $\delta_L(\mathcal{S}) \leq \mu_L(E)$ (resp. $\delta_L(\mathcal{S}) < \mu_L(E)$) the subscheme W is not contained in $\mathcal{S}$.*

We may now apply the above general criterion to both sides of our present setup.

Proposition 4.4. *Let* (Z_+, S_+) *be a building block and* $\mathscr{C}_+$ *a pencil base locus provided in Example 2.7. Let* $E_+ \to Z_+$ *be given by Proposition 3.5, such that*

(i) $c_1(E_+) = -S_+ - G_+ + H_+$, *and*

(ii) E_+ *has a global section whose vanishing locus is a fibre* ℓ_+ *of* $p_1 : \widetilde{\mathscr{C}} \to \mathscr{C}$.

Then $E_{+|S_+}$ *is stable.*

Proof. The bundle $E_{+|S_+}$ can also be seen as a Hartshorne–Serre construction. Indeed, restricting the exact sequence (10), we obtain:

$$0 \longrightarrow \mathcal{O}_{S_+} \longrightarrow E_{+|S_+} \longrightarrow \mathcal{I}_p \otimes \mathcal{O}_{S_+}(B_+ - A_+) \longrightarrow 0, \tag{13}$$

where $p := p_1(\ell)$ is the projection of ℓ on $\mathscr{C}$. To prove stability using Proposition 4.3, we only have to check that S_+ does not contain any effective divisor D of degree

$$\delta_{\mathcal{A}_+}(D) \leq \mu_{\mathcal{A}_+}(E_{+|S_+}) = \frac{(A_+ + B_+) \cdot (B_+ - A_+)}{2} = 1.$$

Suppose such a divisor $D = \alpha A_+ + \beta B_+$ exists; since D is effective, we actually have

$$1 = \delta_{\mathcal{A}_+}(D) = (\alpha A_+ + \beta B_+) \cdot (A_+ + B_+) = 5\beta + 3\alpha.$$

Moreover, D is necessarily a prime divisor, for the sum of two effective divisors would have degree at least 2. By Lemma 3.6 (iii), we also have

$$2\beta^2 + 6\alpha\beta = D^2 \geq -2.$$

Hence $\frac{1-\sqrt{17}}{8} \leq \beta \leq \frac{1+\sqrt{17}}{8}$, and the only integer solution $\beta = 0$ implies $\alpha = \frac{1}{3} \notin \mathbb{Z}$. □

Proposition 4.5. *Let* (Z_-, S_-) *be a building block provided in Example 2.8. Let* E_- *be a bundle on* Z_- *constructed in Proposition 3.9 such that*

(i) $c_1(E_-) = G_-$, *and*

(ii) *E_- has a global section whose vanishing locus is W, where $[W] = h_-$.*

The bundle $E_{-|S_-}$ is stable.

Proof. We proceed as in the proof of Proposition 4.4. The bundle $E_{-|S_-}$ can also be seen as a Hartshorne–Serre construction by restricting (11). Thus we must check that S_- does not contain any effective divisor D of degree

$$\delta_{\mathcal{A}_-}(D) \leq \mu_{\mathcal{A}_-}(E_{-|S_-}) = \frac{(2A_- + B_-) \cdot A_-}{2} = 2.$$

Suppose such $D = \alpha A_- + \beta B_-$ exists; since the intersection form on $\operatorname{Pic} S_-$ is even and D is effective, we have $\delta_{\mathcal{A}_-}(D) = 2$ and so

$$1 = \frac{1}{2}\delta_{\mathcal{A}_-}(D) = \frac{1}{2}(\alpha A_- + \beta B_-) \cdot (2A_- + B_-) = 5\beta + 2\alpha.$$

Moreover, D is also prime, for otherwise its degree would be at least 4, and so

$$2\beta^2 + 8\alpha\beta = D^2 \geq -2.$$

Hence $\frac{1-\sqrt{40}}{18} \leq \beta \leq \frac{1+\sqrt{40}}{18} \Rightarrow \beta = 0 \Rightarrow \alpha = \frac{1}{2} \notin \mathbb{Z}$. □

In the context above, the moduli spaces of the stable bundles $E_{\pm|S_\pm}$ have 'minimal' positive dimension:

Proposition 4.6. *Let $(Z_\pm, S_\pm)$ be the building blocks provided in Examples 2.7 and 2.8, and let $E_\pm \to Z_\pm$ be the asymptotically stable bundles constructed in Propositions 3.5 and 3.9. Let $\mathcal{M}^s_{S_\pm,\mathcal{A}_\pm}(v_\pm)$ be the moduli space of $\mathcal{A}_\pm$-stable bundles on $S_\pm$ with Mukai vector $v_\pm = v(E_{\pm|S_\pm})$. We have:*

$$\dim \mathcal{M}^s_{S_\pm,\mathcal{A}_\pm}(v_\pm) = 2.$$

Proof. That $E_\pm$ are asymptotically stable is the content of the previous Propositions 4.4 and 4.5. Now the claim is a direct application of Theorem 4.1, with $\operatorname{rk} E_{\pm|S_\pm} = 2$, $c_1(E_{+|S_+})^2 = -4$, $c_2(E_{+|S_+}) = 1$, $c_1(E_{-|S_-})^2 = 0$, and $c_2(E_{-|S_-}) = 2$. □

4.2. Proof of Theorem 1.4

Let $Y_+ = \mathbb{P}^1 \times \mathbb{P}^2$ as in Example 2.7, and $S_+ \subset Y_+$ be a smooth anti-canonical K3 divisor. Let $\mathcal{A}_+$ be the ample class $A_+ + B_+$ on S_+, v_{S_+} the Mukai vector $(2, B_+ - A_+, -1)$. The associated moduli space $\mathcal{M}^s_{S_+,\mathcal{A}_+}(v_{S_+})$ is 2-dimensional.

For a smooth curve $\mathscr{C}_+ \in |-K_{Y_+|S_+}|$, let $Z_+ := \mathrm{Bl}_{\mathscr{C}_+} Y_+$ be the building block resulting from Proposition 2.3, and let

$$v_{Z_+} := (2, -S_+ - G_+ + H_+, \ell_+) \in \left(H^0 \oplus H^2 \oplus H^4\right)(Z_+, \mathbb{Z}).$$

Given a bundle $E_+ \to Z_+$ as in Proposition 4.4 with $(\mathrm{rk}\, E_+, c_1(E_+), c_2(E_+)) = v_{Z_+}$, the restriction to S_+ has Mukai vector v_{S_+}, so

$$\mathcal{G} := E_{+|S_+} \in \mathcal{M}^s_{S_+,\mathcal{A}_+}(v_{S_+}).$$

We have now established all the preliminaries for Theorem 1.4, and the rest of this section is devoted to its proof. Since all relevant objects are associated to the block (Z_+, S_+), we omit henceforth the $+$ subscript.

Given $\mathscr{C} \in |-K_{Y|S}|$, we have used the Hartshorne-Serre construction to construct a family of vector bundles $\{E_p \to Z \mid p \in \mathscr{C}\}$ with

$$(\mathrm{rk}\, E, c_1(E), c_2(E)) = v_Z$$

parametrised by $\mathscr{C}$ itself. Proposition 4.4 showed that each E_p is asymptotically stable. Moreover, Proposition 5.9 in the next section will show that E_p is inelastic.

Lemma 4.7. *For each $p \in S$, there exists a rank 2 Hartshorne-Serre bundle $\mathcal{G}_p \to S$ obtained from p such that:*

(i) $c_1(\mathcal{G}_p) = B - A$,

(ii) *$\mathcal{G}_p$ has a unique global section (up to scale) with vanishing locus p.*

(iii) *$\mathcal{G}_p$ is $\mathcal{A}$-μ-stable.*

Proof. By Serre duality, $H^2(S, A - B) = H^0(S, B - A)$, which vanishes since $B - A$ is not an effective divisor. Then a bundle $\mathcal{G}_p$ satisfying (i) is given by Theorem 3.1 and it fits in the exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{G}_p \longrightarrow \mathcal{I}_p \otimes \mathcal{O}_S(B - A) \longrightarrow 0. \tag{14}$$

Again since $B - A$ is not effective, the sheaf $\mathcal{I}_p \otimes \mathcal{O}_S(B - A)$ has no global sections and (ii) follows trivially from (14).

The stability of $\mathcal{G}_p$ is equivalent to the stability of $E_{+|S_+}$ proven in Proposition 4.4, since they are both extensions of $\mathcal{O}_S$ and $\mathcal{I}_p \otimes \mathcal{O}_S(B - A)$. □

One crucial feature of the building block obtained from $Y_+ = \mathbb{P}^1 \times \mathbb{P}^2$ is the fact that the moduli space of bundles over the anti-canonical K3 divisor S is actually isomorphic to S itself:

Proposition 4.8. *The map*

$$\begin{aligned} g : S &\longrightarrow \mathcal{M}^s_{S,\mathcal{A}}(v_S) \\ p &\longmapsto \mathcal{G}_p \end{aligned}$$

defined by Lemma 4.7 is an isomorphism of K3 surfaces.

Proof. It is clear from Lemma 4.7 (ii) that g is injective, so the issue lies in the structure of the image. Our $\mathcal{M}^s_{S,\mathcal{A}}(v_S)$ is an open subset of the moduli space $\mathcal{M}^{ss,G}_{S,\mathcal{A}}(v_S)$ of Gieseker semi-stable sheaves on S, and the latter have first Chern class $B - A$ primitive in $\operatorname{Pic} S$. Hence, by [9, Theorem 6.2.5], $\mathcal{M}^{ss,G}_{S,\mathcal{A}}(v_S)$ is a K3 surface if the polarisation $\mathcal{A}$ is contained in an open chamber (*cf.* [9, Definition 4.C.1]), *i.e.* , if

$$\mathcal{A} \cdot D \neq 0, \quad \forall D \in \operatorname{Div}(S) \text{ such that } -\Delta \leq D^2 < 0,$$

where $\Delta := 2\operatorname{rk}.c_2 - (\operatorname{rk} - 1)c_1^2$ is the discriminant in $\mathcal{M}^{ss,G}_{S,\mathcal{A}}(v_S)$. In our case $\Delta = 4 - (-4) = 8$, and again we argue by contradiction as in Proposition 4.4: suppose there is a divisor $D = \alpha A + \beta B$ such that $-8 \leq D^2 < 0$ but $D \cdot \mathcal{A} = 0$; then

$$\begin{cases} -4 \leq \beta^2 + 3\alpha\beta < 0 \\ 5\beta + 3\alpha = 0 \end{cases} \qquad \Rightarrow \quad 0 < \beta^2 \leq 1.$$

The integer solutions $\beta = \pm 1$ imply $\alpha = \mp\frac{5}{3} \notin \mathbb{Z}$, therefore $\mathcal{M}^{ss,G}_{S,\mathcal{A}}(v_S)$ is a K3 surface. It follows that the map g is a bi-meromorphism of K3 surfaces between S and $\mathcal{M}^{ss,G}_{S,\mathcal{A}}(v_S)$, and every such map is an isomorphism. It follows that $\mathcal{M}^s_{S,\mathcal{A}}(v_S) = \mathcal{M}^{ss,G}_{S,\mathcal{A}}(v_S)$. □

Now let $\mathcal{G} \in \mathcal{M}^s_{S,\mathcal{A}}(v_S)$ and $V \subset H^1(S, \mathscr{E}nd_0(\mathcal{G}))$. From Proposition 4.8, there is $p \in S$ such that $\mathcal{G} = \mathcal{G}_p$ and let $V' = (\mathrm{d}g)_p^{-1}(V)$. Since $-K_{Y|S}$ is very ample (see Example 2.7), Lemma 2.5 allows the choice of a smooth base locus curve $\mathscr{C} \in |-K_{Y|S}|$ such that $p \in \mathscr{C}$ and $T_p\mathscr{C} = V'$. By Proposition 3.5, we can find a family $\{E_q \to Z \mid q \in \mathscr{C}\}$ of bundles parametrised by $\mathscr{C}$, with prescribed topology

$$(\operatorname{rk} E, c_1(E), c_2(E)) = v_Z$$

and $E_{q|S} = \mathscr{G}_q$. The bundle E_p has therefore all the properties claimed in Theorem 1.4 apart the inelasticity which will be proved in the next section.

5. Inelasticity of asymptotically stable Hartshorne-Serre bundles

Definition 5.1. Let (Z, S) be a building block and E a bundle on Z. We say that E is *inelastic* if

$$H^1(Z, \mathscr{E}nd_0(E)(-S)) = 0.$$

This condition means that there are no global deformations of the bundle E which keep fixed the bundle "at infinity" $E_{|S}$. Section 5.1 provides a characterisation of inelasticity in the case of asymptotically stable bundles, for then one may relate the freedom to extend E and the dimension of the moduli space $\mathcal{M}^s_{S,\mathcal{A}}(v_E)$. In Section 5.2 we apply this to Hartshorne-Serre bundles, by computing the dimension of the moduli space in terms of the construction data. These results hold for general building blocks and may be of independent interest.

Section 5.3 contains the computations in cohomology to establish the inelasticity of our bundles $E_\pm$ constructed in Propositions 3.5 and 3.9.

5.1. Inelasticity of asymptotically stable bundles

This section is dedicated to proving the following statement.

Proposition 5.2. *Let (Z, S) be a building block and E an asymptotically stable bundle on Z. Let $\mathcal{M}^s_{S,\mathcal{A}}(v)$ be the moduli space of $\mathcal{A}$-μ-stable bundles on S with Mukai vector $v = v(E_{|S})$. The following statements are equivalent:*

(i) *The bundle E is inelastic.*

(ii) *The sequence*

$$0 \longrightarrow \operatorname{Ext}^1(E,E) \longrightarrow \operatorname{Ext}^1(E_{|S}, E_{|S}) \longrightarrow H^2(Z, \mathscr{E}nd(E)(-S)) \longrightarrow 0\,.$$

(which is self-dual for Serre duality) is exact.

(iii) $\dim \operatorname{Ext}^1(E,E) = \frac{1}{2} \dim \mathcal{M}^s_{S,\mathcal{A}}(v)$.

By Serre duality we have $\chi(\mathscr{E}nd_0(E)(-S)) = -\chi(\mathscr{E}nd_0(E))$. Now, restriction to S gives the exact sequence

$$(15) \qquad 0 \longrightarrow \mathscr{E}nd_0(E)(-S) \longrightarrow \mathscr{E}nd_0(E) \longrightarrow \mathscr{E}nd_0(E)_{|S} \longrightarrow 0,$$

hence, by Maruyama's Theorem 4.1, it is also equivalent to:

$$(16) \qquad 2\chi(\mathscr{E}nd_0(E)(-S)) = -\chi(\mathscr{E}nd_0(E)_{|S}) = \dim \mathcal{M}^s_{S,\mathcal{A}}(v).$$

Moreover, the long exact sequence associated to (15) and the Serre duality show that E being inelastic is equivalent of having the following exact sequence:

$$(17) \qquad \begin{aligned} 0 \longrightarrow H^1(Z, \mathscr{E}nd_0(E)) &\longrightarrow H^1(S, \mathscr{E}nd_0(E)_{|S}) \\ \longrightarrow H^2(Z, \mathscr{E}nd_0(E)(-S)) &\longrightarrow 0. \end{aligned}$$

Lemma 5.3. *The bundle E is simple.*

Proof. The restriction of the class of S to S is trivial. Hence twisting (15) by $\mathcal{O}_Z(-(n-1)S)$, with $n \in \mathbb{N}$, we get:

$$0 \longrightarrow \mathscr{E}nd_0(E)(-nS) \longrightarrow \mathscr{E}nd_0(E)(-(n-1)S) \longrightarrow \mathscr{E}nd_0(E)_{|S} \longrightarrow 0\,.$$

Since $E_{|S}$ is stable, in particular it is simple and so $H^0(\mathscr{E}nd_0(E)_{|S}) = 0$. It follows by induction that

$$H^0(\mathscr{E}nd_0(E)) = H^0(\mathscr{E}nd_0(E)(-nS)), \quad \forall n \in \mathbb{N}.$$

If there could occur $H^0(\mathscr{E}nd_0(E)) \neq 0$, then one would have

$$h^0(\mathscr{E}nd_0(E)) \geq h^0(\mathcal{O}_Z(nS)), \quad \forall n \in \mathbb{N}.$$

Considering the exact sequence

$$0 \longrightarrow \mathcal{O}_Z((n-1)S) \longrightarrow \mathcal{O}_Z(nS) \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

we find, by induction, $h^0(\mathcal{O}_Z(nS)) = n+1$, which would render the dimension $h^0(\mathscr{E}nd_0(E))$ undefined; so indeed it must vanish. □

We now examine the terms on the left-hand side of (16). It follows from (15), Lemma 5.3 and Serre duality that h^0 and h^3 are zero:

$$h^0(\mathscr{E}nd_0(E)(-S)) = \underbrace{h^0(\mathscr{E}nd_0(E))}_{0} = h^3(\mathscr{E}nd_0(E)(-S)),$$

therefore $h^1 = h^2 - \chi(\mathscr{E}nd_0(E)(-S)$. On the other hand, from the exact sequence

$$(18) \qquad 0 \longrightarrow \mathscr{E}nd_0(E) \longrightarrow \mathscr{E}nd(E) \xrightarrow{\text{tr}} \mathcal{O}_Z \longrightarrow 0$$

it follows that $H^1(\mathscr{E}nd_0(E)) = H^1(\mathscr{E}nd(E)) = \mathrm{Ext}^1(E,E)$, and we conclude by Serre duality in h^2:

$$h^1(\mathscr{E}nd_0(E)(-S)) = \dim \mathrm{Ext}^1(E,E) - \frac{1}{2}\dim \mathcal{M}^s_{S,\mathcal{A}}(v).$$

This gives (i) ⇔ (iii).

Moreover, applying Lemma 3.6, exact sequence (18) twisted by $\mathcal{O}_Z(-S)$ also provides $H^2(\mathscr{E}nd_0(E)(-S)) = H^2(\mathscr{E}nd(E)(-S))$. Similarly, from the exact sequence

$$0 \longrightarrow \mathscr{E}nd_0(E_{|S}) \longrightarrow \mathscr{E}nd(E_{|S}) \xrightarrow{\text{tr}} \mathcal{O}_S \longrightarrow 0$$

we have $H^1(\mathscr{E}nd_0(E_{|S})) = H^1(\mathscr{E}nd(E_{|S})) = \mathrm{Ext}^1(E_{|S}, E_{|S})$. Then (17) gives (i) ⇔ (ii).

5.2. Application to Hartshorne-Serre bundles

Here we establish a characterisation of inelasticity in the case of asymptotically stable Hartshorne-Serre bundles E of rank 2, as in the context of Theorem 1.3. In view of Proposition 5.2, we can accomplish this by calculating $\dim \mathrm{Ext}^1(E,E)$, the dimension of the infinitesimal deformations of E.

Proposition 5.4. *Let (Z,S) be a building block, and let $E \to Z$ be an asymptotically stable Hartshorne–Serre bundle obtained from a genus 0 curve $W \subset Z$ and a line bundle $\mathcal{L} \to Z$ as in Theorem 3.1.*

Suppose $H^1(E) = 0$. Then

$$(19) \qquad \dim \mathrm{Ext}^1(E,E) = \dim H^0(W, \mathcal{N}_{W/Z}) + \dim H^1(Z, \mathcal{L}^*) - \dim H^0(Z,E) + 1.$$

Remark 5.5. The Hartshorne-Serre construction produces a vector bundle E together with a section s (up to scale). The degrees of freedom in the construction come from deformations of the curve W, parametrised by $H^0(W, \mathcal{N}_{W/Z})$, and choosing an extension

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow E \longrightarrow \mathcal{I}_W \otimes \mathcal{L} \longrightarrow 0, \tag{20}$$

parametrised by $\mathrm{Ext}^1(\mathcal{I}_W, \mathcal{L}^*)$. Hence one would naively expect the space of pairs (E, s) produced by the construction to have dimension

$$\begin{aligned} &\dim H^0(W, \mathcal{N}_{W/Z}) + \dim \mathrm{Ext}^1(\mathcal{I}_W, \mathcal{L}^*) \\ &\qquad = \dim H^0(W, \mathcal{N}_{W/Z}) + \dim H^1(Z, \mathcal{L}^*) + 1. \end{aligned}$$

Accounting for the choice of s yields the right-hand side of (19), so the proposition amounts to stating that the naive calculation gives the correct result under the given hypotheses.

The remainder of the section is devoted to the proof of Proposition 5.4. By construction, E fits in the exact sequence (20). Applying the functor $\mathrm{End}(\cdot, E)$ we obtain:

$$\begin{aligned} 0 \longrightarrow \mathrm{End}(\mathcal{I}_W \otimes \mathcal{L}, E) \longrightarrow \mathrm{End}(E, E) \longrightarrow H^0(E) \\ \longrightarrow \mathrm{Ext}^1(\mathcal{I}_W \otimes \mathcal{L}, E) \longrightarrow \mathrm{Ext}^1(E, E) \longrightarrow H^1(E). \end{aligned} \tag{21}$$

Since $\mathscr{E}nd(\mathcal{I}_W, \mathcal{O}_Z) = \mathcal{O}_Z$, it follows that

$$\mathrm{End}(\mathcal{I}_W \otimes \mathcal{L}, E) = H^0(E \otimes \mathcal{L}^*).$$

We first show that $H^0(E \otimes \mathcal{L}^*) = 0$. Twisting (20) by $\mathcal{L}^*$ we get

$$0 \longrightarrow \mathcal{L}^* \longrightarrow E \otimes \mathcal{L}^* \longrightarrow \mathcal{I}_W \longrightarrow 0. \tag{22}$$

We know that $H^0(\mathcal{I}_W) = 0$. Since E is asymptotically stable, $\mathcal{L}$ corresponds necessarily to an effective divisor, so $H^0(\mathcal{L}^*) = 0$ and the claim follows.

Moreover, by assumption, $H^1(E) = 0$, so (21) simplifies to

$$0 \to \mathrm{End}(E, E) \to H^0(E) \to \mathrm{Ext}^1(\mathcal{I}_W \otimes \mathcal{L}, E) \to \mathrm{Ext}^1(E, E) \to 0.$$

Using Lemma 5.3, this gives

$$\dim \mathrm{Ext}^1(E, E) = \dim \mathrm{Ext}^1(\mathcal{I}_W \otimes \mathcal{L}, E) - \dim H^0(E) + 1,$$

and it only remains to check that

$$\dim \operatorname{Ext}^1(\mathcal{I}_W \otimes \mathcal{L}, E) = \dim H^0(\mathcal{N}^*_{W/Z} \otimes \mathcal{L}_{|W}) + \dim H^1(Z, \mathcal{L}^*). \tag{23}$$

Lemma 5.6. *In the hypotheses of Proposition 5.4,*

$$H^1(E \otimes \mathcal{L}^*) = H^1(\mathcal{L}^*) \text{ and } H^2(E \otimes \mathcal{L}^*) = 0.$$

Proof. From the exact sequence

$$0 \longrightarrow \mathcal{I}_W \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_W \longrightarrow 0\ ,$$

we see that $H^0(\mathcal{I}_W) = H^1(\mathcal{I}_W) = H^2(\mathcal{I}_W) = 0$. Hence, from (22) we read

$$H^1(E \otimes \mathcal{L}^*) = H^1(\mathcal{L}^*) \quad \text{and} \quad H^2(E \otimes \mathcal{L}^*) = H^2(\mathcal{L}^*).$$

Moreover, the latter is trivial by the hypotheses of the Hartshorne-Serre construction (Theorem 3.1). □

Now, from the spectral sequence

$$E_2^{p,q} := H^p(\mathscr{E}xt^q(\mathcal{I}_W \otimes \mathcal{L}, E)) \Rightarrow E_n := \operatorname{Ext}^n(\mathcal{I}_\ell \otimes \mathcal{L}, E),$$

we obtain the following exact sequence:

$$0 \longrightarrow E_2^{1,0} \longrightarrow E^1 \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0}. \tag{24}$$

Moreover, we have

$$\mathscr{E}xt^q(\mathcal{I}_W \otimes \mathcal{L}, E) = \mathscr{E}xt^q(\mathcal{I}_W, \mathcal{O}_Z) \otimes E \otimes \mathcal{L}^*,$$

with $\mathscr{E}xt^1(\mathcal{I}_W, \mathcal{O}_Z) = \wedge^2 \mathcal{N}_{W/Z}$ (see *e.g.* [1, Section 1]). Hence, (24) provides

$$\begin{aligned} 0 \longrightarrow H^1(E \otimes \mathcal{L}^*) &\longrightarrow \operatorname{Ext}^1(\mathcal{I}_W \otimes \mathcal{L}, E) \\ \longrightarrow H^0(E_{|W} \otimes \wedge^2 \mathcal{N}_{W/Z} \otimes \mathcal{L}^*_W) &\longrightarrow H^2(E \otimes \mathcal{L}^*) \end{aligned}$$

and, by Lemma 5.6, we obtain:

$$\dim \operatorname{Ext}^1(\mathcal{I}_W \otimes \mathcal{L}, E) = \dim H^1(\mathcal{L}^*) + \dim H^0(E_{|W} \otimes \wedge^2 \mathcal{N}_{W/Z} \otimes \mathcal{L}^*_W).$$

Since E is a Hartshorne–Serre bundle obtained from the line W, we have: $\wedge^2 \mathcal{N}_{W/Z} \otimes \mathcal{L}^*_W = \mathcal{O}_W$, thus

$$\operatorname{Ext}^1(\mathcal{I}_W \otimes \mathcal{L}, E) = H^0(E_{|W}).$$

As explained in [1, Section 1], restricting the exact sequence (20) to W, we obtain (23) from

$$E_{|W} = \mathcal{N}^*_{W/Z} \otimes \mathcal{L}_{|W} = \mathcal{N}^*_{W/Z} \otimes \wedge^2 \mathcal{N}_{W/Z} = \mathcal{N}_{W/Z}.$$

Remark 5.7. One way to determine $\mathcal{N}_{W/Z}$ is to find a surface $\mathcal{S}$ such that $W \subset \mathcal{S} \subset Z$, which fits in the exact sequence:

$$0 \longrightarrow \mathcal{N}_{W/\mathcal{S}} \longrightarrow \mathcal{N}_{W/Z} \longrightarrow (\mathcal{N}_{\mathcal{S}/Z})_{|W} \longrightarrow 0 \,.$$

Corollary 5.8. *Let (Z, S) be a building block, and let $E \to Z$ be an asymptotically stable Hartshorne–Serre bundle obtained from a genus 0 curve $W \subset Z$ and a line bundle $\mathcal{L} \to Z$ as in Theorem 3.1. Let $\mathcal{M}^s_{S,\mathcal{A}}(v)$ be the moduli space of $\mathcal{A}$-μ-stable bundles on S with Mukai vector $v = v(E_{|S})$.*

Suppose $H^1(E) = 0$. Then E is inelastic if and only if

$$\frac{1}{2} \dim \mathcal{M}^s_{S,\mathcal{A}}(v) = \dim H^0(W, \mathcal{N}_{W/Z}) + \dim H^1(Z, \mathcal{L}^*) - \dim H^0(Z, E) + 1.$$

5.3. Inelasticity of E_+ and E_-

We will now prove the inelasticity of the bundle E_+ constructed in Proposition 3.5, over the building block Z_+ obtained by blowing up $Y_+ = \mathbb{P}^1 \times \mathbb{P}^2$ from Example 2.7. For tidiness, we omit the + subscript.

Proposition 5.9. *Let $E \to Z = \mathrm{Bl}_{\mathscr{C}} Y$ be the bundle constructed in Proposition 3.5, over the building block from Example 2.7, satisfying:*

(i) *$c_1(E) = -S - G + H$,*

(ii) *E has a global section with vanishing locus given by an exceptional fibre ℓ of $p_1 : \widetilde{\mathscr{C}} \to \mathscr{C}$ over the base locus of the anti-canonical pencil.*

Then the bundle E is inelastic.

Proof. By Corollary 5.8, using Lemmata 3.7 (i) and 3.8 with Proposition 4.6, we only have to check that $\dim H^0(\mathcal{N}_{\ell/Z}) = 1$, but this follows directly from the fact that the lines of class ℓ in Z are parametrised by the curve $\mathscr{C}$. □

NB.: In the light of Remark 5.7, we could also verify that $\mathcal{N}_{\ell/Z} = \mathcal{O}_\ell \oplus \mathcal{O}_\ell(-1)$.

Similarly, we prove the inelasticity of the bundle E_- constructed in Proposition 3.9, over the building block Z_- obtained by blowing up $Y_- \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^2$ from Example 2.8. We also omit the $-$ subscript.

Proposition 5.10. *Let $E \to Z$ be the bundle constructed in Proposition 3.9, over the building block from Example 2.8, satisfying:*

(i) *$c_1(E) = G$,*

(ii) *E has a global section with vanishing locus W such that $[W] = h$ (*cf. *Example 2.8).*

Then the bundle E is inelastic.

Proof. As before, with Corollary 5.8, using Lemmata 3.10 (i) and 3.11 with Proposition 4.6 we have to check that $\dim H^0(\mathcal{N}_{W/Z}) = 2$, which is true since the family of curves of class $h = [W]$ in Z is parametrised by a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ (see Section 2.3). □

NB.: Using Remark 5.7, we could also see that $\mathcal{N}_{W/Z} = \mathcal{O}_W \oplus \mathcal{O}_W$.

6. Proof of Theorem 1.3

Twisting the Mukai vector

$$v'_{S_+} := (2,\, 5A_+ - 3B_+,\, -18)$$

by $\mathcal{O}_{S_+}(-2B_+ + 3A_+)$ gives a natural isomorphism

$$\mathcal{M}^s_{S_+,\mathcal{A}_+}(v_{S_+}) \simeq \mathcal{M}^s_{S_+,\mathcal{A}_+}(v'_{S_+}).$$

Moreover, since

$$\mathcal{O}_{S_+}(-2B_+ + 3A_+) = \mathcal{O}_{Z_+}(-2H_+ + 3G_+)_{|S_+},$$

we can rewrite Theorem 1.4 with $\mathcal{M}^s_{S_+,\mathcal{A}_+}(v_{S'_+})$ instead of $\mathcal{M}^s_{S_+,\mathcal{A}_+}(v_{S_+})$.

Corollary 6.1. *In the context of Example 2.7, for every bundle $\mathcal{G} \in \mathcal{M}^s_{S_+,\mathcal{A}_+}(v'_{S_+})$ and every complex line $V \subset H^1(S_+, \mathscr{E}nd_0(\mathcal{G}))$, there are a smooth curve $\mathscr{C}_+ \in |-K_{Y_+|S_+}|$ and an asymptotically stable and inelastic* *vector bundle* $F_+ \to Z_+$ *with*

$$(\mathrm{rk}, c_1, c_2)(F_+) = (2, 5G_+ - 3H_+,\\ \ell_+ + (H_+ - G_+) \cdot (-2H_+ + 3G_+) + (-2H_+ + 3G_+)^2),$$

such that $F_{+|S_+} = \mathcal{G}$ *and* $\mathrm{res} : H^1(Z_+, \mathscr{E}nd_0(F_+)) \to H^1(S_+, \mathscr{E}nd_0(\mathcal{G}))$ *has image* V*.*

We have a similar result on the block (Z_-, S_-). Twisting the vector

$$v'_{S_-} := (2,\, 5A_- - 2B_-,\, -18)$$

by $\mathcal{O}_{S_-}(-B_- + 2A_-)$ identifies $\mathcal{M}^s_{S_-,\mathcal{A}_-}(v_{S_-}) \simeq \mathcal{M}^s_{S_-,\mathcal{A}_-}(v'_{S_-})$ and, since

$$\mathcal{O}_{S_+}(-B_- + 2A_-) = \mathcal{O}_{Z_-}(-H_- + 2G_-)_{|S_-},$$

we can reformulate Propositions 3.9, 4.5 and 5.10 for

$$F_- := E_- \otimes \mathcal{O}_{Z_-}(-H_- + 2G_-).$$

Corollary 6.2. *In the context of Example 2.8, there exists a family of asymptotically stable and inelastic vector bundles* $\{F_- \to Z_-\}$*, parametrised by the set of the lines in* Y_- *of class* h_-*, such that* $F_{-|S_-} \in \mathcal{M}^s_{S_-,\mathcal{A}_-}(v'_{S_-})$ *and*

$$(\mathrm{rk}, c_1, c_2)(F_-) = (2, 5G_- - 2H_-,\\ h_- + G_- \cdot (-H_- + 2G_-) + (-H_- + 2G_-)^2).$$

Theorem 1.3 is immediate from the following result, which we deduce from Corollaries 6.1 and 6.2.

Theorem 6.3. *Let* $\mathfrak{r} : S_+ \to S_-$ *be a matching between* $Y_+ = \mathbb{P}^1 \times \mathbb{P}^2$ *and* $Y_- \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^2$ *. Then there exist smooth curves* $\mathscr{C}_\pm \in |-K_{Y_\pm|S_\pm}|$ *and holomorphic bundles* $F_\pm \to Z_\pm$ *over the resulting blocks* $Z_\pm := \mathrm{Bl}_{\mathscr{C}_\pm} Y_\pm$*, with*

$$(\mathrm{rk}, c_1, c_2)(F_+) = (2, 5G_+ - 3H_+,\\ \ell + (H_+ - G_+) \cdot (-2H_+ + 3G_+) + (-2H_+ + 3G_+)^2)\\ (\mathrm{rk}, c_1, c_2)(F_-) = (2, 5G_- - 2H_-,\\ h_- + G_- \cdot (-H_- + 2G_-) + (-H_- + 2G_-)^2),$$

satisfying all the hypotheses of Theorem 1.2.

Proof. We fix a representative $F_- \to Z_-$ in the family of holomorphic bundles from Corollary 6.2, to be matched by a bundle $F_+ \to Z_+$ given by Corollary 6.1, so that asymptotic stability and inelasticity hold from the outset.

It remains to address compatibility and transversality. Since the chosen configuration for $\mathfrak{r}$ ensures that $\mathfrak{r}^*$ identifies the Mukai vectors of $F_{\pm|S_\pm}$, it induces a map $\bar{\mathfrak{r}}^* : \mathcal{M}^s_{S_-,\mathcal{A}_-}(v'_{S_-}) \to \mathcal{M}^s_{S_+,\mathcal{A}_+}(v'_{S_+})$. In particular, the target moduli space is 2-dimensional, by Proposition 4.6, and $\mathfrak{r}^*(\operatorname{im}\operatorname{res}_-)$ is 1-dimensional, since the bundles $\{F_-\}$ are parametrised by lines of fixed class h_-. So indeed we apply Corollary 6.1 with $\mathcal{G} = \bar{\mathfrak{r}}^*(F_{-|S_-})$ and any choice of a direct complement subspace V such that

$$V \oplus \bar{\mathfrak{r}}^*(\operatorname{im}\operatorname{res}_-) = H^1(S_+, \mathscr{E}nd_0(\bar{\mathfrak{r}}^*(F_{-|S_-}))).$$

Denoting by $\mathscr{M}_{S_\pm}(v)$ the moduli space of ASD instantons over $S_\pm$ with Mukai vector v, the maps $f_\pm$ in Theorem 1.2 (*cf.* (1)) are the linearisations of the Hitchin-Kobayashi isomorphisms

$$\mathcal{M}^s_{S_\pm,\mathcal{A}_\pm}(v'_{S\pm}) \simeq \mathscr{M}_{S_\pm}(v'_{S\pm}).$$

Therefore, our bundles $F_\pm$ indeed satisfy $A_{\infty,+} = \bar{\mathfrak{r}}^* A_{\infty,-}$ for the corresponding instanton connections. Moreover, by linearity, $\lambda_+(H^1(Z_+, \mathscr{E}nd_0(F_+)))$ is transverse in $T_{A_{\infty,+}}\mathscr{M}_{S_+}(v'_{S_+})$ to the image of the real 2-dimensional subspace $\lambda_-(H^1(Z_-, \mathscr{E}nd_0(F_-))) \subset T_{A_{\infty,-}}\mathscr{M}_{S_-}(v'_{S_-})$ under the linearisation of $\bar{\mathfrak{r}}^*$. □

Remark 6.4. Similar techniques could still be used on blocks with a perpendicular lattice $N_0^\perp$ of rank higher than 2. Indeed, according to Propositions 2.3 and 2.6, we can choose Kähler classes $\mathrm{k}_\pm$ on $Z_\pm$ such that the restrictions $\mathrm{k}_{\pm|S_\pm}$ are arbitrarily close to $\mathcal{A}_\pm$. Hence it is not a problem to consider asymptotic stability with respect to $\mathcal{A}_\pm$ instead of $\mathrm{k}_\pm$.

Acknowledgements

We thank Daniele Faenzi, Marcos Jardim and Thomas Walpuski for many important discussions. In particular, we acknowledge Marcos Jardim for suggesting the Hartshorne-Serre technique to produce bundles parametrised by curves. GM is supported by grant 2014/05733-9, São Paulo Research Foundation (Fapesp), and the Marco Brunella Grant of Burgundy University. HSE is supported by grant 2014/24727-0, São Paulo Research Foundation

(Fapesp), and the Brazilian National Council for Scientific and Technological Development (CNPq) Productivity Grant 312390/2014-9. JN is supported by the Simons Foundation under the Simons Collaboration on Special Holonomy in Geometry, Analysis and Physics (grant #488631, Johannes Nordström).

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Received May 3, 2019
Accepted October 14, 2019