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The boundary rigidity for holomorphic self-maps of some fibered domains

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We prove Burns-Krantz type boundary rigidity theorems for holomorphic self-maps of some fibered domains.

1. Introduction

The classical Schwarz lemma states that if a holomorphic self-map f of the unit disk Δ fixes the origin, then $|f'(0)| \leq 1$ and the equality holds if and only if f is a rotation. In particular, if $f'(0) = 1$ then $f(z) \equiv z$, which is often referred to as the "rigidity" part of the Schwarz lemma.

The rigidity part of the Schwarz lemma was extended to a boundary fixed point by Burns and Krantz [1] as follows:

Theorem 1. *[1, Theorem 4.5] Let Ω be a bounded strongly pseudoconvex domain with C^∞-smooth boundary and F a holomorphic self-map of Ω. Suppose that*

$$F(z) = z + O(\|z - p\|^4), \quad z \to p \in \partial\Omega.$$

Then $F(z) \equiv z$.

In [1, p. 663], Burns and Krantz remarked that $O(\|z - p\|^4)$ in the above theorem can be replaced by $o(\|z - p\|^3)$. Later in [6], Huang gave a "localized" version of the boundary rigidity as follows:

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Theorem 2. *[6, Theorem 2.5] Let Ω be a bounded domain with a C^∞-smooth strongly pseudoconvex boundary point p and F a holomorphic self-map of Ω. Suppose that*

$$F(z) = z + o(\|z-p\|^3), \quad z \to p \in \partial\Omega.$$

Then $F(z) \equiv z$.

In this paper, we prove Burns-Krantz type boundary rigidity theorems for holomorphic self-maps of some fibered domains, without assuming pseudoconvexity or boundary smoothness.

First, consider bounded domains in $\mathbb{C}^{n+1}$ of the form

$$\Omega = \bigcup_{z\in\Delta} \Omega_z, \tag{1}$$

where Ω_z's are bounded complete Reinhardt domains in $\mathbb{C}^n$.

Denote by Δ_r the disk of radius r centered at the origin in $\mathbb{C}$, and $\Delta = \Delta_1$. Let $\delta : \Delta \to \mathbb{R}^+$ be a function such that $\Delta^n_{\delta(z)} \subset \Omega_z$ for any $z \in \Delta$ and $\delta(z) \to 0$ as $z \to 1$. For any $(z,w) \in \Omega$, denote by $D_{z,w}$ the image of the linear mapping:

$$L_{z,w} : \Delta \to \mathbb{C}^{n+1}; \quad \tau \mapsto \left(\tau, \frac{w}{z-1}(\tau-1)\right),$$

which is a graph of Δ through (z,w) and $(1,0)$. We define a *cone* with end $(1,0)$ (of size δ) as

$$C_\delta := \bigcup\{D_{z,w};\ z\in\Delta,\ w \in \Delta^n_{\delta(z)}\}. \tag{2}$$

We say that Ω is a *fibered domain satisfying the cone condition* if it is of the form (1) and contains a cone with end $(1,0)$. Note that polydisks are such special domains.

Theorem 3. *Let Ω be a fibered domain satisfying the cone condition and F a holomorphic self-map of Ω. For $(z,w) \in \Omega$, suppose that*

$$F(z,w) = (z,w) + o(\|(1-z,w)\|^3), \quad (z,w) \to (1,0).$$

Then $F(z,w) \equiv (z,w)$.

Next, consider Ω of the form (1), but with Ω_z's being just bounded and containing the origin. Let $\rho : \Delta \to \mathbb{R}^+$ be a function such that $\Omega_z \subset \Delta^n_{\rho(z)}$ for any $z \in \Delta$ and $\rho \in C^0(\bar{\Delta})$. We say that Ω is a *fibered domain with boundary size zero* if there exists a $\rho(z)$ with

$$\int_{-\pi}^{\pi} \log \rho(e^{i\theta}) d\theta = -\infty. \tag{3}$$

Note that balls and eggs are such special domains (with $\rho(e^{i\theta}) \equiv 0$). More interestingly, polydisks with the "diagonal" disk as its base are also such special domains.

Theorem 4. *Let Ω be a fibered domain with boundary size zero and F a holomorphic self-map of Ω. Suppose that*

$$F(z, w) = (z, w) + o(\|(1 - z, w)\|^3), \quad (z, w) \to (1, 0).$$

Then $F(z, w) \equiv (z, w)$.

In Section 2, we prove Theorem 3 for the bidisk. In Section 3, we prove Theorem 3 for general Ω. The proof of Theorem 4 is given in Section 4. More general versions of Theorem 3 are given in Section 5.

2. Proof of Theorem 3 for the bidisk

To illustrate the key ideas, we first prove Theorem 3 for the bidisk.

For any $z \in \Delta$ and small $\epsilon > 0$, set

$$\Delta_{z,\epsilon} = \{(z, w) \in \Delta^2 : \ |w| < \epsilon(1 - |z|)\}.$$

For any $(z, w) \in \Delta_{z,\epsilon}$, denote by $D_{z,w}$ the intersection of Δ^2 with the complex line through (z, w) and $(1, 0)$. Denote by π the vertical projection from Δ^2 to $D_{z,w}$. Write $F(z, w) = (F_0(z, w), F_1(z, w))$.

Parametrize $D_{z,w}$ as

$$D_{z,w} = \left\{ \left(\tau, \frac{w}{z - 1}(\tau - 1) \right) : \ \tau \in \Delta \right\}.$$

Consider the map $f(\tau) = \pi \circ F|_{D_{z,w}}$.

Then, one readily checks using the assumptions that f is a holomorphic self-map of $D_{z,w}$ with

$$f(\tau) = \tau + o(|1-\tau|^3), \quad \tau \to 1. \tag{4}$$

Thus, by the boundary rigidity theorem of Burns-Krantz, we have $f(\tau) \equiv \tau$. In particular, for $\tau = z$, we have

$$F_0(z,w) = z.$$

Since this holds for every $(z,w) \in \Delta_{z,\epsilon}$, it follows from the identity theorem that $F_0(z,w) \equiv z$.

Now, write $F_1(z,w)$ as

$$F_1(z,w) = a_0(z) + a_1(z)w + a_2(z)w^2 + a_3(z)w^3 + \cdots .$$

Denote $g_z(w) = F_1(z,w)$. Then g_z is a holomorphic self-map of Δ and $g_z'(0) = a_1(z)$. By Schwarz-Pick lemma, it follows that

$$|a_1(z)| \le 1 - |a_0(z)|^2 \le 1.$$

Thus a_1 is a holomorphic self-map of Δ.

At the point $(z,w) = (z, 1-z)$, we have

$$\begin{aligned} F_1(z,w) - w = a_0(z) &+ (1-z)(a_1(z)-1) \\ &+ (1-z)^2 a_2(z) + (1-z)^3 a_3(z) + o(|1-z|^3). \end{aligned}$$

At the point $(z,w) = (z, z-1)$, we have

$$\begin{aligned} F_1(z,w) - w = a_0(z) &- (1-z)(a_1(z)-1) \\ &+ (1-z)^2 a_2(z) - (1-z)^3 a_3(z) + o(|1-z|^3). \end{aligned}$$

At the point $(z,w) = (z, \frac{1}{2}(1-z))$, we have

$$\begin{aligned} F_1(z,w) - w = a_0(z) &+ \frac{1-z}{2}(a_1(z)-1) \\ &+ \frac{(1-z)^2}{4} a_2(z) + \frac{(1-z)^3}{8} a_3(z) + o(|1-z|^3). \end{aligned}$$

At the point $(z,w) = (z, \frac{1}{2}(z-1))$, we have

$$F_1(z,w) - w = a_0(z) - \frac{1-z}{2}(a_1(z)-1) + \frac{(1-z)^2}{4}a_2(z) - \frac{(1-z)^3}{8}a_3(z) + o(|1-z|^3).$$

From the above four expressions and the assumption of Theorem 3, we get

$$|a_1(z) - 1| = o(|1-z|^2), \quad z \to 1.$$

Consider $h(z) = za_1(z)$ as a holomorphic self-map of Δ with $h(0) = 0$ and

$$|h(z) - z| = o(|1-z|^2), \quad z \to 1. \tag{5}$$

Then, it follows from [6, Corollary 1.5] that $h(z) \equiv z$, i.e. $a_1(z) \equiv 1$. Again, using Schwarz-Pick lemma, we get $F_1(z,w) \equiv w$.

3. Proof of Theorem 3

For any $z \in \Delta$ and $0 < \epsilon < \delta(z)$, set

$$\Delta^n_{z,\epsilon} = \{(z,w) \in \Omega : \ w \in \Delta^n_\epsilon\}.$$

For any $(z,w) \in \Delta^n_{z,\epsilon}$, denote by $D_{z,w}$ the intersection of Ω with the complex line through (z,w) and $(1,0)$. Denote by π the vertical projection from Ω to $D_{z,w}$. Write $F(z,w) = (F_0(z,w), F_1(z,w), \dots, F_n(z,w))$.

A similar argument as in the previous section shows that $F_0(z,w) \equiv z$.

Now, for $1 \le j \le n$, write $F_j(z,w)$ as

$$F_j(z,w) = \sum_\alpha a_{j,\alpha}(z) w^\alpha,$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $w^\alpha = w_1^{\alpha_1} \cdots w_n^{\alpha_n}$.

For each $1 \le j \le n$, consider $g_{j,z}(w_j) = F_j(z, 0, \dots, 0, w_j, 0, \dots, 0)$. Then a similar argument as in the previous section shows that

$$a_{j,e_j}(z) \equiv 1, \quad a_{j,le_j}(z) \equiv 0, \ l \ne 1,$$

where e_j denotes the j-th unit vector.

Therefore, we can write $F_j(z,w)$ as

$$F_j(z,w) = w_j + \sum_{1\le k\ne j\le n} a_{j,e_k}(z)w_k + \sum_{|\alpha|\ge 2} a_{j,\alpha}(z)w^\alpha,$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

For any fixed $z \in \Delta$, consider the holomorphic self-map $G_z(w)$ of Ω_z,

$$G_z(w) = (F_1(z,w), \ldots, F_n(z,w)).$$

By a linear change of coordinates, we can suppose that the Jacobian $JG_z(0)$ is in upper triangular form with all diagonal entries equal to 1. Consider the iterations G_z^k and apply Cauchy's estimates, we get that all the off-diagonal entries must be identically zero, i.e. $JG_z(0) = id$. Then, it follows from Cartan's uniqueness theorem that $G_z(w) \equiv w$. This proves Theorem 3.

Similar to [6, Corollary 1.5, Corollary 2.7] (see also [5, Corollary 3]), we have the following

Theorem 5. *Let Ω be as in Theorem 3 and F a holomorphic self-map of Ω. Suppose that*

$$F(z,w) = (z,w) + o(\|(1-z,w)\|^2), \quad (z,w) \to (1,0).$$

Assume further that the fixed point set Γ of F satisfies

$$\Gamma \cap D_{z,w} \ne \emptyset, \ \forall D_{z,w} \subset C_\delta.$$

Then $F(z,w) \equiv (z,w)$.

Proof. The difference here is that in (4) we have $o(|1-\tau|^2)$ instead of $o(|1-\tau|^3)$, and in (5) we have $o(|1-z|)$ instead of $o(|1-z|^2)$. But with the additional assumption on the fixed point set Γ, [6, Corollary 1.5] still applies. The rest of the proof is exactly the same. □

4. Proof of Theorem 4

Consider the z-disk

$$\Delta_z = \{(z,w) \in \Omega : \ w = 0\}.$$

Arguing as in Section 2, we get $\pi \circ F|_{\Delta_z}(\tau) \equiv \tau$, i.e. $F|_{\Delta_z}$ is a graph over Δ_z. Thus, there exists a holomorphic mapping $f : \Delta_z \to \mathbb{C}^n$ such that $F|_{\Delta_z} = (\tau, f(\tau))$.

Write $f = (f_1, \ldots, f_n)$. Then for any $1 \le j \le n$ and $\tau \in \Delta_z$, $|f_j(\tau)| < \rho(\tau)$. Since $\log|f_j(\tau)|$ is subharmonic, we get from (3) that

$$\log|f_j(\tau)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\varphi - \theta)\log\rho(e^{i\theta})d\theta = -\infty, \quad \tau = re^{i\varphi}.$$

Here P_r is the Poisson kernel. Thus we must have $f_j(\tau) \equiv 0$, i.e. F fixes the z-disk Δ_z.

Theorem 4 then follows from [6, Theorem 2.2] (with $\epsilon = 1$ and $\mu = 1$).

Remark 1. Theorem 4 is a generalization of [6, Theorem 0.3], in which the domains are fibered domains with $\rho(e^{i\theta}) \equiv 0$.

Similar to Theorem 5, we also have the following

Theorem 6. *Let Ω be as in Theorem 4 and F a holomorphic self-map of Ω. Suppose that $F(z_0, 0) = (z_0, 0)$ for some $z_0 \in \Delta$ and*

$$F(z, w) = (z, w) + o(\|(1 - z, w)\|^2), \quad (z, w) \to (1, 0).$$

Then $F(z, w) \equiv (z, w)$.

Proof. The difference here is that in (4) we have $o(|1 - \tau|^2)$ instead of $o(|1 - \tau|^3)$, but with $f(z_0) = z_0$. Thus, [6, Corollary 1.5] applies, as does [6, Theorem 2.2]. □

5. More general domains

In Theorem 3, we can allow the base of the domain Ω to be any bounded domain with a strongly pseudoconvex boundary point p.

More precisely, let Ω be a bounded domain of the form

$$\Omega = \bigcup_{z \in D} \Omega_z, \tag{6}$$

where D is a bounded domain in $\mathbb{C}^m$ with a C^∞-smooth strongly pseudoconvex boundary point p, and Ω_z's are bounded complete Reinhardt domains in $\mathbb{C}^n$.

Let $\delta : D \to \mathbb{R}^+$ be a function such that $\Delta^n_{\delta(z)} \subset \Omega_z$ for any $z \in D$ and $\delta(z) \to 0$ as $z \to p$. For any $(z, w) \in \Omega$, denote by $D_{z,w}$ the linear graph over D through (z, w) and $(p, 0)$. We then define a cone with end $(p, 0)$ (of size δ) as in (2).

We again call Ω a *fibered domain satisfying the cone condition* if it is of the form (6) and contains a cone with end $(p, 0)$.

Theorem 7. *Let Ω be a fibered domain satisfying the cone condition and F a holomorphic self-map of Ω. For $(z, w) \in \mathbb{C}^m \times \mathbb{C}^n$, suppose that*

$$F(z, w) = (z, w) + o(\|(p - z, w)\|^3), \quad (z, w) \to (p, 0).$$

Then $F(z, w) \equiv (z, w)$.

Proof. For simplicity, assume $n = 1$. Write $F(z, w) = (F_0(z, w), F_1(z, w))$. Then, arguing as in Section 2 using Huang's rigidity theorem on bounded domains with a C^∞-smooth strongly pseudoconvex boundary point, we get that $F_0(z, w) \equiv z$.

Write $F_1(z, w) = a_0(z) + a_1(z)w + a_2(z)w^2 + a_3(z)w^3 + \cdots$. Then, by Schwarz-Pick lemma, we get that $|a_1(z)| \leq 1$. And a similar argument as in Section 2 (using $\|p - z\|$ instead of $|1 - z|$) shows that

$$|a_1(z) - 1| = o(\|p - z\|^2), \quad z \to p.$$

Let ψ be a change of coordinates in a neighborhood U of p such that $V := \psi(U \cap D)$ is strongly convex near $q := \psi(p)$. Then, there exists a small cone C with end q contained in V, which is the union of one-dimensional simply-connected domains with q on the boundary. For each such a one-dimensional simply-connected domain W, let ϕ be a Riemann mapping from W to the unit disk with $\lim_{k\to\infty} \phi(\zeta_k) = 1$ for some $\zeta_k \to q$.

Set $\tilde{a}_1 := a_1 \circ \psi^{-1}|_W \circ \phi^{-1}$. Since W has C^2-smooth boundary, it follows from Lemma 8 below that

$$|1 - \tau| \sim |q - \phi^{-1}(\tau)|, \quad \tau \to 1.$$

Thus, $\tilde{a}_1$ is a holomorphic self-map of the unit disk satisfying

$$|\tilde{a}_1(\tau) - 1| = o(|1 - \tau|^2), \quad \tau \to 1.$$

Then, arguing exactly as in Section 2, we get that $a_1 \equiv 1$ and $F_1(z, w) \equiv w$. □

The following lemma is probably known. We give a proof for completeness.

Lemma 8. *Let W be a simply-connected domain in $\mathbb{C}$ with C^2-smooth boundary near $q \in \partial W$. Let ϕ be a Riemann mapping from W to Δ with $\lim_{k\to\infty} \phi(\zeta_k) = 1$ for some $\zeta_k \to q$. Then ϕ extends to be bi-Lipschitz near $q \in \overline{W}$.*

Proof. First of all, by [3, Theorem 1.1], ϕ extends to be homeomorphic near $q \in \overline{W}$.

Set $\tau = \phi(\zeta)$. Denote by $d(\zeta, \partial W)$ the distance between $\zeta \in W$ and the boundary ∂W. Since W has C^2-smooth boundary near q, it follows from the Hopf lemma (see e.g. [2]) that

$$d(\tau, \partial\Delta) \sim d(\zeta, \partial W), \quad \tau \to \partial\Delta \text{ near } 1.$$

Denote by $K_W(\zeta; \xi)$ the Kobayashi metric at $\zeta \in W$ with unit vector ξ. Set $\xi' = \phi'(\zeta)\xi/|\phi'(\zeta)|$. Then from the definition of the Kobayashi metric, it follows that

$$K_\Delta(\tau; \xi') \leq K_W(\zeta; \xi)/|\phi'(\zeta)|.$$

Thus,

$$|\phi'(\zeta)| \leq \frac{K_W(\zeta; \xi)}{K_\Delta(\tau; \xi')} \lesssim \frac{d(\tau, \partial\Delta)}{d(\zeta, \partial W)} \sim 1, \quad \zeta \to q.$$

Here, for the second inequality, we used the fact that $K_W(\zeta; \xi) \sim d(\zeta, \partial W)^{-1}$ (see e.g. [4]).

Since the same argument applies to ϕ^{-1}, we actually get that $|\phi'(\zeta)| \sim 1$ as $\zeta \to q$, which proves the lemma. □

Remark 2. It is clear from the above proof that the base D of the fibered domain Ω can be even more general domains, which satisfies the following conditions:
i) The boundary rigidity theorem holds on D at a boundary fixed point p;
ii) D is C^2-smooth near p and (after a change of coordinates centered at p if necessary) there exists a small cone C with end p contained in D, which is the union of one-dimensional simply-connected domains with p on the boundary.

Remark 3. Theorem 5 can also be generalized to fibered domains with more general base D, which satisfies condition ii) above and the following condition:

i') The boundary rigidity theorem with interior fixed point holds on D (i.e. $F(z_0) = z_0$ for some $z_0 \in D$ and $F(z) = z + o(\|z-p\|^2)$ implying $F(z) \equiv z$). For instance, D can be chosen to be those in [5, Corollary 3].

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