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Hörmander Fourier multiplier theorems with optimal regularity in bi-parameter Besov spaces

Jiao Chen†, Liang Huang*, and Guozhen Lu*‡

The main aim of this paper to establish a bi-parameter version of a theorem of Baernstein and Sawyer [1] on boundedness of Fourier multipliers on one-parameter Hardy spaces $H^p(\mathbb{R}^n)$ which improves an earlier result of Calderón and Torchinsky [2]. More precisely, we prove the boundedness of the bi-parameter Fourier multiplier operators on the Lebesgue spaces $L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ $(1<p<\infty)$ and bi-parameter Hardy spaces $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ $(0<p\leq 1)$ with optimal regularity for the multiplier being in the bi-parameter Besov spaces $B_{2,1}^{(\frac{n_1}{2},\frac{n_2}{2})}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ and $B_{2,q}^{(s_1,s_2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$.

The Besov regularity assumption is clearly weaker than the assumption of the Sobolev regularity. Thus our results sharpen the known Hörmander multiplier theorem for the bi-parameter Fourier multipliers using the Sobolev regularity in the same spirit as Baernstein and Sawyer improved the result of Calderón and Torchinsky. Our method is differential from the one used by Baernstein and Sawyer in the one-parameter setting. We employ the bi-parameter Littlewood-Paley-Stein theory and atomic decomposition for the bi-parameter Hardy spaces $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ $(0<p\leq 1)$ to establish our main result (Theorem 1.6). Moreover, the bi-parameter nature involves much more subtlety in our situation where atoms are supported on arbitrary open sets instead of rectangles.

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*Corresponding authors: Liang Huang and Guozhen Lu.
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1. Introduction

We first recall some basics about the Fourier multiplier operators. For $m \in L^\infty(\mathbb{R}^n)$, the Fourier multiplier operator $\widetilde{T}_m$ is defined by

$$\widetilde{T}_m f(x) = \int_{\mathbb{R}^n} m(\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. The Mihlin multiplier theorem [25] says that if $m \in C^{[n/2]+1}(\mathbb{R}^n \setminus \{0\})$ satisfies

$$|\partial_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|}$$

for all $|\alpha| \le [n/2]+1$, then the Fourier multiplier operator $\widetilde{T}_m$ is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a Schwartz function in $\mathbb{R}^d$ (with d changing from time to time as needed) satisfying

(1.1)
$$\text{supp}\psi \subset \left\{\xi \in \mathbb{R}^d : \frac{1}{2} \le |\xi| \le 2\right\},\ \sum_{j\in\mathbb{Z}} \psi(\xi/2^j) = 1 \text{ for all } \xi \in \mathbb{R}^d\backslash\{0\}.$$

For $s \in \mathbb{R}$, the Sobolev space $W^s(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{W^s} \triangleq \|(I-\triangle)^{s/2} f\|_{L^2} < \infty, \tag{1.2}$$

where $(I-\triangle)^{s/2} f = \mathcal{F}^{-1}[(1+|\xi|^2)^{s/2}\hat{f}(\xi)]$ and $\xi \in \mathbb{R}^n$. Then the Hörmander multiplier theorem [18] says

Theorem 1.1. *If $m \in L^\infty(\mathbb{R}^n)$ satisfies*

$$\sup_{j\in\mathbb{Z}} ||m(2^j\cdot)\psi||_{W^s(\mathbb{R}^n)} < \infty \textit{ for all } s > \frac{n}{2},$$

where ψ is the same as in (1.1) when $d = n$ and $W^s(\mathbb{R}^n)$ is the Sobolev space, then the Fourier multiplier operator $\widetilde{T}_m$ defined with the symbol m is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

Calderón and Torchinsky [2] set up the following Hörmander's multiplier theorem on Hardy spaces.

Theorem 1.2. *If $m \in L^\infty(\mathbb{R}^n)$ satisfies*

$$\sup_{j\in\mathbb{Z}} ||m(2^j\cdot)\psi||_{W^s(\mathbb{R}^n)} < \infty \text{ for all } s > \frac{n}{p} - \frac{n}{2}, \tag{1.3}$$

where ψ is the same as in (1.1) when $d = n$ and $W^s(\mathbb{R}^n)$ is the Sobolev space, then the Fourier multiplier operator $\widetilde{T}_m$ defined with the symbol m is bounded from $H^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ for all $0 < p \le 1$.

Baernstein and Sawyer [1] obtained the following sharpened result when $0 < p < 1$ at the limiting case of (1.3), i.e., $s = \frac{n}{p} - \frac{n}{2}$.

Theorem 1.3. *If $m \in L^\infty(\mathbb{R}^n)$ satisfies*

$$\begin{aligned} &\sup_{j\in\mathbb{Z}} ||m(2^j\cdot)\psi||_{B^\alpha_{2,q}(\mathbb{R}^n)} < \infty \\ &\text{for all } \alpha = \frac{n}{p} - \frac{n}{2} \text{ and } q \le p \text{ or } \alpha > \frac{n}{p} - \frac{n}{2} \text{ and } 0 < q < \infty, \end{aligned} \tag{1.4}$$

where ψ is the same as in (1.1) when $d = n$ and $B^\alpha_{2,p}(\mathbb{R}^n)$ is the Besov space(see the definition of Besov space in Section 2), then $\widetilde{T}_m$ is bounded from $H^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ for all $0 < p < 1$.

Remark 1.1. If $p = 1$ and m satisfies (1.4) with $\alpha > \frac{n}{2}$, then T_m is still bounded from $H^1(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$. For $p = 1$, if m satisfies (1.4) and $\alpha = \frac{n}{2}$, a counterexample of Baernstein and Sawyer shows that T_m needs not be bounded on $H^1(\mathbb{R}^n)$.

In the bi-parameter setting, the Fourier multiplier operator is defined by

$$T_m(f)(x_1, x_2) := \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} m(\xi_1, \xi_2)\hat{f}(\xi_1, \xi_2)e^{2\pi i(x_1\cdot\xi_1 + x_2\cdot\xi_2)}d\xi_1 d\xi_2$$

for $f \in \mathcal{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, where $x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

We note that as convolution type singular integral operators in multi-parameter setting, the L^p boundedness for the bi-parameter Fourier multiplier operators follows from the work of R. Fefferman and Stein [17]. We also refer to [26] for L^p boundedness and [23] for H^p boundedness of a class of rather general non-convolutional type of multi-parameter singular integral operators. The Hardy $H^p \to H^p$ boundedness for the bi-parameter Fourier multipliers (thus being convolutional type singular integral operators) follows from the works [4], [5], [15], [24] without knowing the optimal regularity of the multipliers.

We set

$$m_{j_1,j_2}(\xi_1,\xi_2) = m(2^{j_1}\xi_1, 2^{j_2}\xi_2)\psi(\xi_1)\psi(\xi_2), \quad j_1, j_2 \in \mathbb{Z}, \tag{1.5}$$

where $\psi(\xi_1)$ is as in (1.1) with $d = n_1$ and $\psi(\xi_2)$ is as in (1.1) with $d = n_2$.

For $s_1, s_2 \in \mathbb{R}$, the two-parameter Sobolev space $W^{(s_1,s_2)}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined to be the class of all $f \in \mathcal{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ such that

$$\|f\|_{W^{(s_1,s_2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} = \|D_{(s_1,s_2)}f\|_{L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} < \infty, \tag{1.6}$$

where $D_{(s_1,s_2)}f(x_1,x_2) = \mathcal{F}^{-1}[(1+|\xi_1|^2)^{s_1/2}(1+|\xi_2|^2)^{s_2/2}\hat{f}(\xi_1,\xi_2)](x_1,x_2)$.

The following is a bi-parameter version of the Fourier multipliers theorem of Calderón and Torchinsky (see [3] and [10]). This sharpens results in [11, 16, 20]. The numbers $s_1 > n_1(\frac{1}{p} - \frac{1}{2})$ and $s_2 > n_2(\frac{1}{p} - \frac{1}{2})$ are optimal in the sense of the Sobolev regularity (see [3] and [10]).

Theorem 1.4. *Assume that $m(\xi)$ is a function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ satisfying*

$$\sup_{j_1,j_2\in\mathbb{Z}} \|m_{j_1,j_2}\|_{W^{(s_1,s_2)}} < \infty$$

with $s_i > n_i(\frac{1}{p} - \frac{1}{2})$ for $1 \le i \le 2$. Then T_m is bounded from $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ for all $0 < p \le 1$ and

$$\|T_m\|_{H^p\to H^p} \lesssim \sup_{j_1,j_2\in\mathbb{Z}} \|m_{j_1,j_2}\|_{W^{(s_1,s_2)}},$$

where m_{j_1,j_2} is defined by (1.5). Moreover, the smoothness assumption on s_i is optimal in the sense that there exists a multiplier m with some $s_i \le n_i(\frac{1}{p} - \frac{1}{2})$ such that T_m is not bounded on $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

We also refer to the reader to [7] and [19] for the Hörmander multiplier theorem in the anisotropic one and bi-parameter settings.

In the present paper, we shall consider the Fourier multipliers which satisfy the following conditions (1.7) or (1.9) with optimal regularity in the bi-parameter Besov spaces (see the definition in Section 2). We will obtain a limiting case of the above Theorem 1.4 using the bi-parameter Besov spaces regularity instead of using the bi-parameter Sobolev space regularity.

The following is the first main result, which gives the L^p-estimates of the bi-parameter Fourier multipliers for $1<p<\infty$.

Theorem 1.5. *Let $1<p<\infty$. Assume that $m(\xi_1,\xi_2)$ is a function on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ satisfying*

$$\sup_{j_1,j_2\in\mathbb{Z}}\|m_{j_1,j_2}\|_{B_{2,1}^{(n_1/2,n_2/2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}<\infty. \tag{1.7}$$

Then

$$\|T_m\|_{L^p\to L^p}\lesssim\sup_{j_1,j_2\in\mathbb{Z}}\|m_{j_1,j_2}\|_{B_{2,1}^{(n_1/2,n_2/2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.$$

Remark 1.2. We can show that the numbers $n_1/2$, $n_2/2$ in Theorem 1.5 are sharp, see Proposition 5.1.

For $s_1>n_1/2$ and $s_2>n_2/2$, it should be remarked that

$$W^{(s_1,s_2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})\hookrightarrow B_{2,1}^{(n_1/2,n_2/2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})\hookrightarrow L^\infty(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}).$$

Thus, we can obtain a corollary of Theorem 1.5.

Corollary 1.1. *Let $1<p<\infty$. Assume that $m(\xi_1,\xi_2)$ is a function on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ satisfying*

$$\sup_{j_1,j_2\in\mathbb{Z}}\|m_{j_1,j_2}\|_{W^{(s_1,s_2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}<\infty. \tag{1.8}$$

Then

$$\|T_m\|_{L^p\to L^p}\lesssim\sup_{j_1,j_2\in\mathbb{Z}}\|m_{j_1,j_2}\|_{W^{(s_1,s_2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})},$$

where $s_1>n_1/2$ and $s_2>n_2/2$.

The following theorem is the second main result.

Theorem 1.6. *Let $0 < p \le 1$ and $0 < q < \infty$. Assume that $m(\xi_1, \xi_2)$ is a function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ satisfying*

$$\sup_{j_1, j_2 \in \mathbb{Z}} \|m_{j_1,j_2}\|_{B_{2,q}^{(s_1,s_2)}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} < \infty. \tag{1.9}$$

Then

$$\|T_m\|_{H^p \to H^p} \lesssim \sup_{j_1, j_2 \in \mathbb{Z}} \|m_{j_1,j_2}\|_{B_{2,q}^{(s_1,s_2)}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})},$$

where $s_1 > n_1(1/p - 1/2)$ and $s_2 > n_2(1/p - 1/2)$.

Let $0 < p \le 1$, $s_1 > n_1(1/p - 1/2)$, $s_2 > n_2(1/p - 1/2)$ and $q \ge 2$. Notice that

$$W^{(s_1,s_2)} = B_{2,2}^{(s_1,s_2)}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \hookrightarrow B_{2,q}^{(s_1,s_2)}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$$

Thus, Theorem 1.4 is a corollary of Theorem 1.6.

Remark 1.3. One may wonder if Theorem 1.6 is still true when $0 < p \le 1$ and $s_1 = n_1(1/p - 1/2), s_2 = n_2(1/p - 1/2)$ as in the one-parameter case. In establishing our Theorem 1.6, we apply Fefferman's criterion (see Theorem 2.1 below) in the bi-parameter case. We provide an example here to show that the multiplier operators T_m do not satisfy Fefferman's criterion when $s_1 = n_1(1/p - 1/2), s_2 = n_2(1/p - 1/2)$. (see Section 5 for such an example.)

Remark 1.4. Furthermore, if we assume assume

$$\sup_{j_1, j_2 \in \mathbb{Z}} \|m_{j_1,j_2}\|_{B_{2,1}^{(s_1,s_2)}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} < \infty, \tag{1.10}$$

where $s_1 > n_1/2, s_2 > n_2/2$, then T_m is bounded on $H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and the numbers $n_1/2$ and $n_2/2$ are sharp, see Proposition 5.2.

Remark 1.5. We can also prove that the numbers $n_1(\frac{1}{p} - \frac{1}{2})$ and $n_2(\frac{1}{p} - \frac{1}{2})$ in Theorem 1.6 are sharp for $0 < p < 1$.

By duality of the product $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $CMO^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ (see [22]) and the $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ boundedness of T_m, we have

Theorem 1.7. *Assume that $m(\xi_1, \xi_2)$ is a function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ satisfying*

$$\sup_{j_1, j_2 \in \mathbb{Z}} \|m_{j_1,j_2}\|_{B_{2,q}^{(s_1,s_2)}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} < \infty, \tag{1.11}$$

with$s_1 > n_1(\frac{1}{p} - \frac{1}{2})$, $s_2 > n_2(\frac{1}{p} - \frac{1}{2})$ and $0 < p \le 1$ and $0 < q < \infty$.

Then T_m is bounded from $CMO^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $CMO^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Moreover

$$\|T_m\|_{CMO^p \to CMO^p} \lesssim \sup_{j_1, j_2 \in \mathbb{Z}} \|m_{j_1,j_2}\|_{B^{(s_1,s_2)}_{2,q}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}.$$

In the case of $p = 1$, we derive the boundedness of T_m on the bi-parameter $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ under the assumption that the multiplier m satisfies the minimal smoothness $s_1 > n_1/2$ and $s_2 > n_2/2$.

As we pointed out earlier, the following relationship shows that our main results (Theorems 1.5 and 1.6) indeed improve the known Hörmander type multiplier theorem with regularity in the bi-parameter Sobolev spaces (e.g., Theorem 1.4):

$$W^{(s_1,s_2)}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = B^{(s_1,s_2)}_{2,2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \hookrightarrow B^{(s_1,s_2)}_{2,q}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$$

for $0 < p \leq 1$, $q \geq 2$, $s_1 > n_1(1/p - 1/2)$ and $s_2 > n_2(1/p - 1/2)$ and

$$W^{(s_1,s_2)}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \hookrightarrow B^{(n_1/2,n_2/2)}_{2,1}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \hookrightarrow L^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$$

for $s_1 > n_1/2$ and $s_2 > n_2/2$.

We finally mention that Hörmander Fourier multiplier theorems with optimal Besov regularity on Hardy spaces of arbitrary number of parameters have been recently established by the authors [8]. It requires different ideas since the Fefferman's boundedness criterion fails in the case of three or more parameters.

We use the notations $A \approx B$ to denote $C^{-1}B \leq A \leq CB$ for some absolute constant $C \geq 1$ and $A \lesssim B$ to denote $A \leq CB$ for some absolute constant $C > 0$.

The organization of this paper is as follows: In Section 2 we recall some preliminary facts and give some relevant definitions. In Sections 3 and 4, we prove Theorems 1.5 and 1.6 respectively. In Section 5, the sharpness of the conditions of Theorem 1.5 and Theorem 1.6 are discussed and an example is constructed to show T_m does not satisfy Fefferman's criterion.

2. Preliminary results

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the

Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \int_{\mathbb{R}^n} f(\xi)e^{2\pi i x\cdot\xi}d\xi.$$

For $m \in L^\infty(\mathbb{R}^n)$, the linear Fourier multiplier operator $m(D)$ is defined by

$$m(D)f(x) = \mathcal{F}^{-1}[m\hat{f}](x) = \int_{\mathbb{R}^n} m(\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

The Hardy-Littlewood maximal function $\mathcal{M}$ is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy,$$

where f is a locally integral function on $\mathbb{R}^n$.

We recall the definition of Besov spaces. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ be such that

$$\text{(2.12)} \quad \text{supp}\varphi \subset \{\xi \in \mathbb{R}^d : 1/2 \le |\xi| \le 2\}, \quad \sum_{k\in\mathbb{Z}} \varphi(\xi/2^k) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

And see $\varphi_0(\xi) = 1 - \sum_{k=1}^\infty \varphi(\xi/2^k)$, $\varphi_k(\xi) = \varphi(\xi/2^k)$ for $k \ge 1$.

For $0 < p, q \le \infty$ and $s \in \mathbb{R}$, the Besov space $B^s_{p,q}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{k=0}^\infty 2^{ksq}\|\varphi_k(D)f\|^q_{L^p}\right)^{1/q} < \infty.$$

The norm of the Besov space of the product type $B^{(s_1,s_2)}_{p,q}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, $s_1, s_2 \in \mathbb{R}$, for $f \in \mathcal{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is also defined by

$$\|f\|_{B^{(s_1,s_2)}_{p,q}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} = \left(\sum_{k_1,k_2=0}^\infty 2^{(k_1s_1+k_2s_2)q}\|\Phi_{(k_1,k_2)}(D)f\|^q_{L^p}\right)^{1/q} < \infty,$$

where

$$\text{(2.13)} \qquad \begin{aligned}&\Phi_{(k_1,k_2)}(\xi) = (\varphi_{k_1} \otimes \varphi_{k_2})(\xi) = \varphi_{k_1}(\xi_1)\varphi_{k_2}(\xi_2),\\ &\xi = (\xi_1, \xi_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.\end{aligned}$$

Let us recall the definition of bi-parameter Hardy spaces and atomic decomposition of the product Hardy spaces $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. For $\psi(\xi_i) \in \mathcal{S}(\mathbb{R}^{n_i})$

satisfy condition (1.1) for $i = 1, 2$ and set $\Psi_{j_1,j_2}(x_1, x_2) = \psi_{j_1}(x_1)\psi_{j_2}(x_2)$. The product Littlewood-Paley square function of $f \in \mathcal{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined by

$$f^*(x_1, x_2) = \left(\sum_{j_1, j_2 \in \mathbb{Z}} |\Psi_{j_1,j_2}(D_1, D_2) f(x_1, x_2)|^2 \right)^{1/2}.$$

For $0 < p \leq 1$, the product Hardy space $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ can be defined by

$$H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \{f \in \mathcal{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) : f^* \in L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})\}$$

with $\|f\|_{H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} := \|f^*\|_{L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}$.

A function $a(x_1, x_2)$ defined in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is called an $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ atom if $a(x_1, x_2)$ is supported in an open set $\Omega \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with finite measure and satisfies the following conditions:

(i) $\|a\|_{L^2} \leq |\Omega|^{1/2-1/p}$,

(ii) a can further be decomposed as $a(x_1, x_2) = \sum_{R \in \mathrm{M}(\Omega)} a_R(x_1, x_2)$, where a_R are supported on the double of $R = I \times J$ (I a dyadic cube in $\mathbb{R}^{n_1}$, J a dyadic cube in $\mathbb{R}^{n_2}$) and $\mathrm{M}(\Omega)$ is the collection of all maximal dyadic rectangles contained in Ω,

$$\left\{ \sum_{R \in \mathrm{M}(\Omega)} \|a_R\|_{L^2}^2 \right\}^{1/2} \leq |\Omega|^{1/2-1/p},$$

(iii) $\int_{2I} a_R(x_1, x_2) x_1^\alpha dx_1 = 0 \quad$ for all $x_2 \in \mathbb{R}^{n_2}$, $\quad 0 \leq |\alpha| \leq N_{p,n_1}$,
$\int_{2J} a_R(x_1, x_2) x_2^\beta dx_2 = 0 \quad$ for all $x_1 \in \mathbb{R}^{n_1}$, $\quad 0 \leq |\beta| \leq N_{p,n_2}$,
where N_{p,n_1}, N_{p,n_2} are large integers depending on n_1, n_2 and p.

Chang and R. Fefferman [4, 5] proved the atomic decomposition of product Hardy spaces $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Moreover, we also employ in the proof of Theorem 1.6 the following boundedness criterion which was established by R. Fefferman [15].

Theorem 2.1. *Let $0 < p \leq 1$ and T be a bounded linear operator on $L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Suppose that there exist constants $C > 0$ and $\delta > 0$ such that, for any $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ rectangle atom a supported on R,*

$$\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \setminus \gamma R} |Ta(x_1, x_2)|^p dx_1 dx_2 \leq C\gamma^{-\delta} \quad \textit{for all } \gamma \geq 2, \tag{2.14}$$

where γR denotes the concentric γ-fold dilation of R. Then T is a bounded operator from $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

We now recall the definition of the dual space of weighted multi-parameter Hardy spaces $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ introduced in [22] using the Littlewood-Paley-Stein square functions. We only consider the nonweighted case here. It is the so-called Carleson measure space $CMO^p = CMO^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. We refer to [22] for more details.

Definition 2.1. For $0 < p \le 1$, we call $f \in CMO^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ if $f \in (\mathcal{S}_\infty)'(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ with the finite norm defined by

$$\sup_{\Omega}\left\{\frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{j,k\in\mathbb{Z}}\sum_{I_1\times I_2}|\psi_1(D/2^j)\psi_2(D/2^k)f(2^{-jl_1},2^{-kl_2})|^2\times|I_1\times I_2|\right\}^{1/2}$$

for all open sets Ω in $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ with finite measures, here I_1 are dyadic cubes in $\mathbb{R}^{n_1}$ with the side length 2^{-j} and the left lower corners of I_1 is $2^{-j}l_1, l_1\in\mathbb{Z}^{n_1}$ and I_2 are dyadic cubes in $\mathbb{R}^{n_2}$ with the side length 2^{-k} and the left lower corners of I_2 is $2^{-k}l_2, l_2\in\mathbb{Z}^{n_2}$.

We will use Littlewood-Paley-Stein square functions to prove our optimal Fourier multiplier theorem using the Besov space regularity. We remark the multi-parameter local Hardy space theory has also been developed using the Littlewood-Paley-Stein square functions and boundedness of multi-parameter singular integrals and pseudo-differential operators on such spaces has been established (see [12], [13], [14], [6]).

We also need the definition of the strong maximal operator $\mathcal{M}_s$ in [21]. Suppose that f is a locally integrable function on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, then $\mathcal{M}_s(f)$ is defined by

$$\mathcal{M}_s f(x_1,x_2) = \sup_{r_1,r_2>0}\frac{1}{{r_1}^n}\frac{1}{{r_2}^n}\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}|f(y_1,y_2)|dy_1dy_2, \tag{2.15}$$

where $R=\{(y_1,y_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : |y_1-x_1|<r_1, |y_2-x_2|<r_2\}$. It is well known that $\mathcal{M}_s$ is bounded on $L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ for all $1<p<\infty$ (see [9]).

The following lemma and proposition will be used later, we can find them in [27].

Lemma 2.1. *Let $s_1, s_2 > 0$, $1\le p\le\infty$ and $0\le q<\infty$. Then there exists a constant C such that the estimate*

$$\|f\cdot g\|_{B^{(s_1,s_2)}_{p,q}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \le C\|f\|_{B^{(s_1,s_2)}_{p,q}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\|g\|_{B^{(s_1,s_2)}_{\infty,q}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \tag{2.16}$$

holds for all f in $B^{(s_1,s_2)}_{p,q}$ and all g in $B^{(s_1,s_2)}_{\infty,q}$.

Proposition 2.1. *Let $s_1, s_2 > 0$, $1 \le p \le \infty$ and $0 \le q < \infty$. Then the following inequality holds:*

$$\|f(2^{l_1}\cdot, 2^{l_2}\cdot)\|_{B^{(s_1,s_2)}_{p,q}} \lesssim \max\{1, 2^{l_1 s_1}\}2^{-l_1 n_1/2} \max\{1, 2^{l_2 s_2}\}2^{-l_2 n_2/2}\|f\|_{B^{(s_1,s_2)}_{p,q}}$$

holds for all f in $B^{(s_1,s_2)}_{p,q}$ and all g in $B^{(s_1,s_2)}_{\infty,q}$.

We also need the following result which will be used in the proof of Theorem 1.5.

Theorem 2.2. *([17]) Let $1 < p < \infty$, and let $\psi_1 \in \mathcal{S}(R^{n_1}), \psi_2 \in \mathcal{S}(R^{n_2})$ be such that supp$\psi_1 \subset \{\xi \in \mathbb{R}^{n_1} : 1/a \le |\xi| \le a\}$ for some $a > 1$, supp$\psi_2 \subset \{\eta \in \mathbb{R}^{n_2} : 1/b \le |\eta| \le b\}$ for some $b > 1$. Then there exists a constant $C > 0$ such that*

$$\left\|\left\{\sum_{j,k\in\mathbb{Z}} |\psi_1(D/2^j)\psi_2(D/2^k)f|^2\right\}^{1/2}\right\|_{L^p} \le C\|f\|_{L^p} \text{ for all } f \in L^p(\mathbb{R}^{n_1+n_2}) \tag{2.17}$$

where $[\psi_1(D/2^j)\psi_2(D/2^k)f](\xi_1,\xi_2) = \mathcal{F}^{-1}\left[\psi_1(\cdot/2^j)\psi_2(\cdot/2^k)\hat{f}(\cdot,\cdot)\right](\xi_1,\xi_2)$.

Moreover, if $\sum_{j\in\mathbb{Z}} \psi_i(\xi_i/2^j) = 1$ for all $\xi_i \ne 0$, for $i = 1, 2$, then

$$\left\|\left\{\sum_{j,k\in\mathbb{Z}} |\psi_1(D/2^j)\psi_2(D/2^k)f|^2\right\}^{1/2}\right\|_{L^p} \approx \|f\|_{L^p} \text{ for all } f \in L^p(\mathbb{R}^{n_1+n_2}). \tag{2.18}$$

Lemma 2.2. *Let $s_1 > 0, s_2 > 0$, $q > 0$, and let $\psi'(\xi_i) \in \mathcal{S}(\mathbb{R}^{n_i})$ be such that supp$\psi'(\xi_i)$ is a compact subset of $\mathbb{R}^{n_i} \setminus \{0\}$ for $i = 1, 2$. Assume that $\phi \in C^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \setminus \{0\})$ satisfies*

$$|\partial^{\alpha_1}_{\xi_1}\partial^{\alpha_2}_{\xi_2}\phi(\xi_1,\xi_2)| \le C_{\alpha_1,\alpha_2}(|\xi_1| + |\xi_2|)^{-(|\alpha_1|+|\alpha_2|)}$$

for all multi-indices α_1, α_2. Then there exists a constant $C > 0$ such that

$$\sup_{t_1,t_2>0} \|m(t_1\cdot, t_2\cdot)\phi(t_1\cdot, t_2\cdot)\psi'\psi'\|_{B^{(s_1,s_2)}_{2,q}} \le C \sup_{j_1,j_2\in\mathbb{Z}} \|m_{j_1,j_2}\|_{B^{(s_1,s_2)}_{2,q}}$$

for all $m \in L^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ satisfying $\sup_{j_1,j_2\in\mathbb{Z}} \|m_{j_1,j_2}\|_{B^{(s_1,s_2)}_{2,q}} < \infty$, where m_{j_1,j_2} is defined by (1.5).

Proof. We may assume that $\mathrm{supp}\psi'(\xi_1) \subset \{\xi_1 \in \mathbb{R}^{n_1} : 1/2^{l_1} \le |\xi_1| \le 2^{l_1}\}$ and $\mathrm{supp}\psi'(\xi_2) \subset \{\xi_2 \in \mathbb{R}^{n_2} : 1/2^{l_2} \le |\xi_2| \le 2^{l_2}\}$ for some $l_1, l_2 \in \mathbb{N}$. Given $t_1, t_2 > 0$, take $j_1, j_2 \in \mathbb{Z}$ satisfying $2^{j_1-1} \le t_1 \le 2^{j_1}$ and $2^{j_2-1} \le t_2 \le 2^{j_2}$. Then, since $1 < 2^{j_1}/t_1 \le 2$ and $1 < 2^{j_2}/t_2 \le 2$, by a change of variables and Proposition 2.1,

$$\begin{aligned}&\|m(t_1\cdot, t_2\cdot)\phi(t_1\cdot, t_2\cdot)\psi'\psi'\|_{B^{(s_1,s_2)}_{2,q}}\\ &\qquad \le C\|m(2^{j_1}\cdot, 2^{j_2}\cdot)\phi(2^{j_1}\cdot, 2^{j_2}\cdot)\psi'(2^{j_1}t_1^{-1}\cdot)\psi'(2^{j_2}t_2^{-1}\cdot)\|_{B^{(s_1,s_2)}_{2,q}}.\end{aligned}$$

Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be as in (1.1) with $d = n_1$ and $d = n_2$, and note that

$$\mathrm{supp}\psi(\xi_1/2^{k_1}) \subset \{2^{k_1-1} \le |\xi_1| \le 2^{k_1+1}\}$$

and

$$\mathrm{supp}\psi(\xi_2/2^{k_2}) \subset \{2^{k_2-1} \le |\xi_2| \le 2^{k_2+1}\}.$$

Set $\Psi(\xi_1, \xi_2) = \psi(\xi_1)\psi(\xi_2)$ and $\Psi'(\xi_1, \xi_2) = \psi'(\xi_1)\psi'(\xi_2)$. Using

$$\mathrm{supp}\psi'(2^{j_1}t_1^{-1}\xi_1) \subset \{1/2^{l_1+1} \le |\xi_1| \le 2^{l_1}\}$$

and

$$\mathrm{supp}\psi'(2^{j_2}t_2^{-1}\xi_2) \subset \{1/2^{l_2+1} \le |\xi_2| \le 2^{l_2}\},$$

we have by Lemma 2.1

$$\begin{aligned}&\|m(2^{j_1}\cdot, 2^{j_2}\cdot)\phi(2^{j_1}\cdot, 2^{j_2}\cdot)\Psi'(2^{j_1}t_1^{-1}\cdot, 2^{j_2}t_2^{-1}\cdot)\|_{B^{(s_1,s_2)}_{2,q}}\\ &\quad \le \sum_{k_1=-(l_1+1)}^{l_1}\sum_{k_2=-(l_2+1)}^{l_2} \|m(2^{j_1}\cdot, 2^{j_2}\cdot)\phi(2^{j_1}\cdot, 2^{j_2}\cdot)\\ &\qquad\qquad \times \Psi'(2^{j_1}t_1^{-1}\cdot, 2^{j_2}t_2^{-1}\cdot)\Psi(\cdot/2^{k_1}, \cdot/2^{k_2})\|_{B^{(s_1,s_2)}_{2,q}}\end{aligned}$$

$$\begin{aligned}&\le C\sum_{k_1=-(l_1+1)}^{l_1}\sum_{k_2=-(l_2+1)}^{l_2} \|m(2^{j_1}\cdot, 2^{j_2}\cdot)\Psi(\cdot/2^{k_1}, \cdot/2^{k_2})\|_{B^{(s_1,s_2)}_{2,q}}\\ &\qquad\qquad \times \|\phi(2^{j_1}\cdot, 2^{j_2}\cdot)\Psi'(2^{j_1}t_1^{-1}\cdot, 2^{j_2}t_2^{-1}\cdot)\|_{B^{(s_1,s_2)}_{\infty,q}}\\ &\le C\sum_{k_1=-(l_1+1)}^{l_1}\sum_{k_2=-(l_2+1)}^{l_2} \|m(2^{j_1+k_1}\cdot, 2^{j_2+k_2}\cdot)\Psi\|_{B^{(s_1,s_2)}_{2,q}}\|\phi(t_1\cdot, t_2\cdot)\Psi'\|_{B^{(s_1,s_2)}_{\infty,q}}\\ &\le C\left(\sup_{j_1,j_2\in\mathbb{Z}}\|m_{j_1,j_2}\|_{B^{(s_1,s_2)}_{2,q}}\right)\left(\sup_{t_1,t_2>0}\|\phi(t_1\cdot, t_2\cdot)\Psi'\|_{B^{(s_1,s_2)}_{\infty,q}}\right).\end{aligned}$$

Since $|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\phi(t_1\xi_1, t_2\xi_2)| \le C_{\alpha_1,\alpha_2}(|\xi_1| + |\xi_2|)^{-(|\alpha_1|+|\alpha_2|)}$, and $\mathrm{supp}\Psi'$ does not contain the origin, we have $|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}(\phi(t_1\cdot, t_2\cdot)\Psi')| \le C_{\alpha_1,\alpha_2}$ for all α_1, α_2 and t_1, t_2, and consequently,

$$\sup_{t_1,t_2>0} \|\phi(t_1\cdot, t_2\cdot)\Psi'\|_{B_{\infty,q}^{(s_1,s_2)}} < \infty.$$

The proof is complete. □

Remark 2.1. By Lemma 2.2, we have

$$\sum_{k_1,k_2} 2^{(k_1s_1+k_2s_2)q}\|\varphi_{k_1}\varphi_{k_2}\mathcal{F}^{-1}[\xi_1^{\alpha_1}\xi_2^{\alpha_2}m_{j_1,j_2}(\xi_1,\xi_2)]\|_{L^2}^q \le \sup_{j_1,j_2\in\mathbb{Z}} \|m_{j_1,j_2}\|_{B_{2,q}^{(s_1,s_2)}}^q,$$

where $s_1, s_2 > 0$ and m_{j_1,j_2} is defined by (1.5). In fact, since

$$\mathcal{F}^{-1}[\xi_1^{\alpha_1}\xi_2^{\alpha_2}m_{j_1,j_2}(\xi_1,\xi_2)] = \mathcal{F}^{-1}[m(2^{j_1}\cdot, 2^{j_2}\cdot)\xi_1^{\alpha_1}\xi_2^{\alpha_2}\Psi],$$

the estimate follows from Lemma 2.2 with $\phi = 1$ and $\psi'\psi' = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\Psi$.

We first prove the following lemma which is needed in our proof of Theorem 1.5.

Lemma 2.3. *Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ be such that $\phi(x) = \phi(-x)$, $x \in \mathbb{R}^d$, and $\phi(x) = 1$ on $\{x \in \mathbb{R}^d : |x| \le 2\}$. Then*

$$\begin{aligned}|T_{m(\cdot/2^{j_1},\cdot/2^{j_2})}(f)(x)| \lesssim \sum_{k_1,k_2=0}^{\infty} & 2^{(k_1n_1+k_2n_2)/2}\|\Phi_{(k_1,k_2)}(D)m\|_{L^2} \\ & \times [(|\phi|^2)_{(k_1-j_1)}(|\phi|^2)_{(k_2-j_2)}] * |f|^2(x)^{1/2}\end{aligned}$$

for $x=(x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ and $j_1, j_2\in\mathbb{Z}$, where $(|\phi|^2)_{(k)}(y)=2^{-kd}|\phi(2^{-k}y)|^2$ with $d = n_1$, $d = n_2$ and $\Phi_{(k_1,k_2)}$ is defined by (2.13).

Proof. Let $\{\varphi_{k_i}\}_{k_i=0}^{\infty}$ be the partition of unity appearing in the definition of Besov spaces of the product type for $i = 1, 2$. Then

$$\begin{aligned}&T_{m(\cdot/2^{j_1},\cdot/2^{j_2})}(f)(x) \\ &\quad= 2^{j_1n_1+j_2n_2}\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \mathcal{F}^{-1}m(2^{j_1}(x_1-y_1), 2^{j_2}(x_2-y_2))f(y_1,y_2)dy_1dy_2 \\ &\quad= 2^{j_1n_1+j_2n_2}\sum_{k_1,k_2=0}^{\infty}\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \varphi_{k_1}(2^{j_1}(y_1-x_1))\varphi_{k_2}(2^{j_2}(y_2-x_2)) \\ &\qquad\times \widehat{m}(2^{j_1}(y_1-x_1), 2^{j_2}(y_2-x_2))f(y_1,y_2)dy_1dy_2.\end{aligned}$$

Since $\mathrm{supp}\varphi_{k_i} \subset \{y_i \in \mathbb{R}^{n_i} : |y_i| \le 2^{k_i+1}\}$, we have $\varphi_{k_i}(y_i) = \varphi_{k_i}(y_i)\phi(y_i/2^{k_i})$. Hence, using Schwartz's inequality, a change of variables and $\phi(y_i) = \phi(-y_i)$, we have

$$\begin{aligned}
&|T_{m(\cdot/2^{j_1},\cdot/2^{j_2})}(f)(x)| \\
&\quad \lesssim 2^{j_1n_1+j_2n_2} \sum_{k_1,k_2=0}^{\infty} \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} [\Phi_{(k_1,k_2)}\widehat{m}](2^{j_1}(y_1-x_1), 2^{j_2}(y_2-x_2)) \\
&\qquad \times \phi\left(\frac{2^{j_1}(y_1-x_1)}{2^{k_1}}\right)\phi\left(\frac{2^{j_2}(y_2-x_2)}{2^{k_2}}\right) f(y_1,y_2)dy_1dy_2 \\
&\quad \le 2^{j_1n_1+j_2n_2} \sum_{k_1,k_2=0}^{\infty} \|[\Phi_{(k_1,k_2)}\widehat{m}](2^{j_1}(y_1-x_1), 2^{j_2}(y_2-x_2))\|_{L^2} \\
&\qquad \times \left\|\phi\left(\frac{y_1-x_1}{2^{k_1-j_1}}\right)\phi\left(\frac{y_2-x_2}{2^{k_2-j_2}}\right) f(y_1,y_2)\right\|_{L^2} \\
&\quad = \sum_{k_1,k_2=0}^{\infty} 2^{(k_1n_1+k_2n_2)/2}\|\Phi_{(k_1,k_2)}(D)m\|_{L^2} \\
&\qquad \times [(|\phi|^2)_{(k_1-j_1)}(|\phi|^2)_{(k_2-j_2)}] * |f|^2(x)^{1/2}.
\end{aligned}$$

This completes the proof of Lemma 2.3. □

3. The proof of main theorem: Theorem 1.5

In this section, we prove Theorem 1.5.

Proof. First, we consider $2 < p < \infty$. We obtain that

$$\|T_m(f)\|_{L^p} \lesssim \left\|\left(\sum_{j_1,j_2\in\mathbb{Z}} |\Delta_{j_1,j_2}T_m(f)|^2\right)^{1/2}\right\|_{L^p}.$$

Here, we use $\mathbf{A}$ to denote the set of $\psi \in \mathcal{S}(\mathbb{R}^d)$ for which $\mathrm{supp}\psi$ is a compact subset of $\mathbb{R}^d \setminus \{0\}$. Then we can find functions $\widetilde{\psi} \in \mathbf{A}$ independent of j_1, j_2 such that

$$\begin{aligned}
&m(\xi_1,\xi_2)\psi(\xi_1/2^{j_1})\psi(\xi_2/2^{j_2}) \\
&\qquad = m(\xi_1,\xi_2)\psi(\xi_1/2^{j_1})\psi(\xi_2/2^{j_2})\widetilde{\psi}(\xi_1/2^{j_1})\widetilde{\psi}(\xi_2/2^{j_2}),
\end{aligned}$$

where we used the fact that $\mathrm{supp}\psi(\xi_i) \subset \{\xi_i \in \mathbb{R}^{n_i} : 1/2 \le |\xi_i| \le 2\}$. Hence, we see that

$$\begin{aligned}
&\Delta_{j_1,j_2} T_m(f)(x) \\
&\quad = \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} m(\xi_1,\xi_2)\psi(\xi_1/2^{j_1})\psi(\xi_2/2^{j_2})\widehat{f}(\xi_1,\xi_2)e^{2\pi i(x_1\xi_1+x_2\xi_2)}d\xi_1 d\xi_2 \\
&\quad = T_{m_{j_1,j_2}(\cdot/2^{j_1},\cdot/2^{j_2})}(\widetilde{\Delta}_{j_1,j_2} f)(x),
\end{aligned}$$

where $\widetilde{\Delta}_{j_1,j_2} f = [\widetilde{\psi}(D/2^{j_1})\widetilde{\psi}(D/2^{j_2})]f$.

By Lemma 2.3, we have

$$\begin{aligned}
&|T_{m_{j_1,j_2}(\cdot/2^{j_1},\cdot/2^{j_2})}(\widetilde{\Delta}_{j_1,j_2} f)(x)| \\
&\quad \lesssim \sum_{k_1,k_2=0}^{\infty} 2^{(k_1n_1+k_2n_2)/2}\|\Phi_{(k_1,k_2)}(D)m_{j_1,j_2}\|_{L^2} \\
&\qquad\qquad \times [(|\phi|^2)_{(k_1-j_1)}(|\phi|^2)_{(k_2-j_2)}] * |\widetilde{\Delta}_{j_1,j_2} f|^2(x)^{1/2} \\
&\quad =: E_{j_1,j_2}(x).
\end{aligned}$$

It follows from Schwartz's inequality that $E_{j_1,j_2}(x)$ is estimated by

$$\begin{aligned}
E_{j_1,j_2}(x) \le{}& \left(\sum_{k_1,k_2=0}^{\infty} 2^{(k_1n_1+k_2n_2)/2}\|\Phi_{(k_1,k_2)}(D)m_{j_1,j_2}\|_{L^2}\right)^{1/2} \\
&\times \Bigg\{\sum_{k_1,k_2=0}^{\infty} 2^{(k_1n_1+k_2n_2)/2}\|\Phi_{(k_1,k_2)}(D)m_{j_1,j_2}\|_{L^2} \\
&\quad \times [(|\phi|^2)_{(k_1-j_1)}(|\phi|^2)_{(k_2-j_2)}] * |\widetilde{\Delta}_{j_1,j_2} f|^2(x)\Bigg\}^{1/2} \\
\le{}& \|m_{j_1,j_2}\|^{1/2}_{B_{2,1}^{(n_1/2,n_2/2)}} \Bigg\{\sum_{k_1,k_2=0}^{\infty} 2^{(k_1n_1+k_2n_2)/2}\|\Phi_{(k_1,k_2)}(D)m_{j_1,j_2}\|_{L^2} \\
&\times [(|\phi|^2)_{(k_1-j_1)}(|\phi|^2)_{(k_2-j_2)}] * |\widetilde{\Delta}_{j_1,j_2} f|^2(x)\Bigg\}^{1/2}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
\|T_m(f)\|_{L^p} &\lesssim \left\| \left(\sum_{j_1,j_2\in\mathbb{Z}} |\Delta_{j_1,j_2} T_m(f)|^2 \right)^{1/2} \right\|_{L^p} \\
&\lesssim \left(\sup_{j_1,j_2} \|m_{j_1,j_2}\|_{B_{2,1}^{(n_1/2,n_2/2)}} \right)^{1/2} \\
&\quad \times \left\| \left(\sum_{j_1,j_2\in\mathbb{Z}} \sum_{k_1,k_2=0}^{\infty} 2^{(k_1n_1+k_2n_2)/2} \|\Phi_{(k_1,k_2)}(D) m_{j_1,j_2}\|_{L^2} \right.\right. \\
&\quad\quad \left.\left. \times [(|\phi|^2)_{(k_1-j_1)}(|\phi|^2)_{(k_2-j_2)}] * |\widetilde{\Delta}_{j_1,j_2} f|^2(x) \right)^{1/2} \right\|_{L^p}.
\end{aligned}
$$

In order to estimate the above L^p-norm, we use a duality argument. Let $g \in \mathcal{S}$ be such that $\|g\|_{L^{(p/2)'}} = 1$. Then, we have

$$
\begin{aligned}
\|T_m(f)\|_{L^p} &\lesssim \left(\sup_{j_1,j_2} \|m_{j_1,j_2}\|_{B_{2,1}^{(n_1/2,n_2/2)}} \right)^{1/2} \\
&\quad \times \left| \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \left(\sum_{j_1,j_2\in\mathbb{Z}} \sum_{k_1,k_2=0}^{\infty} 2^{(k_1n_1+k_2n_2)/2} \|\Phi_{(k_1,k_2)}(D) m_{j_1,j_2}\|_{L^2} \right.\right. \\
&\quad\quad \left.\left. \times [(|\phi|^2)_{(k_1-j_1)}(|\phi|^2)_{(k_2-j_2)}] * |\widetilde{\Delta}_{j_1,j_2} f|^2(x) \right) g(x)\,dx_1dx_2 \right|^{1/2} \\
&\lesssim \left(\sup_{j_1,j_2} \|m_{j_1,j_2}\|_{B_{2,1}^{(n_1/2,n_2/2)}} \right)^{1/2} \\
&\quad \times \left\{ \sum_{j_1,j_2\in\mathbb{Z}} \sum_{k_1,k_2=0}^{\infty} 2^{(k_1n_1+k_2n_2)/2} \|\Phi_{(k_1,k_2)}(D) m_{j_1,j_2}\|_{L^2} \right. \\
&\quad\quad \left. \times \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} |\widetilde{\Delta}_{j_1,j_2} f|^2(x) [(|\phi|^2)_{(k_1-j_1)}(|\phi|^2)_{(k_2-j_2)}] * g(x)\,dx_1dx_2 \right\}^{1/2} \\
&\lesssim \left(\sup_{j_1,j_2} \|m_{j_1,j_2}\|_{B_{2,1}^{(n_1/2,n_2/2)}} \right)^{1/2} \\
&\quad \times \left\{ \sup_{j_1,j_2\in\mathbb{Z}} \sum_{k_1,k_2=0}^{\infty} 2^{(k_1n_1+k_2n_2)/2} \|\Phi_{(k_1,k_2)}(D) m_{j_1,j_2}\|_{L^2} \right\}^{1/2} \\
&\quad \times \left\{ \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \left(\sum_{j_1,j_2\in\mathbb{Z}} |\widetilde{\Delta}_{j_1,j_2} f|^2(x) \right) \mathcal{M}_s g(x)\,dx_1dx_2 \right\}^{1/2}
\end{aligned}
$$

$$\lesssim \left(\sup_{j_1,j_2} \|m_{j_1,j_2}\|_{B_{2,1}^{(n_1/2,n_2/2)}}\right)$$
$$\times \left\{ \left\| \sum_{j_1,j_2\in\mathbb{Z}} |\widetilde{\Delta}_{j_1,j_2} f|^2(x) \right\|_{L^{p/2}} \|\mathcal{M}_s g(x)\|_{L^{(p/2)'}} \right\}^{1/2}$$
$$\lesssim \left(\sup_{j_1,j_2} \|m_{j_1,j_2}\|_{B_{2,1}^{(n_1/2,n_2/2)}}\right) \|f\|_{L^p} \|g\|_{L^{(p/2)'}}^{1/2}.$$

By taking the supremum over all g as above, we have

$$\|T_m(f)\|_{L^p} \lesssim (\sup_{j_1,j_2} \|m_{j_1,j_2}\|_{B_{2,1}^{(n_1/2,n_2/2)}}) \|f\|_{L^p}. \tag{3.19}$$

For the proof of $1 < p < 2$, we use duality. Then the dual space of $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is $L^{p'}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Therefore exists a function $g \in L^{p'}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ such that

$$\begin{aligned}
\|T_m(f)\|_{L^p} &= \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} T_m(f)(x) g(x) dx_1 dx_2 \\
&= \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} f(x) T_m(g)(x) dx_1 dx_2 \\
&\leq \|f\|_{L^p} \|T_m(g)\|_{L^{p'}} \\
&\lesssim \left(\sup_{j_1,j_2} \|m_{j_1,j_2}\|_{B_{2,1}^{(n_1/2,n_2/2)}}\right) \|f\|_{L^p} \|g\|_{L^{p'}}.
\end{aligned}$$

By taking the supremum over all g as above, we have

$$\|T_m(f)\|_{L^p} \lesssim \left(\sup_{j_1,j_2} \|m_{j_1,j_2}\|_{B_{2,1}^{(n_1/2,n_2/2)}}\right) \|f\|_{L^p}. \tag{3.20}$$

For $p = 2$, by Plancherel's theorem and the embedding theorem, we have

$$\|T_m(f)\|_{L^2} \lesssim \left(\sup_{j_1,j_2} \|m_{j_1,j_2}\|_{B_{2,1}^{(n_1/2,n_2/2)}}\right) \|f\|_{L^2}. \tag{3.21}$$

A combination of (3.19), (3.20) and (3.21) yields

$$\|T_m(f)\|_{L^p} \lesssim \left(\sup_{j_1,j_2} \|m_{j_1,j_2}\|_{B_{2,1}^{(n_1/2,n_2/2)}}\right) \|f\|_{L^p}$$

for all $1 < p < \infty$. This completes the proof. □

4. The proof of main theorem: Theorem 1.6

In this section, we prove Theorem 1.6.

First we write $*_2$ for the convolution operational symbols in variables x_2 and $\widehat{f}^2$ for the Fourier transform acting only on x_2 variables.

For $0 < p \leq 1$ and $0 < q_1 \leq q_2 < \infty$, then

$$B_{2,q_1}^{(s_1,s_2)}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \hookrightarrow B_{2,q_2}^{(s_1,s_2)}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}).$$

Therefore, we just consider $\|m_{j_1,j_2}\|_{B_{2,q}^{(s_1,s_2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} < \infty$ for $q > p$.

Since T_m is a convolution operator, we have

$$\|T_m\|_{L^2\to L^2} \lesssim \|m\|_{L^\infty}.$$

By the Besov embedding theorem, we then have

$$\|m\|_{L^\infty} \lesssim \sup_{j_1,j_2} \|m_{j_1,j_2}\|_{B_{2,q}^{(s_1,s_2)}},$$

when $s_1 > n_1/2, s_1 > n_1/2$.

Therefore, to establish Theorem 1.6, by Fefferman criterion, we just need to prove the following: if a is an $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ rectangle atom $(0 < p \leq 1)$ supported on $R = I \times J$, we have

$$\int\int_{(\gamma R)^c} |T_m^*(a)(x_1,x_2)|^p dx_1 dx_2 \lesssim \sup_{j_1,j_2} \|m_{j_1,j_2}\|^p_{B_{2,q}^{(s_1,s_2)}} \gamma^{-\delta p} \quad \text{for all } \gamma \geq 2,$$

where $s_1 > n_1(1/p - 1/2), s_2 > n_2(1/p - 1/2)$ and some fixed $\delta > 0$. By a translation, we only consider an atom a supported in R which is centered at $(0,0)$. By the Besov embedding theorem, it is sufficient to consider the case

$$n_i(1/p - 1/2) < s_i < [n_i(1/p - 1)] + n_i/2 + 1 \text{ for all i} = 1,2.$$

Let a be a rectangle atom supported in R. We decompose $(\gamma R)^c := \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \setminus \gamma R$ into the following three subsets:

$$\begin{aligned} V_1 &= \{(\xi_1,\xi_2) : (\gamma R)^c \setminus (V_2 \cup V_3)\}, \\ V_2 &= \{(\xi_1,\xi_2) : \xi_1 \in (\gamma I)^c, \xi_2 \in J\}, \\ V_3 &= \{(\xi_1,\xi_2) : \xi_1 \in I, \xi_2 \in (\gamma J)^c\}. \end{aligned}$$

The proof of Theorem 1.6 will then be divided into three steps.

Step 1: Estimate $\|T_m(a)(x_1,x_2)\|^p_{H^p(V_1)}$.
We define

$$K_{j_1,j_2} = \mathcal{F}^{-1}[m(\cdot,\cdot)\psi(\cdot/2^{j_1})\psi(\cdot/2^{j_2})] = \mathcal{F}^{-1}[m_{j_1,j_2}(\cdot/2^{j_1},\cdot/2^{j_2})].$$

If we write $\widetilde{K}_{j_1,j_2} = \mathcal{F}^{-1}[m_{j_1,j_2}]$, then

$$K_{j_1,j_2}(x_1,x_2) = 2^{j_1n_1+j_2n_2}\widetilde{K}_{j_1,j_2}(2^{j_1}x_1, 2^{j_2}x_2).$$

The Littlewood-Paley-Stein square function $T_m^*(a)(x_1,x_2)$ of $T_m(a)(x_1,x_2)$ can be written as

$$\begin{aligned}
&T_m^*(a)(x_1,x_2)\\
&\quad= \left(\sum_{j_1,j_2\in\mathbb{Z}} |(\Psi_{j_1,j_2})^\vee * T_m(a)(x_1,x_2)|^2\right)^{1/2}\\
&\quad= \left(\sum_{j_1,j_2\in\mathbb{Z}} \left|\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} K_{j_1,j_2}(x_1-y_1,x_2-y_2)a(y_1,y_2)dy_1dy_2\right|^2\right)^{1/2}\\
&\quad=: \left(\sum_{j_1,j_2\in\mathbb{Z}} |F_{j_1,j_2}(x_1,x_2)|^2\right)^{1/2}.
\end{aligned}$$

We shall estimate the function $F_{j_1,j_2}(x_1,x_2)$. To this end, for $i=1,2$, writing $\partial_i^{\alpha_i}K_{j_1,j_2}(y_1,y_2) = \partial_{y_i}^{\alpha_i}K_{j_1,j_2}(y_1,y_2)$ and using the moment condition on rectangle atom $a(y_1,y_2)$, we have for nonnegative integers L_1 and L_2 to be chosen below.

$$\begin{aligned}
F_{j_1,j_2}(x_1,x_2) &= \int_{I\times J}\Bigg[K_{j_1,j_2}(x_1-y_1,x_2-y_2)\\
&\qquad - \sum_{|\alpha_1|\le L_1-1}\frac{(-y_1)^{\alpha_1}}{\alpha_1!}\partial_1^{\alpha_1}K_{j_1,j_2}(x_1,x_2-y_2)\Bigg]a(y_1,y_2)dy\\
&= L_1\sum_{|\alpha_1|=L_1}\int_{I\times J}\int_0^1\frac{(-y_1)^{\alpha_1}}{\alpha_1!}(1-t_1)^{L_1-1}\partial_1^{\alpha_1}\\
&\qquad\times K_{j_1,j_2}(x_1-t_1y_1,x_2-y_2)a(y_1,y_2)dt_1dy
\end{aligned}$$

$$
\begin{aligned}
&= L_1 \sum_{|\alpha_1|=L_1} \int_{I\times J}\int_0^1 \frac{(-y_1)^{\alpha_1}}{\alpha_1!}(1-t_1)^{L_1-1} \\
&\quad \times \Bigg[\partial_1^{\alpha_1} K_{j_1,j_2}(x_1-t_1y_1, x_2-y_2) \\
&\qquad - \sum_{|\alpha_2|\le L_2-1} \frac{(-y_2)^{\alpha_2}}{\alpha_2!}\partial_2^{\alpha_2}\partial_1^{\alpha_1} K_{j_1,j_2}(x_1-t_1y_1, x_2-y_2)\Bigg] \\
&\quad \times a(y_1,y_2)dt_1dy \\
&= L_1L_2 \sum_{|\alpha_1|=L_1}\sum_{|\alpha_2|=L_2} \int_{I\times J}\int_0^1\int_0^1 \frac{(-y_1)^{\alpha_1}}{\alpha_1!}\frac{(-y_2)^{\alpha_2}}{\alpha_2!}(1-t_1)^{L_1-1} \\
&\quad \times (1-t_2)^{L_2-1}\partial_1^{\alpha_1}\partial_2^{\alpha_2} K_{j_1,j_2}(x_1-t_1y_1, x_2-t_2y_2)] \\
&\quad \times a(y_1,y_2)dtdy,
\end{aligned}
$$

where $0 \le L_1 \le [n_1(1/p-1)]+1,\ 0 \le L_2 \le [n_2(1/p-1)]+1,\ dy = dy_1dy_2$ and $dt = dt_1dt_2$.

Thus, by Hölder inequality, we have

$$
\begin{aligned}
&|F_{j_1,j_2}(x_1,x_2)| \\
&\quad \lesssim |I|^{L_1/n_1}|J|^{L_2/n_2} \sum_{|\alpha_1|=L_1}\sum_{|\alpha_2|=L_2}\left(\int_{I\times J}|a(y_1,y_2)|^2dy_1dy_2\right)^{1/2} \\
&\qquad \times \left(\int_{I\times J}\int_0^1\int_0^1 |\partial_1^{\alpha_1}\partial_2^{\alpha_2}K_{j_1,j_2}(x_1-t_1y_1,x_2-t_2y_2)|^2dydt\right)^{1/2} \\
&\quad \lesssim \sum_{|\alpha_1|=L_1}\sum_{|\alpha_2|=L_2} |I|^{1/2-1/p+L_1/n_1}|J|^{1/2-1/p+L_2/n_2} \\
&\qquad \times \left(\int_{I\times J}\int_0^1\int_0^1 |\partial_1^{\alpha_1}\partial_2^{\alpha_2}K_{j_1,j_2}(x_1-t_1y_1,x_2-t_2y_2)|^2dydt\right)^{1/2}.
\end{aligned}
$$

By the subadditivity of the p-th power of the L^p-norm, $0<p\le 1$ and by Hölder inequality, we have

$$
\int_{V_1}|T_m^*(a)(x_1,x_2)|^pdx_1dx_2 \le \sum_{j_1,j_2\in\mathbb{Z}}\int_{V_1}|F_{j_1,j_2}(x_1,x_2)|^pdx_1dx_2
$$

$$\lesssim \sum_{j_1,j_2\in\mathbb{Z}} \sum_{|\alpha_1|=L_1} \sum_{|\alpha_2|=L_2} |I|^{p/2-1+L_1p/n_1}|J|^{p/2-1+L_2p/n_2}$$

$$\times \int_{V_1}\left(\int_{I\times J}\int_0^1\int_0^1 |\partial_1^{\alpha_1}\partial_2^{\alpha_2}K_{j_1,j_2}(x_1-t_1y_1,x_2-t_2y_2)|^2dydt\right)^{p/2} dx_1dx_2$$

$$=: \sum_{j_1,j_2\in\mathbb{Z}} \sum_{|\alpha_1|=L_1} \sum_{|\alpha_2|=L_2} |I|^{p/2-1+L_1p/n_1}|J|^{p/2-1+L_2p/n_2}G_{j_1,j_2}.$$

Next, we will estimate G_{j_1,j_2} pointwisely. We set $A_{k_1}=\{x_1\in\mathbb{R}^{n_1}: 2^{k_1}\le |x_1|\le 2^{k_1+1}\}$ and $A_{k_2}=\{x_2\in\mathbb{R}^{n_2}: 2^{k_2}\le |x_2|\le 2^{k_2+1}\}$. Since $s_i>n_i(\frac{1}{p}-\frac{1}{2})$, we can choose that $s_i'=\big(s_i+n_i(\frac{1}{p}-\frac{1}{2})\big)/2$ for $i=1,2$. Hence, using Hölder's inequality and a change of variables, we have

$$G_{j_1,j_2} = \int_{V_1}\left(\int_{I\times J}\int_0^1\int_0^1 |\partial_1^{\alpha_1}\partial_2^{\alpha_2}K_{j_1,j_2}(x_1-t_1y_1,x_2-t_2y_2)|^2dydt\right)^{p/2} dx_1dx_2$$

$$\le \left(\int_{V_1} |x_1|^{-s_1'\frac{2p}{2-p}}|x_2|^{-s_2'\frac{2p}{2-p}}dx_1dx_2\right)^{1-p/2}$$

$$\times\left(\int_{V_1}\int_{I\times J}\int_0^1\int_0^1 |\partial_1^{\alpha_1}\partial_2^{\alpha_2}K_{j_1,j_2}(x_1-t_1y_1,x_2-t_2y_2)|^2\right.$$

$$\left.\times\, dtdy|x_1|^{2s_1'}|x_2|^{2s_2'}dx_1dx_2\right)^{p/2}$$

$$\lesssim \gamma^{-\delta}|I|^{-s_1'p/n_1+1-p/2}|J|^{-s_2'p/n_2+1-p/2}$$

$$\times \sum_{k_1\ge\rho_1,k_2\ge\rho_2}\left\{\int_{A_{k_1}\times A_{k_2}}\int_{I\times J}\int_0^1\int_0^1 |x_1-t_1y_1|^{2s_1'}|x_2-t_2y_2|^{2s_2'}\right.$$

$$\left.\times\,|\partial_1^{\alpha_1}\partial_2^{\alpha_2}K_{j_1,j_2}(x_1-t_1y_1,x_2-t_2y_2)|^2dtdydx_1dx_2\right\}^{p/2}$$

$$\lesssim \sum_{k_1,k_2\in\mathbb{Z}} \gamma^{-\delta}|I|^{-s_1'p/n_1+1}|J|^{-s_2'p/n_2+1}$$

$$\times\left\{\int_{A_{k_1}\times A_{k_2}} ||x_1|^{s_1'}|x_2|^{s_2'}\partial_1^{\alpha_1}\partial_2^{\alpha_2}K_{j_1,j_2}(x_1,x_2)|^2dx_1dx_2\right\}^{p/2}.$$

where $-\delta=\max\{-s_1'p+n_1-n_1p/2,-s_2'p+n_2-n_2p/2\}$, $2^{\rho_1}\approx\gamma|I|$, $2^{\rho_2}\approx\gamma|J|$ for $\rho_1,\rho_2\in\mathbb{Z}$.

Since $K_{j_1,j_2}(x_1,x_2)=2^{j_1n_1+j_2n_2}\widetilde{K}_{j_1,j_2}(2^{j_1}x_1,2^{j_2}x_2)$. We also set $B_{k_i}=\{x_i\in\mathbb{R}^{n_i}: 2^{k_i}\le|x_i|\le 2^{k_i+1}, k_i\ge 1\}$ and $B_{k_i}=\{x_i\in\mathbb{R}^{n_i}: |x_i|\le 2, k_i=0\}$.

Then, by direct calculations and a change of variables, we obtain

$$
\begin{aligned}
G_{j_1,j_2} &\lesssim \gamma^{-\delta}|I|^{-s_1'p/n_1+1}|J|^{-s_2'p/n_2+1} \\
&\quad \times \sum_{k_1,k_2\in\mathbb{Z}} 2^{j_1p(-s_1'+n_1+|\alpha_1|)}2^{j_2p(-s_2'+n_2+|\alpha_2|)} \\
&\quad \times \left\{\int_{A_{k_1}\times A_{k_2}} ||2^{j_1}x_1|^{s_1'}|2^{j_2}x_2|^{s_2'}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\widetilde{K}_{j_1,j_2}(2^{j_1}x_1,2^{j_2}x_2)|^2dx_1dx_2\right\}^{p/2} \\
&= \gamma^{-\delta}|I|^{-s_1'p/n_1+1}|J|^{-s_2'p/n_2+1} \\
&\quad \times \sum_{k_1,k_2\in\mathbb{Z}} 2^{j_1p(-s_1'+n_1/2+L_1)}2^{j_2p(-s_2'+n_2/2+L_2)} \\
&\quad \times \left\{\int_{A_{k_1}\times A_{k_2}} ||x_1|^{s_1'}|x_2|^{s_2'}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\widetilde{K}_{j_1,j_2}(x_1,x_2)|^2dx_1dx_2\right\}^{p/2} \\
&\lesssim \gamma^{-\delta}|I|^{-s_1'p/n_1+1}|J|^{-s_2'p/n_2+1}2^{j_1p(-s_1'+n_1/2+L_1)}2^{j_2p(-s_2'+n_2/2+L_2)} \\
&\quad \times \sum_{k_1,k_2\in\mathbb{N}}\left\{\int_{B_{k_1}\times B_{k_2}} ||x_1|^{s_1'}|x_2|^{s_2'}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\widetilde{K}_{j_1,j_2}(x_1,x_2)|^2dx_1dx_2\right\}^{p/2} \\
&\lesssim \gamma^{-\delta}|I|^{-s_1'p/n_1+1}|J|^{-s_2'p/n_2+1}2^{j_1p(-s_1'+n_1/2+L_1)}2^{j_2p(-s_2'+n_2/2+L_2)} \\
&\quad \times \sum_{k_1,k_2\in\mathbb{N}}\left\{\int_{B_{k_1}\times B_{k_2}} |2^{(k_1s_1'+k_2s_2')}\varphi_{k_1}(x_1)\varphi_{k_2}(x_2)\right. \\
&\qquad \left.\times\, \partial_1^{\alpha_1}\partial_2^{\alpha_2}\widetilde{K}_{j_1,j_2}(x_1,x_2)|^2dx_1dx_2\right\}^{p/2} \\
&\lesssim \gamma^{-\delta}|I|^{-s_1'p/n_1+1}|J|^{-s_2'p/n_2+1}2^{j_1p(-s_1'+n_1/2+L_1)}2^{j_2p(-s_2'+n_2/2+L_2)} \\
&\quad \times \sum_{k_1,k_2\in\mathbb{N}} 2^{(-k_1\theta_1s_1'-k_2\theta_2s_2')p}\{2^{k_1(1+\theta_1)s_1'+k_2(1+\theta_2)s_2'} \\
&\qquad \times \|\varphi_{k_1}\varphi_{k_2}\mathcal{F}^{-1}[\xi_1^{\alpha_1}\xi_2^{\alpha_2}m_{j_1,j_2}(\xi_1,\xi_2)]\|_{L^2}\}^p \\
&\lesssim \gamma^{-\delta}|I|^{-s_1'p/n_1+1}|J|^{-s_2'p/n_2+1}2^{j_1p(-s_1'+n_1/2+L_1)}2^{j_2p(-s_2'+n_2/2+L_2)} \\
&\quad \times \left[\sum_{k_1,k_2\in\mathbb{N}} 2^{(-k_1\theta_1s_1'-k_2\theta_2s_2')p/(1-p/q)}\right]^{1-p/q} \\
&\quad \times \left\{\sum_{k_1,k_2\in\mathbb{N}} 2^{(k_1(1+\theta_1)s_1'+k_2(1+\theta_2)s_2')q}\right. \\
&\qquad \left.\times\, \|\varphi_{k_1}\varphi_{k_2}\mathcal{F}^{-1}[\xi_1^{\alpha_1}\xi_2^{\alpha_2}m_{j_1,j_2}(\xi_1,\xi_2)]\|_{L^2}^q\right\}^{p/q}
\end{aligned}
$$

$$\lesssim \gamma^{-\delta}|I|^{-s_1'p/n_1+1}|J|^{-s_2'p/n_2+1}2^{j_1p(-s_1'+n_1/2+L_1)} \times 2^{j_2p(-s_2'+n_2/2+L_2)}\|m_{j_1,j_2}\|^p_{B^{(s_1,s_2)}_{2,q}},$$

where the last inequality holds by Lemma 2.3 and φ_{k_i} is defined by (2.12), $q>p$ and $(1+\theta_i)s_i'=s_i$ for $i=1,2$.

For $I\times J$, there exist $l_1,l_2\in\mathbb{Z}$ such that $|I|\approx 2^{-l_1n_1}$, $|J|\approx 2^{-l_2n_2}$. Thus, use the estimates of G_{j_1,j_2}, we obtain that

$$\begin{aligned}\int_{V_1}&|T_m^*(a)(x_1,x_2)|^p dx_1dx_2\\ &\lesssim \sum_{j_1,j_2\in\mathbb{Z}}\sum_{|\alpha_1|=L_1}\sum_{|\alpha_2|=L_2}|I|^{(-s_1'+n_1/2+L_1)p/n_1}|J|^{(-s_2'+n_2/2+L_2)p/n_2}\\ &\quad\times\gamma^{-\delta}2^{j_1p(-s_1'+n_1/2+L_1)}2^{j_2p(-s_2'+n_2/2+L_2)}\|m_{j_1,j_2}\|^p_{B^{(s_1,s_2)}_{2,q}}\\ &\lesssim \sup_{j_1,j_2\in\mathbb{Z}}\|m_{j_1,j_2}\|^p_{B^{(s_1,s_2)}_{2,q}}\gamma^{-\delta}\\ &\quad\times\sum_{j_1,j_2\in\mathbb{Z}}2^{(j_1-l_1)p(-s_1'+n_1/2+L_1)}2^{(j_2-l_2)p(-s_2'+n_2/2+L_2)}.\end{aligned}$$

Set

$$\mathbf{B}=\sum_{j_1,j_2\in\mathbb{Z}}2^{(j_1-l_1)p(-s_1'+n_1/2+L_1)}2^{(j_2-l_2)p(-s_2'+n_2/2+L_2)}.$$

In the above summation $\mathbf{B}$, we can choose $L_i=0$, if $j_i\ge l_i$ and $L_i=[n_i/p-n_i]+1$, if $j_i<l_i$ for $i=1,2$. Hence, we have

$$\begin{aligned}\int_{V_1}&|T_m^*(a)(x_1,x_2)|^p dx_1dx_2\lesssim\gamma^{-\delta}\sup_{j_1,j_2\in\mathbb{Z}}\|m_{j_1,j_2}\|^p_{B^{(s_1,s_2)}_{2,q}}\\ &\times\sum_{j_1,j_2\in\mathbb{Z}}2^{(j_1-l_1)p(-s_1'+n_1/2+L_1)}2^{(j_2-l_2)p(-s_2'+n_2/2+L_2)}\\ &\lesssim\gamma^{-\delta}\sup_{j_1,j_2\in\mathbb{Z}}\|m_{j_1,j_2}\|^p_{B^{(s_1,s_2)}_{2,q}}\\ &\quad\times\left\{\sum_{j_1\ge l_1}2^{(j_1-l_1)p(-s_1'+n_1/2)}+\sum_{j_1<l_1}2^{(j_1-l_1)p(-s_1'+n_1/2+[n_1/p-n_1]+1)}\right\}\\ &\quad\times\left\{\sum_{j_2\ge l_2}2^{(j_2-l_2)p(-s_2'+n_2/2)}+\sum_{j_2<l_2}2^{(j_2-l_2)p(-s_2'+n_2/2+[n_2/p-n_2]+1)}\right\}\\ &\lesssim\gamma^{-\delta}\sup_{j_1,j_2\in\mathbb{Z}}\|m_{j_1,j_2}\|^p_{B^{(s_1,s_2)}_{2,q}}\end{aligned}$$

where $-\delta=\max\{-s_1'p+n_1-n_1p/2,-s_2'p+n_2-n_2p/2\}<0$.

Finally, we have concluded

$$\|T_m(a)(x_1,x_2)\|^p_{H^p(V_1)} \lesssim \gamma^{-\delta} \sup_{j_1,j_2\in\mathbb{Z}} \|m_{j_1,j_2}\|^p_{B^{(s_1,s_2)}_{2,q}}. \tag{4.22}$$

Step 2: Estimate $\|T_m(a)(x_1,x_2)\|^p_{H^p(V_2)}$.

As before, we have

$$\begin{aligned}
\int_{V_2} |T^*_m(a)(x_1,x_2)|^p dx_1dx_2 &\le \sum_{j_1\in\mathbb{Z}} \int_{V_2} \Big(\sum_{j_2\in\mathbb{Z}} |F_{j_1,j_2}(x_1,x_2)|^2\Big)^{p/2} dx_1dx_2 \\
&\le \sum_{j_1\in\mathbb{Z}} \Big(\int_{V_2} |x_1|^{-s_1'\frac{2p}{2-p}} dx_1dx_2\Big)^{1-p/2} \\
&\quad\times \Big(\int_{V_2} \sum_{j_2\in\mathbb{Z}} |F_{j_1,j_2}(x_1,x_2)|^2|x_1|^{2s_1'} dx_1dx_2\Big)^{p/2} \\
&\lesssim \gamma^{-\delta'}|I|^{-s_1'p/n_1+1-p/2}|J|^{1-p/2} \\
&\quad\times \sum_{j_1\in\mathbb{Z}}\sum_{k_1\ge\rho_1} \Big(\int_{A_{k_1}\times\mathbb{R}^{n_2}} \sum_{j_2\in\mathbb{Z}} |F_{j_1,j_2}(x_1,x_2)|^2|x_1|^{2s_1'} dx_1dx_2\Big)^{p/2} \\
&= \gamma^{-\delta'}|I|^{-s_1'p/n_1+1-p/2}|J|^{1-p/2} \\
&\quad\times \sum_{j_1\in\mathbb{Z}}\sum_{k_1\ge\rho_1} \|\{|F_{j_1,j_2}(x_1,x_2)||x_1|^{s_1'}\}_{l^2_{j_2}}\|^p_{L^2(A_{k_1}\times\mathbb{R}^{n_2})} \\
&=: \gamma^{-\delta'}|I|^{-s_1'p/n_1+1-p/2}|J|^{1-p/2} \sum_{j_1\in\mathbb{Z}}\sum_{k_1\ge\rho_1} (\star)^p,
\end{aligned}$$

where $-\delta' = -s_1'p + n_1 - n_1p/2$.

In order to estimate $(\star)$, we use a duality argument.

$$(\star) \le \sup_{\|\{h_{j_2}\}_{l^2_{j_2}}\|_{L^2(A_{k_1}\times\mathbb{R}^{n_2})}\le 1} \sum_{j_2\in\mathbb{Z}} \int_{A_{k_1}\times\mathbb{R}^{n_2}} |F_{j_1,j_2}(x_1,x_2)||x_1|^{s_1'} h_{j_2}(x_1,x_2)dx_1dx_2.$$

Similarly, we have

$$\begin{aligned}
F_{j_1,j_2}(x_1,x_2) = L_1 \sum_{|\alpha_1|=L_1} \int_{I\times J}\int_0^1 \frac{(-y_1)^{\alpha_1}}{\alpha_1!}(1-t_1)^{L_1-1} \\
\times \partial_1^{\alpha_1} K_{j_1,j_2}(x_1-t_1y_1, x_2-y_2)a(y_1,y_2)dt_1dy,
\end{aligned}$$

where $0 \le L_1 \le [n_1(1/p-1)]+1$ and $dy = dy_1dy_2$.

Fixed h_{j_2}, we have

$$\begin{aligned}
&\int_{A_{k_1}\times\mathbb{R}^{n_2}} |F_{j_1,j_2}(x_1,x_2)||x_1|^{s_1'} h_{j_2}(x_1,x_2)dx_1dx_2 \\
&\quad = L_1 \sum_{|\alpha_1|=L_1} \int_{A_{k_1}\times\mathbb{R}^{n_2}} \bigg| \int_{I\times J}\int_0^1 \frac{(-y_1)^{\alpha_1}}{\alpha_1!}(1-t_1)^{L_1-1} \\
&\qquad \times \partial_1^{\alpha_1} K_{j_1,j_2}(x_1-t_1y_1, x_2-y_2)a(y_1,y_2)dt_1dy_1dy_2 \bigg| \\
&\qquad \times |x_1|^{s_1'} h_{j_2}(x_1,x_2)dx_1dx_2.
\end{aligned}$$

Fixed x_1, t_1, y_1, j_1, j_2, define

$$S(a)(x_2) = \int_J \partial_1^{\alpha_1} K_{j_1,j_2}(x_1-t_1y_1, x_2-y_2)a(y_1,y_2)dy_2.$$

Set $m_{x_1,t_1,y_1,j_1,j_2}(x_2) = \partial_1^{\alpha_1}\widehat{K}^2_{j_1,j_2}(x_1-t_1y_1,x_2)$ and $\text{supp}\widetilde{\psi} \subset \{\xi : 1/3 \le |\xi| \le 3\}$, $\widetilde{\psi} = 1$ on $\text{supp}\psi$ and $\widetilde{\Delta}_{j_2} g(x_2) = [\widetilde{\psi}(D/2^{j_2})g]$. By Lemma 2.3, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{n_2}} S(a)(x_2)h_{j_2}(x_1,x_2)dx_2 \\
&\quad = \int_{\mathbb{R}^{n_2}} S(a)(x_2)\widetilde{\Delta}_{j_2} h_{j_2}(x_1,x_2)dx_2 \\
&\quad \lesssim \sum_{k_2=0}^{\infty} 2^{k_2n_2/2}\|\varphi_{k_2}\partial_1^{\alpha_1}\widehat{K}^2_{j_1,j_2}(x_1-t_1y_1,x_2)\|_{L^2(\mathbb{R}^{n_2})} \\
&\qquad \times \int_{\mathbb{R}^{n_2}} ((|\phi|^2)_{(k_2-j_2)} *_2 |\widetilde{\Delta}_{j_2}a(y_1,\cdot)|^2)^{1/2}(x_2)|\widetilde{\Delta}_{j_2}h_{j_2}|(x_1,x_2)dx_2.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\left| \int_{A_{k_1}\times\mathbb{R}^{n_2}} F_{j_1,j_2}(x_1,x_2)|x_1|^{s_1'} h_{j_2}(x_1,x_2)dx_1dx_2 \right| \\
&\quad = L_1 \sum_{|\alpha_1|=L_1} \int_{A_{k_1}\times\mathbb{R}^{n_2}} \int_{I\times J}\int_0^1 \frac{(-y_1)^{\alpha_1}}{\alpha_1!}(1-t_1)^{L_1-1} \\
&\qquad \times S(a)(x_2)dt_1dy_1|x_1|^{s_1'} h_{j_2}(x_1,x_2)dx_1dx_2
\end{aligned}$$

$$\lesssim \sum_{|\alpha_1|=L_1} |I|^{L_1/n_1} \int_I \int_{A_{k_1}\times\mathbb{R}^{n_2}} \int_0^1 \sum_{k_2=0}^{\infty} 2^{k_2 n_2/2} |x_1|^{s_1'} \times \|\varphi_{k_2}\partial_1^{\alpha_1}\widehat{K}^2_{j_1,j_2}(x_1-t_1y_1,x_2)\|_{L^2(\mathbb{R}^{n_2})} \times ((|\phi|^2)_{(k_2-j_2)} *_2 |\widetilde{\Delta}_{j_2}a(y_1,\cdot)|^2)^{1/2}(x_2)|\widetilde{\Delta}_{j_2}h_{j_2}|(x_1,x_2)dt_1dx_1dx_2dy_1.$$

Using Schwartz's inequality and a change of variables, we have

$$\begin{aligned}
&\int_{A_{k_1}}\int_0^1 |x_1|^{s_1'}\|\varphi_{k_2}\partial_1^{\alpha_1}\widehat{K}^2_{j_1,j_2}(x_1-t_1y_1,x_2)\|_{L^2_{x_2}}|\widetilde{\Delta}_{j_2}h_{j_2}|(x_1,x_2)dt_1dx_1\\
&\lesssim \int_0^1 \||x_1-t_1y_1|^{s_1'}\varphi_{k_2}\partial_1^{\alpha_1}\widehat{K}^2_{j_1,j_2}(x_1-t_1y_1,x_2)\|_{L^2(A_{k_1}\times\mathbb{R}^{n_2})}dt_1\|\widetilde{\Delta}_{j_2}h_{j_2}\|_{L^2(A_{k_1})}\\
&= \||x_1|^{s_1'}\varphi_{k_2}\partial_1^{\alpha_1}\widehat{K}^2_{j_1,j_2}(x_1,x_2)\|_{L^2(A_{k_1}\times\mathbb{R}^{n_2})}\|\widetilde{\Delta}_{j_2}h_{j_2}\|_{L^2(A_{k_1})}\\
&= 2^{j_1(-s_1'+n_1+L_1)}\||2^{j_1}x_1|^{s_1'}\varphi_{k_2}\partial_1^{\alpha_1}\widehat{K}^2_{j_1,j_2}(2^{j_1}x_1,x_2)\|_{L^2(A_{k_1}\times\mathbb{R}^{n_2})}\|\widetilde{\Delta}_{j_2}h_{j_2}\|_{L^2(A_{k_1})}\\
&= 2^{j_1(-s_1'+n_1/2+L_1)}\||x_1|^{s_1'}\varphi_{k_2}\partial_1^{\alpha_1}\widehat{K}^2_{j_1,j_2}(x_1,x_2)\|_{L^2(A_{k_1}\times\mathbb{R}^{n_2})}\|\widetilde{\Delta}_{j_2}h_{j_2}\|_{L^2(A_{k_1})}\\
&\lesssim 2^{k_1s_1'}\|\varphi_{k_2}\partial_1^{\alpha_1}\widehat{K}^2_{j_1,j_2}(x_1,x_2)\|_{L^2(A_{k_1}\times\mathbb{R}^{n_2})}2^{j_1(-s_1+n_1/2+L_1)}\|\widetilde{\Delta}_{j_2}h_{j_2}\|_{L^2(A_{k_1})}\\
&\lesssim 2^{j_1(-s_1'+n_1/2+L_1)}2^{k_1s_1'}\|\psi_{k_1}\varphi_{k_2}m_{j_1,j_2}\|_{L^2}\|\widetilde{\Delta}_{j_2}h_{j_2}\|_{L^2(A_{k_1})},
\end{aligned}$$

where we have used Lemma 2.2 to obtain the last inequality and ψ is defined by (1.1).

Therefore,

$$\begin{aligned}
&\left|\sum_{j_2\in\mathbb{Z}}\int_{A_{k_1}\times\mathbb{R}^{n_2}} |F_{j_1,j_2}(x_1,x_2)||x_1|^{s_1'}h_{j_2}(x_1,x_2)dx_1dx_2\right|\\
&\lesssim 2^{j_1(-s_1'+n_1/2+L_1)}|I|^{L_1/n_1}\sum_{j_2\in\mathbb{Z}}\sum_{k_2=0}^{\infty}2^{k_2n_2/2}2^{k_1s_1'}\|\psi_{k_1}\varphi_{k_2}m_{j_1,j_2}\|_{L^2}\\
&\quad\times\int_I\int_{\mathbb{R}^{n_2}}((|\phi|^2)_{(k_2-j_2)} *_2 |\widetilde{\Delta}_{j_2}a(y_1,\cdot)|^2)^{1/2}(x_2)\|\widetilde{\Delta}_{j_2}h_{j_2}\|_{L^2(A_{k_1})}dy_1dx_2\\
&\lesssim 2^{j_1(-s_1'+n_1/2+L_1)}|I|^{L_1/n_1}\sup_{j_2}\left\{\sum_{k_2=0}^{\infty}2^{k_2n_2/2}2^{k_1s_1'}\|\psi_{k_1}\varphi_{k_2}m_{j_1,j_2}\|_{L^2}\right\}\\
&\quad\times\sum_{j_2\in\mathbb{Z}}\sup_{k_2}\left\{\int_I\int_{\mathbb{R}^{n_2}}((|\phi|^2)_{(k_2-j_2)} *_2 |\widetilde{\Delta}_{j_2}a(y_1,\cdot)|^2)^{1/2}(x_2)\right.\\
&\quad\left.\times\|\widetilde{\Delta}_{j_2}h_{j_2}\|_{L^2(A_{k_1})}dy_1dx_2\right\}.
\end{aligned}$$

It follows from Schwartz's inequality that the sum concerning j_2 in the last line is estimated by

$$
\begin{aligned}
&\sum_{j_2\in\mathbb{Z}} \sup_{k_2}\left\{\int_{\mathbb{R}^{n_2}} ((|\phi|^2)_{(k_2-j_2)} *_2 |\widetilde{\Delta}_{j_2}a(y_1,\cdot)|^2)^{1/2}(x_2)\|\widetilde{\Delta}_{j_2}h_{j_2}\|_{L^2(A_{k_1})}dx_2\right\}\\
&\quad\le \left(\sum_{j_2\in\mathbb{Z}} \sup_{k_2}\int_{\mathbb{R}^{n_2}} ((|\phi|^2)_{(k_2-j_2)} *_2 |\widetilde{\Delta}_{j_2}a(y_1,\cdot)|^2)(x_2)dx_2\right)^{1/2}\\
&\qquad\times\left(\sum_{j_2\in\mathbb{Z}}\int_{A_{k_1}\times\mathbb{R}^{n_2}} |\widetilde{\Delta}_{j_2}h_{j_2}|^2dx_1dx_2\right)^{1/2}\\
&\quad\le \left(\sum_{j_2\in\mathbb{Z}} \sup_{k_2}\sup_{\|g\|_{L^\infty}=1}\left|\int_{\mathbb{R}^{n_2}} ((|\phi|^2)_{(k_2-j_2)} *_2 |\widetilde{\Delta}_{j_2}a(y_1,\cdot)|^2)(x_2)g(x_2)dx_2\right|\right)^{1/2}\\
&\qquad\times\left(\sum_{j_2\in\mathbb{Z}}\int_{A_{k_1}\times\mathbb{R}^{n_2}} |h_{j_2}|^2dx_1dx_2\right)^{1/2}\\
&\quad\le \left(\sum_{j_2\in\mathbb{Z}} \sup_{k_2}\sup_{\|g\|_{L^\infty}=1}\left|\int_{\mathbb{R}^{n_2}} |\widetilde{\Delta}_{j_2}a(y_1,\cdot)|^2(x_2)(|\phi|^2)_{(k_2-j_2)} *_2 g(x_2)dx_2\right|\right)^{1/2}\\
&\qquad\times\left(\sum_{j_2\in\mathbb{Z}}\int_{A_{k_1}\times\mathbb{R}^{n_2}} |h_{j_2}|^2dx_1dx_2\right)^{1/2}\\
&\quad\le \left(\sup_{\|g\|_{L^\infty}=1}\left|\int_{\mathbb{R}^{n_2}} \sum_{j_2\in\mathbb{Z}}|\widetilde{\Delta}_{j_2}a(y_1,\cdot)|^2(x_2)\mathcal{M}g(x_2)dx_2\right|\right)^{1/2}\\
&\qquad\times\left(\sum_{j_2\in\mathbb{Z}}\int_{A_{k_1}\times\mathbb{R}^{n_2}} |h_{j_2}|^2dx_1dx_2\right)^{1/2}\\
&\quad\le \|a(y_1,\cdot)\|_{L^2(\mathbb{R}^{n_2})}\|\{h_{j_2}\}_{l^2}\|_{L^2(A_{k_1}\times\mathbb{R}^{n_2})}.
\end{aligned}
$$

Since $\|a\|_{L^2}\le |I\times J|^{1/2-1/p}$, we have

$$
\begin{aligned}
&\sum_{j_2\in\mathbb{Z}} \sup_{k_2}\left\{\int_I\int_{\mathbb{R}^{n_2}} ((|\phi|^2)_{(k_2-j_2)} *_2 |\widetilde{\Delta}_{j_2}a(y_1,\cdot)|^2)^{1/2}(x_2)\|\widetilde{\Delta}_{j_2}h_{j_2}\|_{L^2(A_{k_1})}dy_1dx_2\right\}\\
&\quad\le \int_I \|a(y_1,\cdot)\|_{L^2(\mathbb{R}^{n_2})}\|\{h_{j_2}\}_{l^2}\|_{L^2(A_{k_1}\times\mathbb{R}^{n_2})}dy_1\\
&\quad\le |I|^{1/2}\|a\|_{L^2}\|\{h_{j_2}\}_{l^2}\|_{L^2(A_{k_1}\times\mathbb{R}^{n_2})}\\
&\quad\le |I|^{1/2}|I\times J|^{1/2-1/p}\|\{h_{j_2}\}_{l^2}\|_{L^2(A_{k_1}\times\mathbb{R}^{n_2})}.
\end{aligned}
$$

Combining these estimates and for $I \times J$, there exist $l_1 \in \mathbb{Z}$ such that $|I| \approx 2^{-l_1 n_1}$, we have

$$\begin{aligned}
&\int_{V_2} |T_m^*(a)(x_1,x_2)|^p dx_1 dx_2 \\
&\lesssim \gamma^{-\delta'} \sup_{j_1,j_2} \sum_{k_1\in\mathbb{Z}} \left\{ \sum_{k_2=0}^{\infty} 2^{k_2 n_2/2} 2^{k_1 s_1'} \|\varphi_{k_1}\varphi_{k_2} m_{j_1,j_2}\|_{L^2} \right\}^p \\
&\quad \times \sum_{j_1\in\mathbb{Z}} 2^{j_1 p(-s_1'+n_1/2+L_1)} |I|^{p(-s_1'+n_1/2+L_1)/n_1} \\
&\quad \times \sup_{\|\{h_{j_2}\}_{l^2_{j_2}}\|_{L^2(A_{k_1}\times\mathbb{R}^{n_2})}\le 1} \|\{h_{j_2}\}_{l^2}\|^p_{L^2(A_{k_1}\times\mathbb{R}^{n_2})} \\
&\lesssim \gamma^{-\delta'} \sup_{j_1,j_2} \left\{ \sum_{k_1\in\mathbb{N}} \sum_{k_2\in\mathbb{N}} 2^{k_2 s_2' p} 2^{k_1 s_1' p} \|\varphi_{k_1}\varphi_{k_2} m_{j_1,j_2}\|^p_{L^2} \right\} \\
&\quad \times \sum_{j_1\in\mathbb{Z}} 2^{j_1 p(-s_1'+n_1/2+L_1)} |I|^{p(-s_1'+n_1/2+L_1)/n_1} \\
&\lesssim \gamma^{-\delta'} \left(\sup_{j_1,j_2} \|m_{j_1,j_2}\|^p_{B^{(s_1,s_2)}_{2,q}} \right) \sum_{j_1\in\mathbb{Z}} 2^{j_1 p(-s_1+n_1/2+L_1)} |I|^{p(-s_1'+n_1/2+L_1)/n_1} \\
&\lesssim \gamma^{-\delta'} \left(\sup_{j_1,j_2} \|m_{j_1,j_2}\|^p_{B^{(s_1,s_2)}_{2,q}} \right) \\
&\quad \times \left\{ \sum_{j_1\ge l_1} 2^{(j_1-l_1)p(-s_1'+n_1/2)} + \sum_{j_1<l_1} 2^{(j_1-l_1)p(-s_1'+n_1/2+[n_1/p-n_1]+1)} \right\} \\
&\lesssim \gamma^{-\delta'} \sup_{j_1,j_2} \|m_{j_1,j_2}\|^p_{B^{(s_1,s_2)}_{2,q}}.
\end{aligned}$$

Thus, we obtain

$$\|T_m(a)(x_1,x_2)\|^p_{H^p(V_2)} \lesssim \gamma^{-\delta'} \sup_{j_1,j_2\in\mathbb{Z}} \|m_{j_1,j_2}\|^p_{B^{(s_1,s_2)}_{2,q}}, \tag{4.23}$$

with $-\delta' = -s_1'p + n_1 - n_1 p/2$. By symmetry of the situation, the cases V_2 and V_3 are treated in the similar way.

Therefore, we have

$$\|T_m(a)(x_1,x_2)\|^p_{H^p(V_3)} \lesssim \gamma^{-\delta''} \sup_{j_1,j_2\in\mathbb{Z}} \|m_{j_1,j_2}\|^p_{B^{(s_1,s_2)}_{2,q}}, \tag{4.24}$$

with $-\delta'' = -s_2'p + n_2 - n_2 p/2$.

Combining (4.22) with (4.23) and (4.24), we have the desired estimate,

$$\|T_m(a)(x_1,x_2)\|^p_{H^p((\gamma R)^c)} \lesssim \gamma^{-\sigma} \sup_{j_1,j_2\in\mathbb{Z}} \|m_{j_1,j_2}\|^p_{B^{(s_1,s_2)}_{2,q}}, \tag{4.25}$$

where $-\sigma = \max\{-\delta, -\delta', -\delta''\}$.

By Theorem 2.1, we have proved the H^p boundedness for T_m. The proof of Theorem 1.6 is thus completed.

5. The sharpness of the conditions in Theorem 1.5 and 1.6

In this section, we consider the sharpness of Theorems 1.5 and 1.6.

Proposition 5.1. *Let $1 < p < \infty$. Then the estimates*

$$\|T_m(f)\|_{L^p} \le C \sup_{j_1,j_2\in\mathbb{Z}} \|m_{j_1,j_2}\|_{B^{(s_1,s_2)}_{2,1}} \|f\|_{L^p} \tag{5.26}$$

holds only if $s_1 \ge n_1/2, s_2 \ge n_2/2$.

Proof. First, we set $f(x_1,x_2) = f_1(x_1)f_2(x_2)$ where $f_1(x_1) \in L^p(\mathbb{R}^{n_1})$ and $f_2(x_2) \in L^p(\mathbb{R}^{n_2})$. We take functions φ_1 and φ_2 such that

$$\varphi_i \in \mathcal{S}(\mathbb{R}^{n_i}), \quad \varphi_i = 1 \ for \ |\xi_i| \le 1, \qquad \operatorname{supp} \varphi \subset \{\xi_i \in \mathbb{R}^{n_i} : |\xi_i| \le 2\}$$

for all $i = 1,2$. And let $\widehat{\theta}_1$ and $\widehat{\theta}_2$ be smoothing functions, and assume that

$$\widehat{\theta}_i \in \mathcal{S}(\mathbb{R}^{n_i}), \quad \widehat{\theta}_i(\xi_i) = 1 \ for \ |\xi_i| \le 1/2, \qquad \operatorname{supp} \widehat{\theta}_i \subset \{\xi_i \in \mathbb{R}^{n_i} : |\xi_i| \le 1\},$$

for all $i = 1,2$. And define $\widehat{f_i}(\xi_i) = \varphi_i((\xi_i - \eta_i^0)/\epsilon_i)$ and $m_i(\xi_i) = \widehat{\theta}_i((\xi_i - \eta_i^0)/\epsilon_i)$ with $|\eta_i^0| = 1$ for $i = 1,2$.

To prove the necessity of the condition $s_1 \ge n_1/2$ and $s_2 \ge n_2/2$, we set, for sufficiently small $0 < \epsilon_i < 1$,

$$m(\xi_1,\xi_2) = m_1(\xi_1)m_2(\xi_2) = \widehat{\theta}_1((\xi_1 - \eta_1^0)/\epsilon_1)\widehat{\theta}_2((\xi_2 - \eta_2^0)/\epsilon_2).$$

For $m = m_1m_2$ and $f = f_1f_2$, we have

$$T_m(f)(x_1,x_2) = \mathcal{F}^{-1}[\widehat{\theta}_1((\cdot - \eta_1^0)/\epsilon_1)\widehat{f}_1](x_1)\mathcal{F}^{-1}[\widehat{\theta}_1((\cdot - \eta_2^0)/\epsilon_2)\widehat{f}_2(\cdot)](x_2).$$

Fix ϵ_i, we first estimate the norm $\|m\|_{B^{(s_1,s_2)}_{2,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$.

In fact, we have

$$\begin{aligned}\|m\|_{B_{2,1}^{(s_1,s_2)}} &= \|\widehat{\theta}_1((\xi_1-\eta_1^0)/\epsilon_1)\|_{B_{2,1}^{s_1}(\mathbb{R}^{n_1})}\|\widehat{\theta}_2((\xi_2-\eta_2^0)/\epsilon_2)\|_{B_{2,1}^{s_2}(\mathbb{R}^{n_2})}\\ &\le C\epsilon_1^{-s_1+n_1/2}\epsilon_2^{-s_2+n_2/2},\end{aligned}$$

the last inequality follows from Proposition 2.1.

Hence, we have

$$\begin{aligned}&\|T_m(f)(x_1,x_2)\|_{L^p}\\ &\quad\le C\|\mathcal{F}^{-1}[\widehat{\theta}_1((\cdot-\eta_1^0)/\epsilon_1)\widehat{f}_1](x_1)\mathcal{F}^{-1}[\widehat{\theta}_2((\cdot-\eta_2^0)/\epsilon_2)\widehat{f}_2(\cdot)](x_2)\|_{L^p}\\ &\quad\le C\|\widehat{\theta}_1((\xi_1-\eta_1^0)/\epsilon_1)\|_{B_{2,1}^{s_1}(\mathbb{R}^{n_1})}\|\widehat{\theta}_2((\xi_2-\eta_2^0)/\epsilon_2)\|_{B_{2,1}^{s_2}(\mathbb{R}^{n_2})}\|f_1\|_{L^p}\|f_2\|_{L^p}\\ &\quad\le C\epsilon_1^{-s_1+n_1/2}\epsilon_2^{-s_2+n_2/2}\|f_1\|_{L^p}\|f_2\|_{L^p}\\ &\quad\le C\epsilon_1^{-s_1+n_1/2}\epsilon_2^{-s_2+n_2/2}\epsilon_1^{n_1-n_1/p}\epsilon_2^{n_2-n_2/p}.\end{aligned}$$

Moreover, a simple calculation gives

$$\|T_m(f)(x_1,x_2)\|_{L^p}=\|\epsilon_1^{n_1}\varphi_1(\epsilon_1x_1)\epsilon_2^{n_2}\varphi_2(\epsilon_2x_2)\|_{L^p}=C\epsilon_1^{n_1-n_1/p}\epsilon_2^{n_2-n_2/p}.$$

Thus, (5.26) yields the inequality

$$\epsilon_1^{n_1-n_1/p}\epsilon_2^{n_2-n_2/p}\le C\epsilon_1^{-s_1+n_1/2}\epsilon_2^{-s_2+n_2/2}\epsilon_1^{n_1-n_1/p}\epsilon_2^{n_2-n_2/p},$$

which holds only if $s_1\ge n_1/2$ and $s_2\ge n_2/2$. □

Proposition 5.2. *Let $p=1$. Then the estimates*

$$\|T_m(f)\|_{H^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\le C\sup_{j_1,j_2\in\mathbb{Z}}\|m_{j_1,j_2}\|_{B_{2,1}^{(s_1,s_2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\|f\|_{H^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$$

holds only if $s_1>n_1/2, s_2>n_2/2$.

Proof. First of all, we only need to consider the simplest case. In the following, we set $f(x_1,x_2)=f_1(x_1)f_2(x_2)$, where $f_1(x_1)\in H^1(\mathbb{R}^{n_1})$ and $f_2(x_2)\in H^1(\mathbb{R}^{n_2})$, then we have $f(x_1,x_2)\in H^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. Furthermore, we also set $m(\xi_1,\xi_2)=m_1(\xi_1)m_2(\xi_2)$, where $m_1(\xi_1)\in B_{2,1}^{s_1}(\mathbb{R}^{n_1})$ and $m_2(\xi_2)\in B_{2,1}^{s_2}(\mathbb{R}^{n_2})$.

Therefore, we just estimate

$$T_m f(x_1, x_2) = T_{m_1} f_1(x_1) T_{m_2} f_1(x_2).$$

Moreover, we could obtain

$$\|T_m(f)(x_1, x_2)\|_{H^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} = C\|T_{m_1}(f_1)(x_1)\|_{H^1(\mathbb{R}^{n_1})}\|T_{m_2}(f_2)(x_2)\|_{H^1(\mathbb{R}^{n_2})}$$

By the sharpness of Baernstein and Sawyer's Theorem [1], we have that Theorem 1.6 is sharpness in the sense that there exists a multiplier $m \in B_{2,1}^{(s_1,s_2)}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ with $s_1 \le n_1/2$ or $s_2 \le n_2/2$ such that T_m is unbounded on product Hardy spaces $H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ for $p = 1$. Proposition 5.2 is proved. □

Proposition 5.3. *Let* $0 < p < 1$. *Then the estimate*

$$\|T_m(f)\|_{H^p} \le C \sup_{j_1,j_2\in\mathbb{Z}} \|m_{j_1,j_2}\|_{B_{2,q}^{(s_1,s_2)}} \|f\|_{H^p} \tag{5.27}$$

holds only if $s_1 \ge n_1(\frac{1}{p} - \frac{1}{2}), s_2 \ge n_2(\frac{1}{p} - \frac{1}{2})$.

Proof. We take the function θ_i as in the proof of Proposition 5.1.

To prove the necessity of the condition $s_1 \ge n_1(\frac{1}{p} - \frac{1}{2})$ and $s_2 \ge n_2(\frac{1}{p} - \frac{1}{2})$, we set, for sufficiently small $0 < \epsilon_i < 1$,

$$m(\xi_1, \xi_2) = m_1(\xi_1)m_2(\xi_2) = \widehat{\theta}_1((\xi_1 - \eta_1^0)/\epsilon_1)\widehat{\theta}_2((\xi_2 - \eta_2^0)/\epsilon_2).$$

For $m = m_1 m_2$ and $f = f_1 f_2$, we have

$$T_m(f)(x_1, x_2) = \mathcal{F}^{-1}[\widehat{\theta}_1((\cdot - \eta_1^0)/\epsilon_1)\widehat{f_1}](x_1)\mathcal{F}^{-1}[\widehat{\theta}_1((\cdot - \eta_2^0)/\epsilon_2)\widehat{f_2}(\cdot)](x_2).$$

Fix ϵ_i, we first estimate the norm $\|m\|_{B_{2,q}^{(s_1,s_2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$.

In fact, we have

$$\|m\|_{B_{2,q}^{(s_1,s_2)}} \le C\epsilon_1^{-s_1+n_1/2}\epsilon_2^{-s_2+n_2/2}.$$

We take the function $f_i(x_i) = \psi_i(x_i) \in \mathcal{S}(\mathbb{R}^{n_i})$, where $\psi_i(x_i)$ is chosen such that supp$\hat{\psi}_i$ is a compact subset of $\mathbb{R}^{n_i} \setminus \{0\}$, $\hat{\psi}_i(\xi_i) = 1$ in a neighborhood of η_i^0 for $i = 1, 2$.

Hence, we have

$$\begin{aligned}\|T_m(f)(x_1,x_2)\|_{H^p} &\le C\|\widehat{\theta}_1((\xi_1-\eta_1^0)/\epsilon_1)\|_{B^{s_1}_{2,q}(\mathbb{R}^{n_1})}\\ &\quad\times\|\widehat{\theta}_2((\xi_2-\eta_2^0)/\epsilon_2)\|_{B^{s_2}_{2,q}(\mathbb{R}^{n_2})}\|f_1\|_{H^p}\|f_2\|_{H^p}\\ &\le C\epsilon_1^{-s_1+n_1/2}\epsilon_2^{-s_2+n_2/2}\|f_1\|_{H^p}\|f_2\|_{H^p}\\ &\le C\epsilon_1^{-s_1+n_1/2}\epsilon_2^{-s_2+n_2/2}.\end{aligned}$$

Moreover, a simple calculation gives

$$\|T_m(f)(x_1,x_2)\|_{H^p} = C\|\epsilon_1^{n_1}\theta_1(\epsilon_1x_1)\epsilon_2^{n_2}\theta_2(\epsilon_2x_2)\|_{H^p} = C\epsilon_1^{n_1-n_1/p}\epsilon_2^{n_2-n_2/p}.$$

Thus, (5.27) yields the inequality

$$\epsilon_1^{n_1-n_1/p}\epsilon_2^{n_2-n_2/p} \le C\epsilon_1^{-s_1+n_1/2}\epsilon_2^{-s_2+n_2/2},$$

which inequality holds only if $s_1 \ge n_1(\frac{1}{p}-\frac{1}{2})$ and $s_2 \ge n_2(\frac{1}{p}-\frac{1}{2})$. The proof of Proposition 5.3 is complete. □

At last, we give an example of T_m to show that T_m does not satisfy Fefferman's criterion when $s_1 = n_1(\frac{1}{p}-\frac{1}{2}), s_2 = n_2(\frac{1}{p}-\frac{1}{2})$. The construction of our counterexample is based on [1]. First, we must introduce function spaces $K_s^{\alpha,p}(\mathbb{R}^n)$ which was considered in [1]. Suppose that

$$1 \le s \le \infty, 0 \le \alpha < \infty, 0 < p \le \infty.$$

Herz space $K_s^{\alpha,p}(\mathbb{R}^n)$ consists of all functions $f \in L^s_{loc}(\mathbb{R}^n\backslash\{0\}) \cap L^s(\mathbb{R}^n)$ with

$$\|f\|_{K_s^{\alpha,p}(\mathbb{R}^n)} = \|f\|_{L^s(\mathbb{R}^n)} + \|f\|_{\dot{K}_s^{\alpha,p}(\mathbb{R}^n)} < \infty,$$

where

$$\|f\|_{\dot{K}_s^{\alpha,p}(\mathbb{R}^n)} = \left\{\sum_{-\infty}^{\infty}\left(\int_{A_k}|f(x)|^s dx\right)^{p/s}2^{k\alpha s}\right\}^{1/p} \text{ for } A_k = \{2^k \le |x| \le 2^{k+1}\}.$$

Now, we recall a Lemma in [1].

Lemma 5.1. *([1]) Suppose that $s > 0$, $0 < p \leq 1$, $r > n + \alpha$ and that $Q \in L^1(\mathbb{R}^n)$ satisfies*

$$\|Q\|_{L^1(\mathbb{R}^n)} \leq 1, \quad \textit{and} \quad |Q(x)| \leq |x|^{-r} \quad \textit{for} \ \ |x| > 2. \tag{5.28}$$

Then, for $g \in K_2^{\alpha,p}(\mathbb{R}^n)$, we have

$$\|g * Q\|_{K_2^{\alpha,p}(\mathbb{R}^n)} \leq C(s,p,r,n)\|g\|_{K_2^{\alpha,p}(\mathbb{R}^n)}.$$

Notice that $\mathcal{F}$ maps $B^{\alpha}_{2,q}$ isomorphically onto $K_2^{\alpha,q}$ for $\alpha \geq 0$ and $0 < q \leq \infty$, see [1]. For simplicity, we consider the case $n = 1$.

Our main purpose of this section is to give an example of a bi-parameter Fourier multiplier m such that the multiplier operator does not satisfy the Fefferman criterion of boundedness from the bi-parameter Hardy space. To this end, we will first construct such an example in the one-parameter setting.

Proposition 5.4. *For $0 < p \leq 1, s = 1/p - 1/2$, we can find a sequence of constants $\gamma \geq 2$ and a multiplier $m(\xi)$ satisfying*

$$\sup_{j \in \mathbb{Z}} \|m_j\|_{K_2^{(\frac{1}{p}-\frac{1}{2}),p}} < \infty$$

such that

$$\|T_m(a)\|^p_{L^p((\gamma I)^c)} \leq c\gamma^{-\delta} \ \ \textit{fails for any given} \ \ \delta > 0,$$

where $a(x) = e^{ix}\chi_{(0,2\pi)}$ is a $H^p(\mathbb{R})$ atom, $I = (0, 2\pi)$.

Proof. Define f_0 by

$$f_0(x) = \begin{cases} 2^{-k/p}|k|^{-1/p^2}, & \text{for all } x \in A_k, \ k = 2, 4, 6, \dots, \\ 0, & \text{otherwise.} \end{cases} \tag{5.29}$$

Then, we have $f_0 \in K_2^{\frac{1}{p}-\frac{1}{2},p}(\mathbb{R})$. Take $Q \in C^\infty(\mathbb{R})$ satisfying $Q \geq 0$, $Q(0) > 0$, $\|Q\|_{L^1(\mathbb{R})} = 1$, $\widehat{Q} \in C^\infty(\mathbb{R})$ and $\operatorname{supp}\widehat{Q} \subset \{|\xi| \leq \frac{1}{2}\}$. For fixed $r > \frac{1}{p} + \frac{1}{2}$, Q satisfies $|Q(x)| \leq C|x|^{-r}$ for some C, and so cQ satisfies the hypothesis of Lemma 5.1 for some $c > 0$. Hence,

$$f(x) := e^{ix}(f_0 * Q)(x) \in K_2^{n(\frac{1}{p}-\frac{1}{2}),p}(\mathbb{R}).$$

Define $m = \hat{f}$. Then $\operatorname{supp} m \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$ and $m_\sigma(\xi) = m(\sigma\xi)\psi(\xi) = 0$ unless $\frac{1}{8} \leq \sigma \leq 8$. And the inverse Fourier transform of $m(\sigma\xi)$ is $\sigma^{-1}f(\sigma^{-1}x)$. These functions are all in $K_2^{(\frac{1}{p}-\frac{1}{2}),p}(\mathbb{R})$. From Lemma 5.1 with $Q = c\check{\psi}$, it follows that m satisfies that

$$\sup_\sigma \|m_\sigma\|_{K_2^{(\frac{1}{p}-\frac{1}{2}),p}} \approx 1.$$

Take $x \in A_k$, $k = 2, 4, 6, \ldots$. Then

$$\begin{aligned}(f_0 * Q)(x) = &\int_{|y|<2^{k-1}} Q(x-y)f_0(y)dy \\ &+ \int_{A_k} Q(x-y)f_0(y)dy + \int_{|y|>2^{k+2}} Q(x-y)f_0(y)dy.\end{aligned}$$

Since $|Q(x)| \leq C|x|^{-r}$, the first and the third integrals are controlled by $\frac{C}{2^{kr}}\|f_0\|_{L^1(\mathbb{R})}$. Since $Q(0) > 0$, the middle integral

$$\left|\int_{A_k} Q(x-y)f_0(y)dy\right| \geq C2^{-k/p}|k|^{-1/p^2}.$$

Since $r > \frac{1}{p}$, it follows that for all sufficiently large k

$$(f_0 * Q)(x) \geq C2^{-k/p}|k|^{-1/p^2} \quad \text{for} \quad x \in A_k.$$

Take an H^p atom $a(x) = e^{ix}\chi_{(0,2\pi)}$, where $\frac{1}{2} < p \leq 1$. Hence,

$$\begin{aligned}\|T_m(a)\|^p_{L^p((\gamma I)^c)} &= \int_{(\gamma I)^c}\left|\int_{(0,2\pi)} e^{i(x-y)}f_0 * Q(x-y)a(y)dy\right|^p dx \\ &= \int_{(\gamma I)^c}\left|\int_{(0,2\pi)} f_0 * Q(x-y)dy\right|^p dx \\ &\geq \int_{A_k}\left|\int_{(0,2\pi)} f_0 * Q(x-y)dy\right|^p dx \\ &\geq C|k|^{-1/p},\end{aligned}$$

we can take $\gamma \approx 2^k$, then we deduce

$$|k|^{-1/p} \leq c2^{-k\delta},$$

which is impossible when k trend to ∞. Therefore,

$$\|T_m(a)\|^p_{L^p((\gamma I)^c)} \gneqq c\gamma^{-\delta}. \qquad \square$$

Next, we will construct an example in the bi-parameter setting.

Proposition 5.5. *For $0 < p \leq 1, s = 1/p - 1/2$, we can find a sequence of constants $\gamma \geq 2$ and a multiplier $m(\xi_1, \xi_2)$ satisfying*

$$\sup_{j,k\in\mathbb{Z}} \|m_{jk}\|_{K_2^{(\frac{1}{p}-\frac{1}{2}),p}(\mathbb{R}\times\mathbb{R})} < \infty$$

and a rectangle atom $a(x_1, x_2)$ in $H^p(\mathbb{R}\times\mathbb{R})$ supported in the rectangle $I = (0, 2\pi) \times (0, 2\pi)$ such that

$$\|T_m(a)\|^p_{L^p((\gamma I)^c)} \leq c\gamma^{-\delta} \text{ fails}$$

for any given $\delta > 0$.

Proof. For bi-parameter case, we set $a(x_1, x_2) = a_1(x_1)a_2(x_2)$, where $a_1(x_1) = e^{ix_1}\chi_{(0,2\pi)} \in H^p(\mathbb{R})$ and $a_2(x_2) = e^{ix_2}\chi_{(0,2\pi)} \in H^p(\mathbb{R})$. Furthermore, we also set $m(\xi_1, \xi_2) = m_1(\xi_1)m_2(\xi_2)$, where $m_1(\xi_1) \in K_2^{(\frac{1}{p}-\frac{1}{2}),p}(\mathbb{R}), m_2(\xi_2) \in K_2^{(\frac{1}{p}-\frac{1}{2}),p}(\mathbb{R})$ are the same as in the above example. Therefore, we just estimate

$$T_m f(x_1, x_2) = T_{m_1}a_1(x_1)T_{m_2}a_2(x_2).$$

Moreover, we have

$$\begin{aligned}\|T_m(f)(x_1, x_2)\|^p_{L^p((R)^c)} &\geq C\|T_{m_1}(a_1)\|^p_{L^p((\gamma I)^c)}\|T_{m_2}(a_2)\|^p_{L^p((\gamma I)^c)} \\ &\geq C|k_1|^{-1/p}|k_2|^{-1/p},\end{aligned}$$

where $R = I \times I$ and $I = (0, 2\pi)$.

We can take $\gamma \approx 2^{k_1}2^{k_2}$, then we deduce

$$|k_1|^{-1/p}|k_2|^{-1/p} \leq 2^{-k_1\delta}2^{-k_2\delta} \text{ for } \delta > 0$$

which is impossible when k_1, k_2 tend to ∞. Then,

$$\|T_m(f)(x_1, x_2)\|^p_{L^p((\gamma R)^c)} \nleq c\gamma^{-\delta}.$$

Therefore, T_m does not satisfy Fefferman's criterion when $s_1 = \frac{1}{p} - \frac{1}{2}, s_2 = \frac{1}{p} - \frac{1}{2}$ for $0 < p \leq 1$. $\square$

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References

[1] A. I. Baernstein and E. T. Sawyer, *Embedding and multiplier theorems for $H^p(\mathbb{R}^n)$*, Mem. Amer. Math. Soc. **53** (1985), no. 318.

[2] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution. II*, Adv. Math. **24** (1977), no. 2, 101–171.

[3] A. Carbery and A. Seeger, *H^p- and L^p-variants of multiparameter Calderón-Zygmund theory*, Trans. Amer. Math. Soc. **334** (1992), no. 2, 719–747.

[4] S.-Y. A. Chang and R. Fefferman, *A Continuous version of duality of H^1 with BMO on the bidisc*, Ann. of Math. **112** (1980), 179–201.

[5] S.-Y. A. Chang and R. Fefferman, *Some recent developments in Fourier analysis and H^p theory on product domains*, Bull. Amer. Math. Soc. **12** (1985), 1–43.

[6] J. Chen, W. Ding and G. Lu, Boundedness of multi-parameter pseudo-differential operators on multi-parameter local Hardy spaces. Forum Math. **32** (2020), no. 4, 919-936.

[7] J. Chen and L. Huang, *Hörmander type multipliers on anisotropic Hardy spaces*, Acta Math. Sin. **35** (2019), no. 11, 1841–1853.

[8] J. Chen, L. Huang, and G. Lu, *Hörmander Fourier multiplier theorems with optimal Besov regularity on multi-parameter Hardy spaces*, to appear in Forum Mathematicum.

[9] J. Chen and G. Lu, *Hörmander type theorems for multi-linear and multi-parameter Fourier multiplier operators with limited smoothness*, Nonlinear Anal. **101** (2014), 98–112.

[10] J. Chen and G. Lu, *Hörmander type theorem on Bi-parameter Hardy spaces for Fourier multipliers with optimal smoothness*, Rev. Mat. Iberoam. **34** (2018), no. 4, 1541–1561.

[11] L. K. Chen, *The multiplier operators on the product spaces*, Illinois. J. Math. **38** (1994), no. 3, 420–433.

[12] W. Ding and G. Lu, Boundedness of inhomogeneous Journé's type operators on multi-parameter local Hardy spaces. Nonlinear Anal. **197** (2020), 111816, 31 pp.

[13] W. Ding, G. Lu, and Y. Zhu, Multi-parameter local Hardy spaces. Nonlinear Anal. 184 (2019), 352-380.

[14] W. Ding, G. Lu, and Y. Zhu, Discrete Littlewood-Paley-Stein characterization of multi-parameter local Hardy spaces. Forum Math. **31** (2019), no. 6, 1467-1488.

[15] R. Fefferman, *Harmonic analysis on product spaces*, Ann. of Math. **126** (1987), 109–130.

[16] R. Fefferman and K. C. Lin, *A sharp result on multiplier operators*, Manuscript written in early 1990s, unpublished.

[17] R. Fefferman and E. M. Stein, *Singular integrals on product spaces*, Adv. Math. **45** (1982), 117–143.

[18] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta. Math. **104** (1960), 93–140.

[19] L. Huang and J. Chen, *Hörmander type multiplier theorems on bi-parameter anisotropic Hardy spaces*, Forum Math. **32** (2020), no. 3, 577–594.

[20] V. L. Hung, *Multiplier operators on product spaces*, Studia. Math. **151** (2002), no. 3, 265–275.

[21] B. Jessen, J. Marcinkiewicz, and A. Zygmund, *Note on the differentiability of multiple integrals*, Funda. Math. **25** (1935), 217–234.

[22] G. Lu and Z. Ruan, *Duality theory of weighted Hardy spaces with arbitrary number of parameters*, Forum. Math. **26** (2014), no. 5, 1429–1457.

[23] G. Lu, J. Shen, and L. Zhang, Multi-parameter Hardy space theory and endpoint estimates for multi-parameter singular integrals, to appear in Memoires of Amer. Math. Soc.

[24] G. Lu and Y. Zhu, *Singular integrals and weighted Triebel-Lizorkin and Besov spaces of arbitrary number of parameters*, Acta Math. Sin. **29** (2013), no. 1, 39–52.

[25] S. G. Mihlin, *On the multipliers of Fourier integrals*, Dokl. Akad. Naulc SSSR **109** (1956), 701–703.

[26] B. Street, *Multi-parameter singular integrals*, Annals of Mathematics Studies **189** (2014).

[27] M. Sugimoto, *Pseudo-differential operators on Besov spaces*, Tsukuba J. Math. **12** (1988), no. 1, 43–63.

School of Mathematical Sciences
Chongqing Normal University
Chongqing 400000, China
E-mail address: chenjiaobnu@163.com

School of Science
Xi'an University of Posts and Telecommunications
Xi'an, Shanxi 710121, China
E-mail address: huangliang@mail.bnu.edu.cn

Department of Mathematics
University of Connecticut
Storrs, CT 06269, USA
E-mail address: guozhen.lu@uconn.edu

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