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Regularity estimates for the gradient flow of a spinorial energy functional

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In this article, we establish certain regularity estimates for the spinor flow introduced and initially studied in [AWW16]. Consequently, we prove that $|\nabla^2\phi|_{L^\infty(M)}(t)$ goes to ∞ as t approaches to the finite singular time. This generalizes the blow up criteria obtained in [Sch18] for surfaces to general dimensions. As another application of the estimates, we also obtain a lower bound for the existence time in terms of the initial data. Our estimates are based on an observation that, up to pulling back by a one-parameter family of diffeomorphisms, the metric part of the spinor flow is equivalent to a modified Ricci flow.

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1. Introduction

In [AWW16], in order to study parallel spinors, and related special holonomy metrics, including Calabi-Yau, hyper-Kähler, torsion free G_2 and Spin(7) metrics, from a uniform variational approach, Ammann, Weiss and Witt introduced the *spinorial energy functional* defined as

$$\begin{aligned} &\mathcal{E} : \mathcal{N} \to \mathbb{R}_{\geq 0}, \\ &(g, \phi) \mapsto \frac{1}{2} \int_M |\nabla^g \phi|_g^2 dv_g, \end{aligned} \tag{1.1}$$

where $\mathcal{N}$ is the union of pairs (g, ϕ) of a Riemannian metric g and a g-spinor of constant length one $\phi \in \Gamma(\Sigma_g M)$ over a closed spin manifold M.

In (1.1), ∇^g denotes the connection on the spinor bundle $\Sigma_g M$ induced by the Levi-Civita connection of g, and $|\cdot|_g$ the pointwise norm on $T^*M \otimes \Sigma_g M$ and dv_g the volume form induced by g. For the simplicity of notations, in the following, we will omit the superscript and subscript g in ∇^g and $|\cdot|_g$ once no ambiguity is caused.

Note that to compare spinor bundles associated to two different metric spin structures and then to calculate the variation of (1.1), one needs a connection on the Fréchet vector bundle $\mathcal{N} \to \mathcal{M}$, where $\mathcal{M}$ is the space of Riemnnian metrics on M. A natural one is the Bourguignon-Gauduchon (partial) connection introduced in [BG92] (also see [AWW16]). The evolution of the spinor ϕ is taken to be the vertical part with respect to this connection. In [Wan91], Wang studied the deformation of parallel spinors under variation of metrics in a different manner, where the spinor bundle is fixed and the variation of metrics is reflected as a variation of spinor connections.

The variation formula of the functional (1.1) has been derived in [AWW16]. Consequently, in dimension of $M \geq 3$, they obtained that the critical points of (1.1) are metrics with parallel spinors, and metrics with Killing spinors are certain critical points of (1.1) subject to the constraint of fixed volume. The classification of holonomy groups of Riemannian manifolds with parallel spinors was done by Wang in [Wan89] and [Wan95], and for the corresponding special holonomy metrics, one can refer to [Joy00].

Moreover, the negative gradient flow of (1.1) called the *spinor flow* is given by the coupled system

$$\begin{cases} \dfrac{\partial}{\partial t} g_{jk} = \dfrac{1}{4} T_{jk} + \dfrac{1}{2} \langle \nabla_j \phi, \nabla_k \phi \rangle - \dfrac{1}{4} |\nabla \phi|^2 g_{jk}, \\ \dfrac{\partial}{\partial t} \phi = \Delta \phi + |\nabla \phi|^2 \phi, \end{cases} \tag{1.2}$$

where

$$T_{jk} = g^{li}\nabla_l \tilde{T}_{ijk} \tag{1.3}$$

is the divergence over the first variable of the following 3-tensor

$$\tilde{T}(X,Y,Z) = \frac{1}{2}\langle X\wedge Y\cdot\phi, \nabla_Z\phi\rangle + \frac{1}{2}\langle X\wedge Z\cdot\phi, \nabla_Y\phi\rangle. \tag{1.4}$$

It has been proved in [AWW16] that (1.2) is a weakly parabolic system whose degeneracy is caused by spin-diffeomorphism invariance, and the short-time existence of the initial value problem can be shown by pulling it back to a stricly parabolic system. Behaviour of this flow on surfaces, on Berger spheres and on homogeneous spaces have been studied in [AWW16-2], [Wit16] and [FSW18] respectively, stability of the flow has been studied in [Sch17]. Moreover, a variational approach of investigating special metrics on 7-manifolds has already been carried out in an earlier work [WW12].

To study the long time behaviour of this flow, a fundamental question is to find a criterion for the occurrence of finite time singularities. In has been proved in [Sch18] that on a closed 2-dimensional surface, if $|\nabla^2\phi|$ stays uniformly bounded, then no finite time singularity will occur. We show that this result is true in general dimensions.

Our approach is to pull-back a solution of (1.2) by a family of diffeomorphisms to an equivalent system which looks like a modified Ricci flow coupled with an evolving spinor (see (3.2) below). Then it is sufficient to establish estimates for (3.2). Recall that pointwise regularity estimates for the Ricci flow have been obtained by Hamilton [Ham82] and by Shi [Shi89] using maximum principle, and these estimates have played a fundamental role in the analysis of Ricci flow. In this article we obtain the following Bernstein-Bando-Shi type estimates for the spinor flow in L^2-integral form.

Theorem 1.1. *Let $(M^n, g(t), \phi(t))$ be a solution of the spinor flow. For any constants $K>0$, $a>0$, suppose $|Rm|\le K$, $|\nabla^2\phi|\le K$ and $|\nabla\phi|^2\le K$ on $B_{g(0)}(p,\frac{r}{\sqrt{K}})\times[0,\frac{a}{K}]$. Then for any integer $k\ge 0$ there exists a constant $C(n,a,k,r)$ such that*

$$\fint_{B_{\frac{r}{2\sqrt{K}}}} |\nabla^k Rm|^2 dv(t) \le \frac{CK^2}{t^k}, \tag{1.5}$$

$$\fint_{B_{\frac{r}{2\sqrt{K}}}} |\nabla^{2+k}\phi|^2 dv(t) \le \frac{CK^2}{t^k}, \tag{1.6}$$

for $t \in (0, \frac{a}{K}]$*. Here,* $B_{\frac{r}{2\sqrt{K}}}$ *denotes the geodesic ball centered at* p *with radius* $\frac{r}{2\sqrt{K}}$ *with respect to the initial metric* $g(0)$*,* $dv(t)$ *denotes the volume form associated to the Riemannian metric* $g(t)$*, and as usual* $⨍$ *denotes the average, e.g.*

$$⨍_{B_{\frac{r}{2\sqrt{K}}}} |\nabla^k Rm|^2 dv(t) := \left(\mathrm{Vol}_{g(t)}\left(B_{\frac{r}{2\sqrt{K}}}\right)\right)^{-1} \int_{B_{\frac{r}{2\sqrt{K}}}} |\nabla^k Rm|^2 dv(t).$$

Pointwise estimates then follow from standard embedding theorems. In view of the similarity of (3.2) to the Ricci flow, we can adapt a method of [KNM16] to control the norm of the Riemann tensor in terms of the initial data and the first two derivatives of the spinor, see Lemma 4.1 below. Hence as our first application of the estimates in Theorem 1.1, we have

Theorem 1.2. *Let* $(g(t), \phi(t))$ *be a solution to (1.2) or (3.2) on a closed manifold* M *and on time interval* $[0, T)$*, for some* $T < \infty$*. If* $\sup_{M\times[0,T)} |\nabla^2\phi| < \infty$*, then the flow can be extended to a larger time interval.*

Note that in Theorem 1.2 we do not need to assume uniform bounds for $|\nabla\phi|$ along the spinor flow, since it is controlled by $|\nabla^2\phi|$ by Lemma 7.1 below.

As another application of Theorem 1.1, we also obtain a lower estimate for the existence time in terms of initial data.

Theorem 1.3. *Let* $g(t), \phi(t)$ *be a solution of (3.2) on a closed manifold* M^n *with* $n \geq 3$*. Suppose* $\sup_M |Rm|_{g(0)} \leq LK$ *and* $\sup_M |\nabla^2\phi|_{g(0)} \leq K$*. There are constants* Λ *and* δ *depending on* n*,* L *and*

$$⨍_M K^{-2-i}|\nabla^i Rm|^2 dv(g(0)), \quad ⨍_M K^{-2-i}|\nabla^{2+i}\phi|^2 dv(g(0)),$$

where $i = 0, 1, 2, \ldots, [\frac{n}{2}] + 1$*, such that the existence time interval contains* $[0, \frac{\delta}{K}]$*, and we have*

$$\sup_{M\times[0,\frac{\delta}{K}]} |\nabla^2\phi| \leq \Lambda K.$$

Finally, we would like to mention that in recent years, Shi-type estimates and long time behaviors have been studied for certain geometric flows aimed at finding torsion-free G_2 structures. These flows work with G2 structures directly, rather than taking a spinorial approach. In [LW17], Lotay and Wei proved Shi-type estimates and their applications for Laplacian flow introduced by Bryant. In [Li18], Li established some local curvature estimates and some consequences for the Lapacian flow. In particular, Li provided a new proof for the finite time blow up criteria obtained in [LW17]. In [Che18], Chen obtained Shi-type estimates and their applications for Laplace co-flow. Moreover, in [DGK19], Dwivedi, Gianniotis and Karigiannis introduced an isometric flow to study G_2 structures and obtained Shi-type estimates and their applications for the flow.

The rest of the article is organized as follows. In §2, we recall some basic facts in spin geometry that we need in the article. In §3, we describe how to pull back the spinor flow system in (1.2) to an equivalent system in (3.2), in which the metric evolving equation behaves similarly to the Ricci flow equation. Therefore, in §4, we are able to obtain C^0 estimate for Riemannian curvature tensor along the flow in (3.2) by adapting the technique developed in [KNM16] for Ricci flow to this spinor flow. Furthermore, in §5 we prove Theorem 1.1. Then as applications of Theorem 1.1, we prove Theorem 1.2 in §6 and Theorem 1.3 in §7. Finally, in the appendix §8 we recall some interpolation inequalities that we need in preceding sections.

2. Preliminaries on spin geometry

In this section, we will briefly review some basic facts on spin geometry. Throughout this article, M^n will be a connected closed spin manifold of dimension $n \geq 2$, and $\mathcal{M}$ is the space of Riemannian metrics on M.

2.1. Metric spin structure and spinor bundle

For more details about this subsection, we refer to e.g. [BFGK91], [Fri00], and [LM89].

Fix a Riemannian metric $g \in \mathcal{M}$ on M, let $P_{\mathrm{SO}}(g)$ denote the oriented orthonormal frame bundle with respect to g. A *metric spin structure* with respect to g is a $\mathrm{Spin}(n)$-principal bundle P_{Spin} over M together with an equivariant two-sheeted covering map

$$\xi : P_{\mathrm{Spin}} \to P_{\mathrm{SO}}(g),$$

which restricts to a non-trivial double covering map $\rho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ on each fiber. This is said to be equivalent to another metric spin structure $\xi' : P'_{\mathrm{Spin}} \to P_{\mathrm{SO}}(g)$ with respect to g, if there exists a principal $\mathrm{Spin}(n)$-bundle isomorphism $\chi : P_{\mathrm{Spin}} \to P'_{\mathrm{Spin}}$ such that there is the commutative diagram

$$\begin{array}{ccc} P_{\mathrm{Spin}} & \xrightarrow{\quad\chi\quad} & P'_{\mathrm{Spin}} \\ & {\scriptstyle\xi}\searrow \quad \swarrow{\scriptstyle\xi'} & \\ & P_{\mathrm{SO}}(g). & \end{array}$$

The *complex* Clifford algebra $\mathbb{C}l_n$ of the standard Euclidean vector space $(\mathbb{R}^n, g_{\mathbb{R}^n})$ may be classified as

$$\mathbb{C}l_n \cong \begin{cases} \mathrm{End}(\Sigma_n), & \text{if} \quad n = \textit{even}, \\ \mathrm{End}(\Sigma_n) \oplus \mathrm{End}(\Sigma_n), & \text{if} \quad n = \textit{odd}, \end{cases} \tag{2.1}$$

where "$\cong$" means algebra isomorphism, and Σ_n is a complex vector space of complex dimension $2^{[n/2]}$. The spin group $\mathrm{Spin}(n)$ is a subgroup of the group of invertible elements in $\mathbb{C}l_n$. Thus the inclusion $\mathrm{Spin}(n) \hookrightarrow \mathbb{C}l_n$ together with the algebra isomorphism in (2.1) gives *the spin representation* of the spin group $\mathrm{Spin}(n)$ on Σ_n, and which is referred as $\mu : \mathrm{Spin}(n) \to \mathrm{End}(\Sigma_n)$. Then *the spinor bundle* is defined to be the associated bundle

$$\Sigma_g M := P_{\mathrm{Spin}} \times_\mu \Sigma_n = (P_{\mathrm{Spin}} \times \Sigma_n)/\sim,$$

where $(\tilde{e}, \tilde{\phi}) \sim (\tilde{e} \cdot g, \mu(g^{-1})\tilde{\phi})$ for $g \in \mathrm{Spin}(n)$. A section of the spinor bundle $\Sigma_g M$ is called a *spinor (field)*.

The (complex) *Clifford bundle* $\mathbb{C}l(g)$ is referred to be the union of complex clifford algebras of $(T_x M, g(x))$ for all $x \in M$. This bundle may be viewed as an associated bundle as

$$\mathbb{C}l(g) = P_{\mathrm{Spin}} \times_{\mathrm{Ad}} \mathbb{C}l_n,$$

where $\mathrm{Ad} : \mathrm{Spin}(n) \to \mathrm{Aut}(\mathbb{C}l_n)$ is given by $\mathrm{Ad}(g)(\varphi) = g\varphi g^{-1}$ for $g \in \mathrm{Spin}(n)$ and $\varphi \in \mathbb{C}l_n$. By this characterization of $\mathbb{C}l(g)$, one can easily check that the Clifford bundle $\mathbb{C}l(g)$ acts on the spinor bundle $\Sigma_g M$. In particular, the tangent bundle $TM \subset \mathbb{C}l(g)$ acts on $\Sigma_g M$, i.e. a vector field X acts on a spinor ϕ, written as $X \cdot \phi$ and this is called *Clifford multiplication*. In terms of a local representation $[\tilde{e}, \tilde{\phi}]$ of ϕ, where for an open set $U \subset M$, $\tilde{\phi} : U \to \Sigma_n$, and $\tilde{e} : U \to P_{\mathrm{Spin}}$ covers a local orthonormal frame $e = (e_1, \ldots, e_n) : U \to P_{\mathrm{SO}}(g)$, we have

$$X \cdot \phi = \sum_{i=1}^{n} \langle X, e_i \rangle e_i \cdot \phi = \sum_{i=1}^{n} \langle X, e_i \rangle [\tilde{e}, E_i \cdot \tilde{\phi}],$$

where $\{E_1, \ldots, E_n\}$ is the standard basis of $\mathbb{R}^n$, and $E_i \cdot \tilde{\phi}$ is the Clifford multiplication induced by the algebra isomorphism in (2.1).

The Clifford multiplication naturally extends to actions of tensor algebras. For example, for any p-form α,

$$\alpha \cdot \phi := \sum_{1 \le i_1 \le \cdots \le i_p \le n} \alpha(e_{i_1}, \ldots, e_{i_p}) e_{i_1} \cdots e_{i_p} \cdot \phi.$$

In particular,

$$(X \wedge Y) \cdot \phi = X \cdot Y \cdot \phi + g(X, Y)\phi. \tag{2.2}$$

Here we identify TM and T^*M by using the Riemannian metric g.

The Levi-Civita connection on the orthonormal frame bundle $P_{\mathrm{SO}}(g)$ naturally induces a connection on P_{Spin} and further on $\Sigma_g M$. In a local expression as above, the spinor covariant derivative of ϕ in the direction of a vector field X can be written as

$$\nabla_X \phi = \left[\tilde{e}, X\tilde{\phi} + \frac{1}{4} \sum_{i,j=1}^{n} g(\nabla_X e_i, e_j) E_i \cdot E_j \cdot \tilde{\phi} \right].$$

In other words,

$$\nabla_X \phi = X(\phi) + \frac{1}{4} g(\nabla_X e_i, e_j) e_i \cdot e_j \cdot \phi.$$

Here and also in the following the Einstein summation convention is used.

Define the spinor curvature operator as

$$\mathcal{R}(X, Y)\phi := \nabla^2_{X,Y}\phi - \nabla^2_{Y,X}\phi = \nabla_X(\nabla_Y \phi) - \nabla_Y(\nabla_X \phi) - \nabla_{[X,Y]}\phi.$$

By direct calculation

$$\mathcal{R}(e_i, e_j)\phi = \frac{1}{4} R_{ijkl} e_l \cdot e_k \cdot \phi.$$

Using the first Bianchi identity one can derive a formula for the Ricci curvature as

$$e_i \cdot \mathcal{R}(e_j, e_i)\phi = -\frac{1}{2} R_{jk} e_k \cdot \phi. \tag{2.3}$$

Recall that there is a positive definite real inner product on the spinor bundle $\Sigma_g M$ which is invariant under Clifford multiplication by unit vectors and compatible with the spinor covariant differentiation, i.e.

$$\begin{aligned} \langle X\cdot\phi, X\cdot\psi\rangle &= \langle\phi,\psi\rangle, \quad \text{if} \quad |X| = 1, \\ X\langle\phi,\psi\rangle &= \langle\nabla_X\phi,\psi\rangle + \langle\phi,\nabla_X\psi\rangle. \end{aligned}$$

Using this inner product we get a formula for the Ricci tensor

$$R_{jk} = -2\langle e_i\cdot\mathcal{R}(e_j,e_i)\phi, e_k\cdot\phi\rangle, \quad \text{if} \quad |\phi| = 1. \tag{2.4}$$

Hence the Ricci curvature tensor can be completely determined by the second covariant derivative of any unit spinor field, we will keep this fact in mind in the following of this article.

2.2. Topological spin structure and universal spinor bundle

In order to compare spinors with respect to different Riemannian metrics, one needs to gather all spinor bundles $\Sigma_g M$ for all $g\in\mathcal{M}$ together into a single bundle, which will be the so called universal spinor bundle. For defining a universal spinor bundle, one needs a topological spin structure without referring to any specific Riemannian metric. For more details about these materials, see e.g. [AWW16], [BG92], and [Swi93].

Fix an orientation for the manifold M^n. Let P_{GL^+} be the oriented frame bundle over M, which is a principal $\mathrm{GL}(n)^+$-bundle over M. Then a *topological spin structure* is a principal $\widetilde{\mathrm{GL}}(n)^+$-bundle $P_{\widetilde{\mathrm{GL}}^+}$ together with an equivariant two-sheeted covering map

$$\theta : P_{\widetilde{\mathrm{GL}}^+} \to P_{\mathrm{GL}^+},$$

which restricts to a non-trivial double covering map $\widetilde{\mathrm{GL}}(n)^+ \to \mathrm{GL}(n)^+$ on each fiber. This is said to be equivalent to another topological spin structure $\theta' : P'_{\widetilde{\mathrm{GL}}^+} \to P_{\mathrm{GL}^+}$ if there exists an equivariant principal $\widetilde{\mathrm{GL}}(n)^+$-bundle isomorphism $\chi : P_{\widetilde{\mathrm{GL}}^+} \to P'_{\widetilde{\mathrm{GL}}^+}$ such that there is the commutative diagram

$$\begin{array}{ccc} P_{\widetilde{\mathrm{GL}}^+} & \xrightarrow{\quad\chi\quad} & P'_{\widetilde{\mathrm{GL}}^+} \\ & {\scriptstyle\theta}\searrow \quad \swarrow{\scriptstyle\theta'} & \\ & P_{\mathrm{GL}^+}. & \end{array}$$

It was shown in [Swi93] that for any fixed metric g the equivalent metric spin structures with respect to g are in one-to-one correspondence with equivalent topological spin structures.

The bundle of positive definite bilinear forms can be viewed as the associated bundle

$$\odot^2_+ T^*M = P_{\mathrm{GL}^+} \times_p (\mathrm{GL}(n)^+/\mathrm{SO}(n)) = P_{\mathrm{GL}}/\mathrm{SO}(n),$$

where p is the natural left action of $\mathrm{GL}(n)^+$ on the quotient space $\mathrm{GL}(n)^+/\mathrm{SO}(n)$. Now we choose and fix a topological spin structure $\theta : P_{\widetilde{\mathrm{GL}}^+} \to P_{\mathrm{GL}^+}$ throughout the rest of the article. Then the bundle $\odot^2_+ T^*M$ can also be viewed as

$$\begin{aligned}\odot^2_+ T^*M &= P_{\widetilde{\mathrm{GL}}^+} \times_{\widetilde{p}} (\mathrm{GL}(n)^+/\mathrm{SO}(n)) \\ &= P_{\widetilde{\mathrm{GL}}^+} \times_{\widetilde{p}'} (\widetilde{\mathrm{GL}}(n)^+/\mathrm{Spin}(n)) \\ &= P_{\widetilde{\mathrm{GL}}^+}/\mathrm{Spin}(n),\end{aligned}$$

where $\widetilde{p}$ and $\widetilde{p}'$ are natural actions of $\widetilde{\mathrm{GL}}(n)^+$ on $\mathrm{GL}(n)^+/\mathrm{SO}(n)$ and $\widetilde{\mathrm{GL}}(n)^+/\mathrm{Spin}(n)$, respectively.

Thus the projection $P_{\widetilde{\mathrm{GL}}^+} \to \odot^2_+ T^*M$ is a principal $\mathrm{Spin}(n)$-bundle. The *universal spinor bundle* is then defined as the associated vector bundle

$$\pi : \Sigma M := P_{\widetilde{\mathrm{GL}}^+} \times_\mu \Sigma_n \to \odot^2_+ T^*M.$$

By composing π with the projection of the bundle $\odot^2_+ T^*M \to M$, one can also view ΣM as a fiber bundle over M with fiber $(\widetilde{\mathrm{GL}}^+ \times \Sigma_n)/\mathrm{Spin}(n)$. Then a section $\Phi \in \Gamma(\Sigma M)$ of this bundle over M determines a Riemannian metric g_Φ and a spinor $\phi_\Phi \in \Gamma(\Sigma_{g_\Phi} M)$ and vice versa. Therefore, we identify sections of the universal bundle $\Sigma M \to M$ with the corresponding pairs (g, ϕ).

The Bourguignon-Gauduchon (partial) connection (or horizontal distribution) in [BG92], or equivalently generalized cylinder construction in [BGM05], produces the decomposition

$$T_{(g,\phi)}\Sigma M \cong \odot^2_+ T^*_x M \oplus \Sigma_{g,x} M, \tag{2.5}$$

where $(g, \phi) \in \Sigma M$ has base-point $x \in M$, and $\Sigma_{g,x} M$ is the fiber of the spinor bundle $\Sigma_g M$ with respect to the metric g over $x \in M$. In the decomposition (2.5), the first factor is the horizontal part and the second factor the vertical part. For details of the decomposition (2.5), we also refer to [AWW16] and references therein.

The universal bundle of unit spinors $S(\Sigma M)$ is given by

$$S(\Sigma M) := \{(g,\phi) \in \{g\} \times \Sigma_g M \subset \Sigma M \mid |\phi|_g = 1\} \subset \Sigma M.$$

As in [AWW16], let $\mathcal{F}$ and $\mathcal{N}$ denote respectively the spaces of smooth sections

$$\mathcal{F} := \Gamma(\Sigma M) \quad \text{and} \quad \mathcal{N} := \Gamma(S(\Sigma M)).$$

They can been considered as Fréchet fiber bundles over $\mathcal{M}$. Then it follows from the decomposition (2.5) that

$$T_{(g,\phi)}\mathcal{F} = \mathcal{H}_{(g,\phi)} \oplus T_{(g,\phi)}\mathcal{F}_g \cong \Gamma(\odot^2 T^*M) \oplus \Gamma(\Sigma_g M),$$

and

$$\begin{aligned} T_{(g,\phi)}\mathcal{N} &= \Gamma(\odot^2 T^*M) \oplus \Gamma(\phi^\perp) \\ &= \{(h,\varphi) \in \Gamma(\odot^2 T^*M) \oplus \Gamma(\Sigma_g M) \mid \langle \varphi(x), \phi(x)\rangle = 0, \forall x \in M\}. \end{aligned}$$

Again the first factor in the decomposition is the horizontal part, and the second factor the vertical part. As mentioned in Introduction, in the spinor evolving equation in the spinor flow system, $\frac{\partial}{\partial t}\phi$ takes values in the vertical part.

3. Pull-back to a modified Ricci flow

In this section, we will pull-back the spinor flow (1.2) to a modified Ricci flow coupled with an evolving spinor as in (3.2). This helps us eliminate the term involving the second order derivative of spinor on the right hand side of the metric evolving equation in (1.2), and so that we can apply the technique developed in [KNM16] to do C^0-estimate for Riemann tensor in §4.

For simplicity we denote the covariant derivative in the direction of e_i as ∇_i. Using the fact that $|\phi| \equiv 1$ and the formula in (2.2), we have (recall tensors $\tilde{T}$ and T defined in (1.4) and (1.3), respectively,)

$$\tilde{T}_{ijk} = \frac{1}{2}\langle e_i \cdot \phi, e_j \cdot \nabla_k \phi + e_k \cdot \nabla_j \phi\rangle.$$

Hence for any vectors X and Y, we can write T as

$$\begin{aligned} T(X,Y) = &\frac{1}{2}\langle e_i \wedge X \cdot \nabla_i \phi, \nabla_Y \phi\rangle + \frac{1}{2}\langle e_i \wedge Y \cdot \nabla_i \phi, \nabla_X \phi\rangle \\ &+ \frac{1}{2}\langle e_i \wedge X \cdot \phi, \nabla_i \nabla_Y \phi\rangle + \frac{1}{2}\langle e_i \wedge Y \cdot \phi, \nabla_i \nabla_X \phi\rangle. \end{aligned}$$

Let $\mathbf{D}$ be the Dirac operator defined by

$$\mathbf{D}\phi := e_i \cdot \nabla_i \phi.$$

Lemma 3.1. *For any tangent vectors X and Y, we have*

$$\begin{aligned} T(X,Y) = & -\frac{1}{2}Ric(X,Y) - 2\langle \nabla_X\phi, \nabla_Y\phi\rangle \\ & + \frac{1}{2}\langle \mathbf{D}\phi, X\cdot\nabla_Y\phi + Y\cdot\nabla_X\phi\rangle \\ & + \frac{1}{2}\langle \phi, X\cdot\nabla_Y\mathbf{D}\phi + Y\cdot\nabla_X\mathbf{D}\phi\rangle. \end{aligned}$$

Proof. For simplicity we use the standard trick to choose an orthonormal frame e_i, $i = 1, 2, \ldots, n$, and calculate at a point where $\nabla_i e_j = 0$. Use the formula in (2.2), and group $e_i\cdot$ and ∇_i whenever possible to introduce the Dirac operator to the expression of

$$\begin{aligned} T_{jk} = & \frac{1}{2}\langle \mathbf{D}\phi, e_j\cdot\nabla_k\phi + e_k\cdot\nabla_j\phi\rangle - \langle\nabla_j\phi, \nabla_k\phi\rangle \\ & + \frac{1}{2}\langle \phi, e_j\cdot\mathbf{D}\nabla_k\phi + e_k\cdot\mathbf{D}\nabla_j\phi + \nabla_j\nabla_k\phi + \nabla_k\nabla_j\phi\rangle. \end{aligned}$$

Differentiating $|\phi| \equiv 1$ twice yields

$$\langle\nabla_j\phi, \nabla_k\phi\rangle = -\langle\phi, \nabla_j\nabla_k\phi\rangle.$$

Hence

$$\begin{aligned} T_{jk} = & \frac{1}{2}\langle \mathbf{D}\phi, e_j\cdot\nabla_k\phi + e_k\cdot\nabla_j\phi\rangle + \frac{1}{2}\langle\phi, e_j\cdot\mathbf{D}\nabla_k\phi + e_k\cdot\mathbf{D}\nabla_j\phi\rangle \\ & - 2\langle\nabla_j\phi, \nabla_k\phi\rangle. \end{aligned}$$

By using (2.3), one can easily see

$$\mathbf{D}\nabla_k\phi = e_i\cdot\nabla_i\nabla_k\phi = e_i\cdot(\nabla_k\nabla_i\phi + \mathcal{R}(e_i,e_k)\phi) = \nabla_k\mathbf{D}\phi + \frac{1}{2}R_{kl}e_l\cdot\phi.$$

We can rewrite the formula of T as

$$\begin{aligned}
T_{jk} &= \frac{1}{2}\langle \mathbf{D}\phi, e_j \cdot \nabla_k \phi + e_k \cdot \nabla_j \phi\rangle + \frac{1}{2}\langle \phi, e_j \cdot \nabla_k \mathbf{D}\phi + e_k \cdot \nabla_j \mathbf{D}\phi\rangle \\
&\quad + \frac{1}{2}\left\langle \phi, \frac{1}{2}R_{kl} e_j \cdot e_l \cdot \phi + \frac{1}{2}R_{jl} e_k \cdot e_l \cdot \phi \right\rangle - 2\langle \nabla_j \phi, \nabla_k \phi\rangle \\
&= \frac{1}{2}\langle \mathbf{D}\phi, e_j \cdot \nabla_k \phi + e_k \cdot \nabla_j \phi\rangle + \frac{1}{2}\langle \phi, e_j \cdot \nabla_k \mathbf{D}\phi + e_k \cdot \nabla_j \mathbf{D}\phi\rangle \\
&\quad - \frac{1}{2}R_{jk} - 2\langle \nabla_j \phi, \nabla_k \phi\rangle
\end{aligned}$$

In the second equality, in order to deal with the third term, we use that $\langle \phi, e_j \cdot e_l \cdot \phi\rangle = -\delta_{jl}$, which can be easily derived from the fact the Clifford action of a unit vector on spinor is skew-symmetric with respect to the real inner product $\langle \cdot, \cdot \rangle$, and $|\phi| = 1$. □

The above lemma reveals that the main term in the metric evolution equation is the Ricci tensor. However, there is a term involving second order derivatives of the spinor which is undesirable. To cancel it we pull-back the flow by a family of spin-diffeomorphisms. Define a vector field

$$X := \langle \phi, e_i \cdot \mathbf{D}\phi\rangle e_i.$$

Let $F(t)$ be the 1-parameter family of diffeomorphism generated by $-\frac{1}{8}X$, i.e.

$$\begin{cases} \frac{d}{dt}F(t) = -\frac{1}{8}X, \\ F(0) = id. \end{cases} \tag{3.1}$$

$F(t)$ is a family of spin diffeomorphisms since they are isotopic to the identity. Suppose $(g(t), \phi(t))$ is a solution to (1.2), pull-back $g(t), \phi(t)$ by $F(t)$. For simplicity we denote F^*g and $F^*\phi$ still by g and ϕ. By the formula for the Lie derivative and the *metric Lie derivative* obtained in Proposition 17 in [BG92] (also see (8) in [AWW16])

$$\begin{aligned}
L_X g_{ij} &= -\langle e_i \cdot \nabla_j \phi + e_j \cdot \nabla_i \phi, \mathbf{D}\phi\rangle \\
&\quad + \langle \phi, e_i \cdot \nabla_j \mathbf{D}\phi + e_j \cdot \nabla_i \mathbf{D}\phi\rangle, \\
\tilde{L}_X \phi &= \nabla_X \phi - \frac{1}{4} dX^b \cdot \phi \\
&= \langle \phi, e_i \cdot \mathbf{D}\phi\rangle \nabla_i \phi - \frac{1}{4}(\langle \nabla_j \phi, e_i \cdot \mathbf{D}\phi\rangle + \langle \phi, e_i \cdot \nabla_j \mathbf{D}\phi\rangle) e_j \wedge e_i \cdot \phi,
\end{aligned}$$

we have

$$
\begin{cases}
\dfrac{\partial}{\partial t} g_{jk} = -\dfrac{1}{8} R_{jk} - \dfrac{1}{4}|\nabla\phi|^2 g_{jk} \\
\qquad\qquad + \dfrac{1}{4}\langle \mathbf{D}\phi, e_j \cdot \nabla_{e_k}\phi + e_k \cdot \nabla_{e_j}\phi\rangle, \\
\dfrac{\partial}{\partial t}\phi = \Delta\phi + |\nabla\phi|^2\phi - \dfrac{1}{8}\langle\phi, e_i \cdot \mathbf{D}\phi\rangle\nabla_i\phi \\
\qquad\qquad + \dfrac{1}{32}(\langle\nabla_j\phi, e_i \cdot \mathbf{D}\phi\rangle + \langle\phi, e_i \cdot \nabla_j \mathbf{D}\phi\rangle) e_j \wedge e_i \cdot \phi.
\end{cases} \tag{3.2}
$$

Hence by pulling back the flow (1.2) by a 1-parameter family of diffeomorphisms we simplified the evolution equation of the Riemmanian metric, at the expense of complicating the evolution equation of the spinor field by introducing a new second order term

$$\frac{1}{32}\langle\phi, e_i \cdot \nabla_j \mathbf{D}\phi\rangle e_j \wedge e_i \cdot \phi.$$

Recall that under a smooth deformation of the metric $\frac{\partial}{\partial t} g = h$, we have

$$
\begin{aligned}
\frac{\partial}{\partial t} R_{ijlk} = \frac{1}{2}(&\nabla_i\nabla_k h_{jl} + \nabla_j\nabla_l h_{ik} - \nabla_i\nabla_l h_{jk} \\
&- \nabla_j\nabla_k h_{il} + R_{ijpk}h_{pl} + R_{ijlp}h_{kp}).
\end{aligned}
$$

Denote

$$\mathcal{V}(h)_{ijlk} := \nabla_i\nabla_k h_{jl} + \nabla_j\nabla_l h_{ik} - \nabla_i\nabla_l h_{jk} - \nabla_j\nabla_k h_{il}$$

Under the flow (3.2) we have

$$
\begin{aligned}
\frac{\partial}{\partial t} Rm = -\frac{1}{16}\mathcal{V}(Ric) + \frac{1}{8} Rm(Ric) + Rm * \nabla\phi * \nabla\phi \\
+ \nabla^3\phi * \nabla\phi + \nabla^2\phi * \nabla^2\phi.
\end{aligned} \tag{3.3}
$$

Here and in the rest of the article, "$*$" is allowed to involve three types of operations: contraction by the metric g, clifford multiplication by unit vectors, and contraction by the spinor inner product. For simplicity, we sometimes omit the coefficients of terms containing "$*$", as long as the eventual estimates are not affected.

By the second Bianchi identity we can derive the heat-type evolution equation of the Riemann curvature tensor.

$$\frac{\partial}{\partial t}Rm = \frac{1}{16}\Delta Rm + Rm * Rm + Rm * \nabla\phi * \nabla\phi + \nabla^3\phi * \nabla\phi + \nabla^2\phi * \nabla^2\phi. \tag{3.4}$$

After tracing we have the evolution equation of the Ricci tensor

$$\frac{\partial}{\partial t}Ric = \frac{1}{16}\Delta Ric + Rm * Ric + Rm * \nabla\phi * \nabla\phi + \nabla^3\phi * \nabla\phi + \nabla^2\phi * \nabla^2\phi. \tag{3.5}$$

For covariant derivatives of the Riemann tensor we have evolution equations

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{1}{16}\Delta\right)\nabla^k Rm = &\sum_{l=0}^{k}\sum_{r=0}^{l}\nabla^{1+r}\phi * \nabla^{1+l-r}\phi * \nabla^{k-l}Rm \\ &+ \sum_{l=0}^{k}\nabla^l Rm * \nabla^{k-l}Rm \\ &+ \sum_{l=0}^{k+2}\nabla^{1+l}\phi * \nabla^{3+k-l}\phi,\end{aligned} \tag{3.6}$$

where ∇^k denote the k-th covariant derivative. For the spinor we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\nabla^k\phi = &\frac{1}{32}\langle\phi, e_p\cdot\nabla^k\nabla_q\mathbf{D}\phi\rangle e_q\wedge e_p\cdot\phi \\ &+ \sum_{p=0}^{k}\sum_{q=0}^{k+1-p}\nabla^{1+p}\phi * \nabla^q\phi * \nabla^{k+1-p-q}\phi \\ &+ \sum_{0\le l\le k-1}\nabla^l Rm * \nabla^{k-l}\phi + \sum_{1\le l\le k}\nabla^l Ric * \nabla^{k-l}\phi.\end{aligned} \tag{3.7}$$

4. Estimates of the Riemann tensor

In this section we show that the first two derivative of ϕ controls the growth of $|Rm|$ along the flow (3.2). Since this is trivial in dimension ≤ 3, we can assume here the dimension is $n \ge 4$. We use the method of [KNM16]. In this and the next section, Cauchy-Schwarz inequality is used frequently, and oftentimes we use it in the form

$$A^pB^q \leq \epsilon A^{p+r}B^{q+s} + \frac{1}{4\epsilon}A^{p-r}B^{q-s},$$

where the constants ϵ, r and s have to be chosen properly each time we use it.

Lemma 4.1. *Let $(g(t), \phi(t))$ be a solution of (3.2) with $|\nabla\phi|^2, |\nabla^2\phi| \leq K$ on $B(r) \times [0, T]$, where we can take $B(r)$ to be a geodesic ball with respect to $g(0)$ with radius r. Then we have*

$$\sup_{B(r/2)\times[0,T]} |Rm| \leq C\left(n, K, r^{-1}, T, \sup_{B(r)} |Rm|(g(0))\right).$$

Proof. We use C to denote constants that may depend on n, K, r^{-1} and take $p \geq 4$. For simplicity, we allow C to vary from term to term. Let η be a cut-off function independent of t, and $|\nabla\eta|^2 \leq Cr^{-1}\eta$.

Note that

$$\frac{\partial}{\partial t}|Rm|^2 = 2\left\langle Rm, \frac{\partial}{\partial t}Rm\right\rangle + \frac{\partial}{\partial t}g * Rm * Rm,$$

since the norm is taken with respect to the evolving metric g. And the volume form is evolving by

$$\frac{\partial}{\partial t}dv = \frac{1}{2}tr_g\left(\frac{\partial}{\partial t}g\right)dv.$$

By (3.2) and (3.3) we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\int \eta^p|Rm|^p dv(t)\\
&\quad = -\frac{1}{16}\int p\eta^p|Rm|^{p-2}\langle Rm, \mathcal{V}(Ric)\rangle + \int p\eta^p|Rm|^{p-2}Rm * \nabla^3\phi * \nabla\phi\\
&\qquad + \int p\eta^p|Rm|^{p-2}Rm * (Rm * Ric + Rm * \nabla\phi * \nabla\phi + \nabla^2\phi * \nabla^2\phi),
\end{aligned}$$

where for simplicity we have absorbed terms such as $|Rm|^p tr_g(Ric)$ by the more general term $|Rm|^{p-2}Rm * Rm * Ric$.

For the first two terms, we use integration by parts to get rid of $\mathcal{V}(Ric)$ and $\nabla^3\phi$.

$$\begin{aligned}&\int \eta^p|Rm|^{p-2}\langle Rm, \mathcal{V}(Ric)\rangle\\ &\quad= -\int p\eta^{p-1}|Rm|^{p-2}\nabla\eta * Rm * \nabla Ric + \eta^p|Rm|^{p-2}\nabla Rm * \nabla Ric\\ &\qquad -\int (p-2)\eta^p|Rm|^{p-4}Rm * Rm * \nabla Rm * \nabla Ric,\end{aligned}$$

$$\begin{aligned}&\int \eta^p|Rm|^{p-2}Rm * \nabla^3\phi * \nabla\phi\\ &\quad= -\int p\eta^{p-1}|Rm|^{p-2}\nabla\eta * Rm * \nabla^2\phi * \nabla\phi + \eta^p|Rm|^{p-2}\nabla Rm * \nabla^2\phi * \nabla\phi\\ &\qquad -\int (p-2)\eta^p|Rm|^{p-4}Rm * Rm * \nabla Rm * \nabla^2\phi * \nabla\phi\\ &\qquad -\int \eta^p|Rm|^{p-2}Rm * \nabla^2\phi * \nabla^2\phi.\end{aligned}$$

By Cauchy-Schwarz inequality

$$\begin{aligned}&\int \eta^{p-1}|Rm|^{p-2}\nabla\eta * Rm * \nabla Ric\\ &\quad\leq C\int \eta^{p-2}|\nabla\eta|^2|Rm|^{p-2}|\nabla Ric|^2 + C\int \eta^p|Rm|^p.\end{aligned}$$

Then using the assumptions we can estimate

$$\begin{aligned}(4.1)\qquad &\frac{\partial}{\partial t}\int \eta^p|Rm|^p dv(t)\\ &\leq C\int \eta^{p-2}|\nabla\eta|^2|Rm|^{p-2}|\nabla Ric|^2 + \eta^p|Rm|^{p-2}|\nabla Rm||\nabla Ric|\\ &\quad + C\int \eta^p|Rm|^p + \eta^p|Rm|^{p-1} + \eta^{p-1}|\nabla\eta||Rm|^{p-1}\\ &\quad + \eta^p|Rm|^{p-2}|\nabla Rm|.\end{aligned}$$

In this section we can allow the constants to depend on p.

By the heat type equations (3.4) and (3.5), we have

$$\begin{aligned}\frac{1}{8}|\nabla Rm|^2 \leq &\left(\frac{1}{16}\Delta - \frac{\partial}{\partial t}\right)|Rm|^2 + C(n)(|Rm|^3 + |Rm||\nabla^2\phi|^2)\\ &+ C|Rm|^2 + Rm * \nabla^3\phi * \nabla\phi,\end{aligned}$$

and

$$\frac{1}{8}|\nabla Ric|^2 \leq \left(\frac{1}{16}\Delta - \frac{\partial}{\partial t}\right)|Ric|^2 + C(n)(|Rm||Ric|^2 + |Rm||Ric||\nabla\phi|^2) + C(n)|Ric||\nabla^2\phi|^2 + Ric * \nabla^3\phi * \nabla\phi.$$

Note that $|\nabla^2\phi| \leq K$ implies $|Ric| \leq nK$. We need to deal with the following (other terms are easy to handle)

$$I = \int \eta^{p-1}|Rm|^{p-2}|\nabla Ric|^2,$$
$$II = \int \eta^p|Rm|^{p-2}|\nabla Rm||\nabla Ric|,$$
$$III = \int \eta^p|Rm|^{p-2}|\nabla Rm|.$$

$$I \leq 8\int \eta^{p-1}|Rm|^{p-2}\left(\frac{1}{16}\Delta - \frac{\partial}{\partial t}\right)|Ric|^2 + C\eta^{p-1}|Rm|^{p-1} + C\int \eta^{p-1}|Rm|^{p-2} + \eta^{p-1}|Rm|^{p-2}Ric * \nabla^3\phi * \nabla\phi.$$

Use integration by parts to get rid of $\Delta|Ric|^2$ and $\nabla^3\phi$ in the first and the last term respectively,

$$\frac{1}{2}\int \eta^{p-1}|Rm|^{p-2}\Delta|Ric|^2 = -\int p\eta^{p-2}|Rm|^{p-2}\nabla\eta * \nabla Ric * Ric + \int (p-2)\eta^{p-1}|Rm|^{p-4}Rm * \nabla Rm * \nabla Ric * Ric,$$

$$\int \eta^{p-1}|Rm|^{p-2}Ric * \nabla^3\phi * \nabla\phi = -\int (p-1)\eta^{p-2}|Rm|^{p-2}\nabla\eta * Ric * \nabla^2\phi * \nabla\phi - \int (p-2)\eta^{p-1}|Rm|^{p-4}Rm * \nabla Rm * Ric * \nabla^2\phi * \nabla\phi - \int \eta^{p-1}|Rm|^{p-2}\nabla Ric * \nabla^2\phi * \nabla\phi + \eta^{p-1}|Rm|^{p-2}Ric * \nabla^2\phi * \nabla^2\phi.$$

Then by the assumptions of the lemma we have

$$
\begin{aligned}
I \leq & -8\int \eta^{p-1}|Rm|^{p-2}\frac{\partial}{\partial t}|Ric|^2 \\
& + C\int \eta^{p-1}|Rm|^{p-3}|\nabla Rm||\nabla Ric| + C\int \eta^{p-2}|\nabla\eta||Rm|^{p-2}|\nabla Ric| \\
& + C\int \eta^{p-1}|Rm|^{p-1} + \eta^{p-1}|Rm|^{p-2} + \eta^{p-2}|\nabla\eta||Rm|^{p-2} \\
& + C\int \eta^{p-1}|Rm|^{p-3}|\nabla Rm| + C\int \eta^{p-1}|Rm|^{p-2}|\nabla Ric|.
\end{aligned}
$$

By Cauchy-Schwarz inequality we can control each term on the RHS containing $|\nabla Ric|$ by $\frac{1}{6}I$ plus another term, for example

$$
C\int \eta^{p-1}|Rm|^{p-3}|\nabla Rm||\nabla Ric| \leq \frac{1}{6}I + \frac{6}{4}C^2\int \eta^{p-1}|Rm|^{p-4}|\nabla Rm|^2.
$$

This will produce a total of $\frac{1}{2}I$ on the RHS, so moving it to the LHS will give us the estimate

$$
\begin{aligned}
I \leq & \, C\int \eta^{p-1}|Rm|^{p-4}|\nabla Rm|^2 + \eta^{p-1}|Rm|^{p-1} + \eta^{p-2}|Rm|^{p-2} \\
& - 16\frac{\partial}{\partial t}\int \eta^{p-1}|Rm|^{p-2}|Ric|^2 + 16\int \eta^{p-1}|Ric|^2\frac{\partial}{\partial t}|Rm|^{p-2}.
\end{aligned} \tag{4.2}
$$

The first term in the above is

$$
\begin{aligned}
& \int \eta^{p-1}|Rm|^{p-4}|\nabla Rm|^2 \\
& \leq 8\int \eta^{p-1}|Rm|^{p-4}\left(\frac{1}{16}\Delta - \frac{\partial}{\partial t}\right)|Rm|^2 + C\int \eta^{p-1}(|Rm|^{p-1} + |Rm|^{p-2}) \\
& \quad + C\int \eta^{p-1}|Rm|^{p-3} + C\int \eta^{p-1}|Rm|^{p-4}Rm * \nabla^3\phi * \nabla\phi.
\end{aligned}
$$

After integration by parts we have

$$
\begin{aligned}
& \int \eta^{p-1}|Rm|^{p-4}\Delta|Rm|^2 \\
& = -\int (p-1)\eta^{p-2}|Rm|^{p-4}\nabla\eta * \nabla|Rm|^2 \\
& \quad - \left(\frac{p}{2} - 2\right)\eta^{p-1}|Rm|^{p-6}|\nabla|Rm|^2|^2 \\
& \leq C\int \eta^{p-2}|Rm|^{p-3}|\nabla\eta||\nabla Rm|,
\end{aligned}
$$

and

$$\begin{aligned}
&\int \eta^{p-1}|Rm|^{p-4}Rm * \nabla^3\phi * \nabla\phi \\
&\quad = -\int (p-1)\eta^{p-2}|Rm|^{p-4}\nabla\eta * Rm * \nabla^2\phi * \nabla\phi \\
&\qquad - \int (p-4)\eta^{p-1}|Rm|^{p-6}\nabla Rm * Rm * Rm * \nabla^2\phi * \nabla\phi \\
&\qquad - \int \eta^{p-1}|Rm|^{p-4}\nabla Rm * \nabla^2\phi * \nabla\phi + \eta^{p-1}|Rm|^{p-4}Rm * \nabla^2\phi * \nabla^2\phi.
\end{aligned}$$

Hence by the assumption $|\nabla\phi|^2, |\nabla^2\phi| \leq K$ we have

$$\begin{aligned}
&\int \eta^{p-1}|Rm|^{p-4}|\nabla Rm|^2 \\
&\quad \leq -8(\frac{p}{2}-1)^{-1}\frac{\partial}{\partial t}\int \eta^{p-1}|Rm|^{p-2} + C\int \eta^{p-2}|Rm|^{p-3}|\nabla\eta||\nabla Rm| \\
&\qquad + C\int \eta^{p-1}(|Rm|^{p-1} + |Rm|^{p-2} + |Rm|^{p-3}) \\
&\qquad + C\int \eta^{p-1}|Rm|^{p-4}|\nabla Rm|.
\end{aligned}$$

By Cauchy-Schwarz inequality we have

$$\begin{aligned}
&C\int \eta^{p-2}|Rm|^{p-3}|\nabla\eta||\nabla Rm| \\
&\quad \leq \frac{1}{4}\int \eta^{p-1}|Rm|^{p-4}|\nabla Rm|^2 + C^2\int \eta^{p-3}|\nabla\eta|^2|Rm|^{p-2},
\end{aligned}$$

and

$$\begin{aligned}
&C\int \eta^{p-1}|Rm|^{p-4}|\nabla Rm| \\
&\quad \leq \frac{1}{4}\int \eta^{p-1}|Rm|^{p-4}|\nabla Rm|^2 + C^2\int \eta^{p-1}|Rm|^{p-4}.
\end{aligned}$$

Then by the assumption $|\nabla\eta|^2 \leq Cr^{-1}\eta$, we have

$$
\begin{aligned}
(4.3)\qquad &\int \eta^{p-1}|Rm|^{p-4}|\nabla Rm|^2\\
&\leq -C\frac{\partial}{\partial t}\int \eta^{p-1}|Rm|^{p-2}\\
&\quad + C\int \eta^{p-1}(|Rm|^{p-1}+|Rm|^{p-2}+|Rm|^{p-3}+|Rm|^{p-4})\\
&\quad + \eta^{p-2}|Rm|^{p-2}
\end{aligned}
$$

For the last term in (4.2), we have

$$
\begin{aligned}
&\int \eta^{p-1}|Ric|^2\frac{\partial}{\partial t}|Rm|^{p-2}\\
&\quad = \int (p-2)\eta^{p-1}|Ric|^2|Rm|^{p-4}\left(\left\langle Rm, \frac{\partial}{\partial t}Rm\right\rangle + \frac{\partial}{\partial t}g * Rm * Rm\right)\\
&\quad \leq (p-2)\int \eta^{p-1}|Ric|^2|Rm|^{p-4}\left\langle Rm, \frac{\partial}{\partial t}Rm\right\rangle + C\int \eta^{p-1}|Rm|^{p-2}.
\end{aligned}
$$

Use (3.3) and integration by parts to get

$$
\begin{aligned}
&\int \eta^{p-1}|Ric|^2|Rm|^{p-4}\left\langle Rm, \frac{\partial}{\partial t}Rm\right\rangle\\
&\quad = \frac{1}{16}\int (p-1)\eta^{p-2}|Ric|^2|Rm|^{p-4}\nabla\eta * Rm * \nabla Ric\\
&\qquad + \frac{1}{16}\int \eta^{p-1}|Rm|^{p-4}Rm * Ric * \nabla Ric * \nabla Ric\\
&\qquad + \frac{1}{16}\int (p-4)\eta^{p-1}|Ric|^2|Rm|^{p-6}Rm * Rm * \nabla Rm * \nabla Ric\\
&\qquad + \frac{1}{16}\int \eta^{p-1}|Ric|^2|Rm|^{p-4}\nabla Rm * \nabla Ric\\
&\qquad + \int \eta^{p-1}|Ric|^2|Rm|^{p-4}Rm * (Rm * Ric + Rm * \nabla\phi * \nabla\phi + \nabla^2\phi * \nabla^2\phi)\\
&\qquad - \int (p-1)\eta^{p-2}|Ric|^2|Rm|^{p-4}\nabla\eta * Rm * \nabla^2\phi * \nabla\phi\\
&\qquad - \int 2\eta^{p-1}|Rm|^{p-4}Rm * Ric * \nabla Ric * \nabla^2\phi * \nabla\phi\\
&\qquad - \int (p-4)\eta^{p-1}|Ric|^2|Rm|^{p-6}\nabla Rm * Rm * Rm * \nabla^2\phi * \nabla\phi\\
&\qquad - \int \eta^{p-1}|Ric|^2|Rm|^{p-4}\nabla Rm * \nabla^2\phi * \nabla\phi.
\end{aligned}
$$

Then by the assumptions of the lemma we have

$$\begin{aligned}
&\int \eta^{p-1}|Ric|^2 \frac{\partial}{\partial t}|Rm|^{p-2} \\
&\quad \leq C \int \eta^{p-2}|Rm|^{p-3}|\nabla Ric||\nabla \eta| + \eta^{p-1}|Rm|^{p-3}|\nabla Ric|^2 \\
&\qquad + C \int \eta^{p-1}|Rm|^{p-4}|\nabla Rm||\nabla Ric| + \eta^{p-1}|Rm|^{p-2} + \eta^{p-1}|Rm|^{p-3} \\
&\qquad + C \int \eta^{p-2}|Rm|^{p-3}|\nabla \eta| + \eta^{p-1}|Rm|^{p-3}|\nabla Ric| + \eta^{p-1}|Rm|^{p-4}|\nabla Rm|.
\end{aligned}$$

Then the idea is to control each term by a small portion of I and by terms involving only lower powers of $|Rm|$. For each $0 < \epsilon < 1$, by Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\int \eta^{p-1}|Ric|^2 \frac{\partial}{\partial t}|Rm|^{p-2} \\
&\quad \leq \epsilon I + \frac{C}{\epsilon} \int \eta^{p-3}|\nabla \eta|^2|Rm|^{p-4} + \eta^{p-1}|Rm|^{p-4}|\nabla Ric|^2 \\
&\qquad + C \int \eta^{p-1}\Big(|Rm|^{p-4}|\nabla Rm||\nabla Ric| \\
&\qquad\qquad + |Rm|^{p-2} + |Rm|^{p-3} + \frac{1}{\epsilon}|Rm|^{p-4}\Big) \\
&\qquad + C \int \eta^{p-3}|\nabla \eta|^2|Rm|^{p-3} + \eta^{p-1}|Rm|^{p-4}|\nabla Rm|^2 \\
&\quad \leq C\left(1 + \frac{1}{\epsilon}\right) \int \eta^{p-1}|Rm|^{p-4}|\nabla Rm|^2 \\
&\qquad + \eta^{p-2}(|Rm|^{p-2} + |Rm|^{p-3} + |Rm|^{p-4}) + \epsilon I.
\end{aligned} \tag{4.4}$$

Here the last inequality follows from $|\nabla Ric| \leq (n-1)|\nabla Rm|$, $|\nabla \eta|^2 \leq Cr^{-1}\eta$ and $\eta \leq 1$.

Therefore

$$\begin{aligned}
I \leq &-C\frac{\partial}{\partial t} \int \eta^{p-1}|Rm|^{p-2} - C\frac{\partial}{\partial t} \int \eta^{p-1}|Rm|^{p-2}|Ric|^2 \\
&+ C \int \eta^{p-1}|Rm|^{p-1} + \eta^{p-2}(|Rm|^{p-2} + |Rm|^{p-3} + |Rm|^{p-4}).
\end{aligned} \tag{4.5}$$

Now we estimate the term

$$
\begin{aligned}
II &= \int \eta^p |Rm|^{p-2} |\nabla Rm||\nabla Ric| \\
&\leq C \int \eta^p |Rm|^{p-1} |\nabla Ric|^2 + C \int \eta^p |Rm|^{p-3} |\nabla Rm|^2 \\
&= II_1 + II_2.
\end{aligned}
$$

$$
\begin{aligned}
II_1 \leq{}& C \int \eta^p |Rm|^{p-1} \left(\frac{1}{16}\Delta - \frac{\partial}{\partial t}\right) |Ric|^2 + \eta^p |Rm|^p + \eta^p |Rm|^{p-1} \\
&+ \int \eta^p |Rm|^{p-1} Ric * \nabla^3 \phi * \nabla \phi \\
\leq{}& C \int \eta^{p-1} (|\nabla \eta| + 1) |Rm|^{p-1} |\nabla Ric| + \eta^p |Rm|^{p-2} |\nabla Rm||\nabla Ric| \\
&+ C \int \eta^p (|Rm|^p + |Rm|^{p-1}) \\
&+ C \int \eta^{p-1} |\nabla \eta||Rm|^{p-1} + \eta^p |Rm|^{p-2} |\nabla Rm| \\
&- C \frac{\partial}{\partial t} \int \eta^p |Rm|^{p-1} |Ric|^2 + C \int \eta^p |Ric|^2 \frac{\partial}{\partial t} |Rm|^{p-1}.
\end{aligned}
$$

Apply the Cauchy-Schwarz inequality to estimate terms in the first and the third line after the last inequality, we have for all $\epsilon > 0$,

$$
\begin{aligned}
& C \int \eta^{p-1} (|\nabla \eta| + 1) |Rm|^{p-1} |\nabla Ric| \\
&\quad \leq \epsilon II_1 + \frac{C}{\epsilon} \int \eta^{p-1} (|\nabla \eta|^2 + 1) |Rm|^{p-1}, \\
& C \int \eta^p |Rm|^{p-2} |\nabla Rm||\nabla Ric| \leq \epsilon II_1 + \frac{C}{\epsilon} \int \eta^{p+1} |Rm|^{p-3} |\nabla Rm|^2, \\
& \int \eta^{p-1} |\nabla \eta||Rm|^{p-1} \leq \int \eta^{p-2} |\nabla \eta|^2 |Rm|^{p-1} + \eta^p |Rm|^{p-1}, \\
& \int \eta^p |Rm|^{p-2} |\nabla Rm| \leq \int \eta^p |Rm|^{p-1} + II_2.
\end{aligned}
$$

Hence by choosing $\epsilon = \frac{1}{8}$ we have

$$
\begin{aligned}
II_1 \leq{}& \frac{1}{4} II_1 + C II_2 + C \int \eta^p |Rm|^p + \eta^{p-1} (|Rm|^{p-1} + |Rm|^{p-2}) \\
&- C \frac{\partial}{\partial t} \int \eta^p |Rm|^{p-1} |Ric|^2 + C \int \eta^p |Ric|^2 \frac{\partial}{\partial t} |Rm|^{p-1}.
\end{aligned}
$$

Similarly as in the derivation of (4.4), we derive

$$\int \eta^p |Ric|^2 \frac{\partial}{\partial t}|Rm|^{p-1} \le II_2 + C\int \eta^{p-1}(|Rm|^{p-1} + |Rm|^{p-2} + |Rm|^{p-3}) + C\int \eta^p |Rm|^p + \frac{1}{4C} II_1.$$

Hence

$$II_1 \le CII_2 - C\frac{\partial}{\partial t}\int \eta^p |Rm|^{p-1}|Ric|^2 + C\int \eta^{p-1}(|Rm|^{p-3} + |Rm|^{p-1}).$$

Using the same method as in the derivation of (4.3), we have

$$\begin{aligned} II_2 &\le C\int \eta^p (|Rm|^{p-3}\left(\frac{1}{16}\Delta - \frac{\partial}{\partial t}\right)|Rm|^2 + |Rm|^p + |Rm|^{p-2}) \\ &\quad + C\int \eta^p |Rm|^{p-3} Rm * \nabla^3\phi * \nabla\phi \\ &\le -C\frac{\partial}{\partial t}\int \eta^p |Rm|^{p-1} \\ &\quad + C\int \eta^p |Rm|^p + \eta^{p-1}(|Rm|^{p-1} + |Rm|^{p-2} + |Rm|^{p-3}). \end{aligned}$$

Hence we have the estimate

$$\begin{aligned} II &\le -C\frac{\partial}{\partial t}\int \eta^p |Rm|^{p-1}|Ric|^2 - C\frac{\partial}{\partial t}\int \eta^p |Rm|^{p-1} \\ &\quad + C\int \eta^p |Rm|^{p-1} + \eta^{p-1}(|Rm|^{p-1} + |Rm|^{p-2} + |Rm|^{p-3}). \end{aligned} \tag{4.6}$$

Clearly, by Cauchy-Schwarz inequality,

$$III \le CII_2 + C\int \eta^p |Rm|^{p-1}. \tag{4.7}$$

Note that by Cauchy-Schwarz inequality

$$\begin{aligned} 2\eta^{p-1}|Rm|^{p-1} &\le \eta^p |Rm|^p + \eta^{p-2}|Rm|^{p-2}, \\ 2\eta^{p-2}|Rm|^{p-2} &\le \eta^p |Rm|^p + \eta^{p-4}|Rm|^{p-4}, \\ 2\eta^{p-1}|Rm|^{p-3} &\le \eta^p |Rm|^{p-2} + \eta^{p-2}|Rm|^{p-4}, \end{aligned}$$

and $\eta \le 1$.

Therefore, combining (4.1) (4.5) (4.6), and (4.7), we have shown

$$\begin{aligned}\frac{\partial}{\partial t}\int \eta^p|Rm|^p dv(t) \leq & -C\frac{\partial}{\partial t}\int \eta^{p-1}|Rm|^{p-2}|Ric|^2 - C\frac{\partial}{\partial t}\int \eta^{p-1}|Rm|^{p-2} \\ & - C\frac{\partial}{\partial t}\int \eta^{p}|Rm|^{p-1}|Ric|^2 - C\frac{\partial}{\partial t}\int \eta^{p}|Rm|^{p-1} \\ & + C\int \eta^p|Rm|^p + \eta^{p-4}|Rm|^{p-4}\end{aligned}$$

Using Gronwall's inequality, and using Young's inequality to interpolate, yields an estimate of $\int |Rm|^p dv(t)$,

$$\int \eta^p|Rm|^p dv(g(t)) \leq e^{Ct}\left(\int \eta^p|Rm|^p dv(g(0)) + CV_{g(0)}(\mathrm{spt}(\eta))t\right),$$

where the constant C depends on n, p, K and $\sup \eta^{-1}|\nabla\eta|^2$, $V_{g(0)}(\mathrm{spt}(\eta))$ is the volume of the support of η with respect to the metric $g(0)$. Note that by the assumptions of the lemma we have the equivalence of metrics

$$e^{-C(n)Kt}g(0) \leq g(t) \leq e^{-C(n)Kt}g(0), \quad 0 \leq t \leq T,$$

hence by (8.1) in the Appendix we have a uniform Sobolev inequality along the flow. Then the Nash-Moser iteration will give us the pointwise estimate,

$$\begin{aligned}&\sup_{B(r/2)\times[0,T]} |Rm| \\ &\quad \leq C(n)e^{c(n,K)T}((C(n,p)r^2 \sup_{[0,T]} |Rm|_{L^p(r)}(t))^{\alpha(n,k,p)} + r^{-\beta(n,k)}) \\ &\quad \times \left(V(r)^{-1}|Rm|_{L^k(B(r)\times[0,T])} + \sup_{B(r)} |Rm|(g(0))\right),\end{aligned}$$

for any $k > 1$. Note the only non-standard step here is to use integration by parts and absorption arguments to get rid of $\nabla^3\phi$. See for example [Li12] for details of Nash-Moser iteration and Theorem 1.2 of [LT91] where the initial value was taken into account. □

5. L^2 estimates of derivatives

In this section we use the L^2 method to obtain derivative estimates for (3.2). We assume $|\nabla^2\phi|$, $|\nabla\phi|^2$, $|Rm| \leq K$ in the support of a cut-off function η, with $|\nabla\eta|^2 \leq L\eta$.

Lemma 5.1. *For each integer $k \geq 0$ and $m \geq 2k$, there exists a constant C depending on n, k, m, K, L, such that*

$$\begin{aligned}&\frac{\partial}{\partial t}\int \eta^m|\nabla^k Rm|^2\\ &\quad\leq -\frac{1}{128}\int \eta^m|\nabla^{k+1}Rm|^2 + C\int(\eta^m|\nabla^k Rm|^2 + \eta^m|\nabla^{k+2}\phi|^2)\\ &\qquad + C\int_{\eta>0}|Rm|^2 + |\nabla^2\phi|^2 + |\nabla\phi|^2 + |\phi|^2.\end{aligned}$$

Proof. By the heat type equations (3.6),

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\int \eta^m|\nabla^k Rm|^2\\ &\quad\leq \int \eta^m\left\langle \nabla^k Rm, \frac{1}{16}\Delta\nabla^k Rm\right\rangle\\ &\qquad + \int \eta^m\left\langle \nabla^k Rm, \sum_{l=0}^{k}\nabla^l Rm * \nabla^{k-l}Rm\right\rangle\\ &\qquad + \int \eta^m\left\langle \nabla^k Rm, \sum_{l=0}^{k}\sum_{r=0}^{l}\nabla^{1+r}\phi * \nabla^{1+l-r}\phi * \nabla^{k-l}Rm\right\rangle\\ &\qquad + \int \eta^m\left\langle \nabla^k Rm, \sum_{l=0}^{k+2}\nabla^{1+l}\phi * \nabla^{3+k-l}\phi\right\rangle + C\int \eta^m|\nabla^k Rm|^2\\ &\quad= I_1 + I_2 + I_3 + I_4 + C\int \eta^m|\nabla^k Rm|^2.\end{aligned}$$

$$\begin{aligned}I_1 &= \int \eta^m\left\langle \nabla^k Rm, \frac{1}{16}\Delta\nabla^k Rm\right\rangle\\ &= -\frac{1}{16}\int \eta^m|\nabla^{k+1}Rm|^2 + m\eta^{m-1}\langle\nabla\eta\otimes\nabla^k Rm, \nabla^{k+1}Rm\rangle\\ &\leq -\frac{1}{32}\int \eta^m|\nabla^{k+1}Rm|^2 + Cm^2\int \eta^{m-1}|\nabla^k Rm|^2.\end{aligned}$$

To handle the term $\int \eta^{m-1}|\nabla^k Rm|^2$, we use integration by parts and Cauchy-Schwarz inequality to get

$$
\begin{aligned}
(5.1)\qquad \int \eta^{m-1}|\nabla^k Rm|^2 &= \int \eta^{m-1}\langle \nabla^{k-1}Rm, \nabla^{k+1}Rm\rangle \\
&\quad + \int \langle \nabla \eta^{m-1} \otimes \nabla^{k-1}Rm, \nabla^k Rm\rangle \\
&\leq \epsilon \int \eta^m |\nabla^{k+1}Rm|^2 + \frac{1}{2}\int \eta^{m-1}|\nabla^k Rm|^2 \\
&\quad + C\epsilon^{-1}\int \eta^{m-2}|\nabla^{k-1}Rm|^2,
\end{aligned}
$$

then iterate the above inequality to get

$$
\int \eta^{m-1}|\nabla^k Rm|^2 \leq \epsilon \int \eta^m|\nabla^{k+1}Rm|^2 + C\epsilon^{-1}\int_{\eta>0}|Rm|^2.
$$

Now we can choose ϵ properly to get

$$
I_1 \leq -\frac{1}{64}\int \eta^m|\nabla^{k+1}Rm|^2 + C\int_{\eta>0}|Rm|^2.
$$

Then we apply Lemma 8.2 in the Appendix with tensors Rm and $\nabla\phi$ to estimate

$$
\begin{aligned}
I_2 &= \int \eta^m \langle \nabla^k Rm, \sum_{l=0}^{k} \nabla^l Rm * \nabla^{k-l}Rm\rangle \\
&\leq C\left(\int \eta^m|\nabla^k Rm|^2 + \int_{\eta>0}|Rm|^2\right), \\
I_3 &= \int \eta^m \left\langle \nabla^k Rm, \sum_{l=0}^{k}\sum_{r=0}^{l} \nabla^{1+r}\phi * \nabla^{1+l-r}\phi * \nabla^{k-l}Rm \right\rangle \\
&\leq C\left(\int \eta^m|\nabla^k Rm|^2 + \int_{\eta>0}|Rm|^2\right) \\
&\quad + C\left(\int \eta^m|\nabla^{1+k}\phi|^2 + \int_{\eta>0}|\nabla\phi|^2\right).
\end{aligned}
$$

Moreover, similarly as in the handling of (5.1), we can obtain

$$
\int \eta^m|\nabla^{1+l}\phi|^2 \leq \int \eta^{m-1}|\nabla^{1+l}\phi|^2 \leq C\int \eta^m|\nabla^{2+k}\phi|^2 + C\int_{\eta>0}|\phi|^2.
$$

$$\begin{aligned}
I_4 &= \int \eta^m \left\langle \nabla^k Rm, \sum_{l=0}^{k+2} \nabla^{1+l}\phi * \nabla^{3+k-l}\phi \right\rangle \\
&= \int \eta^m \left\langle \nabla^k Rm, \sum_{j=0}^{k} \nabla^{2+j}\phi * \nabla^{2+k-j}\phi \right\rangle \\
&\quad + \int \eta^m \langle \nabla^k Rm, \nabla^{k+3}\phi * \nabla\phi \rangle,
\end{aligned}$$

first integrate by parts to lower the order of $\nabla^{3+k}\phi$,

$$\begin{aligned}
I_4 = &\int m\eta^{m-1}\nabla\eta * \nabla^k Rm * \nabla^{k+2}\phi * \nabla\phi \\
&+ \eta^m \nabla^{k+1} Rm * \nabla^{k+2}\phi * \nabla\phi \\
&+ \int \eta^m \nabla^k Rm * \nabla^{k+2}\phi * \nabla^2\phi \\
&+ \int \eta^m \left\langle \nabla^k Rm, \sum_{j=0}^{k} \nabla^{2+j}\phi * \nabla^{2+k-j}\phi \right\rangle,
\end{aligned} \tag{5.2}$$

then use Cauchy-Schwarz inequality to estimate the first term on the RHS of (5.2)

$$\eta^m \nabla^{k+1} Rm * \nabla^{k+2}\phi * \nabla\phi \leq \frac{1}{128}\eta^m |\nabla^{k+1} Rm|^2 + C\eta^m |\nabla^{k+2}\phi|^2 |\nabla\phi|^2,$$

for the other terms in (5.2) we apply Lemma 8.2 with tensors Rm and $\nabla^2\phi$ to get

$$\begin{aligned}
I_4 \leq &\frac{1}{128}\int \eta^m |\nabla^{k+1} Rm|^2 + C\left(\int \eta^m |\nabla^k Rm|^2 + \int_{\eta>0} |Rm|^2\right) \\
&+ C\left(\int \eta^m |\nabla^{k+2}\phi|^2 + \int_{\eta>0} |\nabla^2\phi|^2\right),
\end{aligned}$$

where we used the same argument as in the estimate of I_1 to control the term $\int \eta^{m-1}|\nabla^k Rm|^2$. By the above estimates we have the lemma. □

Lemma 5.2. *For any integer $k \geq 1$, $m > 2k$, there exists a constant $C(n,k,m,K,L)$ such that*

$$\frac{\partial}{\partial t}\int \eta^m|\nabla^k\phi|^2 \leq -\frac{1}{4}\int \eta^m|\nabla^{k+1}\phi|^2 + C\int \eta^m|\nabla^k\phi|^2 + C\int \eta^m|\nabla^{k-1}Rm|^2 + C\int_{\eta>0}|\nabla\phi|^2+|\phi|^2+|Rm|^2.$$

Proof. By the heat-type equation (3.7),

$$\begin{aligned}\frac{\partial}{\partial t}\int \eta^m|\nabla^k\phi|^2 &= 2\int \eta^m\langle\nabla^k\phi, \Delta\nabla^k\phi\rangle \\ &\quad + \frac{1}{16}\int \eta^m\langle\phi, e_p\cdot\nabla^k\nabla_q\mathbf{D}\phi\rangle\langle\nabla^k\phi, e_q\wedge e_p\cdot\phi\rangle \\ &\quad + \sum_{p=0}^{k}\sum_{q=0}^{k+1-p}\int \eta^m\langle\nabla^k\phi, \nabla^{1+p}\phi*\nabla^q\phi*\nabla^{k+1-p-q}\phi\rangle \\ &\quad + \sum_{0\leq l\leq k-1}\int \eta^m\langle\nabla^k\phi, \nabla^l Rm*\nabla^{k-l}\phi\rangle \\ &\quad + \sum_{1\leq l\leq k}\int \eta^m\langle\nabla^k\phi, \nabla^l Ric*\nabla^{k-l}\phi\rangle \\ &= II_1+II_2+II_3+II_4+II_5.\end{aligned}$$

Calculate similarly as in the previous lemma,

$$\begin{aligned}II_1 &= 2\int \eta^m\langle\nabla^k\phi, \Delta\nabla^k\phi\rangle \\ &= -2\int \eta^m|\nabla^{k+1}\phi|^2 + 2\int \eta^{m-1}\langle\nabla\eta\otimes\nabla^k\phi, \nabla^{k+1}\phi\rangle \\ &\leq -\int \eta^m|\nabla^{k+1}\phi|^2 + C\int \eta^{m-1}|\nabla^k\phi|^2 \\ &\leq -\frac{1}{2}\int \eta^m|\nabla^{k+1}\phi|^2 + C\int \eta^{m-2}|\nabla^{k-1}\phi|^2 \\ &\leq -\frac{1}{2}\int \eta^m|\nabla^{k+1}\phi|^2 + C\int \eta^m|\nabla^k\phi|^2 + \int_{\eta>0}|\phi|^2.\end{aligned} \tag{5.3}$$

In the following estimates of II_2, II_3, II_4 and II_5, we shall keep the coefficients in front of the highest order term $\int \eta^m|\nabla^{k+1}\phi|^2$ small, so it can be absorbed by the good term $-\frac{1}{2}\int \eta^m |\nabla^{k+1}\phi|^2$ in (5.3).

$$II_2 = \frac{1}{16}\int \eta^m \langle \phi, e_p \cdot \nabla^k \nabla_q \mathbf{D}\phi\rangle \langle \nabla^k \phi, e_q \wedge e_p \cdot \phi\rangle.$$

By commuting covariant derivatives and the Dirac operator, we have

$$\nabla^k \nabla_q \mathbf{D}\phi = \nabla_q \nabla^k \mathbf{D}\phi + \sum_{i=0}^{k-1} \nabla^i Rm * \nabla^{k-1-i}\mathbf{D}\phi.$$

$$\mathbf{D}\nabla^k \phi - \nabla^k \mathbf{D}\phi = \sum_{i=0}^{k-1} \nabla^i Ric * \nabla^{k-1-i}\phi. \tag{5.4}$$

Hence

$$\begin{aligned} II_2 = &\frac{1}{16}\int \eta^m \langle \phi, e_p \cdot \nabla_q \nabla^k \mathbf{D}\phi\rangle \langle \nabla^k \phi, e_q \cdot e_p \cdot \phi + \delta_{pq}\phi\rangle \\ &+ \int \eta^m \sum_{i=0}^{k-1} \nabla^i Rm * \nabla^{k-1-i}\mathbf{D}\phi * \nabla^k \phi * \phi * \phi. \end{aligned}$$

Use integration by parts to lower the order of derivatives in the term $\nabla_q \nabla^k \mathbf{D}\phi$, we get

$$\begin{aligned} &\int \eta^m \langle \phi, e_p \cdot \nabla_q \nabla^k \mathbf{D}\phi\rangle \langle \nabla^k \phi, e_q \cdot e_p \cdot \phi + \delta_{pq}\phi\rangle \\ &\quad = -\int m\eta^{m-1}\nabla_q \eta \langle \phi, e_p \cdot \nabla^k \mathbf{D}\phi\rangle \langle \nabla^k \phi, e_q \wedge e_p \cdot \phi\rangle \\ &\qquad - \int \eta^m \langle \nabla_q \phi, e_p \cdot \nabla^k \mathbf{D}\phi\rangle \langle \nabla^k \phi, e_q \cdot e_p \cdot \phi + \delta_{pq}\phi\rangle \\ &\qquad - \int \eta^m \langle \phi, e_p \cdot \nabla^k \mathbf{D}\phi\rangle \langle \nabla_q \nabla^k \phi, e_q \cdot e_p \cdot \phi + \delta_{pq}\phi\rangle \\ &\qquad - \int \eta^m \langle \phi, e_p \cdot \nabla^k \mathbf{D}\phi\rangle \langle \nabla^k \phi, e_q \cdot e_p \cdot \nabla_q \phi + \delta_{pq}\nabla_q \phi\rangle, \end{aligned}$$

note that in the third line after the last equation, we have

$$\langle \nabla_q \nabla^k \phi, e_q \cdot e_p \cdot \phi\rangle = -\langle e_q \cdot \nabla_q \nabla^k \phi, e_p \cdot \phi\rangle = -\langle \mathbf{D}\nabla^k \phi, e_p \cdot \phi\rangle,$$

and all the terms in the second and the last line can be written in the general form

$$\int \eta^m \nabla^k \mathbf{D}\phi * \nabla^k \phi * \nabla\phi * \phi,$$

we will omit the coefficients in front of these terms since eventually we have to introduce a constant C when estimating them. Hence we have

$$\begin{aligned} II_2 = &-\frac{1}{16}\int \eta^m \langle \phi, e_p \cdot \nabla^k \mathbf{D}\phi\rangle (\langle -\mathbf{D}\nabla^k \phi, e_p \cdot \phi\rangle + \langle \nabla_q \nabla^k \phi, \delta_{pq}\phi\rangle) \\ &+ \frac{1}{16}\int m\eta^{m-1}\nabla_q \eta \langle \phi, e_p \cdot \nabla^k \mathbf{D}\phi\rangle \langle \nabla^k \phi, e_q \wedge e_p \cdot \phi\rangle \\ &+ \int \eta^m \nabla^k \mathbf{D}\phi * \nabla^k \phi * \nabla \phi * \phi \\ &+ \sum_{i=0}^{k-1} \int \eta^m \nabla^i Rm * \nabla^{k-1-i}\mathbf{D}\phi * \nabla^k \phi * \phi * \phi. \end{aligned}$$

Note that

$$\langle \phi, e_p \cdot \nabla^k \mathbf{D}\phi\rangle = -\langle \nabla^k \mathbf{D}\phi, e_p \cdot \phi\rangle,$$

and by (5.4) we have

$$\langle -\mathbf{D}\nabla^k \phi, e_p \cdot \phi\rangle = \langle -\nabla^k \mathbf{D}\phi + \sum_{i=0}^{k-1} \nabla^i Ric * \nabla^{k-1-i}\phi, e_p \cdot \phi\rangle,$$

hence

$$\begin{aligned} II_2 = &-\frac{1}{16}\int \eta^m |\langle e_p \cdot \phi, \nabla^k \mathbf{D}\phi\rangle|^2 \\ &+ \frac{1}{16}\int \eta^m \langle \phi, e_p \cdot \nabla^k \mathbf{D}\phi\rangle \langle \nabla_p \nabla^k \phi, \phi\rangle \\ &+ \frac{1}{16}\int m\eta^{m-1}\nabla_q \eta \langle \phi, e_p \cdot \nabla^k \mathbf{D}\phi\rangle \langle \nabla^k \phi, e_q \wedge e_p \cdot \phi\rangle \\ &+ \int \eta^m \nabla^k \mathbf{D}\phi * \nabla^k \phi * \nabla \phi * \phi \\ &+ \sum_{i=0}^{k-1} \int \eta^m \nabla^i Rm * \nabla^{k-1-i}\mathbf{D}\phi * \nabla^k \phi * \phi * \phi \\ &+ \sum_{i=0}^{k-1} \int \eta^m \nabla^k \mathbf{D}\phi * \nabla^i Ric * \nabla^{k-1-i}\phi * \phi * \phi. \end{aligned} \tag{5.5}$$

For the second term in (5.5) we apply Cauchy-Schwarz inequality to obtain

$$\begin{aligned} &\frac{1}{16}\int \eta^m \langle \phi, e_p \cdot \nabla^k \mathbf{D}\phi\rangle \langle \nabla_p \nabla^k \phi, \phi\rangle \\ &\quad \le \frac{1}{16}\int \eta^m |\langle e_p \cdot \phi, \nabla^k \mathbf{D}\phi\rangle|^2 + \frac{1}{64}\int \eta^m |\nabla^{k+1}\phi|^2, \end{aligned}$$

where in the last term we also used the fact $|\phi| = 1$. For the second and the third line in (5.5), note that $|\nabla^k \mathbf{D}\phi| \le c(n)|\nabla^{k+1}\phi|$, and use Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
&\frac{1}{16}\int m\eta^{m-1}\nabla_q\eta\langle\phi, e_p\cdot\nabla^k\mathbf{D}\phi\rangle\langle\nabla^k\phi, e_q\wedge e_p\cdot\phi\rangle\\
&\qquad\le \frac{1}{64}\int \eta^m|\nabla^{k+1}\phi|^2 + C_1(m,n,L)\int\eta^{m-1}|\nabla^k\phi|^2,\\
&\int \eta^m\nabla^k\mathbf{D}\phi * \nabla^k\phi * \nabla\phi * \phi\\
&\qquad\le \frac{1}{64}\int \eta^m|\nabla^{k+1}\phi|^2 + C_1(n,K)\int\eta^{m-1}|\nabla^k\phi|^2,
\end{aligned}$$

where we have used the assumption $|\nabla\phi|^2 \le K$. By the same method as in (5.1) (with $\epsilon = \frac{1}{64}C_1^{-1}$), we get

$$C_1\int\eta^{m-1}|\nabla^k\phi|^2 \le \frac{1}{64}\int\eta^m|\nabla^{k+1}\phi|^2 + C(m,n,L)\int_{\eta>0}|\phi|^2.$$

The rest of the terms in (5.5) contain at most k^{th}-derivative of ϕ and $(k-1)^{th}$-derivative of Rm, and can be estimated by Lemma 8.2. Therefore we have the estimate

$$\begin{aligned}
II_2 &\le \frac{1}{16}\int\eta^m|\nabla^{k+1}\phi|^2 + C\int\eta^m|\nabla^k\phi|^2 + \eta^m|\nabla^{k-1}Rm|^2\\
&\quad + C\int_{\eta>0}|\nabla\phi|^2 + |Rm|^2 + |\phi|^2.\\
II_3 &= \sum_{p=0}^{k}\sum_{q=0}^{k+1-p}\int\eta^m\langle\nabla^k\phi, \nabla^{1+p}\phi * \nabla^q\phi * \nabla^{k+1-p-q}\phi\rangle\\
&= \sum_{p=1}^{k-1}\sum_{q=1}^{k+1-p}\int\eta^m\langle\nabla^{k-1}\nabla\phi, \nabla^p\nabla\phi * \nabla^{q-1}\nabla\phi * \nabla^{k-p-q}\nabla\phi\rangle\\
&\quad + \sum_{q=1}^{k}\int\eta^m\langle\nabla^{k-1}\nabla\phi, \nabla\phi * \nabla^{q-1}\nabla\phi * \nabla^{k-q}\nabla\phi\rangle\\
&\quad + \int\eta^m\langle\nabla^k\phi, \nabla^{k+1}\phi * \nabla\phi * \phi\rangle\\
&\quad + \sum_{p=1}^{k-1}\int\eta^m\langle\nabla^k\phi, \nabla^{1+p}\phi * \phi * \nabla^{k+1-p}\phi\rangle\\
&\le \frac{1}{16}\int\eta^m|\nabla^{k+1}\phi|^2 + C\left(\int\eta^m|\nabla^k\phi|^2 + \int_{\eta>0}|\nabla\phi|^2 + |\phi|^2\right),
\end{aligned}$$

where the purpose of the second equation is to single out terms containing the highest order derivative of ϕ for applying Cauchy-Schwarz inequality, lower order terms are handled by Lemma 8.2, note that the $p=0, q=0$ term appeared as

$$\int \eta^m \langle \nabla^k \phi, \nabla^{k+1}\phi * \nabla\phi * \phi \rangle.$$

The last term

$$II_5 = \sum_{1\le l\le k} \int \eta^m \langle \nabla^k \phi, \nabla^l Ric * \nabla^{k-l}\phi \rangle.$$

Note that $\nabla^k Ric$ only appears in the $l=k$ term, integrate by parts to get rid of it, then we have

$$\begin{aligned} II_5 &= \int m\eta^{m-1} \langle \nabla\eta \otimes \nabla^k \phi, \nabla^{k-1} Ric * \phi\rangle + \eta^m \langle \nabla^{k+1}\phi, \nabla^{k-1} Ric * \phi\rangle \\ &\quad + \int \eta^m \langle \nabla^k\phi, \nabla^{k-1} Ric * \nabla\phi\rangle + \sum_{1\le l\le k-1} \int \eta^m \langle \nabla^k \phi, \nabla^l Ric * \nabla^{k-l}\phi\rangle \\ &\le \frac{1}{16}\int \eta^m |\nabla^{k+1}\phi|^2 + C\int \eta^m |\nabla^k\phi|^2 + C\int_{\eta>0} |\nabla\phi|^2 + |\phi|^2 \\ &\quad + C\int \eta^m |\nabla^{k-1} Ric|^2 + C\int_{\eta>0} |Ric|^2, \end{aligned}$$

where we used similar method as in the estimate of II_2 and II_3 to obtain the last inequality.

Similarly we can estimate

$$\begin{aligned} II_4 &= \sum_{0\le l\le k-1} \int \eta^m \langle \nabla^k\phi, \nabla^l Rm * \nabla^{k-l}\phi\rangle \\ &\le C\int \eta^m |\nabla^{k-1} Rm|^2 + \eta^m |\nabla^k \phi|^2 + C\int_{\eta>0} |Rm|^2 + |\nabla\phi|^2. \end{aligned}$$

□

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We can rescale the flow parabolically such that $|Rm| \le 1$, $|\nabla^2\phi| \le 1$ and $|\nabla\phi|^2 \le 1$ on $B_{g(0)}(r) \times [0, a]$, for simplicity we still denote it as $(M, g(t), \phi(t))$. It is easy to see the volume can be controlled under the flow

$$e^{-C(n)a}V \le Vol_{g(t)}(B_{g(0)}(r)) \le e^{C(n)a}V, \tag{5.6}$$

where V can be taken as $Vol_{g(s)}(B_{g(0)}(r))$ for any fixed $s \in [0,a]$. Choose a cut-off function η supported on $B_{g(0)}(3r/4)$, with $\eta = 1$ on $B_{g(0)}(r/2)$ and $|\nabla \eta| \le 4/r$.

We will prove the theorem by induction on k. When $k = 0$ it is trivial. Without loss of generality we can suppose the result holds for k on a larger ball with radius $\frac{3r}{4}$. Take $m = 2k+4$. Define

$$F_i(t) = \alpha \int \eta^m |\nabla^i Rm|^2 dv(t) + \int \eta^m |\nabla^{i+2}\phi|^2 dv(t), \tag{5.7}$$

for some constant α, $i = 0, 1, 2, \dots, k+1$.

By Lemma 5.1 and 5.2 we can choose α large enough (depending on n, k, r) such that $\frac{\alpha}{2}$ times of the good term $-\frac{1}{128}\int \eta^m |\nabla^{k+1} Rm|^2$ from Lemma 5.1 cancels out the corresponding term from Lemma 5.2, thus we have

$$\frac{d}{dt} F_k \le -\beta_k F_{k+1} + C_k F_k + C_k V \tag{5.8}$$

for constants β_k and C_k depending on n, a, r and k, where the dependence on a comes in from (5.6). Then let

$$Q_k = t^{k+1} F_{k+1} + \gamma \sum_{i=0}^{k} \frac{(k+1)\cdot k \cdots (k-i+1)}{\beta_k \cdot \beta_{k-1} \cdots \beta_{k-i}} t^{k-i} F_{k-i},$$

where we take $\beta_0 = 1$. Take derivative of Q_k and apply (5.8), we have

$$\begin{aligned} \frac{d}{dt} Q_k \le& -\beta_{k+1} t^{k+1} F_{k+2} + \left(k+1 + C_{k+1} t - (k+1)\gamma\right) t^k F_{k+1} \\ &+ C_{k+1} t^{k+1} V \\ &+ \gamma \sum_{i=0}^{k} \frac{(k+1)k \cdots (k-i+1)}{\beta_k \beta_{k-1} \cdots \beta_{k-i}} t^{k-i} (C_{k-i} F_{k-i} + C_{k-i} V). \end{aligned} \tag{5.9}$$

Thus γ can be properly chosen such that $k + 1 + aC_{k+1} - (k+1)\gamma = -1 < 0$, so the first and the second term in (5.9) are good. Notice that the remaining terms contain at most $t^k F_k$, by induction hypothesis

$$t^{k-i} F_{k-i} \le C, \quad i = 0, 1, \dots, k,$$

hence

$$\frac{d}{dt}Q_k \leq C(1+V)$$

for $t \in [0,a]$, C is a constant depending on n, a, k, r. Since

$$Q_k(0) = \gamma(k+1)F_0(0) \leq CV$$

for some constant C depending on n, k, r, integrate the above inequality yields

$$Q_k(t) \leq C(1+V)(1+a),$$

for $t \in [0,a]$, note that V is bounded from above by a constant depending on n and r by the volume comparison theorem, thus

$$t^{k+1}F_{k+1}(t) \leq C_1$$

for some contant C_1 depending on n, a, r and k, and we finish the induction proof. Notice that $t^k F_k$ scales in the same way as $|Rm|^2$ under parabolic scaling of the flow, so C_1 is the desired constant in the theorem. □

6. Long time existence

In this section, we will show that the norm of the second order covariant derivative of the spinor field becoming unbounded is the only obstruction of long-time existence of the spinor flow by proving Theorem 1.2. We will assume that the dimension of the manifold $n \geq 3$, since the result on 2-dimensional surfaces has been shown in [Sch18].

6.1. Convergence of metrics

Let $(g(t), \phi(t))$, $t \in [0,T)$, for some $T < \infty$, be a solution to the system (3.2) on a closed manifold M^n satisfying

$$\sup_{M \times [0,T)} |\nabla^2 \phi| < \infty.$$

Then by the equation (2.4), we have

$$\sup_{M \times [0,T)} |Ric| < \infty.$$

By Lemma 7.1 below, we also have

$$\sup_{M\times[0,T)} |\nabla\phi| < \infty.$$

Hence by Lemma 4.1 the norm of the Riemann curvature tensor is bounded

$$\sup_{M\times[0,T)} |Rm| < \infty.$$

Therefore the left hand side of the metric evolving equation in (3.2) is uniformly bounded, i.e. we have

$$\sup_{M\times[0,T)} \left|\frac{\partial}{\partial t} g\right| < \infty.$$

Thus there exists a constant $C < \infty$ such that

$$e^{-C} g(0) \leq g(t) \leq e^{C} g(0) \tag{6.1}$$

on M for all $t \in [0,T)$. Moreover, as $t \nearrow T$, the metrics $g(t)$ converge uniformly to a continuous metric $g(T)$ such that

$$e^{-C} g(0) \leq g(T) \leq e^{C} g(0)$$

on M.

As direct consequences of the equivalence of metrics in (6.1), we have the following uniform Sobolev inequalities. For any given $1 \leq p < n$, there exists a constant C independent of t, such that

$$\|f\|_{L^q(M,g(t))} \leq C\left(\|\nabla f\|_{L^p(M,g(t))} + \|f\|_{L^p(M,g(t))}\right), \tag{6.2}$$

for any smooth function $f \in C^\infty(M \times [0,T))$, $q \leq \frac{np}{n-p}$, and all $t \in [0,T)$.

Furthermore, for any given $p \geq 1$ satisfying $1 - \frac{n}{p} > 0$, there exists a constant C independent of t, such that

$$\|f\|_{C^0(M)}(t) \leq C\left(\|\nabla f\|_{L^p(M,g(t))} + \|f\|_{L^p(M,g(t))}\right), \tag{6.3}$$

for any smooth function $f \in C^\infty(M \times [0,T))$ and all $t \in [0,T)$.

Now we can use these uniform Sobolev inequalities and the L^2-estimates in Theorem 1.1 to show the following pointwise estimates.

Lemma 6.1. *Let $(g(t), \phi(t))$, $t \in [0,T)$, for some $T < \infty$, be a solution to the system (3.2) satisfying $\sup_{M\times[0,T)} |\nabla^2\phi| < \infty$. Then for each non-negative integer k, there exists a constant C_k, such that*

$$\sup_{M\times[0,T)} |\nabla^k Rm| \le C_k, \tag{6.4}$$

$$\sup_{M\times[0,T)} |\nabla^{2+k}\phi| \le C_k. \tag{6.5}$$

Proof. We will only prove the estimate in (6.4). The estimate in (6.5) can be obtained similarly.

We will use C_k to denote constants that may depend on k, $\sup_{M\times[0,T)} |Rm|$, and $\sup_{M\times[0,T)} |\nabla^2\phi|$, which we assume to be finite. For simplicity, we allow C_k to vary from term to term. We first choose a time $\beta \in (0,T)$. Then for each non-negative integer k, there exists a constant C_k, such that

$$\sup_{M\times[0,\beta]} |\nabla^k Rm| \le C_k, \tag{6.6}$$

since $|\nabla^k Rm|$ is a smooth function on the compact space $M \times [0,\beta]$ (note the manifold M is closed).

By Theorem 1.1 and the metric equivalence condition in (6.1), for each non-negative integer k, there exists a constant C_k, such that

$$\|\nabla^k Rm\|_{L^2(M,g(t))} \le C_k,$$

for all $t \in [\beta,T)$. Then by setting $f = |\nabla^k Rm|$ in the uniform Sobolev inequality (6.2) with $p = 2$, we have

$$\begin{aligned} \|\nabla^k Rm\|_{L^{\frac{2n}{n-2}}(M,g(t))} &\le C(\|\nabla|\nabla^k Rm|\|_{L^2(M,g(t))} + \|\nabla^k Rm\|_{L^2(M,g(t))}) \\ &\le C(\|\nabla^{k+1} Rm\|_{L^2(M,g(t))} + \|\nabla^k Rm\|_{L^2(M,g(t))}) \\ &\le C_k, \end{aligned} \tag{6.7}$$

for all $t \in [\beta,T)$. Recall we assume $n \ge 3$. Here C is the uniform Sobole constant, which is independent of t. In the second inequality, we used the Kato's inequality $|\nabla|\nabla^k Rm|| \le |\nabla^{k+1} Rm|$.

Now if $\frac{2n}{n-2} > n$ (actually only for $n = 3$), and so $1 - \frac{n}{(\frac{2n}{n-2})} > 0$, then the inequality obtained in (6.7) and the uniform Sobolev inequality in (6.3) imply that there exists a constant C_k such that

$$\|\nabla^k Rm\|_{C^0(M,g(t))}(g) \le C_k, \tag{6.8}$$

for all $t \in [\beta, T)$. In other words,

$$\sup_{M\times[\beta,T)} |\nabla^k Rm| \le C_k. \tag{6.9}$$

Then combining estimates in (6.6) and (6.9) completes the proof of the estimate in (6.4).

If $\frac{2n}{n-2} \le n$ (for $n \ge 4$), then by applying the uniform Sobolev inequality in (6.2) again with $f = |\nabla^k Rm|$ and $p = \frac{2n}{n-2}$, as in (6.7), we have

$$\|\nabla^k Rm\|_{L^q(M,g(t))} \le C_k,$$

for all $t \in [\beta, T)$, and $q = \frac{pn}{n-p} > p$. Now if $q = \frac{pn}{n-p} > n$, then as the case of $\frac{2n}{n-2} > n$, we complete the proof.

Otherwise, we repeat the process by using the uniform Sobolev inequality (6.2) to increase the index q on the left hand side of the inequality (6.2). After finitely many times, which depends on the dimension n of the manifold M, we can obtain that for some $q > n$ there exists a constant C_k such that

$$\|\nabla^k Rm\|_{L^q(M,g(t))} \le C_k,$$

for all $t \in [\beta, T)$. As before, this completes the proof. □

From the pointwise estimates in Lemma 6.1, we can obtain the following metrics convergence result, by using essentially the same argument as showing the blow up condition for Ricci flow (see details in e.g. §7 in Chapter 6 in [CK04]), except replacing $-2Ric$ by $-\frac{1}{8}R_{jk} - \frac{1}{4}|\nabla\phi|^2 g_{jk} + \frac{1}{4}\langle \mathbf{D}\phi, e_j \cdot \nabla_{e_k}\phi + e_k \cdot \nabla_{e_j}\phi\rangle$, whose derivatives are uniformly bounded by Lemma 6.1.

Proposition 6.2. *Let $(g(t), \phi(t))$, $t \in [0, T)$, for some $T < \infty$, be a solution to the system (3.2) satisfying $\sup_{M\times[0,T)} |\nabla^2\phi| < \infty$. Then there exists a smooth metric $g(T)$ on M^n such that $g(t) \to g(T)$ in any C^k norm as $t \nearrow T$.*

6.2. Convergence of spinors

Now in order to extend the solution $(g(t), \phi(t))$, $t \in [0, T)$, crossing the time $t = T$, we also need to choose a spinor $\phi(T) \in \Gamma(\Sigma_{g(T)}M)$, which will be the limit of $\phi(t)$ in certain sense as $t \nearrow T$.

Let $P_{SO}(t)$, $t \in [0,T]$ denote orthonormal frame bundles with respect to the metric $g(t)$. For each $t \in [0,T]$, there exists a unique $A(t) \in \mathrm{End}(TM)$ that gives a principal $\mathrm{SO}(n)$-bundle isomorphism

$$\begin{array}{rcl} A(t) : P_{SO}(t) & \longrightarrow & P_{SO}(T) \\ e = (e_1, \ldots, e_n) & \longmapsto & A(t)e = (A(t)e_1, \ldots, A(t)e_n). \end{array}$$

In particular, $A(T) = id_{TM}$.

Let $\xi_t : P_{Spin}(t) \to P_{SO}(t)$, $t \in [0,T]$, be *metric* spin structures with respect to the metric $g(t)$ corresponding to a fixed *topology* spin structure on M. Then the isomorphism $A(t)$ can be lifted to be an isomorphism $\tilde{A}(t) : P_{Spin}(t) \to P_{Spin}(T)$, for each $t \in [0,T)$, and such that there is the following commutative diagram

$$\begin{array}{ccc} P_{Spin}(t) & \xrightarrow{\tilde{A}(t)} & P_{Spin}(T) \\ \xi_t \downarrow & & \downarrow \xi_T \\ P_{SO}(t) & \xrightarrow{A(t)} & P_{SO}(T). \end{array}$$

Furthermore, $\tilde{A}(t)$ induce isometries, which are still denoted by $A(t)$, between spinor bundles given by

$$\begin{array}{rcl} A(t) : \Sigma_t M := P_{Spin}(t) \times_\mu \Sigma_n & \longrightarrow & \Sigma_T M := P_{Spin}(T) \times_\mu \Sigma_n \\ \phi = [\tilde{e}, \tilde{\phi}] & \longmapsto & A(t)\phi = [\tilde{A}(t)\tilde{e}, \tilde{\phi}]. \end{array}$$

As shown in Proposition 2 in [BG92], for any fixed point $x \in M$, the isometries $A(t)$ coincide with the fiber-wise parallel transport from $\Sigma_{t,x}M$ to $\Sigma_{T,x}M$ along the path $g_s = (1-s)g(t) + sg(T)$, $0 \le s \le 1$, associated with the Bourguignon-Gauduchon connection for the universal spinor bundle.

Let ∇^t denote the Levi-Civita connection with respect to the metric $g(t)$ for all $t \in [0,T]$. Define a connection

$$\overline{\nabla}^t := A(t)^{-1} \circ \nabla^T \circ A(t)$$

on the tangent bundle for each $t \in [0,T]$. This connection is compatible with the metric $g(t)$ and has torsion, for any $t \in [0,T)$, given by

$$\begin{aligned} \overline{T}^t(X,Y) &:= \overline{\nabla}^t_X Y - \overline{\nabla}^t_Y X - [X,Y] \\ &= A(t)^{-1}(\nabla^T_X A(t))Y - A(t)^{-1}(\nabla^T_Y A(t))X. \end{aligned}$$

Then the difference between ∇^t and $\overline{\nabla}^t$ is given in terms of the torsion as

$$
\begin{aligned}
(6.10)\qquad & 2g(t)(\overline{\nabla}^t_X Y - \nabla^t_X Y, Z) \\
&= g(t)(\overline{T}^t(X,Y),Z) - g(t)(\overline{T}^t(X,Z),Y) \\
&\quad - g(t)(\overline{T}^t(Y,Z),X).
\end{aligned}
$$

Let $\{e_1(t),\ldots,e_n(t)\}$ be a local orthonormal frame with respect to $g(t)$, and $\omega(t)$ and $\overline{\omega}(t)$ be the connection 1-forms for ∇^t and $\overline{\nabla}^t$ respectively, whose components are

$$
(6.11)\qquad \begin{aligned}
\omega(t)_{ij} &= g(t)(\nabla^t e_i(t), e_j(t)), \\
\overline{\omega}(t)_{ij} &= g(t)(\overline{\nabla}^t e_i(t), e_j(t)).
\end{aligned}
$$

Metric connections ∇^t and $\overline{\nabla}^t$ naturally induce connections on the spinor bundle $\Sigma_t M$ associated to the metric $g(t)$, which are still denoted by ∇^t and $\overline{\nabla}^t$, respectively. Then by the formula for the induced spinor connections, we have

$$
\overline{\nabla}^t_X \phi(t) - \nabla^t_X \phi(t) = \frac{1}{4}\sum_{i,j=1}^{n} (\overline{\omega}^t_{ij} - \omega^t_{ij})(X) e_i \cdot e_j \cdot \phi(t).
$$

Then combining with equations (6.10) and (6.11), we have

$$
(6.12)\qquad (\overline{\nabla}^t - \nabla^t)\phi(t) = A(t)^{-1} * (\nabla^T A(t)) * \phi(t).
$$

Lemma 6.3. *Let $(g(t),\phi(t))$, $t\in[0,T)$, for some $T<\infty$, be a solution to the system (3.2) satisfying $\sup\limits_{M\times[0,T)} |\nabla^2\phi| < \infty$, and $g(T)$ be the limit smooth metric obtained in Proposition 6.2. Then for each nonnegative integer k, there exists a constant C_k independent of t such that*

$$
\|(\nabla^T)^k (A(t)\phi(t))\|_{L^2(\Sigma_T M, g(T))} \le C_k.
$$

Proof.

$$\begin{aligned}
&\|(\nabla^T)^k(A(t)\phi(t))\|^2_{L^2(\Sigma_T M, g(T))} \\
&\quad = \int_M |(\nabla^T)^k(A(t)\phi(t))|^2_{g(T)} dv(T) \\
&\quad \le C \int_M |A(t)^{-1}(\nabla^T)^k(A(t)\phi(t))|^2_{g(t)} dv(t) \\
&\quad = C \int_M |(\overline{\nabla}^t)^k \phi(t)|^2_{g(t)} dv(t) \\
&\quad = C \int_M |(\nabla^t + \overline{\nabla}^t - \nabla^t)^k \phi(t)|^2_{g(t)} dv(t) \\
&\quad \le C \sum_{i=0}^{k} \int_M |(\nabla^t)^i \phi(t)|^2_{g(t)} dv(t) \le C_k^2.
\end{aligned}$$

In above steps the constants C may vary along steps but all of them are independent of t. The first inequality follows from the uniform equivalence of metrics $g(t)$ and $g(T)$ for all $t \in [0, T]$. The second inequality follows from the expression (6.12) and the fact $|(\nabla^t)^m \nabla^T A(t)^{-1}|$ and $|(\nabla^t)^m A(t)|$ are uniformly bounded for all $t \in [0, T]$, since essentially $A(t) = (g(T)^{-1} g(t))^{\frac{1}{2}}$ and all partial derivatives of $g(t)$ are uniformly bounded (this shall be shown while proving the convergence of metrics, for details, we refer to Proposition 6.48 in [CK04] and its proof). Finally, the last inequality follows from L^2 estimates in Theorem 1.1. □

From the L^2 estimates in Lemma 6.3 and standard Sobolev inequality with respect to metric $g(T)$, one immediately obtains uniform pointwise estimates. Furthermore, we have

Proposition 6.4. *Let $(g(t), \phi(t))$, $t \in [0, T)$, for some $T < \infty$, be a solution to the system (3.2) satisfying $\sup_{M \times [0,T)} |\nabla^2 \phi| < \infty$, and $g(T)$ be the limit smooth metric obtained in Proposition 6.2. Then there exists a spinor $\phi(T) \in \Gamma(\Sigma_T M)$ such that $A(t)\phi(t) \to \phi(T)$ in any C^k norm as $t \nearrow T$.*

Now one can solve the system (3.2) with the initial data $(g(T), \phi(T))$ obtained in Propositions 6.2 and 6.4 to extend the flow crossing the time $t = T$. This completes the proof of Theorem 1.2.

7. Lower bound estimate for the existence time

First we show the easy fact that $|\nabla\phi|$ is naturally controlled by $|\nabla^2\phi|$ under the normalization $|\phi|\equiv 1$. Hence to obtain a lower bound for the existence time, we only need to control $|\nabla^2\phi|$.

Lemma 7.1. *Let ϕ be a spinor field with $|\phi|\equiv 1$ on a domain Ω, then*

$$|\nabla\phi|^2_{\infty,\Omega}\le|\Delta\phi|_{\infty,\Omega}.$$

Proof. Take Laplacian of both sides of the equation $\langle\phi,\phi\rangle=1$, we have

$$|\nabla\phi|^2=-\langle\Delta\phi,\phi\rangle.$$

Hence by Cauchy-Schwarz inequality we have the claim. □

Proof of Theorem 1.3. For $\Lambda>K$ to be determined later, suppose $[0,T]$ is the maximal time interval such that

$$\sup_{M\times[0,T]}|\nabla^2\phi|\le\Lambda.$$

By Lemma 4.1 we have $|Rm|\le C(n,L,\Lambda)$ on $M\times[0,T]$ (note that we can take $\eta\equiv 1$ in its proof since M is closed).

Let $F_i(t)$ be the quantities defined in (5.7). Let $k=\left[\frac{n}{2}\right]+1$, and set

$$P_k(t)=F_k(t)+\sum_{i=1}^{k}\frac{C_kC_{k-1}\cdots C_{k+1-i}}{\beta_{k-1}\beta_{k-2}\cdots\beta_{k-i}}F_{k-i}(t),$$
$$P_{k-1}(t)=F_{k-1}(t)+\sum_{i=1}^{k-1}\frac{C_{k-1}\cdots C_{k-i}}{\beta_{k-2}\cdots\beta_{k-1-i}}F_{k-1-i}(t),$$

where the constants are from (5.8). By (5.8) we have

$$\begin{aligned}\frac{d}{dt}P_k&=\frac{d}{dt}F_k+\sum_{i=1}^{k}\frac{C_kC_{k-1}\cdots C_{k+1-i}}{\beta_k\beta_{k-2}\cdots\beta_{k-i}}\frac{d}{dt}F_{k-i}\\&\le-\beta_kF_{k+1}+C_kF_k+C_kV\\&\quad+\sum_{i=1}^{k}\frac{C_kC_{k-1}\cdots C_{k+1-i}}{\beta_{k-1}\beta_{k-2}\cdots\beta_{k-i}}(-\beta_{k-i}F_{k-i+1}+C_{k-i}F_{k-i}+C_{k-i}V)\end{aligned}$$

$$
\begin{aligned}
&= -\beta_k F_{k+1} + C_k F_k + C_k V - C_k F_k - \sum_{i=2}^{k} \frac{C_k \cdots C_{k+1-i}}{\beta_{k-1} \cdots \beta_{k+1-i}} F_{k+1-i} \\
&\quad + \sum_{i=1}^{k-1} \frac{C_k \cdots C_{k-i}}{\beta_{k-1} \cdots \beta_{k-i}} F_{k-i} + \frac{C_k \cdots C_0}{\beta_{k-1} \cdots \beta_0} F_0 + \sum_{i=1}^{k} \frac{C_k \cdots C_{k-i}}{\beta_{k-1} \cdots \beta_{k-i}} V \\
&= -\beta_k F_{k+1} + C_k V + \frac{C_k \cdots C_0}{\beta_{k-1} \cdots \beta_0} F_0 + \sum_{i=1}^{k} \frac{C_k \cdots C_{k-i}}{\beta_{k-1} \cdots \beta_{k-i}} V \\
&\leq C_k V + \frac{C_k \cdots C_0}{\beta_{k-1} \cdots \beta_0} F_0 + \sum_{i=1}^{k} \frac{C_k \cdots C_{k-i}}{\beta_{k-1} \cdots \beta_{k-i}} V \\
&\leq C(n, L, \Lambda) V
\end{aligned}
$$

Then integrating this differential inequality gives

$$P_k(t) \leq P_k(0) + C(n, L, \Lambda) V t.$$

Similarly, one also has

$$P_{k-1}(t) \leq P_{k-1}(0) + C(n, L, \Lambda) V t.$$

Recall here $V = \mathrm{Vol}_{g(0)} M$. In particular,

$$\int |\nabla^{k+2} \phi|^2 dv(t) \leq P_k(0) + C(n, L, \Lambda) V t, \tag{7.1}$$

and

$$\int |\nabla^{k+1} \phi|^2 dv(t) \leq P_{k-1}(0) + C(n, L, \Lambda) V t. \tag{7.2}$$

Moreover, by setting $k = 0$ in (5.8), dropping the term $-\beta_0 F_1$ on the right hand side of the inequality, and then solving the differential inequality, one can easily get

$$F_0(t) \leq F_0(0) + (e^{C_0 t} - 1) V.$$

In particular,

$$\int |\nabla^2 \phi|^2 dv(t) \leq F_0(0) + (e^{C_0 t} - 1) V. \tag{7.3}$$

Since the Sobolev constants are uniform along the flow up to a factor $e^{C(n,L,\Lambda)t}$, we can use Lemmas 8.1 and 8.3 to get pointwise estimate for

$|\nabla^2\phi|$ as follows. In the rest of the proof, we use C to denote constants only depending on n, C_1 only depending on n and the initial metric $g(0)$, and C_2 also depending on Λ. In the following derivations, constants C, C_1, and C_2 may vary along steps.

By setting $i = 1$, $\eta \equiv 1$, and $A = \nabla\phi$ in Lemma 8.1, we have

$$\begin{aligned}
(7.4)\quad \|\nabla^2\phi\|_{L^{2k}(g(t))} &\le C\left(|\nabla\phi|_\infty(t)\right)^{1-\frac{1}{k}}\left(\|\nabla^{k+1}\phi\|_{L^2(g(t))} + \|\nabla\phi\|_{L^2(g(t))}\right)^{\frac{1}{k}} \\
&\le C\left(|\nabla^2\phi|_\infty(t)\right)^{1-\frac{1}{k}} \\
&\quad \times\left(\|\nabla^{k+1}\phi\|_{L^2(g(t))} + \left(\int|\Delta\phi|dv(t)\right)^{\frac{1}{2}}\right)^{\frac{1}{k}} \\
&\le C\left(|\nabla^2\phi|_\infty(t)\right)^{1-\frac{1}{k}} \\
&\quad \times\left(\|\nabla^{k+1}\phi\|_{L^2(g(t))} + \left(\int\sqrt{n}|\nabla^2\phi|dv(t)\right)^{\frac{1}{2}}\right)^{\frac{1}{k}} \\
&\le C\left(|\nabla^2\phi|_\infty(t)\right)^{1-\frac{1}{k}} \\
&\quad \times\left(\|\nabla^{k+1}\phi\|_{L^2(g(t))} + \sqrt{nV}e^{C_2t} + \|\nabla^2\phi\|_{L^2(g(t))}\right)^{\frac{1}{k}}.
\end{aligned}$$

Similarly, by setting $A = \nabla^2\phi$ in Lemma 8.1, we have

$$(7.5)\quad \|\nabla^3\phi\|_{L^{2k}(g(t))} \le C\left(|\nabla^2\phi|_\infty(t)\right)^{1-\frac{1}{k}}\left(\|\nabla^{k+2}\phi\|_{L^2(g(t))} + \|\nabla^2\phi\|_{L^2(g(t))}\right)^{\frac{1}{k}}$$

By setting $p = 2k$ and $u = |\nabla^2\phi|$ in Lemma 8.3, and combining with the above estimates, we have

$$\begin{aligned}
\left(|\nabla^2\phi|_\infty(t)\right)^{\frac{2k}{\alpha}} &\le C(C_S(g(t)))^{2k}\left(\|\nabla^2\phi\|_{L^m(g(t))}\right)^{\frac{2k}{\alpha}-2k} \\
&\quad \times\left(\|\nabla^3\phi\|_{L^{2k}(g(t))} + C_1e^{C_2t}\|\nabla^2\phi\|_{L^{2k}(g(t))}\right)^{2k} \\
&\le C(C_S(g(t)))^{2k}\left(\|\nabla^2\phi\|_{L^m(g(t))}\right)^{\frac{2k}{\alpha}-2k}2^{2k-1} \\
&\quad \times\left(\|\nabla^3\phi\|^{2k}_{L^{2k}(g(t))} + C_1e^{C_2t}\|\nabla^2\phi\|^{2k}_{L^{2k}(g(t))}\right) \\
&\le C(C_S(g(t)))^{2k}\left(\|\nabla^2\phi\|_{L^m(g(t))}\right)^{\frac{2k}{\alpha}-2k}2^{2k-1}C_1e^{C_2t} \\
&\quad \times\left(\|\nabla^3\phi\|^{2k}_{L^{2k}(g(t))} + \|\nabla^2\phi\|^{2k}_{L^{2k}(g(t))}\right)
\end{aligned}$$

Here in the second inequality, we used binomial formula and Young's inequality for products. Note that we can take $C_1 \ge 1$. So the last inequality follows. In the following, constants C_1 and C_2 may also depend on m. Then by plugging the inequality

$$\|\nabla^2\phi\|_{L^m(g(t))} \leq |\nabla^2\phi|_\infty(t)(V(g(t)))^{\frac{1}{m}} \leq |\nabla^2\phi|_\infty(t)(e^{C_2t}V)^{\frac{1}{m}}$$

and the inequalities in (7.4) and (7.5) into the above, and absorbing 2^{2k-1} into C_1 (recall $k = [\frac{n}{2}] + 1$), one obtains

$$\begin{aligned}
\left(|\nabla^2\phi|_\infty(t)\right)^{\frac{2k}{\alpha}} &\leq C_1e^{C_2t}(C_S(g(t)))^{2k}V^{\frac{1}{m}\left(\frac{2k}{\alpha}-2k\right)}\left(|\nabla^2\phi|_\infty(t)\right)^{\frac{2k}{\alpha}-2} \\
&\quad \times \left(\int |\nabla^{k+2}\phi|^2dv(t) + \int |\nabla^{k+1}\phi|^2dv(t)\right. \\
&\quad \left. + \int |\nabla^2\phi|^2dv(t) + nVe^{C_2t}\right) \\
&\leq C_1e^{C_2t}(C_S(g(t)))^{2k}V^{\frac{1}{m}\left(\frac{2k}{\alpha}-2k\right)}\left(|\nabla^2\phi|_\infty(t)\right)^{\frac{2k}{\alpha}-2} \\
&\quad \times (P_k(0) + P_{k-1}(0) + F_0(0) + C_2Vt \\
&\quad + (e^{C_0t} - 1)V + nVe^{C_2t}).
\end{aligned}$$

Then dividing by $\left(|\nabla^2\phi|_\infty(t)\right)^{\frac{2k}{\alpha}-2}$, we have

$$\begin{aligned}
(|\nabla^2\phi|_\infty(t))^2 \leq C_1e^{C_2t}(C_S(g(t)))^{2k}V^{\frac{1}{m}\left(\frac{2k}{\alpha}-2k\right)}&(P_k(0) + P_{k-1}(0) \\
&+ F_0(0) + C_2Vt + (e^{C_0t} - 1)V + nVe^{C_2t}).
\end{aligned}$$

Recall that $C_S(g(0)) = C_1V^{-\frac{1}{n}}$, Sobolev constants $C_S(g(t))$ are uniform along the flow up to a factor e^{C_2t}, and $\frac{1}{\alpha} = \left(\frac{1}{n} - \frac{1}{2k}\right)m + 1$, thus

$$\begin{aligned}
(|\nabla^2\phi|_\infty(t))^2 &\leq C_1e^{C_2t}V^{\frac{1}{m}\left(\frac{2k}{\alpha}-2k\right)-\frac{2k}{n}}(P_k(0) + P_{k-1}(0) \\
&\quad + F_0(0) + C_2Vt + (e^{C_0t} - 1)V + nVe^{C_2t}) \\
&\leq C_1e^{C_2t}V^{-1}(P_k(0) + P_{k-1}(0) \\
&\quad + F_0(0) + C_2Vt + (e^{C_0t} - 1)V + nVe^{C_2t}) \\
&\leq C_1e^{C_2t}(V^{-1}P_k(0) + V^{-1}P_{k-1}(0) \\
&\quad + V^{-1}F_0(0) + C_2t + (e^{C_0t} - 1) + ne^{C_2t})
\end{aligned} \tag{7.6}$$

In the rest of the proof, constants C_1 and C_2 will be taken from the last step in (7.6). Now we take a constant Λ as

$$\Lambda^2 = \max\{C_1(V^{-1}P_k(0) + V^{-1}P_{k-1}(0) + V^{-1}F_0(0) + n), K^2\}.$$

This choice of Λ guarantees $|\nabla^2\phi|_\infty(0) \le \Lambda$. Then we consider the function $f(t)$ given by

$$f(t) := C_1 e^{C_2 t}\big(V^{-1}P_k(0) + V^{-1}P_{k-1}(0) + V^{-1}F_0(0) + C_2 t + (e^{C_0 t} - 1) + n e^{C_2 t}\big).$$

Clearly, $f(t)$ is a monotone increasing function in t, and $\lim_{t\to+\infty} f(t) = +\infty$. By the choice of Λ in the above, $f(0) \le \Lambda^2$. Thus there exists $\delta > 0$ such that

$$4\Lambda^2 = f(\delta) = C_1 e^{C_2\delta}\big(V^{-1}P_k(0) + V^{-1}P_{k-1}(0) + V^{-1}F_0(0) + C_2\delta + (e^{C_0\delta} - 1) + n e^{C_2\delta}\big).$$

Then by (7.6),

$$|\nabla^2\phi|_\infty(t) \le f(t)^{1/2} \le 2\Lambda,$$

for all $t \in [0,\delta]$. By Theorem 1.2, if the flow cease to exist at a finite time T, then we must have

$$\limsup_{t\to T} |\nabla^2\phi|_\infty = \infty,$$

hence one can see that the existence time interval contains $[0,\delta]$. The dependence of δ is evident from its definition. $\square$

8. Appendix: Interpolation lemma

The following interpolation lemma was proved in the Appendix of [KS02].

Lemma 8.1. *For integers $k > 0$, $1 \le i \le k$ and $m \ge 2k$ we have the inequality*

$$\left(\int \eta^m |\nabla^i A|^{\frac{2k}{i}}\right)^{\frac{i}{2k}} \le C|A|_\infty^{1-\frac{i}{k}} \left(\left(\int \eta^m |\nabla^k A|^2\right)^{\frac{1}{2}} + |A|_{L^2,\eta>0}\right)^{\frac{i}{k}},$$

where the constant C depends only on n, m, k and $|\nabla\eta|_\infty$.

As a corollary we have the following lemma which is needed in our derivation of L^2 derivative estimates.

Lemma 8.2. *Let $1 \le i_1, i_2, \ldots, i_l \le k$ and $i_1 + i_2 + \cdots + i_l = 2k$, $m \ge 2k$. Then for (vector valued) tensors $A_1, A_2, \ldots, A_r$ and a cutoff function η, we have*

$$\int \eta^m \nabla^{i_1} A_1 * \nabla^{i_2} A_2 * \cdots * \nabla^{i_l} A_l * A_{l+1} * \cdots * A_r$$
$$\le C\left(\prod_{i=l+1}^{r} |A_i|_\infty\right) \sum_{j=1}^{l} |A_j|_\infty^{\frac{2k}{i_j}-2} \left(\int \eta^m |\nabla^k A_j|^2 + \int_{\eta>0} |A_j|^2\right)$$

where the constant C depends only on n, m, k, r and $|\nabla\eta|_\infty$.

Proof.

$$\int \eta^m \nabla^{i_1} A_1 * \nabla^{i_2} A_2 * \cdots * \nabla^{i_l} A_l * A_{l+1} * \cdots * A_r$$
$$\le C\left(\prod_{i=l+1}^{r} |A_i|_\infty\right) \prod_{j=1}^{l} \left(\int \eta^m |\nabla^{i_j} A_j|^{\frac{2k}{i_j}}\right)^{\frac{i_j}{2k}}$$
$$\le C\left(\prod_{i=l+1}^{r} |A_i|_\infty\right) \prod_{j=1}^{l} |A_j|_\infty^{1-\frac{i_j}{k}} \left(\int \eta^m |\nabla^k A_j|^2 + \int_{\eta>0} |A_j|^2\right)^{\frac{i_j}{2k}}$$
$$\le C\left(\prod_{i=l+1}^{r} |A_i|_\infty\right) \sum_{j=1}^{l} |A_j|_\infty^{\frac{2k}{i_j}-2} \left(\int \eta^m |\nabla^k A_j|^2 + \int_{\eta>0} |A_j|^2\right)$$

where in the last step we used Young's inequality. □

Recall that ([Sal92]) on any complete Riemannian manifold with $Ric \ge -K$ we have a Sobolev inequality

$$\left(\int_{B(r)} u^{\frac{2\mu}{\mu-2}}\right)^{\frac{\mu-2}{\mu}} \le C(n) \frac{r^2 e^{\sqrt{K}r}}{V(r)^{\frac{2}{\mu}}} \int_{B(r)} (|\nabla u|^2 + r^{-2}u^2), \tag{8.1}$$

for any C^1 function compactly supported on a geodesic ball $B(r)$, where $\mu = n$ when $n \ge 3$ and $2 < \mu < \infty$ when $n = 2$. Using this Sobolev inequality we can prove a multiplicative version by the same argument as in [KS02] (Theorem 5.6).

Lemma 8.3. *Let $u \in C_c^1(B(r))$, where $B(r)$ is a geodesic ball with radius r on a n-dimensional Riemannian manifold with $Ric \ge -K$. For any $p > n$,*

$m \geq 0$ and $0 < \alpha \leq 1$ with $\frac{1}{\alpha} = (\frac{1}{n} - \frac{1}{p})m + 1$, there is a constant C depending only on n, m, p such that

$$|u|_\infty \leq C C_S^\alpha |u|_{L^m}^{1-\alpha} (|\nabla u|_{L^p} + r^{-1}|u|_{L^p})^\alpha,$$

where $C_S = C(n)\frac{re^{\sqrt{K}r/2}}{V(r)^{\frac{1}{n}}}$.

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