

# Average size of the automorphism group of smooth projective hypersurfaces over finite fields

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In this paper we show that the average size of the automorphism group over  $\mathbb{F}_q$  of a smooth degree  $d$  hypersurface in  $\mathbb{P}_{\mathbb{F}_q}^n$  is equal to 1 as  $d \rightarrow \infty$ . We also discuss some consequence of this result for the moduli space of smooth degree  $d$  hypersurfaces in  $\mathbb{P}^n$ .

## 1. Introduction

**Definition.** Let  $\mathcal{S}_{n,d}$  denote the set of smooth degree  $d$  hypersurfaces in  $\mathbb{P}_{\mathbb{F}_q}^n$ .

The automorphism group of a projective smooth degree  $d$  hypersurface  $X$ ,  $\text{Aut}(X)$ , in  $\mathbb{P}_{\mathbb{F}_q}^n$  over the algebraic closure  $\overline{\mathbb{F}_q}$  has been an object of intense study over the past decades. It is known that for most  $(n, d)$ , i.e.  $(n, d) \neq (2, 3), (3, 4)$  all of the automorphisms of the hypersurface  $X$  are induced by an automorphism  $\mathbb{P}_{\mathbb{F}_q}^n$ . This has been proven by Matsumura and Monsky in [MaMo] for  $n \geq 2$  and Chang in [Ch] for  $n = 2$ .

Another classical fact is that  $\text{Aut}(X)$  is finite for  $n \geq 2, d \geq 3$ . This was shown for  $n \geq 3$  by Matsumura and Monsky in [MaMo]. For  $n = 2$  and  $d \geq 4$ , the genus of a smooth degree  $d$  curve is  $g = (d - 1)(d - 2)/2$  is at least 2 so we can use an old result of Schmid [Sc]. Finally, the argument for  $d = 3$  can be found in [Po3].

We also know that for  $(n, d) \neq (2, 3)$  there is a open subset  $U_{n,d}$  of  $\mathcal{S}_{n,d}$  such that  $\text{Aut}(X) = \{1\}$ ; here we include  $(n, d) = (3, 4)$  with the caveat that we are only considering linear automorphisms, i.e., induced by automorphisms of  $\mathbb{P}^n$ . This result can be found in work of Katz and Sarnak [KaSa, Lemma 11.8.5]. The fact that  $U_{n,d}$  is nonempty follows from work of [MaMo] for  $n \geq 3, d \geq 3$ , and this can be adapted to work for the cases  $n \geq 2, d \geq 4$ . An alternative proof of the case  $n = 2$  and  $d \geq 4$  can be found in [KaSa, Lemma 10.6.18].

For our purposes we are only going restrict to the set of automorphisms defined over the base field of the hypersurface,  $\mathbb{F}_q$ , and prove a quantitative version of the above.

**Theorem 1.** *We have that*

$$\lim_{d \rightarrow \infty} \frac{\sum_{X \in \mathcal{S}_{n,d}} |\text{Aut}_{\mathbb{F}_q}(X)|}{|\mathcal{S}_{n,d}|} = 1.$$

We also have an error term version of Theorem 1 that will be used in the last section on the moduli space of smooth degree  $d$  hypersurfaces in  $\mathbb{P}^n$ .

**Theorem 2.** *For  $(n, d) \neq (2, 3), (3, 4)$  the number of smooth degree  $d$  hypersurfaces  $X$  in  $\mathbb{P}^n$  with  $\text{Aut}_{\mathbb{F}_q}(X) \neq \{1\}$  is less than  $q^{C\binom{n+d}{d} + (n+1)^2 + 1}$ , where  $C = 1 - \frac{1}{2^n}$ .*

**Remark.** The above theorem can be used to obtain stronger version of Theorem 1. Namely one can show that the average size of  $|\text{Aut}_{\mathbb{F}_{q^r}}(X)|$  is 1 for any  $r \leq C_1 \binom{n+d}{d}$  where  $C_1$  is a fixed constant.

## 2. Proof of the main theorems

We will not write anymore explicitly the restriction  $(n, d) \neq (2, 3), (3, 4)$ , but the reader should bear it in mind for all the proofs. We start with the following result which appears in [Po1]:

**Theorem 3 (Poonen).** *We have that*

$$\lim_{d \rightarrow \infty} \frac{|\mathcal{S}_{n,d}|}{q^{\binom{d+n}{n}}} = \zeta_{\mathbb{P}^n}(n+1)^{-1},$$

where  $\zeta_{\mathbb{P}^n}$  is the zeta function of projective space; more precisely  $\zeta_{\mathbb{P}^n}(n+1)^{-1} = \prod_{i=1}^{n+1} (1 - q^{-i})$ .

This implies that a positive proportion of hypersurfaces are smooth once  $d$  is large enough compared to  $q$ .

We are left to estimate  $\sum_{X \in \mathcal{S}_{n,d}} |\text{Aut}_{\mathbb{F}_q}(X)|$ . We know that any automorphism of the hypersurface  $X$  is induced from  $\text{PGL}_{n+1}$ . Thus we can write

$$\sum_{X \in \mathcal{S}_{n,d}} |\text{Aut}_{\mathbb{F}_q}(X)| = \sum_{X \in \mathcal{S}_{n,d}} \sum_{\substack{A \in \text{PGL}_{n+1}(\mathbb{F}_q) \\ A \in \text{Aut}(X)}} 1 = \sum_{A \in \text{PGL}_{n+1}(\mathbb{F}_q)} \sum_{\substack{X \in \mathcal{S}_{n,d} \\ A \in \text{Aut}(X)}} 1.$$

Our estimates will not take into account the smoothness of the curves and rather prove a general bound. This leads to introducing the following definition and notations. Note that we have be careful about scaling the matrix, since we will pick a representative for each element of  $\text{PGL}_{n+1}$  in  $\text{GL}_{n+1}$ .

**Definition 4.** Let  $\mathcal{P}_d$  be the vector space of homogenous polynomials of degree  $d$  in  $\mathbb{F}_q[x_1, x_2, \dots, x_{n+1}]$ .

For an element  $A = (a_{i,j}) \in \text{GL}_{n+1}(\mathbb{F}_q)$  and  $\lambda \in \mathbb{F}_q$  consider

$$\mathcal{P}^{A,\lambda} = \{f \in \mathcal{P}_d \mid \lambda f = f \circ A\}$$

where  $f \circ A$  denotes the polynomial where the variables  $(x_1, x_2, \dots, x_{n+1})$  are changed to  $A(x_1, x_2, \dots, x_{n+1})^t$ .

It is easy to see that  $\mathcal{P}^{A,\lambda}$  is a linear subspace. The key to the proof is the following estimate

**Theorem 5.** *If  $A \neq \lambda I_{n+1}$  then  $\dim(\mathcal{P}^{A,\lambda}) \leq \binom{d+n}{n} - \binom{d - \lfloor d/2 \rfloor + n}{n}$ .*

**Remark.** This inequality is not sharp. We believe that diagonal automorphisms defined on the generators by  $x_i \rightarrow -x_i, x_j \rightarrow x_j$  for  $j \neq i$ , and  $1 \leq i \leq n$  should give the right estimate for the upper bound. The above bound is asymptotically equal to  $\left(1 - \frac{1}{2^n}\right) \binom{d+n}{n}$  and the bound one can obtain by considering only diagonal automorphisms is asymptotically equal to  $\left(1 - \frac{d}{2(d+n)}\right) \binom{d+n}{n}$ .

*Proof.* Let assume that we have a matrix  $A$  for which  $\dim(\mathcal{P}^{A,\lambda}) \geq \binom{d+n}{n} - \binom{d - \lfloor d/2 \rfloor + n}{n} + 1$ . We will show then that  $A = \lambda I_n$ , for some scalar  $\lambda$ .

Let  $\mathcal{P}^{r,x_i}$  be the vector space of homogenous polynomials of degree  $d$  that can be written as  $x_i^r h(x_1, x_2, \dots, x_{n+1})$ . Note that the dimension of this vector subspace is  $\binom{d-r+n}{n}$  since obviously the degree of  $h$  must be  $d-r$ .

If  $\dim(\mathcal{P}^{r,x_i}) + \dim(\mathcal{P}^{A,\lambda}) \geq \binom{d+n}{n} + 1$  then their intersection must be nonempty. Thus there is a nonzero  $f_i = x_i^r h_i(x_1, x_2, \dots, x_{n+1})$  such that we have

$$x_i^r h_i(x_1, \dots, x_{n+1}) = (a_{i,1}x_1 + a_{i,2}x_2 + \dots + a_{i,n+1}x_{n+1})^r \cdot (h_i \circ A).$$

Note that the valuation of  $x_i$  on the right hand side must be at least  $r$ . For  $r > d/2$  since  $h_i \neq 0$  we have that the valuation of  $x_i$  in  $h$  is at most  $d-r < r$ . Thus we must have  $x_i | a_{i,1}x_1 + a_{i,2}x_2 + \dots + a_{i,n+1}x_{n+1}$  so  $a_{i,k} = 0$  for  $k \neq i$ . According to our hypothesis we can choose  $r = \lfloor d/2 \rfloor + 1$ , and thus our matrix  $A$  must be diagonal.

To finish off the proof note that for any matrix  $A'$  in the conjugacy class of the matrix  $A$  in  $GL_n(\mathbb{F}_q)$  we have that  $\mathcal{P}^{A,\lambda} \cong \mathcal{P}^{A',\lambda}$ ; more precisely this isomorphism is induced by changing the coordinates on projective space  $(x, y, z) \rightarrow S(x, y, z)$ , where  $A' = SAS^{-1}$ . Using the above argument any such matrix  $A'$  is also diagonal. It is an elementary linear algebra fact that a diagonal matrix  $A$  such that all the matrices similar to it are also diagonal has to be a scalar multiple of the identity, and our claim is proved.  $\square$

**Proof of Theorems 1 and 2**

It is easy to observe that

$$|\mathcal{S}_{n,d}| \leq \sum_{X \in \mathcal{S}_{n,d}} |\text{Aut}_{\mathbb{F}_q}(X)| \leq |\mathcal{S}_{n,d}| + \sum_{(A,\lambda), A \neq \lambda I_{n+1}} q^{\dim(\mathcal{P}^{A,\lambda})}.$$

Now for the upper bound since the size of  $GL_{n+1}(\mathbb{F}_q)$  is

$$(q^{n+1} - 1)(q^{n+1} - q) \dots (q^{n+1} - q^n)$$

we have

$$\begin{aligned} \sum_{(A,\lambda), A \neq \lambda I_{n+1}} q^{\dim(\mathcal{P}^{A,\lambda})} &\leq (q-1)q^{\binom{d+n}{n} - \binom{d-\lfloor d/2 \rfloor + n - 1}{n}} (q^{n+1} - 1) \dots (q^{n+1} - q^n) \\ &< q^{\binom{d+n}{n} - \binom{d-\lfloor d/2 \rfloor + n - 1}{n} + (n+1)^2 + 1}, \end{aligned}$$

using the bound we proved in Theorem 5.

Thus we get

$$|\mathcal{S}_{n,d}| \leq \sum_{X \in \mathcal{S}_{n,d}} |\text{Aut}_{\mathbb{F}_q}(X)| < |\mathcal{S}_{n,d}| + q^{\binom{d+n}{n} - \binom{d-\lfloor d/2 \rfloor + n-1}{n} + (n+1)^2 + 1}$$

and using Theorem 3 we are done. For theorem 2 just note that

$$\begin{aligned} \sum_{X \in \mathcal{S}_d} |\text{Aut}_{\mathbb{F}_q}(X)| &= \sum_{X \in \mathcal{S}_{n,d}, \text{Aut}_{\mathbb{F}_q}(X) = \{1\}} 1 + \sum_{X \in \mathcal{S}_{n,d}, \text{Aut}_{\mathbb{F}_q}(X) \neq \{1\}} |\text{Aut}_{\mathbb{F}_q}(X)| \\ &\geq \sum_{X \in \mathcal{S}_{n,d}, \text{Aut}_{\mathbb{F}_q}(X) = \{1\}} 1 + \sum_{X \in \mathcal{S}_{n,d}, \text{Aut}_{\mathbb{F}_q}(X) \neq \{1\}} 2 \\ &= |\mathcal{S}_{n,d}| + \#\{X \in \mathcal{S}_{n,d} \mid \text{Aut}_{\mathbb{F}_q}(X) \neq \{1\}\}. \end{aligned}$$

□

### 3. Outlook on the moduli space of smooth hypersurfaces

The goal is to discuss some of the implications of theorem 2. First let us introduce some notation. Let  $\mathcal{H}_{n,d}$  be the moduli space of degree  $d$  hypersurfaces in  $\mathbb{P}^n$ ; more precisely, the quotient  $\mathcal{S}_{n,d}/\text{PGL}_{n+1}$ . It is well known that it is a Deligne-Mumford stack over  $\text{Spec}(\mathbb{Z})$ ; see [Ben, Theorem 1.6]. We also have the following fiber bundle

$$\begin{array}{ccc} \text{PGL}_{n+1} & \longrightarrow & \mathcal{S}_{n,d} \\ & & \downarrow \\ & & \mathcal{H}_{n,d} \end{array}$$

which translates into the equality  $[\text{PGL}_{n+1}] \cdot [\mathcal{H}_{n,d}] = [\mathcal{S}_{n,d}]$  in the Grothendieck ring of stacks,  $K_0(\text{St})$ .

The set  $|\mathcal{H}_{n,d}(\mathbb{F}_q)|$  encodes the number isomorphism classes of smooth projective hypersurfaces of degree  $d$  in  $\mathbb{P}^n$ . We can pick one representative for each of the isomorphism classes and thus identify this with  $|\mathcal{H}_{n,d}(\mathbb{F}_q)|$ . Note that we have a natural probability measure on this space where each hypersurface is weighted by  $\frac{1}{|\text{Aut}_{\mathbb{F}_q}(X)|}$ . This is well understood to be most natural way to count objects which have automorphisms and we obviously get the equality  $|\mathcal{H}_{n,d}(\mathbb{F}_q)| = \sum \frac{1}{|\text{Aut}_{\mathbb{F}_q}(X)|}$ , where the sum is taken over the representatives we've chosen.

Using theorem 2 we obtain the following

$$|\mathcal{H}_{n,d}(\mathbb{F}_q)| = \frac{|S_{n,d}|}{|\mathrm{Gl}_{n+1}(\mathbb{F}_q)|} + O(q^{C\binom{n+d}{d}+(n+1)^2+1}),$$

where  $C = 1 - \frac{1}{2^n}$ . We can see this as an incarnation of the above equality in the Grothendieck ring of stacks, but this is neither implied, or implies the equality in  $K_0(\mathrm{St})$ .

We will use a special case of [BuKe, Theorem 1.2] which gives an error term for the Bertini theorems over finite fields.

**Theorem 6 (Bucur-Kedlaya).** *We have that*

$$\left| \frac{|S_{n,d}|}{q^{\binom{d+n}{n}}} - \zeta_{\mathbb{P}^n}(n+1) \right| \leq C_1 q^{-\delta} + C_2 q^{-\frac{d}{\max\{n+1,p\}}}$$

where  $C_1$  and  $C_2$  are explicit constants and  $1 + \frac{\log_q(d)}{n} > \delta > \frac{\log_q(d)}{n} - 2$ .

This implies that for  $d$  large enough we have the following theorem.

**Theorem 7.** *The number isomorphism classes of smooth projective hypersurfaces of degree  $d$  in  $\mathbb{P}^n$  is*

$$|\mathcal{H}_{n,d}(\mathbb{F}_q)| = q^D - q^{D-1} + O(q^{D-\delta})$$

where  $D = \binom{d+n}{n} - (n+1)^2 + 1 = \dim(\mathcal{H}_{n,d})$ .

Let us make a few remarks about the above result. We can think about the above as being a stabilization result for the point counts on the moduli space  $\mathcal{H}_{n,d}$ , and it offers a different perspective on the stabilization results proved in [VaWo]. We can also try to count the number of points on  $\mathcal{H}_{n,d}$  using the Lefschetz trace formula for Deligne Mumford Stacks proved by Behrend [Beh]. Tomassi [To] has proven that the singular cohomology of  $\mathcal{H}_{n,d}$  vanishes in degrees  $2 \leq k \leq \frac{d+1}{2}$ , thus stabilizing. We know that  $\mathcal{H}_{n,d}$  has a compactification such that the boundary is a normal crossings divisor with respect to  $\mathrm{Spec}(\mathbb{Z})$ , and thus we can compare the etale cohomology and the singular cohomology and conclude that etale cohomology vanishes also in these degrees. As far as we know there are no results proven about

the unstable cohomology of  $\mathcal{H}_{n,d}$ , so the above theorem cannot be simply deduced from this cohomological side.

If we use the same heuristic as in [AcErKeMaZB] about the eigenvalues of Frob acting on  $H_{et}^*$ , we can conjecture that there should not be too many unstable classes coming from algebraic cycles in each degree close to the dimension  $2D$ . It is well known that the eigenvalues of Frobenius attached to algebraic cycles are integral powers of  $q$ . The classes that are not coming from algebraic cycles of weight close to the dimension  $2D$  can either be: few in each weight so their contribution is small, or they can be modeled using a random unitary matrix, and using a result of Diaconis and Shahshahani [DiSh] this matrix has bounded trace with high probability. Thus we can heuristically think about the contribution of non-algebraic classes in the Grothendieck-Lefschetz trace formula as being negligible as  $d \rightarrow \infty$ .

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