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On the formal degree conjecture for simple supercuspidal representations

Yoichi Mieda

We prove the formal degree conjecture for simple supercuspidal representations of symplectic groups and quasi-split even special orthogonal groups over a p-adic field, under the assumption that p is odd. The essential part is to compute the Swan conductor of the exterior square of an irreducible local Galois representation with Swan conductor 1. It is carried out by passing to an equal characteristic local field and using the theory of Kloosterman sheaves.

1. Introduction

Let F be a p-adic field, and G a connected reductive group over F. For an irreducible discrete series representation π of $G(F)$, we can consider an invariant $\deg(\pi) \in \mathbb{R}_{>0}$ called the formal degree of π. It is in some sense a generalization of the dimension (or the degree) of a finite-dimensional representation. On the other hand, by the local Langlands correspondence, irreducible smooth representations of $G(F)$ are conjecturally parametrized by pairs (ϕ, ρ), where $\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ is an L-parameter, and ρ is an irreducible representation of a finite group $\mathcal{S}_\phi$ determined by ϕ. The formal degree conjecture, which was proposed by Hiraga-Ichino-Ikeda [8], predicts that $\deg(\pi)$ can be described by using the pair (ϕ_π, ρ_π) attached to π. For more precise formulation, see Section 3. This conjecture has been solved for general linear groups [8], odd special orthogonal groups [9] and unitary groups [2], but it seems still open for many other groups.

In this article, we will focus on a very special class of discrete series representations, simple supercuspidal representations. They are introduced in [7] and [14], and characterized among irreducible smooth representations by the property that they have minimal positive depth. The local Langlands correspondence for simple supercuspidal representations of quasi-split classical groups has been investigated in Oi's work [13] very precisely. As an application of his results, he obtained the following theorem:

Theorem 1.1 ([13, Theorem 9.3]). *Let $n \geq 1$ be an integer and write $2n = p^e n'$ with $p \nmid n'$. Assume $p \neq 2$ and either $p \nmid 2n$ or $n' \mid p-1$. Let G be one of the following groups:*

- Sp_{2n},
- *the quasi-split* SO_{2n} *attached to a ramified quadratic extension of* F,
- *the split* SO_{2n+2},
- *or the quasi-split* SO_{2n+2} *attached to an unramified quadratic extension of* F.

Then, the formal degree conjecture holds for simple supercuspidal representations of $G(F)$.

The goal of this article is to remove the condition "either $p \nmid 2n$ or $n' \mid p-1$" in the theorem above. Here is our main theorem:

Theorem 1.2 (Theorem 3.5 and Remark 3.8). *Assume $p \neq 2$. Let G be one of the groups in Theorem 1.1. Then, the formal degree conjecture holds for simple supercuspidal representations of $G(F)$.*

By the same method as in [13], Theorem 1.2 is easily reduced to the following:

Theorem 1.3 (Theorem 2.1). *Let τ be a $2n$-dimensional irreducible smooth representation of W_F whose Swan conductor $\mathrm{Sw}\,\tau$ equals 1. Then we have $\mathrm{Sw}(\wedge^2\tau) = n-1$.*

In the case $p \mid 2n$ and $n' \mid p-1$ (which is more difficult than the case $p \nmid 2n$), Oi used an explicit description of τ in [10] to obtain the theorem above. Our strategy to Theorem 1.3 is totally different. First we use Deligne's result [5] to reduce Theorem 1.3 to the case where F is an equal characteristic local field. In the equal characteristic case, every irreducible smooth representation of W_F with Swan conductor 1 is essentially obtained as the localization at $\infty \in \mathbb{P}^1$ of a Kloosterman sheaf Kl (see [4, Sommes. trig.] and [12]). The Swan conductor of the localization at ∞ of $\wedge^2 \mathrm{Kl}$ can be computed by using the Grothendieck-Ogg-Shafarevich formula and the Grothendieck-Lefschetz trace formula.

The outline of this paper is as follows. In Section 2, we will compute the Swan conductor of the exterior square of an irreducible smooth representation τ of W_F with Swan conductor 1. Although in Theorem 1.3 we assumed that $\dim \tau$ is even (in fact only this case is needed to prove Theorem 1.2), we will also treat the case where $\dim \tau$ is odd. In Section 3, after recalling the formal degree conjecture, we deduce Theorem 1.2 from Theorem 1.3.

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Notation. Every representation is considered over $\mathbb{C}$, unless otherwise noted.

2. Exterior square of local Galois representations with Swan conductor 1

Let p be a prime number and F a finite extension of $\mathbb{Q}_p$. We write k for the residue field of F and q for the cardinality of k. We fix an algebraic closure $\overline{F}$ of F and put $\Gamma_F = \mathrm{Gal}(\overline{F}/F)$. Let W_F denote the Weil group of F, that is, the subgroup of Γ_F consisting of elements which induce integer powers of the Frobenius automorphism on the residue field $\overline{k}$ of $\overline{F}$. It is a locally compact group containing the inertia group I_F as an open subgroup.

Recall that the Galois group Γ_F is equipped with the upper numbering ramification filtration $\{\Gamma_F^j\}_{j\in\mathbb{R}_{\geq 0}}$, which is a descending filtration consisting of closed normal subgroups of Γ_F. The subgroup Γ_F^0 equals the inertia group I_F, and Γ_F^{0+} equals the wild inertia group P_F. Here Γ_F^{j+} denotes the closure of $\bigcup_{j'>j}\Gamma_F^{j'}$, as usual.

Let V be a finite-dimensional smooth representation of W_F. It is known that there exists a unique direct sum decomposition $V = \bigoplus_{j\in\mathbb{R}_{\geq 0}} V_j$ as a representation of P_F, called the break decomposition, such that

- $V_0 = V^{P_F}$, and
- $V_j^{\Gamma_F^j} = 0$ and $V_j^{\Gamma_F^{j+}} = V_j$ for each $j \in \mathbb{R}_{>0}$

(see [12, Proposition 1.1, Lemma 1.4]). The numbers j with $V_j \neq 0$ are called the breaks of V. The Swan conductor $\mathrm{Sw}\, V$ of V is defined by

$$\mathrm{Sw}\, V = \sum_{j\in\mathbb{R}_{\geq 0}} j \dim V_j.$$

Note that $\mathrm{Sw}\, V$ depends only on the restriction of V to P_F.

Let $n \geq 1$ be an integer. In this section, we prove the following result.

Theorem 2.1. *Let (τ, V) be an n-dimensional irreducible smooth representation of W_F such that $\operatorname{Sw}\tau = 1$. Then we have*

$$\operatorname{Sw}(\wedge^2\tau) = \begin{cases} m-1 & \text{if } n = 2m \text{ is even,} \\ m & \text{if } n = 2m+1 \text{ is odd.} \end{cases}$$

If $n = 1$, this theorem is obvious. Therefore, we assume $n \geq 2$ in the following. First we notice a simple lemma.

Lemma 2.2. *Let (τ, V) be as in Theorem 2.1. Then we have $V^{P_F} = 0$. Moreover, V has only one break $1/n$ and $V|_{I_F}$ is irreducible.*

Proof. Since P_F is a normal subgroup of W_F, V^{P_F} is a W_F-subrepresentation of V. The condition $\operatorname{Sw} V = 1$ implies that $V^{P_F} \neq V$. Therefore we have $V^{P_F} = 0$ by the irreducibility of V. By [12, Lemma 1.11], V has only one break $1/n$ and $V|_{I_F}$ is irreducible. □

To prove Theorem 2.1, we will pass to the equal characteristic case. Put $F' = k((T))$, which is an equal characteristic local field. By Deligne's result [5], we can prove the following:

Lemma 2.3. *Let (τ, V) be as in Theorem 2.1. Then there exists an n-dimensional irreducible smooth representation (τ', V') of $W_{F'}$ such that*

$$\operatorname{Sw}\tau' = \operatorname{Sw}\tau = 1, \qquad \operatorname{Sw}(\wedge^2\tau') = \operatorname{Sw}(\wedge^2\tau).$$

Proof. By [5, §3.5], there exists an isomorphism $\Gamma_F/\Gamma_F^1 \cong \Gamma_{F'}/\Gamma_{F'}^1$, which is canonical up to inner automorphisms. By construction, it preserves the upper numbering ramification filtrations of Γ_F and $\Gamma_{F'}$. Further, [5, Proposition 3.6.1] tells us that it induces an isomorphism $W_F/\Gamma_F^1 \cong W_{F'}/\Gamma_{F'}^1$. By using this isomorphism, we can construct a functor

- from the category of finite-dimensional smooth representations of W_F whose breaks are less than 1
- to the category of finite-dimensional smooth representations of $W_{F'}$ whose breaks are less than 1.

Clearly this functor maps irreducible representations to irreducible representations, commutes with exterior products, and preserves the Swan conductors.

By Lemma 2.2, τ has only one break $1/n$, which is less than 1. Therefore we can take (τ', V') as the image of (τ, V) under this functor. □

Now we use the Kloosterman sheaves introduced in [4, Sommes. trig.] and [12]. Let us recall their construction briefly. Take a prime number $\ell \neq p$. We fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ and identify them. Note that an irreducible finite-dimensional continuous representation of W_F over $\overline{\mathbb{Q}}_\ell$ is automatically smooth, hence can be identified with an irreducible smooth representation of W_F over $\mathbb{C}$. Let $\mathbb{P}^1$ denote the projective line over k, and put $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$, $\mathbb{G}_m = \mathbb{P}^1 \setminus \{0, \infty\}$. We consider the diagram

$$\mathbb{G}_m \xleftarrow{\text{mult}} \mathbb{G}_m^n \xrightarrow{\text{add}} \mathbb{A}^1,$$

where the maps mult and add are given by $(x_1, \ldots, x_n) \mapsto x_1 \cdots x_n$ and $(x_1, \ldots, x_n) \mapsto x_1 + \cdots + x_n$, respectively. We fix a non-trivial additive character $\psi\colon k \to \mathbb{C}^\times$, and write $\mathcal{L}_\psi$ for the Artin-Schreier sheaf on $\mathbb{A}^1$ corresponding to ψ. For multiplicative characters $\chi_1, \ldots, \chi_n\colon k^\times \to \mathbb{C}^\times$, we can construct the Kummer sheaf $\mathcal{K}_{\chi_1}, \ldots, \mathcal{K}_{\chi_n}$ on $\mathbb{G}_m$. We put

$$\mathrm{Kl}(\chi_1, \ldots, \chi_n) = R\,\mathrm{mult}_!\big((\mathcal{K}_{\chi_1} \boxtimes \cdots \boxtimes \mathcal{K}_{\chi_n}) \otimes \mathrm{add}^* \mathcal{L}_\psi\big)[-n+1].$$

If $\chi_1 = \cdots = \chi_n = 1$, we simply write Kl_n for $\mathrm{Kl}(\chi_1, \ldots, \chi_n)$. It is known that $\mathrm{Kl}(\chi_1, \ldots, \chi_n)$ is a smooth sheaf on $\mathbb{G}_m$ of rank n. Further, it enjoys the following properties:

- $\mathrm{Kl}(\chi_1, \ldots, \chi_n)_{\overline{0}}$, which is a representation of $\Gamma_{\mathrm{Frac}\,\widehat{\mathcal{O}}_{\mathbb{P}^1,0}}$, is tamely ramified.
- $\mathrm{Kl}(\chi_1, \ldots, \chi_n)_{\overline{\infty}}$, which is a representation of $\Gamma_{\mathrm{Frac}\,\widehat{\mathcal{O}}_{\mathbb{P}^1,\infty}}$, is totally wildly ramified with Swan conductor 1 (in particular it is irreducible by [12, Lemma 1.11]).

Here $\widehat{\mathcal{O}}_{\mathbb{P}^1,x}$ denotes the completion of the local ring $\mathcal{O}_{\mathbb{P}^1,x}$ at $x \in \mathbb{P}^1$. See [4, Sommes. trig., Théorème 7.8] and [12, Theorem 4.1.1] for detail.

In the following, we fix an isomorphism $k[[T]] \cong \widehat{\mathcal{O}}_{\mathbb{P}^1,\infty}$ and identify them. Then $\mathrm{Kl}(\chi_1, \ldots, \chi_n)_{\overline{\infty}}$ can be regarded as an n-dimensional irreducible smooth representation of $W_{F'}$.

Lemma 2.4. *Let τ' be an n-dimensional irreducible smooth representation of $W_{F'}$ with Swan conductor* 1. *Then we have* $\mathrm{Sw}(\wedge^2 \tau') = \mathrm{Sw}(\wedge^2 \mathrm{Kl}_{n,\overline{\infty}})$.

Proof. By replacing τ' by its unramified twist, we may assume that τ' extends to a smooth representation of $\Gamma_{F'}$. Note that τ' is defined over a finite extension E_λ of $\mathbb{Q}_\ell$ contained in $\overline{\mathbb{Q}}_\ell$. By the theorem of Katz-Gabber ([11, Theorem 1.5.6]), there exists a smooth E_λ-sheaf $\mathcal{F}$ on $\mathbb{G}_m$ of rank n such that

$\mathcal{F}_{\overline{0}}$ is tamely ramified and $\mathcal{F}_{\overline{\infty}}$ is isomorphic to τ'. By [12, Theorem 8.7.1] (see also the proof of [12, Corollary 8.7.2]), there exist a finite extension k' of k, an element $a' \in k'^{\times}$ and multiplicative characters $\chi'_1, \dots, \chi'_n \colon k'^{\times} \to \mathbb{C}^{\times}$ such that

$$\mathcal{F} \otimes_k k' \cong \iota_{a'}^* \operatorname{Kl}(\chi'_1, \dots, \chi'_n),$$

where $\iota_{a'} \colon \mathbb{G}_m \otimes_k k' \to \mathbb{G}_m \otimes_k k'$ is the multiplication by a' and $\operatorname{Kl}(\chi'_1, \dots, \chi'_n)$ is the Kloosterman sheaf over $\mathbb{G}_m \otimes_k k'$ with respect to the additive character $\psi \circ \operatorname{tr}_{k'/k}$ of k'. Since the base change from k to k' and the pull-back by $\iota_{a'}$ do not affect the Swan conductor at ∞, we conclude that $\operatorname{Sw}(\wedge^2 \tau') = \operatorname{Sw}(\wedge^2 \operatorname{Kl}(\chi'_1, \dots, \chi'_n)_{\overline{\infty}})$. On the other hand, by [12, Proposition 10.1], the restriction of $\operatorname{Kl}(\chi'_1, \dots, \chi'_n)_{\overline{\infty}}$ to $P_{k'((T))} = P_{F'}$ is independent of $\chi'_1, \dots, \chi'_n$. Hence we have

$$\begin{aligned}\operatorname{Sw}(\wedge^2 \operatorname{Kl}(\chi'_1, \dots, \chi'_n)_{\overline{\infty}}) &= \operatorname{Sw}(\wedge^2 \operatorname{Kl}(1', \dots, 1')_{\overline{\infty}}) = \operatorname{Sw}(\wedge^2 (\operatorname{Kl}_n \otimes_k k')_{\overline{\infty}}) \\ &= \operatorname{Sw}(\wedge^2 \operatorname{Kl}_{n,\overline{\infty}}),\end{aligned}$$

where $1'$ denotes the trivial character of $k'^{\times}$. This concludes the proof. □

By Lemma 2.4, we may focus on computing $\operatorname{Sw}(\wedge^2 \operatorname{Kl}_{n,\overline{\infty}})$. Since $\wedge^2 \operatorname{Kl}_{n,\overline{0}}$ is tame, the Grothendieck-Ogg-Shafarevich formula [15, Exposé X, Théorème 7.1] tells us that

$$\operatorname{Sw}(\wedge^2 \operatorname{Kl}_{n,\overline{\infty}}) = -\chi_c(\mathbb{G}_m, \wedge^2 \operatorname{Kl}_n) := -\sum_{i=0}^{2} (-1)^i \dim H_c^i(\mathbb{G}_m \otimes_k \overline{k}, \wedge^2 \operatorname{Kl}_n).$$

We shall determine the Euler characteristic $\chi_c(\mathbb{G}_m, \wedge^2 \operatorname{Kl}_n)$ by computing the L-function

$$L(\mathbb{G}_m, \wedge^2 \operatorname{Kl}_n, X) = \exp\left(\sum_{r=1}^{\infty}\left(\sum_{a \in k_r^{\times}} \operatorname{tr}(\operatorname{Frob}_a, \wedge^2 \operatorname{Kl}_{n,\overline{a}})\right)\frac{X^r}{r}\right),$$

where $k_r = \mathbb{F}_{q^r}$ denotes the degree r extension of $k = \mathbb{F}_q$.

Proposition 2.5. *We have*

$$L(\mathbb{G}_m, \wedge^2 \operatorname{Kl}_n, X) = \begin{cases} \dfrac{(1-qX)(1-q^3X)\cdots(1-q^{2m-1}X)}{1-q^{2m}X} & \\ \qquad \text{if } n = 2m \text{ is even,} & \\ (1-qX)(1-q^3X)\cdots(1-q^{2m-1}X) & \\ \qquad \text{if } n = 2m+1 \text{ is odd.} & \end{cases}$$

Proof. First note that

$$\sum_{a\in k_r^\times} \mathrm{tr}(\mathrm{Frob}_a, \wedge^2 \mathrm{Kl}_{n,\overline{a}}) = \frac{1}{2}\left(\sum_{a\in k_r^\times} \mathrm{tr}(\mathrm{Frob}_a, \mathrm{Kl}_{n,\overline{a}} \otimes \mathrm{Kl}_{n,\overline{a}}) - \sum_{a\in k_r^\times} \mathrm{tr}(\mathrm{Frob}_a^2, \mathrm{Kl}_{n,\overline{a}})\right).$$

By the proof of [12, Proposition 10.4.1], we have

$$\sum_{a\in k_r^\times} \mathrm{tr}(\mathrm{Frob}_a, \mathrm{Kl}_{n,\overline{a}} \otimes \mathrm{Kl}_{n,\overline{a}}) = S_r(n,1,1) = \begin{cases} -1 - q^r - q^{2r} - \cdots - q^{(n-1)r} + q^{nr} & \text{if } n \text{ is even or } p=2, \\ -1 - q^r - q^{2r} - \cdots - q^{(n-1)r} & \text{if } n \text{ is odd and } p \neq 2 \end{cases}$$

(note that "if $\alpha(-\alpha)^n\beta = 1$" in the end of p. 173 of [12] should be "if $\alpha(-\alpha)^n\beta = -1$"). On the other hand, by [12, (4.2.1.3), (4.2.1.5)], we have

$$\sum_{a\in k_r^\times} \mathrm{tr}(\mathrm{Frob}_a^2, \mathrm{Kl}_{n,\overline{a}}) = (-1)^{n-1}\frac{1}{q^{2r}-1} \sum_{\rho\in(k_{2r}^\times)^\vee} g(\psi\circ \mathrm{tr}_{k_{2r}/k}, \rho^{q^r-1})^n,$$

where $(k_{2r}^\times)^\vee$ denotes the set of characters of $k_{2r}^\times$ and

$$g(\psi\circ \mathrm{tr}_{k_{2r}/k}, \rho^{q^r-1}) = \sum_{a\in k_{2r}^\times} \psi(\mathrm{tr}_{k_{2r}/k}(a))\rho(a)^{q^r-1}$$

denotes the Gauss sum. Further, by [12, (4.2.1.13)], we have

$$g(\psi\circ \mathrm{tr}_{k_{2r}/k}, \rho^{q^r-1}) = \begin{cases} q^r\rho(-1) & \text{if } \rho^{q^r-1}\neq 1, \\ -1 & \text{if } \rho^{q^r-1} = 1. \end{cases}$$

Since

$$\#\{\rho\in(k_{2r}^\times)^\vee \mid \rho^{q^r-1} = 1\} = q^r - 1,$$

$$\#\{\rho\in(k_{2r}^\times)^\vee \mid \rho^{q^r-1} \neq 1, \rho(-1) = 1\} = \begin{cases} \dfrac{(q^r-1)^2}{2} & \text{if } p\neq 2, \\ q^{2r} - q^r & \text{if } p = 2, \end{cases}$$

$$\#\{\rho\in(k_{2r}^\times)^\vee \mid \rho^{q^r-1} \neq 1, \rho(-1) = -1\} = \begin{cases} \dfrac{q^{2r}-1}{2} & \text{if } p\neq 2, \\ 0 & \text{if } p = 2, \end{cases}$$

we have

$$\sum_{a\in k_r^\times} \mathrm{tr}(\mathrm{Frob}_a^2, \mathrm{Kl}_{n,\overline{a}})$$
$$= \begin{cases} \dfrac{1}{q^r+1}\Big(\dfrac{(-1)^{n-1}q^{nr}(q^r-1)}{2} - \dfrac{q^{nr}(q^r+1)}{2} - 1\Big) & \text{if } p \neq 2, \\ \dfrac{(-1)^{n-1}q^{(n+1)r}-1}{q^r+1} & \text{if } p = 2. \end{cases}$$

Now we assume that $n = 2m$ is even. Then we have

$$\sum_{a\in k_r^\times} \mathrm{tr}(\mathrm{Frob}_a^2, \mathrm{Kl}_{n,\overline{a}}) = -\frac{q^{(n+1)r}+1}{q^r+1} = -1 + q^r - q^{2r} + \cdots - q^{2mr},$$

hence

$$\sum_{a\in k_r^\times} \mathrm{tr}(\mathrm{Frob}_a, \wedge^2 \mathrm{Kl}_{n,\overline{a}})$$
$$= \frac{1}{2}\Big((-1 - q^r - q^{2r} - \cdots - q^{(2m-1)r} + q^{2mr})$$
$$- (-1 + q^r - q^{2r} + \cdots + q^{(2m-1)r} - q^{2mr})\Big)$$
$$= -q^r - q^{3r} - \cdots - q^{(2m-1)r} + q^{2mr}.$$

Therefore we conclude that

$$L(\mathbb{G}_m, \wedge^2 \mathrm{Kl}_n, X) = \frac{(1-qX)(1-q^3X)\cdots(1-q^{2m-1}X)}{1-q^{2m}X}.$$

Next we consider the case where $n = 2m+1$ is odd and $p \neq 2$. We have

$$\sum_{a\in k_r^\times} \mathrm{tr}(\mathrm{Frob}_a^2, \mathrm{Kl}_{n,\overline{a}}) = -\frac{q^{nr}+1}{q^r+1} = -1 + q^r - q^{2r} + \cdots - q^{2mr},$$

hence

$$\sum_{a\in k_r^\times} \mathrm{tr}(\mathrm{Frob}_a, \wedge^2 \mathrm{Kl}_{n,\overline{a}})$$
$$= \frac{1}{2}\Big((-1 - q^r - q^{2r} - \cdots - q^{(2m-1)r} - q^{2mr})$$
$$- (-1 + q^r - q^{2r} + \cdots + q^{(2m-1)r} - q^{2mr})\Big)$$
$$= -q^r - q^{3r} - \cdots - q^{(2m-1)r}.$$

Therefore we conclude that

$$L(\mathbb{G}_m, \wedge^2 \mathrm{Kl}_n, X) = (1-qX)(1-q^3X)\cdots(1-q^{2m-1}X).$$

Finally we assume that $n = 2m+1$ is odd and $p = 2$. Then we have

$$\sum_{a \in k_r^\times} \mathrm{tr}(\mathrm{Frob}_a^2, \mathrm{Kl}_{n,\overline{a}}) = \frac{q^{(n+1)r}-1}{q^r+1} = -1 + q^r - q^{2r} + \cdots + q^{(2m+1)r},$$

hence

$$\begin{aligned}
&\sum_{a \in k_r^\times} \mathrm{tr}(\mathrm{Frob}_a, \wedge^2 \mathrm{Kl}_{n,\overline{a}}) \\
&\quad = \frac{1}{2}\Big((-1 - q^r - q^{2r} - \cdots - q^{(2m-1)r} - q^{2mr} + q^{(2m+1)r}) \\
&\qquad - (-1 + q^r - q^{2r} + \cdots - q^{2mr} + q^{(2m+1)r})\Big) \\
&\quad = -q^r - q^{3r} - \cdots - q^{(2m-1)r}.
\end{aligned}$$

Therefore we conclude that

$$L(\mathbb{G}_m, \wedge^2 \mathrm{Kl}_n, X) = (1-qX)(1-q^3X)\cdots(1-q^{2m-1}X). \qquad \square$$

By the Grothendieck-Lefschetz trace formula, we have

$$L(\mathbb{G}_m, \wedge^2 \mathrm{Kl}_n, X) = \prod_{i=0}^{2} \det(1 - X\,\mathrm{Frob}; H_c^i(\mathbb{G}_m \otimes_k \overline{k}, \wedge^2 \mathrm{Kl}_n))^{(-1)^{i+1}}.$$

In particular we have $\deg L(\mathbb{G}_m, \wedge^2 \mathrm{Kl}_n, X) = -\chi_c(\mathbb{G}_m, \wedge^2 \mathrm{Kl}_n)$. Hence we obtain the following corollary:

Corollary 2.6. *We have*

$$\mathrm{Sw}(\wedge^2 \mathrm{Kl}_{n,\overline{\infty}}) = -\chi_c(\mathbb{G}_m, \wedge^2 \mathrm{Kl}_n) = \begin{cases} m-1 & \text{if } n = 2m \text{ is even,} \\ m & \text{if } n = 2m+1 \text{ is odd.} \end{cases}$$

Now Theorem 2.1 follows from Lemmas 2.3, 2.4 and Corollary 2.6.

Remark 2.7. The referee pointed out the following shorter proof of Corollary 2.6. By Lemma 2.2, all breaks of $\wedge^2 \mathrm{Kl}_{n,\overline{\infty}}$ and $\mathrm{Sym}^2 \mathrm{Kl}_{n,\overline{\infty}}$ are less than or equal to $1/n$. Therefore, by the integrality of the Swan conductor (see [11, Proposition 1.9, Remark 1.10]), we have

$$\mathrm{Sw}(\wedge^2 \mathrm{Kl}_{n,\overline{\infty}}) \leq \left\lfloor \frac{\dim \wedge^2 \mathrm{Kl}_{n,\overline{\infty}}}{n} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor,$$
$$\mathrm{Sw}(\mathrm{Sym}^2 \mathrm{Kl}_{n,\overline{\infty}}) \leq \left\lfloor \frac{\dim \mathrm{Sym}^2 \mathrm{Kl}_{n,\overline{\infty}}}{n} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor.$$

On the other hand, [11, Proposition 10.4.1] tells us that

$$\mathrm{Sw}(\mathrm{Kl}_{n,\overline{\infty}} \otimes \mathrm{Kl}_{n,\overline{\infty}}) = \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Hence we conclude that $\mathrm{Sw}(\wedge^2 \mathrm{Kl}_{n,\overline{\infty}}) = \lfloor \frac{n-1}{2} \rfloor$ and $\mathrm{Sw}(\mathrm{Sym}^2 \mathrm{Kl}_{n,\overline{\infty}}) = \lfloor \frac{n+1}{2} \rfloor$.

3. The formal degree conjecture for simple supercuspidal representations

In this section, we deduce the formal degree conjecture for simple supercuspidal representations of symplectic groups and quasi-split even special orthogonal groups from Theorem 2.1. Let us first recall the conjecture quickly in the case of symplectic groups. For more detail, see [8].

In the following, we put $G = \mathrm{Sp}_{2n}$ for an integer $n \geq 2$. Let us fix a non-trivial additive character $\psi\colon F \to \mathbb{C}^\times$. By using ψ, we normalize a Haar measure on $G(F)$ as in [8, §1]. Let (π, V) be an irreducible discrete series representation of $G(F)$. We fix a $G(F)$-invariant inner product $(\ ,\ )\colon V \times V \to \mathbb{C}$. Then, there exists a unique positive real number $\deg(\pi)$, the formal degree of π, satisfying

$$\int_{G(F)} (\pi(g)v, w)\overline{(\pi(g)v', w')}dg = \deg(\pi)^{-1}(v, v')\overline{(w, w')}$$

for every $v, w, v', w' \in V$. It depends on the Haar measure on $G(F)$, but is independent of the inner product $(\ ,\ )$. The formal degree conjecture predicts that $\deg(\pi)$ can be described by using the local Langlands correspondence.

By the local Langlands correspondence due to Arthur [1], discrete series representations of $G(F)$ are parametrized by pairs (ϕ, ρ), where

- $\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}(\mathbb{C}) = \mathrm{SO}_{2n+1}(\mathbb{C})$ is an L-parameter such that the centralizer group $S_\phi = \mathrm{Cent}_{\widehat{G}(\mathbb{C})}(\mathrm{Im}\,\phi)$ is finite,
- and ρ is an irreducible representation of $\pi_0(S_\phi) = S_\phi$.

The pair attached to a discrete series representation π is denoted by (ϕ_π, ρ_π).

Here is the statement of the formal degree conjecture for Sp_{2n}:

Conjecture 3.1 ([8, Conjecture 1.4]). For an irreducible discrete series representation π of $G(F)$, we have

$$\deg(\pi) = \frac{\dim \rho_\pi}{\# S_{\phi_\pi}} |\gamma(0, \mathrm{Ad} \circ \phi_\pi, \psi)|.$$

Here

- $\mathrm{Ad} \circ \phi_\pi$ is the composite of

$$W_F \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\phi_\pi} \widehat{G}(\mathbb{C}) \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathrm{Lie}\, \widehat{G}(\mathbb{C})),$$

- and

$$\gamma(s, \mathrm{Ad} \circ \phi_\pi, \psi) = \varepsilon(s, \mathrm{Ad} \circ \phi_\pi, \psi) \frac{L(1-s, \mathrm{Ad} \circ \phi_\pi)}{L(s, \mathrm{Ad} \circ \phi_\pi)}$$

  denotes the local γ-factor.

Remark 3.2. In the case $G = \mathrm{Sp}_{2n}$, S_{ϕ_π} is known to be an elementary 2-group, hence $\dim \rho_\pi = 1$.

The formal degree conjecture for a general connected reductive group is formulated similarly, but one needs slight modification if the group is not simply connected. See [8] for detail. The formal degree conjecture has been proved for general linear groups [8, Theorem 3.1], odd special orthogonal groups [9], and unitary groups [2], but it seems still open for many other groups, such as symplectic groups and even special orthogonal groups.

In this paper, we focus on a very special class of discrete series representations, called simple supercuspidal representations. They are constructed by using the compact induction from a compact open subgroup of $G(F)$. Here we choose a symplectic form given by the skew-symmetric $2n \times 2n$ matrix

$$\begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & \iddots & & \\ & 1 & & & \\ -1 & & & & \end{pmatrix}$$

to define $G = \mathrm{Sp}_{2n}$. We define a sequence of compact open subgroups $G(F) \supset I \rhd I^+ \rhd I^{++}$ as follows:

$$I = \begin{pmatrix} \mathcal{O}_F & & \mathcal{O}_F \\ & \ddots & \\ \mathfrak{p}_F & & \mathcal{O}_F \end{pmatrix}, \qquad I^+ = \begin{pmatrix} 1+\mathfrak{p}_F & & \mathcal{O}_F \\ & \ddots & \\ \mathfrak{p}_F & & 1+\mathfrak{p}_F \end{pmatrix},$$

$$I^{++} = \begin{pmatrix} 1+\mathfrak{p}_F & \mathfrak{p}_F & & \mathcal{O}_F \\ & \ddots & \ddots & \\ & \mathfrak{p}_F & \ddots & \mathfrak{p}_F \\ \mathfrak{p}_F^2 & & & 1+\mathfrak{p}_F \end{pmatrix}.$$

Here $\mathcal{O}_F$ denotes the ring of integers of F, and $\mathfrak{p}_F$ the maximal ideal of $\mathcal{O}_F$. If we fix a uniformizer ϖ of $\mathcal{O}_F$, then we have an isomorphism

$$I^+/I^{++} \xrightarrow{\cong} k^{n+1};$$
$$(a_{ij}) \mapsto (a_{12} \bmod \mathfrak{p}_F, \dots, a_{n,n+1} \bmod \mathfrak{p}_F, \varpi^{-1}a_{2n,1} \bmod \mathfrak{p}_F).$$

A character of $I^+/I^{++} \cong k^{n+1}$ is said to be affine generic if it is non-trivial on each factor of k^{n+1}. Let χ be a character of $\pm I^+$ such that $\chi|_{I^{++}}$ is trivial and $\chi|_{I^+}$ induces an affine generic character of I^+/I^{++}. Then, the compact induction $\text{c-Ind}_{\pm I^+}^{G(F)} \chi$ is known to be irreducible supercuspidal. Representations obtained in this way are called simple supercuspidal representations.

The parameter (ϕ_π, ρ_π) attached to a simple supercuspidal representation π is investigated by Oi in detail.

Theorem 3.3 ([13, Corollary 5.13, Theorem 7.18]). *Assume $p \neq 2$. Let ι denote the embedding $\widehat{G}(\mathbb{C}) = \mathrm{SO}_{2n+1}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n+1}(\mathbb{C})$. For a simple supercuspidal representation π of $G(F)$, we have the following:*

- *$\iota \circ \phi_\pi = \tau \oplus \omega$, where τ is an irreducible $2n$-dimensional irreducible representation of W_F with Swan conductor 1 and ω is a quadratic character of W_F. Furthermore, τ is orthogonal, that is, there exists a W_F-invariant non-degenerate symmetric bilinear form $\tau \times \tau \to \mathbb{C}$.*
- *$\#S_{\phi_\pi} = 2$.*

Strictly speaking, [13, Theorem 7.18] claims that $\iota \circ \phi_\pi = \tau \oplus \det \circ \tau$, where τ is the Langlands parameter of a simple supercuspidal representation of $\mathrm{GL}_{2n}(F)$. However, it is well-known that such τ is irreducible and has Swan conductor 1; see [3, §2] for example. Further, since τ is orthogonal (see [13, Corollary 5.13]), the character $\omega = \det \circ \tau$ is quadratic.

By using Theorem 3.3, Oi obtained a partial result on the formal degree conjecture.

Theorem 3.4 ([13, Theorem 9.3]). *We write $2n = p^e n'$ with $p \nmid n'$. Assume $p \neq 2$ and either $p \nmid 2n$ or $n' \mid p-1$. Then, Conjecture 3.1 holds for simple supercuspidal representations of $G(F)$.*

In the case $p \mid 2n$ and $n' \mid p-1$, Oi used an explicit description of τ in Theorem 3.3 due to Imai and Tsushima [10], which is extremely complicated. It involves 4 field extensions $F^{\mathrm{ur}} \subset E \subset T \subset M \subset N$ of the maximal unramified extension F^{ur} of F. The extension N/E is always Galois, but the extension N/F^{ur} is not necessarily Galois. The condition $n' \mid p-1$ ensures that N/F^{ur} is a Galois extension, which makes computations much simpler.

The following is our main theorem for symplectic groups (for quasi-split even special orthogonal groups, see Remark 3.8).

Theorem 3.5. *Assume $p \neq 2$. Then, Conjecture 3.1 holds for simple supercuspidal representations of $G(F)$.*

In fact, Theorem 3.5 can be deduced from Theorem 2.1 exactly in the same way as Oi did in [13, §9.3]. We include some of his arguments for reader's convenience. In the following, we assume that $p \neq 2$ and let π be a simple supercuspidal representation of $G(F)$. Let τ and ω be as in Theorem 3.3.

Lemma 3.6. 1) *We have $L(s, \mathrm{Ad} \circ \phi_\pi) = 1$.*

2) *We have $\mathrm{Ar}(\mathrm{Ad} \circ \phi_\pi) = 2n^2 + 2n$, where Ar denotes the Artin conductor.*

Proof. First of all, note that $\mathrm{Ad} \circ \phi_\pi = \wedge^2(\tau \oplus \omega) = \wedge^2\tau \oplus \tau \otimes \omega$.

1) It suffices to show that $(\mathrm{Ad} \circ \phi_\pi)^{I_F} = 0$. Since $p \neq 2$, the quadratic character ω is tamely ramified. Therefore $(\tau \otimes \omega)^{P_F} = \tau^{P_F} \otimes \omega = 0$ by Lemma 2.2. Hence it suffices to prove that $(\wedge^2\tau)^{I_F} = 0$.

By Lemma 2.2, $\tau|_{I_F}$ is irreducible, hence $\dim(\tau \otimes \tau^\vee)^{I_F} = 1$. Since τ is orthogonal by Theorem 3.3, we have

$$1 = \dim(\tau \otimes \tau^\vee)^{I_F} = \dim(\tau \otimes \tau)^{I_F} = \dim(\mathrm{Sym}^2 \tau)^{I_F} \oplus \dim(\wedge^2\tau)^{I_F}$$

and $(\mathrm{Sym}^2 \tau)^{I_F} = (\mathrm{Sym}^2 \tau)_{I_F} \neq 0$. Therefore we obtain $(\wedge^2\tau)^{I_F} = 0$, as desired.

2) Since ω is tame, we have

$$\mathrm{Sw}(\mathrm{Ad}\circ\phi_\pi) = \mathrm{Sw}(\wedge^2\tau) + \mathrm{Sw}(\tau\otimes\omega) = \mathrm{Sw}(\wedge^2\tau) + \mathrm{Sw}(\tau) = n$$

by Theorem 2.1. On the other hand, by the proof of (1), we have

$$\dim(\mathrm{Ad}\circ\phi_\pi)/(\mathrm{Ad}\circ\phi_\pi)^{I_F} = \dim(\mathrm{Ad}\circ\phi_\pi) = \frac{2n(2n+1)}{2} = 2n^2+n.$$

Therefore we conclude that

$$\mathrm{Ar}(\mathrm{Ad}\circ\phi_\pi) = \mathrm{Sw}(\mathrm{Ad}\circ\phi_\pi) + \dim(\mathrm{Ad}\circ\phi_\pi)/(\mathrm{Ad}\circ\phi_\pi)^{I_F} = 2n^2+2n. \qquad \square$$

Remark 3.7. In the proof of Lemma 3.6, we obtained two equalities $(\mathrm{Ad}\circ\phi_\pi)^{I_F}=0$ and $\mathrm{Sw}(\mathrm{Ad}\circ\phi_\pi)=n$. These mean that $\phi_\pi\colon W_F\to\widehat{G}(\mathbb{C})$ is a simple wild parameter in the sense of [7, §6], which was predicted in [7, §9.5].

Proof of Theorem 3.5. Let $\mathbf{St}$ denote the Steinberg representation of $G(F)$. We may choose ψ so that the following equalities hold:

$$\frac{\deg(\pi)}{\deg(\mathbf{St})} = \frac{q^{n^2+n}}{2\gamma(0,\mathrm{Ad}\circ\phi_{\mathbf{St}},\psi)},$$
$$|\varepsilon(0,\mathrm{Ad}\circ\phi_\pi,\psi)| = q^{\frac{1}{2}\mathrm{Ar}(\mathrm{Ad}\circ\phi_\pi)}.$$

See [7, (72)] for the first equality, and [7, (10) and Proposition 2.3] for the second. Together with Lemma 3.6, we obtain $|\gamma(0,\mathrm{Ad}\circ\phi_\pi,\psi)| = q^{n^2+n}$ and

$$\deg(\pi) = \left|\frac{\deg(\mathbf{St})\gamma(0,\mathrm{Ad}\circ\phi_\pi,\psi)}{2\gamma(0,\mathrm{Ad}\circ\phi_{\mathbf{St}},\psi)}\right|.$$

On the other hand, by [8, §3.3], the formal degree conjecture for $\mathbf{St}$ is known:

$$\deg(\mathbf{St}) = |\gamma(0,\mathrm{Ad}\circ\phi_{\mathbf{St}},\psi)|.$$

Hence we have

$$\deg(\pi) = \frac{1}{2}|\gamma(0,\mathrm{Ad}\circ\phi_\pi,\psi)| = \frac{\dim\rho_\pi}{\#S_{\phi_\pi}}|\gamma(0,\mathrm{Ad}\circ\phi_\pi,\psi)|,$$

as desired (recall that $\dim\rho_\pi=1$ by Remark 3.2 and $\#S_{\phi_\pi}=2$ by Theorem 3.3). $\square$

Remark 3.8. As remarked in [13, §9], by using the results in [6], we can deduce from Theorem 3.5 the formal degree conjecture for simple supercuspidal representations of quasi-split even special orthogonal groups, under the assumption $p \neq 2$. Alternatively, by the same argument as above, we can also conclude it directly from Theorem 2.1 and Oi's description [13, Theorem 7.17, Theorem 8.8] of the L-parameters of simple supercuspidal representations of quasi-split even special orthogonal groups.

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Graduate School of Mathematical Sciences
The University of Tokyo
3–8–1 Komaba, Meguro-ku, Tokyo, 153–8914, Japan
E-mail address: `mieda@ms.u-tokyo.ac.jp`

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