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A new proof of Bowers-Stephenson conjecture

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Inversive distance circle packing on surfaces was introduced by Bowers-Stephenson [7] as a generalization of Thurston's circle packing and conjectured to be rigid. The infinitesimal and global rigidity of circle packing with nonnegative inversive distance were proved by Guo [19] and Luo [25] respectively. The author [34] proved the global rigidity of circle packing with inversive distance in $(-1, +\infty)$. In this paper, we give a new variational proof of the Bowers-Stephenson conjecture for inversive distance in $(-1, +\infty)$, which simplifies the existing proof in [19, 25, 34] and could be generalized to three dimensional case. The new proof also reveals more properties of the inversive distance circle packing on surfaces.

1. Introduction

In the study of hyperbolic structure on 3-dimensional manifolds, Thurston [31] introduced circle packing with non-obtuse intersection angles on surfaces, which generalized the circle packing studied by Andreev [1, 2] and Koebe [24]. Thurston proved the Andreev-Thurston theorem, which includes the existence part and the rigidity part. The Andreev-Thurston rigidity theorem states that the circle packing is globally determined by the discrete curvature on the triangulated surface, which is defined to be 2π less the cone angle at a vertex. Recently, Andreev-Thurston theorem was generalized by Zhou [37] to the case of obtuse angles. For a proof of Andreev-Thurston Theorem, see [8, 9, 22, 27, 28, 31, 37].

Inversive distance circle packing was introduced by Bowers-Stephenson [7] as a generalization of Thurston's circle packing on surfaces, allowing the adjacent circles to separate. Suppose M is a surface with a triangulation $\mathcal{T} = \{V, E, F\}$, where V, E, F are the sets of vertices, edges and faces respectively. We use i, $\{ij\}$, $\{ijk\}$ to denote a vertex, an edge and a face respectively, where i, j, k are natural numbers. A weight on the triangulated surface is a map $I : E \to (-1, +\infty)$. We use I_{ij} to denote $I(\{ij\})$ for simplicity. A weighted triangulated surface is denoted by $(M, \mathcal{T}, I)$ in this paper.

Definition 1.1. Suppose $(M, \mathcal{T}, I)$ is a weighted triangulated surface. An inversive distance circle packing metric is a map $r : V \to (0, +\infty)$ such that

(1): The edge length l_{ij} of $\{ij\}$ is

$$l_{ij} = \sqrt{r_i^2 + r_j^2 + 2r_ir_jI_{ij}} \tag{1.1}$$

for Euclidean background geometry and

$$l_{ij} = \cosh^{-1}(\cosh r_i \cosh r_j + I_{ij} \sinh r_i \sinh r_j) \tag{1.2}$$

for hyperbolic background geometry;

(2): With the assignment of edge lengths l_{ij}, l_{jk}, l_{ik} by (1.1) (respectively (1.2)), the triangle $\{ijk\}$ could be embedded in 2-dimensional Euclidean space $\mathbb{E}^2$ (respectively 2-dimensional hyperbolic space $\mathbb{H}^2$) as a nondegenerate triangle.

The condition (2) in Definition 1.1 is called a nondegenerate condition. If two circles C_i and C_j with radii r_i and r_j respectively are put in the plane with l_{ij} as the distance of the centers of C_i, C_j, then the inversive distance of the two circles is I_{ij}. If $I_{ij} \in [0, 1]$ for any edge $\{ij\} \in E$, the inversive distance circle packing is reduced to Thurston's circle packing [31]. If $I_{ij} \in (-1, 1]$ for any edge $\{ij\} \in E$, the inversive distance circle packing is reduced to the circle packing studied by Zhou [37]. If $I_{ij} \in [0, +\infty)$ for any edge $\{ij\} \in E$, the inversive distance circle packing was studied by Guo [19] and Luo [25]. For more information on inversive distance circle packing, see [6, 7, 19, 28].

Bowers-Stephenson [7] conjectured that the inversive distance circle packing on surfaces is rigid. The infinitesimal rigidity and global rigidity were proved by Guo [19] and Luo [25] respectively for circle packings with nonnegative inversive distance, which generalize the Andreev-Thurston rigidity theorem. Following the proof in [19, 25], the author [34] proved the rigidity of circle packing for inversive distance in $(-1, +\infty)$ recently.

Theorem 1.1 ([19, 25, 34]). *Suppose $(M, \mathcal{T}, I)$ is a weighted triangulated surface with the weight $I : E \to (-1, +\infty)$ satisfying the structure condition*

$$I_{ij} + I_{ik}I_{jk} \geq 0, I_{ik} + I_{ij}I_{jk} \geq 0, I_{jk} + I_{ij}I_{ik} \geq 0, \quad \forall \{ijk\} \in F. \tag{1.3}$$

Then the inversive distance circle packing metric on $(M, \mathcal{T}, I)$ *is uniquely determined by the discrete curvature (up to scaling for the Euclidean background geometry).*

The basic strategy in [19, 25, 34] to prove Theorem 1.1 is to apply the variational principle introduced by de Verdière [9] to inversive distance circle packing, which could be separated into the following three steps. The first step is to prove the admissible space of inversive distance packing metrics for a single triangle is simply connected; The second step is to prove that the Jacobian matrix of the inner angles of a triangle in terms of some appropriate parametrization of the circle radii is symmetric and negative semi-definite (or negative definite), which ensures the definition of a locally concave function; The third step is to extend the locally concave function to be a globally defined concave function, which has been systematically studied in [3, 25], and use this concave function to prove the rigidity.

In this paper, we give a new proof of Theorem 1.1. In the first step, the proof in [19, 25, 34] for simply connectivity of the admissible space for a single triangle is based on the triangle inequalities, which can not be generalized to three or higher dimensional cases. In this paper, we give a new proof of the simply connectivity using the Cayley-Menger determinant, which could be used to characterize the nondegeneracy of a simplex in any dimension. This proof enables us to prove the simply connectivity of admissible space of Thurston's sphere packing metrics for a single tetrahedron in three dimension [20, 21]. In the second step, the arguments in [19, 34] to prove the negative semi-definiteness (or negative definiteness) of the Jacobian matrix of inner angles in a triangle is based on a lengthy estimate of the eigenvalues of the matrix under the nondegenerate condition. In this paper, we give a new and short proof of the negative definiteness involving only the rank of the Jacobian matrix and connectivity of the parameterized admissible space for a triangle, which greatly simplifies the arguments in [19, 34]. The third step is the same as that in [19, 25, 34].

In this paper, we only study the rigidity of inversive distance circle packing in Euclidean and hyperbolic background geometry. For the rigidity of inversive distance circle packing in spherical background geometry, see [4, 5, 26]. Deformation of inversive distance circle packing metrics on surfaces by discrete curvature flows was also studied recently, see [11–13, 15]. Inversive distance circle packing has lots of practical applications, see [6, 23, 35, 36].

This paper is organized as follows. In Section 2, we give a new proof of Theorem 1.1 in Euclidean background geometry. In Section 3, we give a new proof of Theorem 1.1 in hyperbolic background geometry.

2. Rigidity of Euclidean inversive distance circle packing

2.1. Admissible space of Euclidean inversive distance circle packing metrics for a single triangle

Suppose $\sigma = \{123\} \in F$ is a topological triangle in $(M, \mathcal{T}, I)$. The corresponding edge set of the triangle is denoted by $E_\sigma = \{\{12\}, \{13\}, \{23\}\}$. We denote η as the restriction of the weight $I : E \to (-1, +\infty)$ on the edge set E_σ. Given a weight η on the edge set E_σ satisfying the structure condition (1.3), the admissible space $\Omega^E_{123}(\eta)$ of Euclidean inversive distance circle packing metrics for the triangle $\{123\}$ is defined to be the set of Euclidean inversive distance circle packing metrics $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$ such that the triangle $\{123\}$ with the edge lengths given by (1.1) exists in 2-dimensional Euclidean space $\mathbb{E}^2$.

To simplify the notations, we set

$$I_i = I_{jk},\ \{i, j, k\} = \{1, 2, 3\}.$$

Then $l_{ij}^2 = r_i^2 + r_j^2 + 2r_i r_j I_k$. Set

$$G_0(l) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & l_{12}^2 & l_{13}^2 \\ 1 & l_{12}^2 & 0 & l_{23}^2 \\ 1 & l_{13}^2 & l_{23}^2 & 0 \end{pmatrix}$$

to be the Cayley-Menger 4×4-matrix. Recall the following result characterizing the nondegeneracy of a Euclidean triangle $\{123\}$ with positive edge lengths l_{12}, l_{13}, l_{23}.

Lemma 2.1 ([29], Proposition 2.4.1). *A triangle with positive edge lengths l_{12}, l_{13}, l_{23} exists in $\mathbb{E}^2$ if and only if* $\det G_0(l) < 0$.

Remark 2.1. By direct calculations, we have

$$\det G_0(l) = -(l_{12} + l_{13} + l_{23})(l_{12} + l_{13} - l_{23})(l_{12} + l_{23} - l_{13})(l_{13} + l_{23} - l_{12}),$$

which implies $\det G_0(l) < 0$ is equivalent to the triangle inequalities. This was also observed in [19]. The advantage of using $\det G_0(l) < 0$ to characterize the nondegeneracy is that we just need one inequality $\det G_0(l) < 0$ instead of three triangle inequalities. Furthermore, this characterization of nondegeneracy of simplex could be generalized to high dimensional case [29].

Submitting (1.1) into $\det G_0(l)$, we have

$$\begin{aligned}\det G_0(l) =& -4\Big[r_1^2r_2^2(1-I_3^2)+r_1^2r_3^2(1-I_2^2)+r_2^2r_3^2(1-I_1^2)\\ &+2r_1^2r_2r_3(I_1+I_2I_3)+2r_1r_2^2r_3(I_2+I_1I_3)+2r_1r_2r_3^2(I_3+I_1I_2)\Big].\end{aligned}$$

Set

$$\gamma_i = I_i + I_jI_k,\ \kappa_i = r_i^{-1}.$$

Note that $\gamma_i \geq 0$ due to the structure condition (1.3). Denote

$$Q = \kappa_1^2(1-I_1^2)+\kappa_2^2(1-I_2^2)+\kappa_3^2(1-I_3^2)+2\kappa_1\kappa_2\gamma_3+2\kappa_1\kappa_3\gamma_2+2\kappa_2\kappa_3\gamma_1,$$

then we have

$$\det G_0(l) = -4r_1^2r_2^2r_3^2Q.$$

Lemma 2.2 ([19, 34, 37]). *A Euclidean triangle* $\{123\}$ *with edge lengths* l_{12}, l_{13}, l_{23} *given by* (1.1) *exists in* $\mathbb{E}^2$ *if and only if* $Q > 0$.

Set

$$\begin{aligned}h_1 =&\kappa_1(1-I_1^2)+\kappa_2\gamma_3+\kappa_3\gamma_2,\\ h_2 =&\kappa_2(1-I_2^2)+\kappa_1\gamma_3+\kappa_3\gamma_1,\\ h_3 =&\kappa_3(1-I_3^2)+\kappa_1\gamma_2+\kappa_2\gamma_1.\end{aligned} \tag{2.1}$$

By Lemma 2.2, $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$ generates a degenerate Euclidean triangle if and only if

$$Q = \kappa_1h_1+\kappa_2h_2+\kappa_3h_3 \leq 0. \tag{2.2}$$

Remark 2.2. For a nondegenerate inversive distance circle packing metric of the triangle $\{123\}$, there exists a geometric center C_{123} of the triangle $\{123\}$ ([17] Proposition 4), which has the same circle power to the circles attached to the vertices $\{1, 2, 3\}$. Here the circle power of a point x to a circle with center p and radius r is defined to be $\pi_p(x) = d^2(x, p) - r^2$, where $d(x, p)$ is the Euclidean distance between x and p. h_i in (2.1) is a positive multiplication of the signed distance $h_{jk,i}$ of C_{123} to the edge $\{jk\}$, which is defined to be positive if C_{123} is on the same side of the line determined by $\{jk\}$ as the triangle $\{123\}$ and negative otherwise (or zero if C_{123} is on the line). By direct calculations, we have

$$h_{ij,k} = \frac{r_1^2 r_2^2 r_3^2}{2l_{ij}A_{123}}[\kappa_k^2(1-I_k^2)+\kappa_j\kappa_k\gamma_i+\kappa_i\kappa_k\gamma_j] = \frac{r_1^2 r_2^2 r_3^2}{2l_{ij}A_{123}}\kappa_k h_k, \tag{2.3}$$

where A_{123} is the area of the triangle $\{123\}$. Note that h_1, h_2, h_3 are well-defined for any $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$, while $h_{12,3}, h_{13,2}, h_{23,1}$ are defined for non-degenerate inversive distance circle packing metrics. Refer to [16–18, 30] for more information on the geometric center of triangles.

Suppose $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$ is a degenerate inversive distance circle packing metric, then one of the following two cases happens by (2.2).

(1): At least one of h_1, h_2, h_3 is zero;

(2): None of h_1, h_2, h_3 is zero.

We will prove that case (1) never happens. Furthermore, we will prove that only one of h_i, h_j, h_k is negative and the others are positive in case (2).

Note that $Q \leq 0$ is equivalent to the following quadratic inequality of κ_i

$$A_i\kappa_i^2 + B_i\kappa_i + C_i \geq 0, \tag{2.4}$$

where

$$\begin{aligned} A_i =& I_i^2 - 1,\\ B_i =& -2(\kappa_j\gamma_k + \kappa_k\gamma_j) \leq 0,\\ C_i =& \kappa_j^2(I_j^2-1) + \kappa_k^2(I_k^2-1) - 2\kappa_j\kappa_k\gamma_i. \end{aligned} \tag{2.5}$$

with $\{i,j,k\} = \{1,2,3\}$. By direct calculations, the determinant $\Delta_i = B_i^2 - 4A_iC_i$ is given by

$$\Delta_i = 4(I_1^2 + I_2^2 + I_3^2 + 2I_1I_2I_3 - 1)(\kappa_j^2 + \kappa_k^2 + 2\kappa_j\kappa_k I_i). \tag{2.6}$$

Lemma 2.3. *Suppose $\eta = (I_1, I_2, I_3)$ is a weight on the edges of a triangle $\{123\}$ satisfying the structure condition* (1.3)*. If $I_i > 1$, then $\Delta_i > 0$.*

Proof. It is straight forward to check that $\kappa_j^2 + \kappa_k^2 + 2\kappa_j\kappa_k I_i > 0$. We just need to check $I_1^2 + I_2^2 + I_3^2 + 2I_1I_2I_3 - 1 > 0$ by (2.6). If $I_j \geq 0, I_k \geq 0$, then

$$I_1^2 + I_2^2 + I_3^2 + 2I_1I_2I_3 - 1 > I_j^2 + I_k^2 + 2I_1I_2I_3 \geq 0.$$

If $I_j < 0$, then $I_j \in (-1, 0)$ and

$$I_1^2 + I_2^2 + I_3^2 + 2I_1I_2I_3 - 1 = (I_k + I_iI_j)^2 + (1 - I_j^2)(I_i^2 - 1) > 0.$$

Similar argument applies for the case $I_k < 0$. Therefore, under the structure condition (1.3) and $I_i > 1$, we have $I_1^2 + I_2^2 + I_3^2 + 2I_1I_2I_3 - 1 > 0$. □

Now we can prove that the case (1) never happens.

Lemma 2.4. *Suppose $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$ is a degenerate Euclidean inversive distance circle packing metric for a triangle $\{123\}$ with a weigh $\eta : E_\sigma \to (-1, +\infty)$ satisfying the structure condition* (1.3)*, then none of h_1, h_2, h_3 is zero.*

Proof. We prove the lemma by contradiction. By the degenerate condition (2.2), if one of h_1, h_2, h_3 is zero, then there is another one of h_1, h_2, h_3 that is nonpositive. Without loss of generality, we assume $h_1 = 0, h_2 \leq 0$ for a degenerate Euclidean inversive distance circle packing metric $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$.

By $h_1 = 0$, we have $\kappa_1(I_1^2 - 1) = \kappa_2\gamma_3 + \kappa_3\gamma_2$, which implies $I_1 \geq 1$ by the structure condition (1.3). If $I_1 > 1$, we can rewrite $Q \leq 0$ as a quadratic inequality in κ_1:

$$A_1\kappa_1^2 + B_1\kappa_1 + C_1 \geq 0, \tag{2.7}$$

where A_1, B_1, C_1 are given by (2.5) with $A_1 = I_1^2 - 1 > 0$. By Lemma 2.3, we have $\Delta_1 > 0$. Then (2.7) implies

$$\kappa_1 \geq \frac{-B_1 + \sqrt{\Delta_1}}{2A_1} \quad \text{or} \quad \kappa_1 \leq \frac{-B_1 - \sqrt{\Delta_1}}{2A_1},$$

which is equivalent to

$$-2h_1 = 2A_1\kappa_1 + B_1 \geq \sqrt{\Delta_1} > 0 \quad \text{or} \quad -2h_1 = 2A_1\kappa_1 + B_1 \leq -\sqrt{\Delta_1} < 0.$$

This contradicts $h_1 = 0$. Therefore, $I_1 = 1$. By $h_1 = 0$ again, we have $\gamma_2 = \gamma_3 = 0$, which implies $I_2 + I_3 = 0$.

By $h_2 = \kappa_2(1 - I_2^2) + \kappa_1\gamma_3 + \kappa_3\gamma_1 \leq 0$, we have $I_2 \geq 1$, which implies $I_3 = -I_2 \leq -1$. This is impossible. □

By Lemma 2.4, if $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$ is a degenerate Euclidean inversive distance circle packing metric for a triangle $\{123\}$, at least one of h_1, h_2, h_3 is negative and the others are nonzero. Furthermore, we have the following result.

Lemma 2.5. *Suppose* $\{123\}$ *is a triangle with a weigh* $\eta : E_\sigma \to (-1, +\infty)$ *satisfying the structure condition* (1.3) *and* $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$. *Then there exists no subset* $\{i, j\} \subset \{1, 2, 3\}$ *such that* $h_i < 0$ *and* $h_j < 0$.

Proof. Without loss of generality, we consider the case $h_1 < 0, h_2 < 0$. By $h_1 < 0, h_2 < 0$, we have

$$(I_1^2 - 1)\kappa_1 > \kappa_2\gamma_3 + \kappa_3\gamma_2, \quad (I_2^2 - 1)\kappa_2 > \kappa_1\gamma_3 + \kappa_3\gamma_1,$$

which implies $I_1 > 1, I_2 > 1$ and $(I_1^2 - 1)(I_2^2 - 1) > \gamma_3^2$ by the structure condition (1.3). Note that

$$(I_1^2 - 1)(I_2^2 - 1) - \gamma_3^2 = -I_1^2 - I_2^2 - I_3^2 - 2I_1I_2I_3 + 1 < 0$$

by the proof of Lemma 2.3. This is a contradiction. □

Remark 2.3. No matter $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$ is a nondegenerate or degenerate inversive distance circle packing metric for the triangle $\{123\}$, Lemma 2.5 is valid. For nondegenerate inversive distance circle packing metrics, Lemma 2.5 implies that the geometric center does not lie in some special regions in the plane relative to the triangle.

Now we can prove the main result of this subsection.

Proposition 2.1 ([19, 34]). *Suppose* $\sigma = \{123\} \in F$ *is a triangle in* $(M, \mathcal{T})$ *with a weight* $\eta : E_\sigma \to (-1, +\infty)$ *satisfying the structure condition* (1.3). *Then the admissible space* $\Omega^E_{123}(\eta)$ *of Euclidean inversive distance circle packing metrics* $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$ *is nonempty and simply connected. Furthermore, the set of degenerate inversive distance circle packing metric is a disjoint union* $\cup_{i\in P} V_i$, *where* $P = \{i \in \{1, 2, 3\} | I_i > 1\}$ *and*

$$V_i = \left\{ (r_1, r_2, r_3) \in \mathbb{R}^3_{>0} | \kappa_i \geq \frac{-B_i + \sqrt{\Delta_i}}{2A_i} \right\}$$

is bounded by an analytic graph on $\mathbb{R}^2_{>0}$.

Proof. Suppose $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$ is a degenerate inversive distance circle packing metric for the triangle $\{123\}$, then we have $Q = \kappa_1 h_1 + \kappa_2 h_2 + \kappa_3 h_3 \leq 0$. By Lemma 2.4 and Lemma 2.5, one of h_1, h_2, h_3 is negative and the others are positive.

Suppose $h_i < 0$ and $h_j > 0, h_k > 0$ with $\{i, j, k\} = \{1, 2, 3\}$. Then we have $I_i > 1$ by $h_i < 0$. Rewrite $Q \leq 0$ as a quadratic inequality in κ_i: $A_i\kappa_i^2 +$

$B_i\kappa_i + C_i \geq 0$, where A_i, B_i, C_i are defined by (2.5) with $A_i = I_i^2 - 1 > 0$. By Lemma 2.3, we have $\Delta_i > 0$. Then $A_i\kappa_i^2 + B_i\kappa_i + C_i \geq 0$ implies

$$\kappa_i \geq \frac{-B_i + \sqrt{\Delta_i}}{2A_i} \quad \text{or} \quad \kappa_i \leq \frac{-B_i - \sqrt{\Delta_i}}{2A_i}.$$

Note that $h_i < 0$ is equivalent to $\kappa_i > \frac{-B_i}{2A_i}$. This implies $\kappa_i \geq \frac{-B_i+\sqrt{\Delta_i}}{2A_i}$. Therefore, the set of degenerate inversive distance circle packing metrics is contained in $\cup_{i\in P}V_i$.

On the other hand, if $I_i > 1$, then for any $(r_1, r_2, r_3) \in V_i$, we have $Q \leq 0$, which implies any element $(r_1, r_2, r_3) \in V_i$ is a degenerate inversive distance circle packing metric. Therefore, $\Omega_{123}^E(\eta) = \mathbb{R}^3_{>0} \setminus \cup_{i\in P}V_i$, where $P = \{i \in \{1,2,3\}|I_i > 1\}$.

For any $(r_1, r_2, r_3) \in V_i$, we have $\kappa_i > \frac{-B_i}{2A_i}$, which is equivalent to $h_i < 0$. This implies $V_i \cap V_j = \emptyset$ if $I_i > 1$ and $I_j > 1$ by Lemma 2.5.

Note that V_i is bounded by an analytic graph on $\mathbb{R}^2_{>0}$. In fact

$$V_i = \left\{(r_1, r_2, r_3) \in \mathbb{R}^3_{>0} | r_i \leq \frac{2A_i}{-B_i + \sqrt{\Delta_i}}\right\}.$$

This implies $\Omega_{123}^E(\eta) = \mathbb{R}^3_{>0} \setminus \cup_{i\in P}V_i$ is homotopy equivalent to $\mathbb{R}^3_{>0}$. Therefore, $\Omega_{123}^E(\eta)$ is simply connected. $\square$

Remark 2.4. Suppose $\sigma = \{123\} \in F$ is a triangle with a weight $\eta : E_\sigma \to (-1, +\infty)$ satisfying the structure condition (1.3). For $(r_1, r_2, r_3) \in V_i$, we have $h_i < 0$ and $h_j > 0, h_k > 0$.

Remark 2.5. The simply connectivity of the admissible space of nondegenerate inversive distance circle packing metrics was first proved by Guo [19] for nonnegative inversive distance and then by the author [34] for inversive distance in $(-1, +\infty)$ satisfying the structure condition (1.3). The proof of simply connectivity presented here is motivated by the proof of simply connectivity of admissible space of sphere packing metrics of a tetrahedron in 3-dimension [10, 14, 33]. The advantage of the proof of Proposition 2.1 is that we have a precise description of the boundary of the admissible space, each connected component of which is an analytic graph on $\mathbb{R}^2_{>0}$, and the proof could be generalized to 3-dimensional case to prove the simply connectivity of admissible space of Thurston's sphere packing metrics for a tetrahedron [20, 21].

In a single triangle $\{123\}$, we denote θ_i as the angle at vertex i. We have the following result.

Lemma 2.6 ([25, 34]). *Suppose* $\{123\} \in F$ *is a triangle with a weight* $\eta : E_\sigma \to (-1, +\infty)$ *satisfying the structure condition* (1.3)*. Then* $\theta_1, \theta_2, \theta_3$ *defined for* $(r_1, r_2, r_3) \in \Omega^E_{123}(\eta)$ *could be extended by constants to be continuous functions* $\widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\theta}_3$ *defined on* $\mathbb{R}^3_{>0}$.

Proof. If $r = (r_1, r_2, r_3) \in \Omega^E_{123}(\eta)$ tends to a point $\overline{r} = (\overline{r}_1, \overline{r}_2, \overline{r}_3)$ in the boundary ∂V_i of V_i in $\mathbb{R}^3_{>0}$, we have the area $A_{123} \to 0$, $l_{ij}(r) \to l_{ij}(\overline{r}) > 0$ and $l_{ik}(r) \to l_{ik}(\overline{r}) > 0$, which implies $\sin \theta_i = \frac{2A_{123}}{l_{ij} l_{ik}} \to 0$. Therefore, $\theta_i \to \pi$ or 0.

Take $u_i = \ln r_i$. By Lemma 2.9, we have

$$\frac{\partial \theta_i}{\partial u_i} = -\frac{\partial \theta_j}{\partial u_i} - \frac{\partial \theta_k}{\partial u_i} = -\frac{h_{ij,k}}{l_{ij}} - \frac{h_{ik,j}}{l_{ik}}.$$

As $h_j > 0, h_k > 0$ for $\overline{r} \in \partial V_i$ by Remark 2.4, we have $\frac{\partial \theta_i}{\partial u_i} < 0$ for $r \in \Omega^E_{123}(\eta)$ around $\overline{r}$ by (2.3). Therefore, if we increase r_i at $\overline{r}$, which results the triangle does not degenerate, we shall have θ_i decrease. This implies $\theta_i \to \pi$ as $(r_1, r_2, r_3) \to (\overline{r}_1, \overline{r}_2, \overline{r}_3)$. By $\theta_i + \theta_j + \theta_k = \pi$, we have $\theta_j, \theta_k \to 0$.

Then we can extend $\theta_1, \theta_2, \theta_3$ defined on $\Omega^E_{123}(\eta)$ to be continuous functions defined on $\mathbb{R}^3_{>0}$ by setting

$$\widetilde{\theta}_i(r_1, r_2, r_3) = \begin{cases} \theta_i, & \text{if } (r_1, r_2, r_3) \in \Omega^E_{123}(\eta); \\ \pi, & \text{if } (r_1, r_2, r_3) \in V_i; \\ 0, & \text{otherwise.} \end{cases}$$

□

Denote

$$\Gamma = \{(I_1, I_2, I_3) \in (-1, +\infty)^3 | \gamma_1 \geq 0, \gamma_2 \geq 0, \gamma_3 \geq 0\}$$

as the space of weights on the edges of a triangle $\{123\}$ satisfying the structure condition (1.3).

Lemma 2.7. Γ *is connected.*

Proof. It is obviously that $[0, +\infty)^3 \subset \Gamma$. By $\gamma_i = I_i + I_j I_k \geq 0, i = 1, 2, 3$, we have $I_1 + I_2 \geq 0, I_1 + I_3 \geq 0, I_2 + I_3 \geq 0$, which implies that at most one of I_1, I_2, I_3 is negative. Without loss of generality, we consider the case $(I_1, I_2, I_3) \in \Gamma$ with $I_1 < 0, I_2 \geq 0, I_3 \geq 0$. It is straight forward to check that $(tI_1, I_2, I_3) \in \Gamma$ for any $t \in [0, 1]$, which implies Γ is connected. □

Using the space Γ, we can further define the following 6-dimensional parameterized admissible space

$$\Omega_{123}^E = \cup_{\eta\in\Gamma}\Omega_{123}^E(\eta).$$

Lemma 2.8. *Ω_{123}^E is connected.*

Proof. Suppose $\eta_0 \in \Gamma$, then there exists $r_0 \in \Omega_{123}^E(\eta_0)$ with $Q(\eta_0, r_0) > 0$ by the nonempty property of $\Omega_{123}^E(\eta_0)$ in Proposition 2.1. Consider the continuous function $Q(\eta, r_0)$ of η. As $Q(\eta_0, r_0) > 0$, there is a connected neighborhood $U_{\eta_0} \subset \Gamma$ of η_0 such that $Q(\eta, r_0) > 0$ for any $\eta \in U_{\eta_0}$. This implies that for any $\eta \in U_{\eta_0}$, any two points $(\eta, r_A) \in \Omega_{123}^E$ and $(\eta_0, r_B) \in \Omega_{123}^E$ could be connected by a path in Ω_{123}^E by Proposition 2.1. In this case, we call the space $\Omega_{123}^E(\eta)$ and $\Omega_{123}^E(\eta_0)$ could be connected by a path in U_{η_0}. Taking $\Omega_{123}^E(\eta)$ as a point. Then for any $\eta_A, \eta_B \in \Gamma$, the existence of a path from $\Omega_{123}^E(\eta_A)$ to $\Omega_{123}^E(\eta_B)$ in Γ follows from the connectivity of Γ and finite covering theorem, which implies that Ω_{123}^E is connected. ☐

2.2. Negative semi-definiteness of the Jacobian matrix

Set $u_i = \ln r_i$. The following result on the matrix $\Lambda_{123}^E = \frac{\partial(\theta_1,\theta_2,\theta_3)}{\partial(u_1,u_2,u_3)}$ is known.

Lemma 2.9 ([8, 19, 31, 34, 37]). *Suppose (r_1, r_2, r_3) is a nondegenerate Euclidean inversive distance circle packing metric for a triangle $\{123\}$ with a weight η satisfying the structure condition* (1.3)*, then*

$$\frac{\partial\theta_i}{\partial u_j} = \frac{\partial\theta_j}{\partial u_i} = \frac{r_1^2r_2^2r_3^2}{2A_{123}l_{ij}^2}\left[\kappa_k^2(1-I_k^2) + \kappa_j\kappa_k\gamma_i + \kappa_i\kappa_k\gamma_j\right] \tag{2.8}$$

for any adjacent vertices i, j, where A_{123} is the area of the triangle $\{123\}$, and

$$\frac{\partial\theta_i}{\partial u_i} = -\frac{\partial\theta_i}{\partial u_j} - \frac{\partial\theta_i}{\partial u_k}. \tag{2.9}$$

Specially, for $\eta = (I_1, I_2, I_3) = (1, 1, 1) \in \Gamma$, the Jacobian matrix $\Lambda_{123}^E = \frac{\partial(\theta_1,\theta_2,\theta_3)}{\partial(u_1,u_2,u_3)}$ is negative semi-definite with a zero eigenvalue and two negative eigenvalues.

Remark 2.6. By (2.3), (2.9) and Remark 2.4, if $\eta \in \Gamma$ and $(r_1, r_2, r_3) \in \Omega_{123}^E(\eta)$ tends to a point $(\overline{r}_1, \overline{r}_2, \overline{r}_3) \in \partial V_i$, we have $\frac{\partial\theta_i}{\partial u_j} \to +\infty$, $\frac{\partial\theta_i}{\partial u_k} \to +\infty$ and $\frac{\partial\theta_i}{\partial u_i} \to -\infty$.

Set $d_{ij} = \frac{\partial l_{ij}}{\partial u_i}$ and $d_{ji} = \frac{\partial l_{ij}}{\partial u_j}$. Then $d_{ij} = \frac{r_i(r_i+r_jI_k)}{l_{ij}}$, $d_{ji} = \frac{r_j(r_j+r_iI_k)}{l_{ij}}$ and $d_{ij} + d_{ji} = l_{ij}$. This is a type of conformal metric studied in [16–18, 30].

Lemma 2.10 ([19, 34]). *Suppose* $r = (r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$ *is a nondegenerate Euclidean inversive distance circle packing metric for a triangle* $\{123\}$ *with a weight* η *satisfying the structure condition* (1.3)*, then the Jacobian matrix* $\Lambda^E_{123}(\eta) = \frac{\partial(\theta_1,\theta_2,\theta_3)}{\partial(u_1,u_2,u_3)}$ *is negative semi-definite with one dimensional kernel* $\{t(1,1,1)|t \in \mathbb{R}\}$*.*

Proof. By the chain rules, we have

$$\Lambda^E_{123} = \frac{\partial(\theta_1, \theta_2, \theta_3)}{\partial(l_{23}, l_{13}, l_{12})} \cdot \frac{\partial(l_{23}, l_{13}, l_{12})}{\partial(u_1, u_2, u_3)}.$$

By direct calculations,

$$\begin{aligned}
\det \frac{\partial(l_{23}, l_{13}, l_{12})}{\partial(u_1, u_2, u_3)} &= \det \begin{pmatrix} 0 & d_{23} & d_{32} \\ d_{13} & 0 & d_{31} \\ d_{12} & d_{21} & 0 \end{pmatrix} \\
&= \frac{r_1 r_2 r_3}{l_{12} l_{13} l_{23}} [2r_1 r_2 r_3(1 + I_1 I_2 I_3) + r_1(r_2^2 + r_3^2)(I_1 + I_2 I_3) \\
&\quad + r_2(r_1^2 + r_3^2)(I_2 + I_1 I_3) + r_3(r_1^2 + r_2^2)(I_3 + I_1 I_2)] \\
&\geq \frac{2r_1^2 r_2^2 r_3^2}{l_{12} l_{13} l_{23}} (1 + I_1 I_2 I_3 + I_1 + I_2 I_3 + I_2 + I_1 I_3 + I_3 + I_1 I_2) \\
&= \frac{2r_1^2 r_2^2 r_3^2}{l_{12} l_{13} l_{23}} (1 + I_1)(1 + I_2)(1 + I_3) > 0,
\end{aligned}$$

which implies the matrix $\frac{\partial(l_{23},l_{13},l_{12})}{\partial(u_1,u_2,u_3)}$ is nondegenerate. Therefore, the rank of Λ^E_{123} is the same as that of the matrix $\frac{\partial(\theta_1,\theta_2,\theta_3)}{\partial(l_{23},l_{13},l_{12})}$, which is known to be 2 for nondegenerate Euclidean triangles.

Taking Λ^E_{123} as a matrix-valued function defined on Ω^E_{123}, then the two nonzero eigenvalues of Λ^E_{123} are continuous functions of $(\eta, r) \in \Omega^E_{123}$. By the connectivity of Ω^E_{123} in Lemma 2.8, the nonzero eigenvalues do not change sign in Ω^E_{123}. Note that $\Lambda^E_{123}(\eta_0)$ is negative semi-definite with two negative eigenvalues for $\eta_0 = (1, 1, 1)$ by Lemma 2.9. This implies that $\Lambda^E_{123}(\eta)$ is negative semi-definite with two negative eigenvalues for any $\eta \in \Gamma$. The kernel of $\Lambda^E_{123}(\eta)$ is $\{t(1, 1, 1)|t \in \mathbb{R}\}$ follows from the scaling invariance of $\theta_1, \theta_2, \theta_3$. □

Remark 2.7. The negative semi-definiteness of the Jacobian matrix Λ^E_{123} for Thurston's circle packing metric is well-known, see [8, 31, 37]. The negative semi-definiteness of Λ^E_{123} for inversive distance circle packing metrics is proved by Guo [19] for nonnegative inversive distance and by the author [34] for inversive distance in $(-1, +\infty)$ satisfying the structure condition (1.3). The proof we give here simplifies the proof in [19, 34].

2.3. Proof of the rigidity for Euclidean inversive distance circle packing

As the rest of the proof for the rigidity is standard and the same as that in [19, 25, 34], we just give a sketch of the proof here. For more details of the proof, see [19, 25, 34].

By Lemma 2.9 and Proposition 2.1, we can define the following function

$$F_{ijk}(u_i, u_j, u_k) = \int_{(\overline{u}_i,\overline{u}_j,\overline{u}_k)}^{(u_i,u_j,u_k)} \theta_i du_i + \theta_j du_j + \theta_k du_k$$

on $\ln(\Omega^E_{ijk}(\eta))$, which is locally concave by Lemma 2.10. Recall the following definition and extension theorem of Luo [25].

Definition 2.1. A differential 1-form $w = \sum_{i=1}^n a_i(x)dx^i$ in an open set $U \subset \mathbb{R}^n$ is said to be continuous if each $a_i(x)$ is continuous on U. A continuous differential 1-form w is called closed if $\int_{\partial\tau} w = 0$ for each triangle $\tau \subset U$.

Theorem 2.1 ([25], Corollary 2.6). *Suppose $X \subset \mathbb{R}^n$ is an open convex set and $A \subset X$ is an open subset of X bounded by a real analytic codimension-1 submanifold in X. If $w = \sum_{i=1}^n a_i(x)dx_i$ is a continuous closed 1-form on A so that $F(x) = \int_a^x w$ is locally convex on A and each a_i can be extended continuous to X by constant functions to a function $\widetilde{a}_i$ on X, then $\widetilde{F}(x) = \int_a^x \sum_{i=1}^n \widetilde{a}_i(x)dx_i$ is a C^1-smooth convex function on X extending F.*

By Proposition 2.1, Lemma 2.6 and Theorem 2.1, F_{ijk} could be extended to be a C^1-smooth concave function

$$\widetilde{F}_{ijk}(u_i, u_j, u_k) = \int_{(\overline{u}_i,\overline{u}_j,\overline{u}_k)}^{(u_i,u_j,u_k)} \widetilde{\theta}_i du_i + \widetilde{\theta}_j du_j + \widetilde{\theta}_k du_k$$

defined on $\mathbb{R}^3$. Using $\widetilde{F}_{ijk}$, we can further define the following C^1 convex function $\widetilde{F}$ on $\mathbb{R}^{|V|}$

$$\widetilde{F}(u_1, \ldots, u_{|V|}) = 2\pi \sum_{i \in V} u_i - \sum_{\{ijk\} \in F} \widetilde{F}_{ijk}(u_i, u_j, u_k),$$

which has gradient $\nabla_{u_i} \widetilde{F} = 2\pi - \sum_{\{ijk\} \in F} \widetilde{\theta}_i^{jk} = \widetilde{K}_i$, where $\widetilde{K}_i$ is a continuous extension of K_i. Then the global rigidity of K on the admissible space of Euclidean inversive distance circle packing metrics for $(M, \mathcal{T}, I)$ follows from the convexity of $\widetilde{F}$ and the null space of $\Lambda_{ijk}^E(\eta)$ is $\{t(1,1,1) | t \in \mathbb{R}\}$. $\square$

3. Rigidity of hyperbolic inversive distance circle packing

3.1. The admissible space of hyperbolic inversive distance circle packing metrics for a single triangle

Similar to the Euclidean case, we can define the admissible space of hyperbolic inversive distance circle packing metrics for a triangle $\sigma = \{123\} \in F$. Given a weight η on the edge set E_σ of $\{123\}$ satisfying the structure condition (1.3), the admissible space $\Omega_{123}^H(\eta)$ of hyperbolic inversive distance circle packing metrics for the triangle $\sigma = \{123\}$ is defined to be the set of hyperbolic inversive distance circle packing metrics $(r_1, r_2, r_3) \in \mathbb{R}_{>0}^3$ such that the triangle with edge lengths given by (1.2) exists in 2-dimensional hyperbolic space $\mathbb{H}^2$.

Set

$$G_-(l) = \begin{pmatrix} -1 & -\cosh l_{12} & -\cosh l_{13} \\ -\cosh l_{12} & -1 & -\cosh l_{23} \\ -\cosh l_{13} & -\cosh l_{23} & -1 \end{pmatrix}.$$

Recall the following result characterizing nondegeneracy of a hyperbolic triangle $\{123\}$ with positive edge lengths l_{12}, l_{13}, l_{23}.

Lemma 3.1 ([29], Proposition 2.4.1). *A triangle with positive edge lengths l_{12}, l_{13}, l_{23} exists in $\mathbb{H}^2$ if and only if $\det G_-(l) < 0$.*

Remark 3.1. By direct calculations, we have

$$\det G_-(l) = -4 \sinh s \sinh(s - l_{12}) \sinh(s - l_{13}) \sinh(s - l_{23}),$$

where $s = \frac{1}{2}(l_{12} + l_{13} + l_{23})$ is the semiperimeter. This implies that $\det G_-(l) < 0$ is equivalent to the triangle inequalities. This was also observed by Guo in [19]. Similar to the Euclidean case, this approach has the advantage that we just need one inequality to characterize the nondegeneracy instead of three triangle inequalities. Furthermore, this approach could be generalized to higher dimension [29].

For simplicity, we set

$$C_i = \cosh r_i, S_i = \sinh r_i.$$

Submitting the definition of l_{ij} (1.2) into $G_-(l)$, we have

$$\begin{aligned}-\det G_-(l) = &\, 2S_1^2S_2^2S_3^2(1 + I_1I_2I_3)\\ &+ S_1^2S_2^2(1 - I_3^2) + S_1^2S_3^2(1 - I_2^2) + S_2^2S_3^2(1 - I_1^2)\\ &+ 2C_2C_3S_1^2S_2S_3\gamma_1 + 2C_1C_3S_1S_2^2S_3\gamma_2 + 2C_1C_2S_1S_2S_3^2\gamma_3,\end{aligned}$$

where $\gamma_i = I_i + I_jI_k \geq 0$ due to the structure condition (1.3). Set

$$\kappa_i = \coth r_i,$$

then

$$-\det G_-(l) = S_1^2S_2^2S_3^2Q,$$

where

$$\begin{aligned}Q = &\, \kappa_1^2(1 - I_1^2) + \kappa_2^2(1 - I_2^2) + \kappa_3^2(1 - I_3^2) + 2\kappa_1\kappa_2\gamma_3 + 2\kappa_1\kappa_3\gamma_2 + 2\kappa_2\kappa_3\gamma_1\\ &+ I_1^2 + I_2^2 + I_3^2 + 2I_1I_2I_3 - 1.\end{aligned}$$

Then we have the following criterion of nondegeneracy for hyperbolic triangles.

Lemma 3.2 ([19, 34]). *A hyperbolic triangle* $\{123\}$ *with edge lengths* l_{12}, l_{13}, l_{23} *given by* (1.2) *exists in* $\mathbb{H}^2$ *if and only if* $Q > 0$.

Similar to the Euclidean case, set

$$\begin{aligned}h_1 &= \kappa_1(1 - I_1^2) + \kappa_2\gamma_3 + \kappa_3\gamma_2,\\ h_2 &= \kappa_2(1 - I_2^2) + \kappa_1\gamma_3 + \kappa_3\gamma_1,\\ h_3 &= \kappa_3(1 - I_3^2) + \kappa_1\gamma_2 + \kappa_2\gamma_1,\end{aligned}$$

we have

$$Q = \kappa_1h_1 + \kappa_2h_2 + \kappa_3h_3 + I_1^2 + I_2^2 + I_3^2 + 2I_1I_2I_3 - 1.$$

Then $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$ is a degenerate hyperbolic inversive distance circle packing metric for a single triangle $\{123\}$ if and only if

$$Q = \kappa_1 h_1 + \kappa_2 h_2 + \kappa_3 h_3 + I_1^2 + I_2^2 + I_3^2 + 2I_1 I_2 I_3 - 1 \leq 0. \tag{3.1}$$

If $I_1, I_2, I_3 \in (-1, 1]$, we have $h_i \geq 1 - I_i^2 + \gamma_j + \gamma_k$ by $\kappa_i = \coth r_i > 1$, which implies

$$\begin{aligned} Q \geq\ & (1 - I_1^2) + \gamma_3 + \gamma_2 + (1 - I_2^2) + \gamma_3 + \gamma_1 \\ & + (1 - I_3^2) + \gamma_2 + \gamma_1 + I_1^2 + I_2^2 + I_3^2 + 2I_1 I_2 I_3 - 1 \\ =\ & 2(1 + I_1)(1 + I_2)(1 + I_3) > 0 \end{aligned}$$

for any $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$. Therefore, if $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$ is a degenerate hyperbolic inversive distance circle packing metric for a triangle $\{123\}$, at least one of I_1, I_2, I_3 is strictly larger than 1, which implies $I_1^2 + I_2^2 + I_3^2 + 2I_1 I_2 I_3 - 1 > 0$ by the proof of Lemma 2.3.

Therefore, if $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$ is a degenerate inversive distance circle packing metric for a triangle $\{123\}$, we have

$$\kappa_1 h_1 + \kappa_2 h_2 + \kappa_3 h_3 < 0$$

by (3.1), which implies one of the following two cases happens.

(1): At least one of h_1, h_2, h_3 is zero;

(2): None of h_1, h_2, h_3 is zero.

Similar to the Euclidean case, we can prove that case (1) never happens. Furthermore, we can prove that only one of h_i, h_j, h_k is negative and the others are positive in case (2).

Similar to the Euclidean case, we can rewrite $Q \leq 0$ as a quadratic inequality in κ_i:

$$A_i \kappa_i^2 + B_i \kappa_i + C_i \geq 0,$$

where

$$\begin{aligned} A_i &= I_i^2 - 1, \\ B_i &= -2(\kappa_j \gamma_k + \kappa_k \gamma_j) \leq 0, \quad \{i, j, k\} = \{1, 2, 3\}, \\ C_i &= \kappa_j^2 (I_j^2 - 1) + \kappa_k^2 (I_k^2 - 1) - 2\kappa_j \kappa_k \gamma_i \\ &\quad - (I_1^2 + I_2^2 + I_3^2 + 2I_1 I_2 I_3 - 1) \end{aligned} \tag{3.2}$$

with $\{i,j,k\}=\{1,2,3\}$. By direct calculations, we have the determinant $\Delta_i = B_i^2 - 4A_iC_i$ is given by

$$\begin{aligned}\Delta_i = {} & 4(\kappa_j^2+\kappa_k^2+2\kappa_j\kappa_k I_i)(I_1^2+I_2^2+I_3^2+2I_1I_2I_3-1) \\ & + 4(I_i^2-1)(I_1^2+I_2^2+I_3^2+2I_1I_2I_3-1).\end{aligned} \tag{3.3}$$

Similar to the Euclidean case, we have the following results.

Lemma 3.3. *If $I_i > 1$ and the structure condition* (1.3) *is satisfied, then $\Delta_i > 0$.*

Lemma 3.4. *Suppose $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$ is a degenerate hyperbolic inversive distance circle packing metric for a triangle $\{123\}$ with a weight $\eta : E_\sigma \to (-1, +\infty)$ satisfying the structure condition* (1.3)*, then none of h_1, h_2, h_3 is zero.*

Lemma 3.5. *Suppose $\{123\} \in F$ is a triangle with a weight $\eta : E_\sigma \to (-1, +\infty)$ satisfying the structure condition* (1.3) *and $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$. Then there exists no subset $\{i, j\} \subset \{1, 2, 3\}$ such that $h_i < 0$ and $h_j < 0$.*

Proposition 3.1 ([19, 34]). *Suppose $\sigma = \{123\} \in F$ is a triangle in $(M, \mathcal{T})$ with a weight $\eta : E_\sigma \to (-1, +\infty)$ satisfying the structure condition* (1.3)*. Then the admissible space $\Omega^H_{123}(\eta)$ of hyperbolic inversive distance circle packing metrics $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$ is nonempty and simply connected. Furthermore, the set of degenerate inversive distance circle packing metric is a disjoint union $\cup_{i\in P} V_i$, where $P = \{i \in \{1, 2, 3\} | I_i > 1\}$ and*

$$V_i = \left\{ (r_1, r_2, r_3) \in \mathbb{R}^3_{>0} | \kappa_i \geq \frac{-B_i + \sqrt{\Delta_i}}{2A_i} \right\}$$

is bounded by an analytic graph on $\mathbb{R}^2_{>0}$ with A_i, B_i, C_i, Δ_i given by (3.2)(3.3).

Lemma 3.3, Lemma 3.4, Lemma 3.5 and Proposition 3.1 could be proved similarly to that of Lemma 2.3, Lemma 2.4, Lemma 2.5 and Proposition 2.1 by repeating the proof line by line. We omit the details of the proof here. Similar to Remark 2.4, we have the following remark.

Remark 3.2. If $(r_1, r_2, r_3) \in V_i$ is a degenerate hyperbolic inversive distance circle packing metric for a triangle $\{123\}$ with a weight $\eta : E_\sigma \to (-1, +\infty)$ satisfying the structure condition (1.3), then $h_i < 0$, $h_j > 0$, $h_k > 0$, where $\{i, j, k\} = \{1, 2, 3\}$.

Similar to the Euclidean case, the inner angles of a hyperbolic triangle could be extended by constants to be globally defined continuous functions.

Lemma 3.6 ([25, 34]). *Suppose* $\{123\} \in F$ *is a triangle with a weight* $\eta : E_\sigma \to (-1, +\infty)$ *satisfying the structure condition* (1.3)*. Then the functions* $\theta_1, \theta_2, \theta_3$ *defined for* $(r_1, r_2, r_3) \in \Omega^H_{123}(\eta)$ *could be extended by constants to be continuous functions* $\widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\theta}_3$ *defined on* $\mathbb{R}^3_{>0}$*.*

Proof. Suppose $(r_1, r_2, r_3) \in \Omega^H_{123}(\eta)$ tends to a point $(\overline{r}_1, \overline{r}_2, \overline{r}_3) \in \partial V_i$. By direct calculations, we have

$$\begin{aligned}
-\det G_-(l) &= 4 \sinh s \sinh(s - l_{ij}) \sinh(s - l_{ik}) \sinh(s - l_{jk}) \\
&= (\cosh(l_{jk} + l_{ik}) - \cosh l_{ij})(\cosh l_{ij} - \cosh(l_{jk} - l_{ik})) \\
&= (\cosh^2 l_{jk} - 1)(\cosh l_{ik}^2 - 1) - (\cosh l_{jk} \cosh l_{ik} - \cosh l_{ij})^2 \\
&= \sinh^2 l_{jk} \sinh^2 l_{ik} - \sinh^2 l_{jk} \sinh^2 l_{ik} \cos^2 \theta_k \\
&= \sinh^2 l_{jk} \sinh^2 l_{ik} \sin^2 \theta_k,
\end{aligned}$$

where $\{i, j, k\} = \{1, 2, 3\}$. As $(r_1, r_2, r_3) \in \Omega^H_{123}(\eta)$ tends to $(\overline{r}_1, \overline{r}_2, \overline{r}_3) \in \partial V_i$, we have $\det G_-(l) \to 0$, which implies $\theta_1, \theta_2, \theta_3 \to 0$ or π.

By Lemma 3.8, we have

$$\frac{\partial \theta_j}{\partial u_i} = \frac{S_i^2 S_j^2 S_k}{2\widetilde{A}_{123} \sinh^2 l_{ij}} [\kappa_k (1 - I_k^2) + \kappa_i \gamma_j + \kappa_j \gamma_i] = \frac{S_i^2 S_j^2 S_k h_k}{2\widetilde{A}_{123} \sinh^2 l_{ij}},$$

where $u_i = \ln \tanh \frac{r_i}{2}$ and $\widetilde{A}_{123} = \frac{1}{2} \sinh l_{ik} \sinh l_{ij} \sin \theta_i$. Note that for $(\overline{r}_1, \overline{r}_2, \overline{r}_3) \in \partial V_i$, we have $h_k > 0$ by Remark 3.2. Therefore, for $(r_1, r_2, r_3) \in \Omega^H_{123}(\eta)$ sufficiently close to $(\overline{r}_1, \overline{r}_2, \overline{r}_3) \in \partial V_i$, we have $\frac{\partial \theta_j}{\partial u_i} > 0$, which implies $\theta_j, \theta_k \to 0$ as $(r_1, r_2, r_3) \to (\overline{r}_1, \overline{r}_2, \overline{r}_3) \in \partial V_i$.

Furthermore, we have the following formula [32] for the area A_{123} of the hyperbolic triangle $\{123\}$

$$\begin{aligned}
\tan^2 \frac{A_{123}}{4} &= \tanh \frac{p}{2} \tanh \frac{p - l_{12}}{2} \tanh \frac{p - l_{13}}{2} \tanh \frac{p - l_{23}}{2} \\
&= \frac{-\det G_-(l)}{64 \cosh^2 \frac{p}{2} \cosh^2 \frac{p - l_{12}}{2} \cosh^2 \frac{p - l_{13}}{2} \cosh^2 \frac{p - l_{23}}{2}},
\end{aligned}$$

where $p = \frac{1}{2}(l_{12} + l_{13} + l_{23})$. This implies $A_{123} \to 0$ as $(r_1, r_2, r_3) \to (\overline{r}_1, \overline{r}_2, \overline{r}_3) \in \partial V_i$. Further note that $A_{123} = \pi - \theta_1 - \theta_2 - \theta_3$ and $\theta_j, \theta_k \to 0$, we have $\theta_i \to \pi$ as $(r_1, r_2, r_3) \to (\overline{r}_1, \overline{r}_2, \overline{r}_3) \in V_i$.

Therefore, we can extend $\theta_1, \theta_2, \theta_3$ defined on $\Omega_{123}^H(\eta)$ to be continuous functions defined on $\mathbb{R}^3_{>0}$ by setting

$$\widetilde{\theta}_i(r_1, r_2, r_3) = \begin{cases} \theta_i, & \text{if } (r_1, r_2, r_3) \in \Omega_{123}^H(\eta); \\ \pi, & \text{if } (r_1, r_2, r_3) \in V_i; \\ 0, & \text{otherwise.} \end{cases}$$

□

Similar to the Euclidean case, we can define the following 6-dimensional parameterized admissible space

$$\Omega_{123}^H = \cup_{\eta \in \Gamma} \Omega_{123}^H(\eta).$$

Lemma 3.7. Ω_{123}^H *is connected.*

The proof of Lemma 3.7 is the same as that of Lemma 2.8, we omit the proof here.

3.2. Negative definiteness of the Jacobian matrix

Set $u_i = \ln \tanh \frac{r_i}{2}$. The following result on the matrix $\Lambda_{123}^H = \frac{\partial(\theta_1, \theta_2, \theta_3)}{\partial(u_1, u_2, u_3)}$ is known.

Lemma 3.8 ([8, 19, 31, 34, 37]). *Suppose* $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$ *is a non-degenerate hyperbolic inversive distance circle packing metric for a triangle* $\{123\}$ *with a weight* η *satisfying the structure condition* (1.3)*, then*

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i} = \frac{S_i^2 S_j^2 S_k}{2\widetilde{A}_{123} \sinh^2 l_{ij}} \left[\kappa_k(1 - I_k^2) + \kappa_j \gamma_i + \kappa_i \gamma_j\right], \tag{3.4}$$

where $\widetilde{A} = \frac{1}{2} \sinh l_{ik} \sinh l_{ij} \sin \theta_i$ *and* $\{i, j, k\} = \{1, 2, 3\}$*. Specially, for* $\eta = (I_1, I_2, I_3) = (1, 1, 1)$*, the matrix* $\Lambda_{123}^H = \frac{\partial(\theta_1, \theta_2, \theta_3)}{\partial(u_1, u_2, u_3)}$ *is negative definite at* $(r_1, r_2, r_3) = (1, 1, 1)$*.*

Remark 3.3. By (3.4) and Remark 3.2, if $\eta \in \Gamma$ and $(r_1, r_2, r_3) \in \Omega_{123}^H(\eta)$ tends to a point $(\overline{r}_1, \overline{r}_2, \overline{r}_3) \in \partial V_i$, we have $\frac{\partial \theta_i}{\partial u_j} \to +\infty$, $\frac{\partial \theta_i}{\partial u_k} \to +\infty$. Recall the following formula obtained in Proposition 9 of [18]

$$\frac{\partial A_{123}}{\partial u_i} = \frac{\partial \theta_j}{\partial u_i}(\cosh l_{ij} - 1) + \frac{\partial \theta_k}{\partial u_i}(\cosh l_{ik} - 1),$$

we have $\frac{\partial A_{123}}{\partial u_i} \to +\infty$, which implies

$$\frac{\partial \theta_i}{\partial u_i} = -\frac{\partial A_{123}}{\partial u_i} - \frac{\partial \theta_j}{\partial u_i} - \frac{\partial \theta_k}{\partial u_i} \to -\infty.$$

Lemma 3.9. *Suppose $r = (r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$ is a nondegenerate hyperbolic inversive distance circle packing metric for a triangle $\{123\}$ with a weight η satisfying the structure condition (1.3), then the matrix $\Lambda^H_{123}(\eta) = \frac{\partial(\theta_1,\theta_2,\theta_3)}{\partial(u_1,u_2,u_3)}$ is negative definite.*

Proof. By the chain rules, we have

$$\Lambda^H_{123} = \frac{\partial(\theta_1, \theta_2, \theta_3)}{\partial(l_{23}, l_{13}, l_{12})} \cdot \frac{\partial(l_{23}, l_{13}, l_{12})}{\partial(u_1, u_2, u_3)}.$$

By direct calculations, we have

$$\begin{aligned}
\det \frac{\partial(l_{23}, l_{13}, l_{12})}{\partial(u_1, u_2, u_3)} &= \frac{S_1 S_2 S_3}{\sinh l_{12} \sinh l_{13} \sinh l_{23}} \\
&\quad \times \big[2C_1C_2C_3S_1S_2S_3(1 + I_1I_2I_3) \\
&\qquad + C_1S_1\gamma_1(C_2^2S_3^2 + C_3^2S_2^2) + C_2S_2\gamma_2(C_1^2S_3^2 + C_3^2S_1^2) \\
&\qquad + C_3S_3\gamma_3(C_1^2S_2^2 + C_2^2S_1^2)\big] \\
&\geq \frac{2C_1C_2C_3S_1^2S_2^2S_3^2}{\sinh l_{12} \sinh l_{13} \sinh l_{23}}(1 + I_1I_2I_3 + \gamma_1 + \gamma_2 + \gamma_3) \\
&= \frac{2C_1C_2C_3S_1^2S_2^2S_3^2}{\sinh l_{12} \sinh l_{13} \sinh l_{23}}(1 + I_1)(1 + I_2)(1 + I_3) > 0,
\end{aligned}$$

which implies the matrix $\frac{\partial(l_{23},l_{13},l_{12})}{\partial(u_1,u_2,u_3)}$ is nondegenerate. Therefore, the rank of Λ^H_{123} is the same as that of the matrix $\frac{\partial(\theta_1,\theta_2,\theta_3)}{\partial(l_{23},l_{13},l_{12})}$, which is 3 for nondegenerate hyperbolic triangles.

Taking Λ^H_{123} as a matrix-valued function defined on Ω^H_{123}, then the three nonzero eigenvalues of Λ^H_{123} are continuous functions of $(\eta, r) \in \Omega^H_{123}$. By the connectivity of Ω^H_{123} in Lemma 3.7, the three nonzero eigenvalues do not change sign on Ω^H_{123}. Note that $\Lambda^H_{123}(\eta_0)$ is negative definite at $(r_1, r_2, r_3) = (1, 1, 1)$ for $\eta_0 = (1, 1, 1)$ by Lemma 3.9, we have $\Lambda^H_{123}(\eta)$ is negative definite for any $\eta \in \Gamma$. □

3.3. Proof of the rigidity for hyperbolic inversive distance circle packing metrics

Similar to the Euclidean case, we just sketch the proof of rigidity for hyperbolic inversive distance circle packing here. For more details of the proof, see [19, 25, 34].

By Lemma 3.8 and Proposition 3.1, we can define the following function

$$F_{ijk}(u_i, u_j, u_k) = \int_{(\overline{u}_i,\overline{u}_j,\overline{u}_k)}^{(u_i,u_j,u_k)} \theta_i du_i + \theta_j du_j + \theta_k du_k$$

on the image of $\Omega_{ijk}^H(\eta)$ under the map $u_i = \ln\tanh\frac{r_i}{2}$, which is locally concave by Lemma 3.9. By Proposition 3.1, Lemma 3.6 and Theorem 2.1, F_{ijk} could be extended to be a C^1-smooth concave function

$$\widetilde{F}_{ijk}(u_i, u_j, u_k) = \int_{(\overline{u}_i,\overline{u}_j,\overline{u}_k)}^{(u_i,u_j,u_k)} \widetilde{\theta}_i du_i + \widetilde{\theta}_j du_j + \widetilde{\theta}_k du_k$$

defined on $\mathbb{R}^3_{<0}$. Using $\widetilde{F}_{ijk}$, we can further define the following C^1 convex function $\widetilde{F}$ on $\mathbb{R}^{|V|}_{<0}$

$$\widetilde{F}(u_1, \ldots, u_{|V|}) = 2\pi \sum_{i\in V} u_i - \sum_{\{ijk\}\in F} \widetilde{F}_{ijk}(u_i, u_j, u_k),$$

which has gradient $\nabla_{u_i}\widetilde{F} = 2\pi - \sum_{\{ijk\}\in F}\widetilde{\theta}_i^{jk} = \widetilde{K}_i$, where $\widetilde{K}_i$ is a continuous extension of K_i. Then the global injectivity of K on the admissible space of hyperbolic inversive distance circle packing metrics for $(M, \mathcal{T}, I)$ follows from the convexity of $\widetilde{F}$. This is equivalent to the global rigidity of the curvature map K. □

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References

[1] E. M. Andreev, *Convex polyhedra in Lobachevsky spaces* (Russian), Mat. Sb. (N.S.) **81** (1970), no. 123, 445–478.

[2] E. M. Andreev, *Convex polyhedra of finite volume in Lobachevsky space* (Russian), Mat. Sb. (N.S.) **83** (1970), no. 125, 256–260.

[3] A. Bobenko, U. Pinkall, and B. Springborn, *Discrete conformal maps and ideal hyperbolic polyhedra*, Geom. Topol. **19** (2015), no. 4, 2155–2215.

[4] J. C. Bowers and P. L. Bowers, *Ma-Schlenker c-octahedra in the* 2-*sphere*, Discrete Comput. Geom. **60** (2018), no. 1, 9–26.

[5] J. C. Bowers, P. L. Bowers, and K. Pratt, *Rigidity of circle polyhedra in the* 2-*sphere and of hyperideal polyhedra in hyperbolic* 3-*space*, Trans. Amer. Math. Soc. **371** (2019), no. 6, 4215–4249.

[6] P. L. Bowers and M. K. Hurdal, *Planar conformal mappings of piecewise flat surfaces*, Visualization and Mathematics III, 3–34, Math. Vis., Springer, Berlin, (2003).

[7] P. L. Bowers and K. Stephenson, *Uniformizing dessins and Belyĭ maps via circle packing*, Mem. Amer. Math. Soc. **170** (2004), no. 805.

[8] B. Chow and F. Luo, *Combinatorial Ricci flows on surfaces*, J. Differential Geometry **63** (2003), 97–129.

[9] Y. C. de Verdière, *Un principe variationnel pour les empilements de cercles*, Invent. Math. **104** (1991), no. 3, 655–669.

[10] H. Ge and B. Hua, 3-*dimensional combinatorial Yamabe flow in hyperbolic background geometry*, Trans. Amer. Math. Soc. **373** (2020), no. 7, 5111–5140.

[11] H. Ge and W. Jiang, *On the deformation of inversive distance circle packings, II*, J. Funct. Anal. **272** (2017), no. 9, 3573–3595.

[12] H. Ge and W. Jiang, *On the deformation of inversive distance circle packings, III*, J. Funct. Anal. **272** (2017), no. 9, 3596–3609.

[13] H. Ge and W. Jiang, *On the deformation of inversive distance circle packings, I*, Trans. Amer. Math. Soc. **372** (2019), no. 9, 6231–6261.

[14] H. Ge, W. Jiang, and L. Shen, *On the deformation of ball packings*, `arXiv:1805.10573 [math.DG]`.

[15] H. Ge and X. Xu, *On a combinatorial curvature for surfaces with inversive distance circle packing metrics*, J. Funct. Anal. **275** (2018), no. 3, 523–558.

[16] D. Glickenstein, *Discrete conformal variations and scalar curvature on piecewise flat two and three dimensional manifolds*, J. Differential Geometry **87** (2011), 201–238.

[17] D. Glickenstein, *Geometric triangulations and discrete Laplacians on manifolds*, `arXiv:math/0508188 [math.MG]`.

[18] D. Glickenstein and J. Thomas, *Duality structures and discrete conformal variations of piecewise constant curvature surfaces*, Adv. Math. **320** (2017), 250–278.

[19] R. Guo, *Local rigidity of inversive distance circle packing*, Trans. Amer. Math. Soc. **363** (2011), 4757–4776.

[20] X. He and X. Xu, *Thurston's sphere packing on* 3*-dimensional manifolds, I*, `arXiv:1904.11122v3 [math.GT]`.

[21] X. He and X. Xu, *Thurston's sphere packing on* 3*-dimensional manifolds, II*, in preparation.

[22] Z.-X. He, *Rigidity of infinite disk patterns*, Ann. of Math. (2) **149** (1999), no. 1, 1–33.

[23] M. K. Hurdal and K. Stephenson, *Discrete conformal methods for cortical brain flattening*, Neuroimage **45** (2009), S86–S98.

[24] P. Koebe, *Kontaktprobleme der konformen Abbildung*, Ber. Sächs. Akad. Wiss. Leipzig, Math. Phys. Kl. **88** (1936), 141–164.

[25] F. Luo, *Rigidity of polyhedral surfaces, III*, Geom. Topol. **15** (2011), 2299–2319.

[26] J. Ma and J. Schlenker, *Non-rigidity of spherical inversive distance circle packings*, Discrete Comput. Geom. **47** (2012), no. 3, 610–617.

[27] A. Marden and B. Rodin, *On Thurston's formulaton and proof of Andreeev's theorem*, in: Computational Methods and Function Theory (Valparaíso, 1989), pp. 103-116, Lecture Notes in Math. **1435**, Springer, Berlin, (1990).

[28] K. Stephenson, Introduction to Circle Packing: The Theory of Discrete Analytic Functions, Cambridge Univ. Press (2005).

[29] Y. U. Taylor and C. T. Woodward, *6j symbols for $U_q(\mathfrak{sl}_2)$ and non-Euclidean tetrahedra*, Selecta Math. (N.S.) **11** (2005), no. 3-4, 539–571.

[30] J. Thomas, *Conformal variations of piecewise constant two and three dimensional manifolds*, Thesis (Ph.D.), The University of Arizona, (2015), 120 pp.

[31] W. Thurston, *Geometry and topology of 3-manifolds*, Princeton Lecture Notes 1976, http://www.msri.org/publications/books/gt3m.

[32] E. B. Vinberg, Geometry. II, Encyclopaedia of Mathematical Sciences **29**, Springer-Verlag, New York, (1988).

[33] X. Xu, *On the global rigidity of sphere packings on 3-dimensional manifolds*, J. Differential Geom. **115** (2020), no. 1, 175–193.

[34] X. Xu, *Rigidity of inversive distance circle packings revisited*, Adv. Math. **332** (2018), 476–509.

[35] W. Zeng and X. Gu, Ricci Flow for Shape Analysis and Surface Registration, Springer Briefs in Mathematics, Springer, New York (2013).

[36] M. Zhang, W. Zeng, R. Guo, F. Luo, and X. Gu, *Survey on discrete surface Ricci flow*, J. Comput. Sci. Tech. **30** (2015), no. 3, 598–613.

[37] Z. Zhou, *Circle patterns with obtuse exterior intersection angles*, arXiv: 1703.01768v3 [math.GT].

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