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The topological period–index conjecture

Benjamin Antieau and Ben Williams

We prove the topological analogue of the period–index conjecture in each dimension away from a small set of primes.

1. Introduction

We prove the following theorem, partly solving the so-called topological period–index conjecture of [2]. For background, see §2.

Theorem A. *Let X be a finite $2d$-dimensional CW complex*[1] *and let $\alpha \in \mathrm{Br}(X) = \mathrm{H}^3(X,\mathbb{Z})_{\mathrm{tors}}$ be a Brauer class. Setting $n = \mathrm{per}(\alpha)$, we have*

$$\mathrm{ind}(\alpha) | n^{d-1} \prod_{p|n} p^{v_p((d-1)!)},$$

where the product ranges over the prime divisors of n and where v_p denotes the p-adic valuation.

Away from $(d-1)!$, the result simplifies.

Corollary B. *If X is a finite $2d$-dimensional CW complex, $\alpha \in \mathrm{Br}(X)$, and $\mathrm{per}(\alpha)$ is prime to $(d-1)!$, then $\mathrm{ind}(\alpha)|\mathrm{per}(\alpha)^{d-1}$.*

The corollary is an exact topological analogue, away from some small primes, of the well-known period–index conjecture for division algebras over function fields (see [10, Section 2.4]).

Conjecture (Function field period–index conjecture). *Let k be algebraically closed and let K be of transcendence degree d over k. If $\alpha \in \mathrm{Br}(K)$, then*

$$\mathrm{ind}(\alpha)|\mathrm{per}(\alpha)^{d-1}.$$

Tsen's theorem implies that $\mathrm{Br}(K) = 0$ when K has transcendence degree 1 over an algebraically closed field (see [14]). In [13], de Jong proved the

[1] More precisely, what we require is for X to have the homotopy type of a retract of a finite CW complex and for $\mathrm{H}^i(X, A) = 0$ for $i > 2d$ and any abelian group A.

conjecture when $d = 2$ for Brauer classes of order relatively prime to the characteristic. Lieblich removed this restriction in [18] thus establishing the $d = 2$ case in full generality. There are no other cases known of the conjecture. More precisely, the period–index conjecture for function fields is not known for a single function field of transcendence degree $d > 2$ over an algebraically closed field.

Nevertheless, work of de Jong and Starr [20] has reduced the conjecture above to the following special cases.

Conjecture (Global period–index conjecture). *Let k be an algebraically closed field and let X be a smooth projective k-scheme of dimension d. If $\alpha \in \mathrm{Br}(X) \subseteq \mathrm{Br}(k(X))$, then*

$$\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^{d-1}.$$

In other words, to prove the period–index conjecture for function fields, it is enough to prove it for unramified classes. We therefore view the period–index conjecture as a global problem. If $k = \mathbb{C}$, then the space of complex points $X(\mathbb{C})$ of a smooth projective $\mathbb{C}$-scheme of dimension d admits the structure of a $2d$-dimensional CW complex. Thus, Corollary B provides evidence for this conjecture.

The topological period–index problem was introduced by the authors in [1] where weak lower bounds were given and where the $d \leqslant 2$ cases were solved. The $d = 3$ case of the topological period–index problem was settled in [2], where it was proved moreover that the bound appearing in Theorem A is sharp in that case. The case of $d = 4$ was solved by Gu in [16, 17], where the upper bound of Theorem A was found independently (in the $d = 4$ case) and where it is shown that the bound appearing in the theorem is sharp for square-free classes. The best possible bound for $d = 4$ and $n = \mathrm{per}(\alpha)$, as proved by Gu, is

$$\mathrm{ind}(\alpha) | \begin{cases} e_3(n)n^3 & \text{if } 4|n \text{ and} \\ e_2(n)e_3(n)n^3 & \text{otherwise,} \end{cases}$$

where $e_p(n) = p$ if $p|n$ and 1 otherwise. In other words, there exist 8-dimensional finite CW complexes where these upper bounds on the index are achieved.

The fact that for $d = 3$ there exist 6-dimensional finite CW complexes with Brauer classes α having $\mathrm{per}(\alpha) = 2$ and $\mathrm{ind}(\alpha) = 8$ leads to the natural question in [2] of whether the global period–index conjecture might be false at the prime 2 for 3-folds over the complex numbers. However, recent work of Crowley–Grant [11] proves (a) that these topological examples can be found among closed orientable 6-manifolds but that (b) these examples cannot be found among closed 6-dimensional Spin^c-manifolds and hence they cannot be found among closed 6-manifolds of the form $X(\mathbb{C})$ for a smooth projective complex 3-fold X. The question of what happens for $d = 4$ is the subject of ongoing work of Crowley–Gu–Haesemeyer who prove in [12] that for closed orientable 8-manifolds X one has $\mathrm{ind}(\alpha)|\mathrm{per}(\alpha)^3$ for $\alpha \in \mathrm{Br}(X)$ unless $\mathrm{per}(\alpha) \equiv 2 \pmod 4$ in which case $\mathrm{ind}(\alpha)|2\mathrm{per}(\alpha)^3$.

Note that Gu's bound $\mathrm{ind}(\alpha)|e_3(n)n^3$ if $4|n$ is better than the bound $\mathrm{ind}(\alpha)|e_2(n)e_3(n)n^3$ arising from Theorem A. We do not further address in this paper the sharpness of the bounds in Theorem A except to make the following conjecture.

Conjecture C. *The bounds of Corollary B are the best possible. That is, for every $d \geqslant 1$ and every natural number n prime to $(d-1)!$, there exists a finite $2d$-dimensional CW complex X and a Brauer class $\alpha \in \mathrm{Br}(X)$ such that $\mathrm{per}(\alpha) = n$ and $\mathrm{ind}(\alpha) = n^{d-1}$.*

The bounds in the period–index conjecture for function fields are known to be sharp, for example by Gabber's appendix to [9].

2. Background and strategy

We quickly review the period–index problem in three settings.

Period–index for fields. The period–index problem originated in the domain of division algebras over fields. Specifically, for a field K, we have the Brauer group $\mathrm{Br}(K) = \mathrm{H}^2_{\text{ét}}(\mathrm{Spec}\, K, \mathbb{G}_m)$. This group is isomorphic to the set of isomorphism classes of finite dimensional division K-algebras with center exactly K. Given $\alpha \in \mathrm{Br}(K)$, we have two numbers: $\mathrm{per}(\alpha)$, which is the order of α in the torsion abelian group $\mathrm{Br}(K)$, and $\mathrm{ind}(\alpha)$ which is the unique positive integer such that $\mathrm{ind}(\alpha)^2 = \dim_K D$ where D is a division algebra with Brauer class $[D] = \alpha$. It is not hard to see that

$$\mathrm{per}(\alpha)|\mathrm{ind}(\alpha)$$

and Noether proved that these two numbers have the same prime divisors. It follows that there is some integer e_α such that $\mathrm{ind}(\alpha)|\mathrm{per}(\alpha)^{e_\alpha}$.

The period–index problem is to find for a fixed field K a number e such that

$$\mathrm{ind}(\alpha)|\mathrm{per}(\alpha)^e$$

for all $\alpha \in \mathrm{Br}(K)$ and, this done, to find the smallest such number. For example, the Albert–Brauer–Hasse–Noether theorem says that if K is a number field, then $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)$ so that $e = 1$ works (see [14, Remark 6.5.6]).

The period–index conjecture for function fields can be rephrased as saying that if K has transcendence degree d over an algebraically closed field, then $e = d - 1$ is the solution. For $d \geqslant 3$, it is not yet known that there is any e that works for all Brauer classes, but some results are known one prime at a time [19].

Period–index for schemes. We introduced the period–index problem in other settings in [1]. For instance, if X is a quasicompact scheme, then the Brauer group

$$\mathrm{Br}(X) \subseteq \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{tors}} \subseteq \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$$

of Azumaya algebras of Grothendieck [15] is a torsion abelian group. Given $\alpha \in \mathrm{Br}(X)$, we again let $\mathrm{per}(\alpha)$ be the order of α in $\mathrm{Br}(X)$. We define

$$\mathrm{ind}(\alpha) = \gcd\{\deg(\mathcal{A})\colon \mathcal{A} \text{ is an Azumaya algebra with } [\mathcal{A}] = \alpha\}.$$

In [3], we showed that even on smooth schemes over the complex numbers, it is necessary to take the greatest common divisor to obtain a good theory. In this setting, we have $\mathrm{per}(\alpha)|\mathrm{ind}(\alpha)$ and the numbers have the same prime divisors by [4]. Thus, one can formulate the period–index problem for X.

Period–index for topological spaces. Finally, if X is a topological space, we have the Brauer group $\mathrm{Br}(X) \subseteq \mathrm{H}^3(X, \mathbb{Z})_{\text{tors}} \subseteq \mathrm{H}^3(X, \mathbb{Z})$ of topological Azumaya algebras, also introduced in [15]. We define $\mathrm{per}(\alpha)$ and $\mathrm{ind}(\alpha)$ as for schemes. Again, $\mathrm{per}(\alpha)|\mathrm{ind}(\alpha)$ and we proved that these have the same prime divisors if X is a finite CW complex in [1] and in general in [4].

The topological period–index problem of [1] asks the following. Given $d \geqslant 1$, find bounds e such that if X is a finite $2d$-dimensional CW complex and $\alpha \in \mathrm{Br}(X)$, then $\mathrm{ind}(\alpha)|\mathrm{per}(\alpha)^e$. We proposed $e = d - 1$ as a straw man in [2], where we immediately proved that for $d = 3$ this fails in general when $2|\mathrm{per}(\alpha)$. Gu has proved this fails for $d = 4$ if 2 or 3 divides $\mathrm{per}(\alpha)$. But, in these low-dimensional cases, these small primes are the only obstruction. We prove in Theorem A that this pattern continues in higher dimensions.

The topological results reveal a pattern which has not yet been discovered in algebra: a dependence on the prime divisors of the period and their relationship to d. This dependence comes, as we will see, from the 'jumps' in the cohomology of the Eilenberg–MacLane spaces $K(\mathbb{Z}/(n), 2)$.

Strategy. If X is a topological space and $\alpha \in \mathrm{H}^3(X,\mathbb{Z})$, Atiyah and Segal constructed in [6] an α-twisted form of complex K-theory $\mathrm{KU}(X)_\alpha$ and in [7] an α-twisted Atiyah–Hirzebruch spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{H}^s(X, \mathbb{Z}(\tfrac{t}{2})) \Rightarrow \mathrm{KU}^{s+t}(X)_\alpha,$$

where $\mathbb{Z}(\frac{t}{2}) \cong \mathbb{Z}$ if t is even and $\mathbb{Z}(\frac{t}{2}) = 0$ if t is odd. The differentials d_r^α have bidegree $(r, 1-r)$. We proved in [1] that if X is a connected finite CW complex, then $\mathrm{ind}(\alpha)$ generates the group

$$\mathrm{E}_\infty^{0,0} \subseteq \mathrm{H}^0(X,\mathbb{Z}) \cong \mathbb{Z}$$

of permanent cycles. Thus, to bound the index, one attempts to bound the orders of the differentials

$$d_{2r+1} \colon \mathrm{E}_r^{0,0} \to \mathrm{E}_r^{2r+1,-2r}.$$

There is a universal case to consider for all order m topological Brauer classes, namely the space $K(\mathbb{Z}/(m),2)$ and a generator α of $\mathrm{H}^3(K(\mathbb{Z}/(m),2),\mathbb{Z}) \cong \mathbb{Z}/(m)$. By studying the orders of the differentials in this particular case, we prove Theorem A.

3. The cohomology of $K(\mathbb{Z}/(n), 2)$

We recall some results of Cartan [8] on the cohomology of Eilenberg–MacLane spaces in the special case of $K(\mathbb{Z}/(n),2)$, which we will use in the next section to give upper bounds on the index of period n classes. We claim no originality in our presentation here, but we hope its inclusion will be useful to the reader.

Write

$$n = p_1^{r_1} \cdots p_k^{r_k},$$

and pick a generator u_i of the subgroup $\mathbb{Z}/(p_i^{r_i})$ of $\mathbb{Z}/(n)$ for $1 \leqslant i \leqslant k$. Let $v_i = p_i^{r_k - 1} u_i$. Cartan [8, Théorème 1] gives a recipe for computing the integral homology (Pontryagin) ring of $K(\mathbb{Z}/(n),2)$.

For each prime p and positive integer f, consider certain words in the 4 symbols:

$$\sigma, \quad \gamma_p, \quad \phi_p, \quad \psi_{p^f}.$$

The symbol ψ_{p^f}, if it appears, is the last symbol in a word. The *height* of a word α is the total number of σ, ϕ_p and ψ_{p^f} appearing. The *degree* of α is defined recursively by letting $\deg(\emptyset) = 0$ and

$$\deg(\sigma\alpha) = 1 + \deg(\alpha), \qquad \deg(\gamma_p\alpha) = p\deg(\alpha),$$
$$\deg(\phi_p\alpha) = 2 + p\deg(\alpha), \qquad \deg(\psi_{p^f}) = 2.$$

For each prime p, an *admissible* p-word α is a word on 3 symbols σ, γ_p, and ϕ_p such that α is non-empty, the first and last letters of α are σ or ϕ_p, and for each letter γ_p or ϕ_p, the number of letters σ appearing to the right in α is even. In addition to the admissible words, we will use the auxiliary words $\sigma^{h-1}\psi_{p^f}$, of height h and degree $h+1$.

Let $E(x, 2q-1)$ denote the exterior graded algebra over $\mathbb{Z}$ with generator x of degree $2q-1$ endowed with the trivial dg-algebra structure $dx = 0$. Let $P(x, 2q)$ be the divided power polynomial algebra over $\mathbb{Z}$ with generator x of degree $2q$ and given the trivial dg algebra structure with $dx = 0$. Cartan calls $E(x, 2q-1)$ and $P(x, 2q)$ *elementary complexes of the first type.* Define tensor dg algebras $E(x, 2q-1) \otimes_{\mathbb{Z}} P(y, 2q)$ by $dx = 0$ and $dy = hx$ for some integer h and $P(x, 2q) \otimes_{\mathbb{Z}} E(y, 2q+1)$ by $dx = 0$ and $dy = hx$ (the integer h is part of the data, even though it is not specified in the notation). These are the *elementary complexes of the second type.* The positive-degree homology groups of $E(x, 2q-1) \otimes_{\mathbb{Z}} P(x, 2q)$ are

$$\mathrm{H}_{2q-1+2qk}(E(x, 2q-1) \otimes_{\mathbb{Z}} P(x, 2q)) = \mathbb{Z}/h \cdot x\gamma_k(y)$$

for $k \geqslant 0$ and and 0 otherwise, where γ_k is the kth divided power operation. For $P(x, 2q) \otimes_{\mathbb{Z}} E(y, 2q+1)$, we get

$$\mathrm{H}_{2qk}(P(x, 2q) \otimes_{\mathbb{Z}} E(y, 2q+1)) = \mathbb{Z}/hk \cdot \gamma_k(x)$$

for $k \geqslant 0$ and all other homology groups are 0.

The height-2 admissible or auxiliary p-words are

$$\sigma^2, \qquad \sigma\gamma_p^k\phi_p, \qquad \phi_p\gamma_p^k\phi_p, \qquad \sigma\psi_{p^f}$$

for $k \geqslant 0$ and $f \geqslant 1$. These are of degrees 2, $1 + 2p^k$, $2 + 2p^{k+1}$, and 3, respectively. Below, the symbols u_i and v_i, are just formal indeterminates to keep track of generators for different dg algebras.

For each p_i, we define a dg algebra X_{p_i} as the following tensor product of elementary complexes of the second type

$$X_{p_i} = P(\sigma^2 u_i, 2) \otimes E(\sigma\psi_{p_i^{r_i}} u_i, 3)$$
$$\bigotimes_{k=0}^{\infty} E(\sigma\gamma_{p_i}^{k+1}\phi_{p_i} v_i, 1 + 2{p_i}^{k+1}) \otimes P(\phi_{p_i}\gamma_{p_i}^{k}\phi_{p_i} v_i, 2 + 2{p_i}^{k+1}).$$

The differentials are

$$d(\sigma\psi_{p_i^{r_i}} u_i) = p_i^{r_i}\sigma^2 u_i \quad \text{and} \quad d(\phi_{p_i}\gamma_{p_i}^{k}\phi_{p_i} v_i) = p_i\sigma\gamma_{p_i}^{k+1}\phi_{p_i} v_i,$$

i.e., $h = p_i^{r_i}$ or $h = p_i$, respectively. (Here the r_i are the exponents appearing in the prime decomposition of n.) Let

$$X = X_{p_1} \otimes \cdots \otimes X_{p_k}.$$

Then, Cartan [8, Théorème 1] gives a surjection, which depends on the choice of the u_i,

$$\mathrm{H}_k(X) \to \mathrm{H}_k(K(\mathbb{Z}/(n), 2), \mathbb{Z}),$$

and the kernel is described. This map induces for each i a surjection

$$\mathrm{H}_k(X_{p_i}) \to \mathrm{H}_k(K(\mathbb{Z}/(n), 2), \mathbb{Z})\{p_i\},$$

the p_i-primary part of homology. The largest possible torsion in $\mathrm{H}_{2k}(X_{p_i})$ comes from the first term $P(\sigma^2 u_2, 2) \otimes E(\sigma\psi_{p_i^{r_i}} u_i, 3)$. Specifically, by using the Künneth theorem, the exponent of $\mathrm{H}_{2k}(X_{p_i})$ is the same as the exponent of $\mathrm{H}_{2k}(P(\sigma^2 u_2, 2) \otimes E(\sigma\psi_{p_i^{r_i}} u_i, 3))$, namely

$$p_i^{r_i} k.$$

It follows that the exponent of $\mathrm{H}_{2k}(X)$ is at most nk. Write $k = k'a$, where a is largest integer dividing k and prime to m. Since $\mathrm{H}_{2k}(K(\mathbb{Z}/(n), 2), \mathbb{Z})$ is n-primary torsion, we see that nk' kills all of $\mathrm{H}_{2k}(K(\mathbb{Z}/(n), 2), \mathbb{Z})$; in fact, the exponent is exactly nk', as follows from Cartan's description of the kernel of $\mathrm{H}_{2k}(X) \to \mathrm{H}_{2k}(K(\mathbb{Z}/(n), 2), \mathbb{Z})$, but we will not need this fact.

Since we will be able to argue prime-by-prime in a moment, it is helpful to record the p_i-primary part of the exponent of $\mathrm{H}_{2k}(K(\mathbb{Z}/(n), 2), \mathbb{Z})$ for a given prime p_i. If $\xi \in \mathrm{H}_{2k}(K(\mathbb{Z}/(n), 2), \mathbb{Z})$ is an element of p_i-primary order, then the order of ξ divides $p_i^{r_i + v_{p_i}(k)}$.

4. Proof of the main theorem

If X is a topological space with finitely many connected components, then the topological Brauer group $\mathrm{Br}(X)$ is a subgroup of $\mathrm{Br}'(X) = \mathrm{H}^3(X,\mathbb{Z})_{\mathrm{tors}}$. Serre showed that if X is compact, then $\mathrm{Br}(X) = \mathrm{Br}'(X)$ (see [15]). This will be the main setting of this paper. We want to make a general remark on a different version of the index to which our methods apply whether or not X is compact.

We may weaken the hypothesis on the finiteness of X in Theorem A at the expense of using the K-theoretic index rather than the topological index. Recall from [1] that the K-theoretic index $\mathrm{ind}_{\mathrm{K}}(\alpha)$ is defined as the (positive) generator of the image of the rank map $\mathrm{KU}^0(X)_\alpha \to \mathbb{Z}$, where $\mathrm{KU}^0(X)_\alpha$ denotes the α-twisted K-theory group. When X is finite-dimensional (and connected), one computes $\mathrm{ind}_{\mathrm{K}}(\alpha)$ as the generator of the group

$$\mathrm{E}_\infty^{0,0} \subseteq \mathrm{E}_2^{0,0} \cong \mathbb{Z}$$

by the convergence of the α-twisted Atiyah–Hirzebruch spectral sequence; hence the differentials $d_{2k+1}\colon \mathrm{E}_{2k+1}^{0,0} \to \mathrm{E}_{2k+1}^{2k+1,-2k}$ of the twisted Atiyah–Hirzebruch spectral sequence control the K-theoretic index. It is this crucial fact we exploit below.

In general, $\mathrm{per}(\alpha)|\mathrm{ind}_{\mathrm{K}}(\alpha)|\mathrm{ind}(\alpha)$ and if X is compact, then $\mathrm{ind}_{\mathrm{K}}(\alpha) = \mathrm{ind}(\alpha)$. Thus, Theorem A follows from the following theorem.

Theorem 4.1. *Let X be a $2d$-dimensional CW complex, and let $\alpha \in \mathrm{Br}'(X)$ have period $m = p_1^{r_1}\cdots p_k^{r_k}$ where the p_i are distinct primes. Then,*

$$\mathrm{ind}_{\mathrm{K}}(\alpha) \Big| \prod_{i=1}^{k} {p_i}^{(d-1)r_i + v_{p_i}(2)+\cdots+v_{p_i}(d-1)} = m^{d-1}\prod_{i=1}^{k} {p_i}^{v_{p_i}((d-1)!)},$$

where v_{p_i} is the p_i-adic valuation.

Proof. According to the main result of [5], it is enough to prove the theorem when m is a prime power. So, suppose that $m = p_1^{r_1} = p^r$. Let β be a generator of $\mathrm{H}^3(K(\mathbb{Z}/(p^r),2),\mathbb{Z}) \cong \mathbb{Z}/(p^r)$. There is some map $\sigma\colon X \to K(\mathbb{Z}/(p^r),2)$ such that $\sigma^*\beta = \alpha$. The twisted Atiyah–Hirzebruch spectral sequence is functorial, so we obtain—in particular—a map on the $(0,0)$-terms in each page:

$$\mathrm{E}_j^{0,0}(K(\mathbb{Z}/(p^r),2)) \to \mathrm{E}_j^{0,0}(X).$$

An elementary induction argument shows that this map is a monomorphism on each page, and so by cellular approximation, $\mathrm{ind}_{\mathrm{K}}(\alpha)$ is bounded above by $\mathrm{ind}_{\mathrm{K}}$ for the restriction of β to a $2d$-skeleton of $K(\mathbb{Z}/(n),2)$.

There is an isomorphism

$$\tilde{\mathrm{H}}^{2j+1}(K(\mathbb{Z}/(p^r),2),\mathbb{Z}) \cong \tilde{\mathrm{H}}_{2j}(K(\mathbb{Z}/(p^r),2),\mathbb{Z}).$$

We have seen in Section 3 that the latter group is $p^r p^{v_p(j)}$-torsion. In the Atiyah–Hirzebruch spectral sequence for the Bockstein β on a $2d$-skeleton of $K(\mathbb{Z}/(p^r),2)$, only the differentials d^{β}_{2j+1} for $1 \leqslant j \leqslant d-1$ are possibly non-zero—in particular, the cohomology group in degree $2d$ of the skeleton, which may differ from that of $K(\mathbb{Z}/(p^r),2)$, plays no part in the calculation. We are interested in the order of the image of

$$d^{\beta}_{2j+1}\colon \mathrm{E}^{0,0}_{2j+1} \to \mathrm{E}^{2j+1,-2j}_{2j+1},$$

where the latter group is a subquotient of $\mathrm{H}^{2j+1}(K(\mathbb{Z}/(p^r),2),\mathbb{Z})$, and hence the order of the image divides $p^{r+v_p(j)}$. This gives the result. □

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Department of Mathematics, Northwestern University
2033 Sheridan Road, Evanston, IL 60208, USA
E-mail address: antieau@northwestern.edu

Department of Mathematics, The University of British Columbia
1984 Mathematics Road, Vancouver, B.C. V6T 1Z2, Canada
E-mail address: tbjw@math.ubc.ca

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