Math. Res. Lett.
Volume 28, Number 5, 1379–1418, 2021

Gelfand-Tsetlin modules for $\mathfrak{gl}(m|n)$

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We address the problem of classifying irreducible Gelfand-Tsetlin modules for $\mathfrak{gl}(m|n)$ and show that it reduces to the classification of Gelfand-Tsetlin modules for the even part. We also give an explicit tableaux construction and the irreducibility criterion for the class of quasi typical and quasi covariant Gelfand-Tsetlin modules which includes all essentially typical and covariant tensor finite dimensional modules. In the quasi typical case new irreducible representations are infinite dimensional $\mathfrak{gl}(m|n)$-modules which are isomorphic to the parabolically induced (Kac) modules.

1. Introduction

Throughout the paper the field is assumed to be the field of complex numbers. Denote by Let $\mathfrak{g}$ be the Lie superalgebra $\mathfrak{gl}(m|n)$ which consists of $(m+n)\times(m+n)$ complex matrices of the block form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1}$$

where A, B, C and D are respectively $m\times m, m\times n, n\times m$ and $n\times n$ matrices. The even subalgebra $(\mathfrak{g}_{m,n})_{\bar{0}} \simeq \mathfrak{gl}(m)\oplus\mathfrak{gl}(n)$ consists matrices with $B=0$ and $C=0$; the odd subspace $(\mathfrak{g}_{m,n})_{\bar{1}}$ consists those with $A=0$ and $D=0$. The Lie superalgebra is then defined by the bracket $[x,y] = xy - (-1)^{|x||y|}yx$, where x and y are homogeneous elements.

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(m|n)$ is the subalgebra of diagonal matrices. We denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$.

A chain of subalgebras of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \ldots \supset \mathfrak{g}^{m+n},$$

such that $\mathfrak{g}^k$ is isomorphic to $\mathfrak{gl}(p|q)$ with $p+q = m+n-k+1$ is called a *complete flag* in $\mathfrak{g}$ if $\mathfrak{h}^k := \mathfrak{h}\cap\mathfrak{g}^k$ is a Cartan subalgebra of $\mathfrak{g}^k$. A complete flag induces a chain of Cartan subalgebras

$$\mathfrak{h} = \mathfrak{h}^1 \supset \mathfrak{h}^2 \supset \ldots \supset \mathfrak{h}^{m+n}.$$

Every complete flag $\mathcal{C}$ in $\mathfrak{g}$ defines the commutative subalgebra $\Gamma_{\mathcal{C}}$ in $U(\mathfrak{g})$ generated by the centers of the members of the chain. We will say that $\Gamma_{\mathcal{C}}$ is the *Gelfand-Tsetlin subalgebra* of $U(\mathfrak{g})$associated with the flag $\mathcal{C}$.

Let $\Gamma = \Gamma_{\mathcal{C}}$ be a Gelfand-Tsetlin subalgebra of $U(\mathfrak{g})$. A finitely generated module M over $\mathfrak{g}$ is called a *Gelfand-Tsetlin module* (with respect to Γ) if

$$M = \bigoplus_{\mathbf{m}\in \operatorname{Specm}\Gamma} M(\mathbf{m})$$

as a Γ-module, where

$$M(\mathbf{m}) \;=\; \{x\in M \mid \mathbf{m}^k x = 0 \quad \text{for some} \quad k\geqslant 0\}$$

and $\operatorname{Specm}\Gamma$ denotes the set of maximal ideals of Γ. Identifying maximal ideals of $\mathbf{m}\in\operatorname{Specm}\Gamma$ with kernels of characters $\chi\in\hat{\Gamma}$, Gelfand-Tsetlin module

M (with respect to Γ) will have the following form:

$$M = \bigoplus_{\chi \in \hat{\Gamma}} M(\chi),$$

where

$$M(\chi) \;=\; \{x \in M \mid (z - \chi(z))^k x = 0 \quad \text{for some} \quad k \geqslant 0 \quad \text{and all} \quad z \in \Gamma\}.$$

Recall that $\mathfrak{g} = \mathfrak{gl}(m|n)$ has a consistent $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. There is a bijection between irreducible $\mathfrak{g}$-supermodules and irreducible $\mathfrak{g}_0$-supermodules [CM]. The purpose of our paper is to address the classification problem of irreducible Gelfand-Tsetlin modules. We show that this problem reduces to the classification of Gelfand-Tsetlin modules for the even part of $\mathfrak{g}$ with respect to the Gelfand-Tsetlin subalgebra $\phi(\Gamma)$, where $\phi : U(\mathfrak{g}) \to U(\mathfrak{g}_0)$ is the projection with the kernel $\mathfrak{g}_{-1}\, U(\mathfrak{g}) + U(\mathfrak{g})\, \mathfrak{g}_1$. Our first result is summarized in the following theorem

Theorem 1.1. *Let Γ be a Gelfand-Tsetlin subalgebra of $U(\mathfrak{g})$,$\Gamma_0 = \phi(\Gamma)$.*

(i) *A $\mathfrak{g}$-module M is locally finite over Γ if and only if it is locally finite over Γ_0.*

(ii) *Let M be an irreducible Gelfand-Tsetlin (with respect to Γ) $\mathfrak{g}$-module. Then M is a unique simple quotient of the induced (Kac) module*

$$K(N) := U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0 \oplus \mathfrak{g}_1)} N$$

for some irreducible Gelfand-Tsetlin (with respect to Γ_0) $\mathfrak{g}_0$-module N (we assume $\mathfrak{g}_1\, N = 0$).

(iii) *For any $\chi_0 \in \hat{\Gamma}_0$ there exists only finitely many non-isomorphic irreducible Gelfand-Tsetlin (with respect to Γ) $\mathfrak{g}$-modules M with $M(\chi_0) \neq 0$.*

(iv) *If Γ' is another Gelfand-Tsetlin subalgebra of $\mathfrak{g}$ such that $\phi(\Gamma') = \Gamma_0$ then the categories of Gelfand-Tsetlin $\mathfrak{g}$ -modules for Γ and Γ' coincide.*

(v) *The subcategory of Gelfand-Tsetlin (with respect to Γ) $\mathfrak{g}$-modules with a fixed typical central character is equivalent to the subcategory of Gelfand-Tsetlin (with respect to Γ_0) $\mathfrak{g}_0$-modules with a suitable central character.*

A classical paper of Gelfand and Tsetlin [GT] gives a construction of simple finite dimensional modules for $\mathfrak{g}_0$ with explicit basis consisting of Gelfand-Tsetlin tableaux and the action of the Lie algebra. Infinite dimensional Gelfand-Tsetlin modules were studied extensively and many remarkable connections and applications were discovered (cf. [DFO], [EMV], [FGRZ], [FO], [H], [KW1], [KW1], [KTWWY], [O], [W] and references therein).

In [FRZ] a new combinatorial method of constructing of Gelfand-Tsetlin modules was developed. It is based on the work of Gelfand and Graev [GG] and of Lemire and Patera [LP] who initiated a study of formal continuations of both the labelling tableaux and the classical Gelfand-Tsetlin formulas. Imposing certain modified conditions on the entries of a tableau new infinite dimensional irreducible modules can be constructed explicitly with the Lie algebra action given by the classical Gelfand-Tsetlin formulas.

Analogs of the Gelfand–Tsetlin bases for a certain class of finte dimensional *essentially typical* representations of the Lie superalgebra $\mathfrak{gl}(m|n)$ were constructed in [Pal1], [Pal2]. Essentially typical representations is a class of typical representations that have consecutive restrictions to the subalgebras of the chain $\mathfrak{gl}(m|1) \subset \ldots \subset \mathfrak{gl}(m|n)$ completely reducible. This allows to apply the classical Gelfand-Tsetlin formulas to obtain explicit construction of all essentially typical representations. A basis is given by special Gelfand-Tsetlin tableaux.

Also alternative explicit construction of finite dimensional irreducible *covariant tensor* modules over $\mathfrak{gl}(m|n)$ was given in [M2], [SV]. Highest weight of such representation is a $m+n$-tuple $\lambda = (\lambda_1, \lambda_2, \ \ldots, \lambda_{m+n})$ of non-negative integers such that

$$\lambda_i - \lambda_{i+1} \in \mathrm{Z}_{\geq 0}, \text{ for } i = 1, \ldots, m+n-1, \quad i \neq m \tag{2}$$

and

$$\lambda_m \geq \#\{i : \lambda_i > 0, \ m+1 \leq i \leq m+n\}. \tag{3}$$

We generalize the constructions of essentially typical and covariant tensor representations using the combinatorial approach developed in [FRZ]. It allows us to obtain large families of *quasi typical* and *quasi covariant* Gelfand-Tsetlin modules respectively, together with their explicit tableau realization. In particular, these families contain all finite dimensional essentially typical and covariant tensor representations. We give necessary and sufficient conditions of the irreducibility of constructed modules. These

results are subjects of Theorem 7.5, Theorem 8.3, Theorem 8.4 and Theorem 9.3. We summarize the statements in the following theorem.

Theorem 1.2. *Let $\mathcal{C}$ be an admissible (respectively, covariant admissible) set of relations.*

(i) *The formulas* (23)-(29) *define a quasi typical (respectively, quasi covariant) $\mathfrak{gl}(m|n)$-module structure on $V_{\mathcal{C}}([l^0])$ for any tableau $[l^0]$ satisfying $\mathcal{C}$ (respectively, for any $\mathcal{C}$-covariant tableau $[l^0]$).*

(ii) *The module $V_{\mathcal{C}}([l^0])$ is irreducible if and only if $\mathcal{C}$ is the maximal set of relations satisfied by $[l^0]$ and $l^0_{m+n,i} \neq l^0_{m+n,j}, 1 \leq i \leq m < j \leq m+n$ in the quasi typical case.*

(iii) *The module $V_{\mathcal{C}}([l^0])$ is irreducible if and only if $\mathcal{C}$ is the maximal set of relations satisfied by $[l^0]$ in the quasi covariant case.*

(iv) *If $V_{\mathcal{C}}([l^0])$ is irreducible quasi typical module then it is isomorphic to the induced module $K(V_{\mathcal{C}}([l^0])^{\mathfrak{g}_1})$.*

2. Preliminaries

2.1. Weight modules

A $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with even bracket $[\bullet, \bullet] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is a Lie superalgebra iff the following conditions hold

$$[a, b] = -(-1)^{p(a)p(b)}[b, a];$$

$$[a, [b, c]] = [[a, b], c] + (-1)^{p(a)p(b)}[a, [b, c]].$$

Let $\mathfrak{g}$ be a finite dimensional Lie superalgebra equipped with a grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_{\pm 1}$ are odd abelian subalgebras and $\mathfrak{g}_0 = g_{\bar{0}}$ is even. Assume that $\mathfrak{g}_0$, $\mathfrak{g}_{\pm 1}$ are semisimple $\mathfrak{g}_0$-modules and $\mathfrak{g}_0$ contains an element d such that $[d, x] = ix$ for any $x \in \mathfrak{g}_i$.

Let $\mathfrak{h}$ denote a Cartan subalgebra of $\mathfrak{g}_0$ and assume that it coincides with its centralizer in $\mathfrak{g}$. Then $\mathfrak{g}$ has a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta \subset \mathfrak{h}^*} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x\}$.

We fix the sets of positive and negative roots $\Delta = \Delta^+ \cup \Delta^-$ and a corresponding triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

where $\mathfrak{n}^\pm = \bigoplus \mathfrak{g}_\alpha$, $\alpha \in \Delta^\pm$.

A $\mathfrak{g}$-module (respectively, $\mathfrak{g}_0$-module) M is called a *weight* module (with respect to $\mathfrak{h}$) if

$$M = \oplus_{\lambda \in \mathfrak{h}^*} M_\lambda,$$

where $M_\lambda = \{v \in M | hv = \lambda(h)v \quad \text{for all} \quad h \in \mathfrak{h}\}$.

Let M be a weight module over $\mathfrak{g}_0$. Denote by $K(M)$ the induced module:

$$K(M) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0 \oplus \mathfrak{g}_1)} M,$$

where $\mathfrak{g}_1 M = 0$. If M is irreducible then $K(M)$ has a unique irreducible quotient which we will denote $L(M)$.

Hence, $\mathbf{K}$, which sends M to $K(M)$, defines a functor from the category of weight $\mathfrak{g}_0$-modules to the category of weight $\mathfrak{g}$-modules. On the other hand, for a weight $\mathfrak{g}$-module V set $R(V) = V^{\mathfrak{g}_1}$ which is a weight $\mathfrak{g}_0$-module. Thus we obtain the functor $\mathbf{R}$, which sends V to $R(V)$, from the category of weight $\mathfrak{g}$-modules to the category of weight $\mathfrak{g}_0$-modules.

The following statement is an immediate consequence of the Frobenius reciprocity for induced modules.

Proposition 2.1. *The pair* $(\mathbf{K}, \mathbf{R})$ *is adjoint.*

Important examples of weight modules are Verma modules and their irreducible quotients. Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, $\lambda \in \mathfrak{h}^*$. A Verma module with highest weight λ is the following induced module

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} Cv_\lambda,$$

where $\mathfrak{n}^+ v = 0$ and $hv = \lambda(h)v$ for all $h \in \mathfrak{h}$. It has a unique irreducible quotient $L(\lambda)$.

For a dominant weight λ denote by $L_0(\lambda)$ the unique irreducible $\mathfrak{g}_0$-module with highest weight λ. Then $K(\lambda) := K(L_0(\lambda))$ is the indecomposable *Kac module* with highest weight λ.

Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} (-1)^{p(\alpha)} \alpha$ and $\lambda \in \mathfrak{h}^*$. A dominant weight λ is *typical* if $(\lambda + \rho, \alpha) \neq 0$ for any odd positive root α. In the case of dominant typical λ, $K(\lambda) := L(\lambda)$.

We have $U(\mathfrak{g}) = (\mathfrak{g}_{-1}\,U(\mathfrak{g}) + U(\mathfrak{g})\,\mathfrak{g}_1) \oplus U(\mathfrak{g}_0)$. Then the projection onto $U(\mathfrak{g}_0)$ defines an analogue of the *Harish-Chandra homomorphism* $\phi : U(\mathfrak{g})^d \to U(\mathfrak{g}_{\bar{0}})$ where $U(\mathfrak{g})^d = \{y \in U(\mathfrak{g}) \mid [d, y] = 0\}$.

Let $Z(\mathfrak{g})$ denote the center of universal enveloping algebra $U(\mathfrak{g})$ [BZV], [K2], [Se]. Then $\phi(Z(\mathfrak{g})) \subset Z(\mathfrak{g}_0)$, where $Z(\mathfrak{g}_0)$ is the center of the universal enveloping algebra $U(\mathfrak{g}_0)$. If ϕ_0 is the classical Harish-Chandra homomorphism for $\mathfrak{g}_0$, then $\phi_0 \circ \phi : U(\mathfrak{g}) \to U(\mathfrak{h}) = \mathrm{Pol}(\mathfrak{h}^*)$.

2.2. The Lie superalgebra $\mathfrak{gl}(m|n)$

The underlying vector space of the Lie superalgebra $\mathfrak{g}_{m,n} = \mathfrak{gl}(m|n)$ is spanned by the standard basis elements $E_{ij}, 1 \le i, j \le m+n$. The Z_2-grading on $\mathfrak{gl}(m|n)$ is defined by $E_{ij} \mapsto \bar{i} + \bar{j}$, where $\bar{i}$ is an element of Z_2 which equals 0 if $i \le m$ and 1 if $i > m$. The commutation relations in this basis are given by

$$[E_{ij},\ E_{kl}] = \delta_{kj}E_{il} - \delta_{il}E_{kj}(-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})}. \tag{4}$$

Alternatively, $\mathfrak{g}_{m,n}$ can be defined as the quotient of the free Lie superalgebra $\hat{\mathfrak{g}}$ over $\mathbb{C}$ generated by $e_i, f_i, (1 \le i \le m+n-1)$, $h_j, (1 \le j \le m+n)$, subject to the following relations (unless stated otherwise, the indices below run over all possible values):

$$[h_i, h_j] = 0; \tag{5}$$
$$[h_i, e_j] = (\delta_{ij} - \delta_{i,j+1})e_j; \tag{6}$$
$$[h_i, f_j] = -(\delta_{ij} - \delta_{i,j+1})f_j; \tag{7}$$
$$[e_i, f_j] = 0 \text{ if } i \ne j; \tag{8}$$
$$[e_i, f_i] = h_i - h_{i+1} \text{ if } i \ne m; \tag{9}$$
$$[e_m, f_m] = h_m + h_{m+1}; \tag{10}$$
$$[e_i, e_j] = [f_i, f_j] = 0 \text{ if } |i-j| > 1; \tag{11}$$
$$[e_m, e_m] = [f_m, f_m] = 0; \tag{12}$$
$$[e_i, [e_i, e_{i\pm1}]] = [f_i, [f_i, f_{i\pm1}]] = 0, \text{ for } i \ne m; \tag{13}$$
$$[e_m[e_{m\pm1}, [e_m, e_{m\mp1}]]] = [f_m[f_{m\pm1}, [f_m, f_{m\mp1}]]] = 0; \tag{14}$$

The isomorphism between two algebras is defined by $e_i \mapsto E_{i,i+1}$, $f_i \mapsto E_{i+1,i}$, $1 \leq i \leq m+n-1$ and $h_i \mapsto E_{ii}, 1 \leq i \leq m+n$. We define the parity function on $\{1, \dots, m+n\}$ by setting

$$\bar{j} = \begin{cases} 0 & j \leq m \\ 1 & j \geq m+1 \end{cases}.$$

Let $\hat{E}$ be the $(m+n) \times (m+n)$ matrix with coefficients in $U(\mathfrak{g})$ whose ij-th entry equals $\hat{E}_{ij} = (-1)^{\bar{j}} E_{ij}$. The *quantum Berezinian* is an element in $U(\mathfrak{g})[[t]]$ defined by

$$B(t) = \sum_{\sigma \in S_m} sgn(\sigma)(1+t\hat{E})_{\sigma(1),1} \cdots (1+t(\hat{E}-m+1))_{\sigma(m),m} \tag{15}$$
$$\times \sum_{\tau \in S_n} sgn(\tau)(1+t(\hat{E}-m+1))^{-1}_{m+1,m+\tau(1)} \cdots (1+t(\hat{E}-m+n))^{-1}_{m+n,m+\tau(n)}.$$

The coefficients of quantum Berezinian generate the center of $U(\mathfrak{gl}_{m,n})$, see [Naz], [M1].

Fix the Borel subalgebra $\mathfrak{g}$ containing $\mathfrak{g}_1$. Let $x_i = \lambda_i - i + 1$ for $i = 1, \dots, m$, $x_j = -\lambda_j + j - 2m$ for $j = m+1, \dots, m+n$. Then $B(t)$ acts on $L(\lambda)$ by the scalar

$$\frac{(1+tx_1) \cdots (1+tx_m)}{(1+tx_{m+1}) \cdots (1+tx_{m+n})}. \tag{16}$$

The even and the odd roots in the standard basis are respectively the following:

$$\Delta_0 = \{(\varepsilon_i - \varepsilon_j) | i,j \leq m\} \cup \{(\delta_i - \delta_j) | i,j \leq n\},$$
$$\Delta_1 = \{\pm(\varepsilon_i - \delta_j) | i \leq m, j \leq n\},$$

where

$$(\varepsilon_i, \delta_j) = 0,\ (\varepsilon_i, \varepsilon_j) = \delta_{ij},\ (\delta_i, \delta_j) = -\delta_{ij}.$$

3. Gelfand-Tsetlin subalgebras

Set $\mathfrak{g}_{k,l} = \mathfrak{gl}(k|l)$ where $k \geq 0$ and $l \geq 0$. We call a *Gelfand-Tsetlin chain* any chain of subalgebras $\mathfrak{g}_{k,l}$ in $\mathfrak{g}_{m|n}$ such that on every step we have either $\mathfrak{g}_{k,l} \supset \mathfrak{g}_{k-1,l}$ or $\mathfrak{g}_{k,l} \supset \mathfrak{g}_{k,l-1}$.

Fix a Gelfand-Tsetlin chain of subalgebras. Let $Z_{k,l} = Z(\mathfrak{g}_{k,l})$ be the center of the universal enveloping algebra $U(\mathfrak{g}_{k,l})$. A commutative subalgebra

$\Gamma \in U(\mathfrak{g}_{m,n})$ generated by the centers $Z_{k,l}$ corresponding to the members of the Gelfand-Tsetlin chain is called a *Gelfand-Tsetlin subalgebra*. We will say that a Gelfand-Tsetlin subalgebra is *special* if it contains a Gelfand-Tsetlin subalgebra of $\mathfrak{gl}(n)$ or $\mathfrak{gl}(m)$.

Consider a Gelfand-Tsetlin chain in $\mathfrak{g}_{m,n}$ and a corresponding Gelfand-Tsetlin subalgebra Γ. The Gelfand-Tsetlin chain of Γ induces the Gelfand-Tsetlin chain in the even part $\mathfrak{g}_0$ with subalgebras $(\mathfrak{g}_{k,l})_0$.

The centers $Z((\mathfrak{g}_{k,l})_0)$ of the universal enveloping algebras $U((\mathfrak{g}_{k,l})_0)$ generate a Gelfand-Tsetlin subalgebra Γ_0 of $U(\mathfrak{g}_0)$.

Recall the homomorphism $\phi : U(\mathfrak{g})^d \to U(\mathfrak{g}_0)$. Since $\Gamma \subset U(\mathfrak{g})^d$ we can consider the restriction $\phi(\Gamma) \subset U(\mathfrak{g}_0)$. Since $\phi(Z_{k,l} \subset Z((\mathfrak{g}_{k,l})_0)$ we get $\phi(\Gamma) \subset \Gamma_0$.

Proposition 3.1. *The restriction of $\phi : \Gamma \to \Gamma_0$ is surjective, that is $\phi(\Gamma) = \Gamma_0$.*

Proof. We will prove the statement by induction on the rank $m + n$. The base of induction is trivial. Suppose that $\mathfrak{g}' = \mathfrak{g}_{m-1,n}$ is the member of the Gelfand-Tsetlin chain, $\Gamma' \subset \Gamma$ is the corresponding Gelfand-Tsetlin subalgebra of $U(\mathfrak{g}')$ and $\Gamma'_0 \subset \Gamma_0$ is the corresponding Gelfand-Tsetlin subalgebra of $U(\mathfrak{g}'_0)$. By induction assumption $\phi(\Gamma') = \Gamma'_0$. This implies, in particular, that $\phi(\Gamma')$ contains $Z(\mathfrak{gl}(n))$. We need now the following

Lemma 3.2. *$Z(\mathfrak{gl}(n))$ and $\phi(Z_{m,n})$ generate the center of $U(\mathfrak{g}_0)$.*

Proof. Recall the description of the image under the Harish-Chandra homomorphism $HC := \phi_0\phi$ given in the the end of the previous section. Using formula (16) and the fact that $HC(Z(\mathfrak{gl}(n)))$ is generated by coefficients of $(1 + tx_{m+1}) \cdots (1 + tx_{m+n})$ we obtain that $HC(Z(\mathfrak{gl}(n)))$ and $HC(Z_{m,n})$ generate $HC(Z(\mathfrak{g}_0))$. Since $\phi_0 : Z(\mathfrak{g}_0) \to S(\mathfrak{h})$ is injective, we obtain the statement of Lemma. □

Hence, $\phi(Z_{m,n})$ and Γ'_0 generate $Z(\mathfrak{g}_0)$ and thus Γ_0. We conclude that ϕ is surjective. □

Note that the restriction $\phi : Z(\mathfrak{g}_{m,n}) \to Z(\mathfrak{g}_0)$ is injective but not surjective. In contrast the restriction of ϕ to Γ is surjective but not injective as one can see from the following example.

Example 3.3. *Let $\mathfrak{g} = \mathfrak{gl}(1|1)$. In this case there exist two Gelfand-Tsetlin chains $\mathfrak{gl}(1|1) \supset \mathfrak{gl}(1|0)$ and $\mathfrak{gl}(1|1) \supset \mathfrak{gl}(0|1)$. Both chains define the same*

Gelfand-Tsetlin subalgebra Γ *generated by* E_{00}, E_{11} *and* $x = E_{10}E_{01} \in \operatorname{Ker}\phi_0$. *Observe that we have the relation*

$$x(x - E_{00} - E_{11}) = 0.$$

The homomorphism $\phi : \Gamma \to \Gamma_0$ induces the dual map

$$\phi^* : \hat{\Gamma}_0 \to \hat{\Gamma},$$

which is injective by the above proposition.

Remark 3.4. *Since* Γ_0 *is a polynomial algebra there exists an injective homomorphism* $\psi : \Gamma_0 \to \Gamma$ *such that* $\phi\psi = \mathrm{id}$ *and* $\psi((\Gamma_{k,l})_0) \subset \Gamma_{k,l}$.

Remark 3.5. *It is not difficult to see that if* $\mathfrak{g} = \mathfrak{gl}(n|1)$ *and* Γ *is the Gelfand-Tsetlin subalgebra associated with the chain*

$$\mathfrak{gl}(1) \subset \mathfrak{gl}(2) \subset \cdots \subset \mathfrak{gl}(n) \subset \mathfrak{gl}(n|1)$$

then Γ *contains* Γ_0. *In general,* Γ_0 *is not a subalgebra of* Γ.

Example 3.6. *Consider the chain of superalgebras:*

$$\mathfrak{gl}(1) \subset \mathfrak{gl}(1|1) \subset \mathfrak{gl}(1|2).$$

Let us see that Γ *does not contain* Γ_0, *moreover, these two subalgebras do not commute. Indeed, by the previous example* $x = E_{10}E_{01}$ *is an element of* Γ *and* $y = E_{32}E_{23}$ *is an element of* Γ_0, *since the Casimir element of* $U(\mathfrak{gl}(2))$ *lies in* $U(\mathfrak{h}) + 2y$. *A simple computation shows that*

$$[x, y] = E_{31}E_{23}E_{12} - E_{21}E_{32}E_{13} \neq 0.$$

4. Gelfand-Tsetlin modules for $\mathfrak{gl}(m|n)$

4.1. Gelfand-Tsetlin modules over $\mathfrak{g}_0$

Let $\mathfrak{g} = \mathfrak{gl}(m|n) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Let Γ_0 be a Gelfand-Tsetlin subalgebra of $U(\mathfrak{g}_0) \simeq U(\mathfrak{gl}(m) \oplus \mathfrak{gl}(n))$. We will denote by $\hat{\Gamma}_0 = Hom(\Gamma_0, \mathbb{C})$ the space of characters of Γ_0.

The following is straightforward.

Proposition 4.1. *There exist a Gelfand-Tsetlin subalgebra* $\Gamma(m)$ *of* $U(\mathfrak{gl}(m))$ *and a Gelfand-Tsetlin subalgebra* $\Gamma(n)$ *of* $U(\mathfrak{gl}(n))$ *such that* $\Gamma_0 \simeq \Gamma(m) \otimes \Gamma(n)$.

Thus combining Gelfand-Tsetlin subalgebras of $U(\mathfrak{gl}(m))$ and $U(\mathfrak{gl}(n))$ we obtain all Gelfand-Tsetlin subalgebras of $U(\mathfrak{g}_0)$.

A finitely generated module M over $\mathfrak{g}_0$ is called a *Gelfand-Tsetlin module* (with respect to Γ_0) if

$$M = \bigoplus_{\chi \in \hat{\Gamma}_0} M(\chi),$$

where

$$M(\chi) \;=\; \{x \in M \mid (z - \chi(z))^k x = 0 \quad \text{for some} \quad k \geqslant 0 \quad \text{and all} \quad z \in \Gamma_0\}.$$

Denote by $\mathcal{M}(\Gamma_0)$ the category of Gelfand-Tsetlin modules over $\mathfrak{g}_0$ with respect to the subalgebra Γ_0. Clearly, all finite dimensional $\mathfrak{g}_0$-modules belong to $\mathcal{M}(\Gamma_0)$.

Theorem 4.2. *Let* V *be a finite dimensional* $\mathfrak{g}_0$*-module. For* $M \in \mathcal{M}(\Gamma_0)$ *denote* $F_V(M) = M \otimes V$. *Then the correspondance* $F_V : M \to F_V(M)$ *defines a endofunctor of the category* $\mathcal{M}(\Gamma_0)$.

Proof. First, note that $F_V(M)$ is finitely generated for any $M \in \mathcal{M}(\Gamma_0)$. Indeed, it suffices to check this for a cyclic module M. But in this case M is a quotient of a free module $U(\mathfrak{g}_0)$. Then $M \otimes V$ is a quotient of a free module of rank $\dim V$ and hence finitely generated.

Let Z_k be the center of $U(\mathfrak{gl}(k))$ and suppose that $Z_k \subset \Gamma_0$. To prove the statement it is sufficient to show that Z_k acts locally finitely on $F_V(M)$. Consider both M and V as $\mathfrak{gl}(k)$-modules. Tensor product with V defines a projective functor on the category of $\mathfrak{gl}(k)$-modules which restricts to the functor on the subcategory with locally finite action of the center Z_k by [[BG], Corollary 2.6] or [[K1], Theorem 5.1]. Since Z_k is locally finite on M, it will also act locally finitely on $F_V(M)$. In particular, $Z(\mathfrak{g}_{\bar{0}})$ acts locally finitely on $V \otimes W$. The statement follows. □

Gelfand-Tsetlin theory for $\mathfrak{gl}(n)$ was developed in [O]. The results can be easily extended to Gelfand-Tsetlin modules over $\mathfrak{g}_{\bar{0}}$.

Proposition 4.3. *Let* V *be some finite-dimensional* $\mathfrak{g}$*-module.*

(i) *Let $\chi \in \hat{\Gamma}_0$. There exists a finite set $S(\chi) \subset \hat{\Gamma}_0$ such that for any Gelfand-Tsetlin module M we have*

$$M(\chi) \otimes V \subset \bigoplus_{\theta \in S(\chi)} [M \otimes V](\theta).$$

(ii) *For every $\chi \in \hat{\Gamma}_0$ the set $S^{-1}(\chi) := \{\theta \mid \chi \in S(\theta)\}$ is finite.*

(iii) *If $\dim M(\chi) < \infty$ for all $\chi \in \hat{\Gamma}_0$ then $\dim[M \otimes V](\theta) < \infty$ for all $\theta \in \hat{\Gamma}_0$*

Proof. Let us prove (i). The proof goes by induction on the rank of $\mathfrak{g}_0$. We assume that the statement is true for the previous term $(\mathfrak{g}_0', \Gamma_0')$ in the Gelfand-Tsetlin chain. Let χ' be the restriction of χ to Γ_0' and ξ be the central character obtained by restriction of χ to $Z(\mathfrak{g}_0)$. Then $M(\chi) = M(\chi') \cap M(\xi)$. We have

$$M(\chi) \otimes V = M(\chi') \otimes V \cap M(\xi) \otimes V.$$

By the Kostant theorem

$$M(\xi) \otimes V = \oplus_{\zeta \in S(\xi)} [M \otimes V](\zeta)$$

for some finite set $S(\xi)$. Hence

$$M(\chi) \otimes V \subset \bigoplus_{\zeta \in S(\xi), \theta' \in S(\chi')} [M \otimes V](\zeta) \cap [M \otimes V](\theta') = \bigoplus_{\theta \in S(\chi)} [M \otimes V](\theta),$$

where $S(\chi)$ consists of all characters θ such that its restriction to Γ_0' lies in $S(\theta')$ and the restriction to $Z(\mathfrak{g}_0)$ lies in $S(\xi)$.

Now (ii) is a consequence of the proof of (i) since $S^{-1}(\xi)$ is finite for any central character ξ.

Finally, (iii) follows immediately for (ii) and (i). □

4.2. Gelfand-Tsetlin $\mathfrak{g}$-modules

Let Γ be a Gelfand-Tsetlin subalgebra of $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. We will denote by $\hat{\Gamma} = Hom(\Gamma, \mathbb{C})$ the space of characters of Γ_0.

A finitely generated $\mathfrak{g}$-module M is called a *Gelfand-Tsetlin module* (with respect to Γ) if

$$M = \bigoplus_{\chi \in \hat{\Gamma}} M(\chi),$$

where

$$M(\chi) \;=\; \{x \in M \mid (z-\chi(z))^k x = 0 \quad \text{for some} \quad k \geqslant 0 \quad \text{and all} \quad z \in \Gamma\}.$$

Denote by $\mathcal{M}(\Gamma)$ the category of Gelfand-Tsetlin modules over $\mathfrak{g}$ with respect to the subalgebra Γ. In particular, all finite dimensional $\mathfrak{g}$-modules belong to $\mathcal{M}(\Gamma)$. Let $\mathcal{F}(\Gamma)$ be the category of $\mathfrak{g}$-modules with locally finite action of Γ and $\mathcal{F}_{\mathfrak{g}}(\Gamma_0)$ be the category of $\mathfrak{g}$-modules with locally finite action of Γ_0. Obviously, $\mathcal{M}(\Gamma)$ is a full subcategory of $\mathcal{F}(\Gamma)$. Moreover, every object of $\mathcal{F}(\Gamma)$ is a direct limit of objects in $\mathcal{M}(\Gamma)$.

Lemma 4.4. *If $Z(\mathfrak{g}_0)$ acts locally finitely on a $\mathfrak{g}$-module M then $Z(\mathfrak{g})$ acts locally finitely on M.*

Proof. Let $N := M^{\mathfrak{g}_1}$. Obviously, N is a $\mathfrak{g}_0$-submodule of M. Since $zv = \phi(z)v$ for any $z \in Z(\mathfrak{g})$ and any $v \in N$, and $\phi(Z(\mathfrak{g})) \subset Z(\mathfrak{g}_0)$, we obtain that $Z(\mathfrak{g})$ acts locally finitely on N and hence on $U(\mathfrak{g})N$.

Consider a filtration

$$0 \subset F^1(M) \subset F^2(M) \subset \cdots \subset F^k(M) \subset \ldots,$$

defined inductively by

$$F^i(M)/F^{i-1}M = U(\mathfrak{g})(M/F^{i-1}M)^{\mathfrak{g}_1}.$$

By repeating the above argument we get that $Z(\mathfrak{g})$ acts locally finitely on $F^i(M)$ for all i.

On the other hand, $\dim U(\mathfrak{g}_1)v \leq 2^{mn}$ for any $v \in M$, hence $M = \cup_{i=1}^{2^{mn}} F^i(M)$ and the statement follows. □

Theorem 4.5. *Let Γ be a Gelfand-Tsetlin subalgebra of $U(\mathfrak{g})$ such that $\Gamma_0 = \phi(\Gamma)$. Then $\mathcal{F}(\Gamma) = \mathcal{F}_{\mathfrak{g}}(\Gamma_0)$ and $\mathcal{M}(\Gamma) = \mathcal{M}_{\mathfrak{g}}(\Gamma_0)$ where $\mathcal{F}_{\mathfrak{g}}(\Gamma_0)$ (resp., $\mathcal{M}_{\mathfrak{g}}(\Gamma_0)$) is the category of $\mathfrak{g}$-modules (resp., finitely generated $\mathfrak{g}$-modules) locally finite over Γ_0.*

Proof. Suppose $M \in \mathcal{F}(\Gamma)$. Consider the filtration $F^i(M)$ defined in the proof of Lemma 4.4. To prove that $M \in \mathcal{F}_{\mathfrak{g}}(\Gamma_0)$ it suffices to show that Γ_0 acts locally finitely on $F_i := F^i(M)/F^{i-1}M$. Indeed, consider F_i as a $\mathfrak{g}_0$-module. Then F_i is a quotient of $\Lambda(\mathfrak{g}_{-1}) \otimes (M/F^{i-1}M)^{\mathfrak{g}_1}$. For any $\gamma \in \Gamma$ and $v \in (M/F^{i-1}M)^{\mathfrak{g}_1}$ we get $\gamma v = \phi(\gamma)v$. Since $\phi : \Gamma \to \Gamma_0$ is surjective by Proposition 3.1, we obtain that Γ_0 acts locally finitely on $(M/F^{i-1}M)^{\mathfrak{g}_1}$.

Furthermore, $F^i(M)/F^{i-1}M$ considered as a $\mathfrak{g}_0$-module is a quotient of $\Lambda(\mathfrak{g}_{-1}) \otimes (M/F^{i-1}M)^{\mathfrak{g}_1}$. Hence by Theorem 4.2 we obtain that Γ_0 acts locally finitely on $F^i(M)/F^{i-1}M$.

Suppose now that $M \in \mathcal{F}_{\mathfrak{g}}(\Gamma_0)$. By Lemma 4.4 we get that $Z(\mathfrak{g}_{k,l})$ acts locally finitely on M. Therefore Γ acts locally finitely on M.

The second assertion is a consequence of the obvious fact that $U(\mathfrak{g})$ is a finitely generated as $U(\mathfrak{g}_0)$-module and hence a $\mathfrak{g}$-module M is finitely generated over $U(\mathfrak{g}_0)$ if and only if it is finitely generated over $U(\mathfrak{g})$. □

As a consequence we obtain the following super analog of Theorem 4.2.

Corollary 4.6. *Let V be a finite dimensional $\mathfrak{g}$-module. For $M \in \mathcal{F}(\Gamma)$ set $\hat{F}_V(M) = M \otimes V$. Then the correspondence $\hat{F}_V : M \to \hat{F}_V(M)$ defines a functor on the category $\mathcal{F}(\Gamma)$. The restriction of $\hat{F}_V$ on $\mathcal{M}(\Gamma)$ defines a functor on the category $\mathcal{M}(\Gamma)$.*

Proof. Indeed, we have $M \in \mathcal{F}(\Gamma) = \mathcal{F}(\Gamma_0)$ by Theorem 4.5. Hence $\hat{F}_V(M) \in \mathcal{F}(\Gamma_0)$ by Theorem 4.2. Applying Theorem 4.5 again we conclude that $\hat{F}_V(M) \in \mathcal{F}(\Gamma)$. □

We also have the following surprizing result

Corollary 4.7. *If Γ and Γ' are two Gelfand-Tsetlin subalgebras of $\mathfrak{g}$ such that $\phi(\Gamma) = \phi(\Gamma')$, then $\mathcal{F}(\Gamma) = \mathcal{F}(\Gamma')$.*

Proof. Indeed, by Theorem 4.5 we have

$$\mathcal{F}(\Gamma) = \mathcal{F}(\phi(\Gamma)) = \mathcal{F}(\phi(\Gamma')) = \mathcal{F}(\Gamma').$$

□

Let M be a $\mathfrak{g}$-module. We will say that $v \in M$ is a Gelfand-Tsetlin vector with respect to Γ of weight χ if $v \in M(\chi)$. We have the following key property of Gelfand-Tsetlin modules.

Proposition 4.8. *Let M be a $\mathfrak{g}$-module, Γ a Gelfand-Tsetlin subalgebra of $U(\mathfrak{g})$, $\chi \in \hat{\Gamma}$, $M(\chi) \neq 0$. Assume that M is generated by $M(\chi)$. Then $M \in \mathcal{F}(\Gamma)$. Furthermore, if $\dim M(\chi) < \infty$ then M is a Gelfand-Tsetlin module with respect to Γ.*

Proof. We prove the statement by induction in $m + n$. Let $\mathfrak{g}' = \mathfrak{g}_{m,n-1}$ or $\mathfrak{g}_{m-1,n}$ be the previous term in the defining chain. By induction assumption $M' := U(\mathfrak{g}')M(\chi) \in \mathcal{F}(\Gamma')$. Furthermore M is a quotient of the induced

module $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}')} M'$. The latter is isomorphic to $S(\mathfrak{g}\,/\,\mathfrak{g}') \otimes M'$ as a $\mathfrak{g}'$-module. By Corollary 4.6 $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}')} M' \in \mathcal{F}(\Gamma')$ and hence $M \in \mathcal{F}(\Gamma')$. On the other hand, $Z_{m,n}$ is locally finite on M. Hence $M \in \mathcal{F}(\Gamma)$.

□

Corollary 4.9. *Let M be a $\mathfrak{g}$-module, $\phi(\Gamma) = \Gamma_0$, $\chi_0 \in \hat{\Gamma}_0$. If $M(\chi_0) \neq 0$ then $M(\chi) \neq 0$ for some $\chi \in \hat{\Gamma}$. If in addition V is generated by $V(\chi_0)$ then $V \in \mathcal{F}(\Gamma)$. Furthermore, if M is an irreducible $\mathfrak{g}$-module and $M(\chi_0) \neq 0$ then $M \in \mathcal{M}(\Gamma)$.*

Proof. Consider the $\mathfrak{g}$-submodule N of M generated by $M(\chi_0)$. Let $N' = U(\mathfrak{g}_0)M(\chi)$. Then N is a quotient of the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} N'$. Hence $N', N \in \mathcal{F}(\Gamma_0)$ by Theorem 4.2. Applying Theorem 4.5 we conclude that $N \in \mathcal{F}(\Gamma)$. Therefore, $N(\chi) \neq 0$ for some $\chi \in \hat{\Gamma}$ implying the first assertion. Two other assertions are straightforward. □

5. Classification of Gelfand-Tsetlin modules

5.1. Restriction and induction functors

Denote by $\mathbf{K_0}$ the restriction of the functor $\mathbf{K}$ introduced above to the category of Gelfand-Tsetlin $\mathfrak{g}_0$-modules $\mathcal{M}(\Gamma_0)$.

Let $N \in \mathcal{M}(\Gamma_0)$ and $M = K(N)$. Then M is generated by the $\mathfrak{g}_0$-submodule $M_0 \subset M^{\mathfrak{g}_1}$ isomorphic to N. For $v \in M_0$ we have $zv = \phi(z)v$ for all $z \in \Gamma$, where Γ is any Gelfand-Tsetlin subalgebra of $\mathfrak{g}$ such that $\Gamma_0 = \phi(\Gamma)$. Hence M_0 is a Gelfand-Tsetlin $\mathfrak{g}_0$-module with respect to Γ_0 and

$$M_0 = \oplus_{\chi \in \hat{\Gamma}} M_0(\chi) = \oplus_{\chi_0 \in \hat{\Gamma}_0} M_0(\chi_0).$$

Applying Corollary 4.9 we conclude that $M \in \mathcal{F}(\Gamma)$. Moreover, since M is finitely generated (as N is finitely generated) we have that $M = K(N)$ is a Gelfand-Tsetlin $\mathfrak{g}$-module with respect to Γ. Hence $\mathbf{K_0}$ defines a functor:

$$\mathbf{K_0} : \mathcal{M}(\mathbf{\Gamma_0}) \to \mathcal{M}(\mathbf{\Gamma}).$$

Moreover we have:

Corollary 5.1. *Let M be an irreducible $\mathfrak{g}$-module and $M(\chi_0) \neq 0$ for some $\chi_0 \in \hat{\Gamma}_0$. Then M is a unique irreducible quotient of $K(N)$ for some irreducible Gelfand-Tsetlin (with respect to Γ_0) $\mathfrak{g}_0$-module N. Furthermore, M is a Gelfand-Tsetlin $\mathfrak{g}$-module with respect to any Γ such that $\Gamma_0 = \phi(\Gamma)$.*

We also have

Corollary 5.2. (i) *Given any $\chi_0 \in \hat{\Gamma}_0$ there exists an irreducible $M \in \mathcal{M}(\Gamma)$ with $M(\chi_0) \neq 0$.*

(ii) *If M_0 is a Gelfand-Tsetlin module for $\mathfrak{g}_0$ with respect to a Gelfand-Tsetlin subalgebra Γ_0, then $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M_0$ is a Gelfand-Tsetlin module for $\mathfrak{g}$ with respect to any Gelfand-Tsetlin subalgebra Γ such that $\Gamma_0 = \phi(\Gamma)$.*

Proof. Let N be an irreducible $\mathfrak{g}_0$-module such that $N(\chi_0) \neq 0$ and M be the unique simple quotient of $K(N)$. Then $M(\chi_0) \neq 0$ and the first statement follows.

Since $M = \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M_0$ is generated by M_0 then it is a Gelfand-Tsetlin $\mathfrak{g}_0$-module with respect to Γ_0 by Theorem 4.2. Then $M \in \mathcal{F}(\Gamma)$ by Theorem 4.5. Moreover, M is finitely generated as M_0 is finitely generated as a $\mathfrak{g}_0$-module. Hence, M is a Gelfand-Tsetlin module for $\mathfrak{g}$ with respect to any Gelfand-Tsetlin subalgebra Γ such that $\Gamma_0 = \phi(\Gamma)$. □

Denote by $\mathbf{R_0}$ the restriction of the functor $\mathbf{R}$ to the Gelfand-Tsetlin category $\mathcal{M}(\Gamma)$. For a $\mathfrak{g}$-module M, $R(M) = M^{\mathfrak{g}_1}$ is a $\mathfrak{g}_0$-module. Since $\phi : \Gamma \to \Gamma_0$ is an epimorphism then $R(M)$ is a Gelfand-Tsetlin $\mathfrak{g}_0$-module with respect to Γ_0, hence

$$\mathbf{R_0} : \mathcal{M}(\Gamma) \to \mathcal{M}(\Gamma_0).$$

By Proposition 2.1, the pair $(\mathbf{K_0}, \mathbf{R_0})$ is adjoint.

5.2. Reduction to the even part

The results obtained in the previous section allow to reduce the classification problem of irreducible Gelfand-Tsetlin $\mathfrak{g}$-modules to the classification of irreducible Gelfand-Tsetlin $\mathfrak{g}_0$-modules.

Theorem 5.3.

(i) *For any irreducible Gelfand-Tsetlin (with respect to Γ) $\mathfrak{g}$-module M there exists a unique irreducible Gelfand-Tsetlin (with respect to Γ_0) $\mathfrak{g}_0$-module N such that M is a unique irreducible quotient of $K(N)$.*

(ii) *Let M be an irreducible Gelfand-Tsetlin module with respect to a Gelfand-Tsetlin subalgebra Γ' such that $\phi(\Gamma') = \Gamma_0$. Then M is a Gelfand-Tsetlin module with respect to any Gelfand-Tsetlin subalgebra Γ such that $\phi(\Gamma) = \Gamma_0$.*

Proof. The first statement follows from Corollary 5.1 and Theorem 4.5. The second statement follows from Corollary 4.7. □

It is shown in [O] (see also [FO, Theorem 4.12(c)]) that Gelfand-Tsetlin multiplicities of any irreducible Gelfand-Tsetlin $\mathfrak{g}_0$-module are finite and uniformly bounded. Then we have

Corollary 5.4. *If M is an irreducibe Gelfand-Tsetlin module. Then Gelfand-Tsetlin multiplicities of M are finite and uniformly bounded.*

Proof. By Theorem 5.3 (i) it suffices to check this property for $K(N)$ for some irreducible Gelfand-Tsetlin module N. The Gelfand-Tsetlin multiplicities of N are finite and uniformly bounded. Since $K(N) \simeq \Lambda(\mathfrak{g}_{-1}) \otimes N$ as a $\mathfrak{g}_0$-module, the statement follows from Proposition 4.3 (iii). □

Finally we have the following finiteness result.

Theorem 5.5. *Let Γ be a Gelfand-Tsetlin subalgebra of $U(\mathfrak{g})$.*

(i) *Given $\chi_0 \in \hat{\Gamma}_0$, there exist finitely many non-isomorphic irreducible Gelfand-Tsetlin (with respect to Γ) $\mathfrak{g}$-modules M such that $M^{\mathfrak{g}_1} \cap M(\chi_0) \neq 0$.*

(ii) *For any $\chi_0 \in \hat{\Gamma}_0$ there exist finitely many non-isomorphic irreducible Gelfand-Tsetlin (with respect to Γ) $\mathfrak{g}$-modules M with $M(\chi_0) \neq 0$.*

Proof. The first statement is an immediate consequence of Theorem 5.3 (i).

By the finiteness theorem for $\mathfrak{g}_0$ ([O]), for any $\theta \in \hat{\Gamma}_0$ there are finitely many up to isomorphism irreducible Gelfand-Tsetlin modules N such that $N(\theta_0) \neq 0$. We claim that there are finitely many N such that $K(N)(\chi_0) \neq 0$. For this we recall that $K(N)$ is isomorphic to $N \otimes V$ for $V = \Lambda(\mathfrak{g}_{-1})$. By Proposition 4.3 (i), $K(N)(\chi_0) \neq 0$ implies that $N(\theta_0) \neq 0$ for some $\theta_0 \in S^{-1}(\chi_0)$. By Proposition 4.3 (ii), the set $S^{-1}(\chi)$ is finite, hence the statement follows from the above mentioned result for $\mathfrak{g}_0$-modules. □

5.3. Equivalence of subcategories

Recall that the category of representations of a basic classical Lie superalgebra of type I with a fixed typical central character is equivalent to the category of representations of its even part with a suitable central character [PS],[G]. Due to Proposition 5.2 this equivalence restricts to the Gelfand-Tsetlin categories $\mathcal{M}(\Gamma)$ and $\mathcal{M}(\Gamma_0)$. Namely we have the following.

Let $\xi \in \hat{Z}(\mathfrak{g})$ and $\mathcal{M}^{\xi}(\Gamma)$ denote the full subcategory of $\mathcal{M}(\Gamma)$ consisting of modules admitting the central character ξ. Let $\xi_0 := \phi^*(\xi)$ and $\mathcal{M}^{\xi_0}(\Gamma_0)$ the subcategory of Gelfand-Tsetlin $\mathfrak{g}_0$-modules which admit the central character ξ_0.

Recall that ξ is typical if ξ is the central character of a Verma module with typical highest weight.

It is shown in [PS] that if ξ is typical then the functors $\mathbf{R}$ and $\mathbf{K}$ define equivalence between the categories of $\mathfrak{g}$-modules admitting ξ and the category of $\mathfrak{g}_0$-modules admitting ξ_0. Since $\mathbf{K}$ and $\mathbf{R}$ restrict to the functors

$$\mathbf{K_0} : \mathcal{M}^{\xi_0}(\mathbf{\Gamma_0}) \to \mathcal{M}^{\xi}(\mathbf{\Gamma}), \quad \mathbf{R_0} : \mathcal{M}^{\xi}(\mathbf{\Gamma}) \to \mathcal{M}^{\xi_0}(\mathbf{\Gamma_0}),$$

we obtain the following theorem.

Theorem 5.6. *If $\xi \in \hat{Z}(\mathfrak{g})$ is typical then $\mathbf{K_0}$ and $\mathbf{R_0}$ define the equivalence between the categories $\mathcal{M}^{\xi}(\Gamma)$ and $\mathcal{M}^{\xi_0}(\Gamma_0)$.*

6. Gelfand-Tsetlin bases

Given a tuple λ of complex numbers of the form $\lambda = (\lambda_1, \lambda_2, \ \ldots, \lambda_{m+n})$, the irreducible highest weight module $L(\lambda)$ of the Lie superalgebra $\mathfrak{gl}(m|n)$ with the highest weight λ is generated by a nonzero vector v satisfying the conditions:

$$\begin{aligned} &h_i v = \lambda_i v, 1 \leq i \leq m+n, \\ &e_i v = 0, 1 \leq i \leq m+n-1. \end{aligned} \tag{17}$$

Recall that the weight λ is dominant if

$$\lambda_i - \lambda_{i+1} \in \mathrm{Z}_{\geq 0}, \text{ for } i = 1, \ldots, m+n-1, \quad i \neq m. \tag{18}$$

The dominant weight λ (and the module $L(\lambda)$) is typical if $(\lambda + \rho, \varepsilon_i - \delta_j) \neq 0$ for $1 \leq i \leq m, 1 \leq j \leq n$.

Set

$$\begin{aligned} &l_i = \lambda_i - i + 1, (1 \leq i \leq m); \\ &l_j = -\lambda_j + j - 2m, (m+1 \leq j \leq m+n). \end{aligned} \tag{19}$$

Since $(\lambda + \rho, \varepsilon_i - \delta_j) = l_i - l_j$, the condition for typicality is equivalent to $l_i \neq l_j$ for $1 \leq i \leq m < j \leq m+n$.

For $a, b \in \mathbb{C}$ such that $b - a$ is nonnegative integer, we set

$$[a; b] = \{a + s \mid 0 \leq s \leq b - a\}.$$

The dominant weight λ (and the module $L(\lambda)$) is called *essentially typical* if

$$\{l_1, l_2, \ldots, l_m\} \cap [l_{m+1}; l_{m+n}] = \emptyset. \tag{20}$$

Clearly, every essentially typical weight is typical.

Our main combinatorial device is an array of complex numbers $\Lambda = (\lambda_{ij})$ presented in the following form:

$$\begin{array}{ccccccc}
\lambda_{m+n,1} & \cdots & \lambda_{m+n,m} & \lambda_{m+n,m+1} & \cdots & \lambda_{m+n-1,m+n} & \lambda_{m+n,m+n} \\
\lambda_{m+n-1,1} & \cdots & \lambda_{m+n-1,m} & \lambda_{m+n-1,m+1} & \cdots & \lambda_{m+n-1,m+n-1} & \\
\vdots & \vdots & \vdots & \vdots & \iddots & & \\
\lambda_{m+1,1} & \cdots & \lambda_{m+1,m} & \lambda_{m+1,m+1} & & & \\
\lambda_{m1} & \cdots & \lambda_{mm} & & & & \\
\lambda_{m-1,1} & \cdots & & & & & \\
\vdots & \iddots & & & & & \\
\lambda_{11} & & & & & &
\end{array} \tag{21}$$

We will call Λ a *Gelfand-Tsetlin tableau*. Given $\Lambda = (\lambda_{ij})$ we set

$$l_{ki} = \lambda_{ki} - i + 1, (1 \leq i \leq m); \quad l_{kj} = -\lambda_{kj} + j - 2m, (m + 1 \leq j \leq k). \tag{22}$$

For an essentially typical highest weight λ a basis of the irreducible module $L(\lambda)$ parameterized by Gelfand-Tsetlin tableaux was constructed in [Pal1]. We will use the following modified version of these formulas.

Theorem 6.1. *Let λ be an essentially typical highest weight. Then $L(\lambda)$ admits a basis ξ_Λ parameterized by all Gelfand-Tsetlin tableaux Λ which satisfy the following conditions:*

1) $\lambda_{m+n,i} = \lambda_i, \quad 1 \leq i \leq m + n;$

2) $\lambda_{k,i} - \lambda_{k-1,i} \equiv \theta_{k-1,i} \in \{0, 1\}, 1 \leq i \leq m; m + 1 \leq k \leq m + n;$

3) $\lambda_{ki} - \lambda_{k,i+1} \in \mathrm{Z}_{\geq 0}, 1 \leq i \leq m - 1; m + 1 \leq k \leq m + n - 1;$

4) $\lambda_{k+1,i} - \lambda_{ki} \in \mathrm{Z}_{\geq 0}$ *and* $\lambda_{k,i} - \lambda_{k+1,i+1} \in \mathrm{Z}_{\geq 0}$, $1 \leq i \leq k \leq m - 1$ *or* $m + 1 \leq i \leq k \leq m + n - 1$.

The action of the generators of $\mathfrak{gl}_{m,n}$ is given by the formulas

$$h_k\xi_\Lambda = \left(\sum_{i=1}^{k}\lambda_{kj} - \sum_{j=1}^{k-1}\lambda_{k-1,j}\right)\xi_\Lambda, \quad 1 \le k \le m+n; \tag{23}$$

$$e_k\xi_\Lambda = -\sum_{i=1}^{k}\frac{\Pi_{j=1}^{k+1}(l_{k+1,j}-l_{ki})}{\Pi_{j\neq i,j=1}^{k}(l_{kj}-l_{ki})}\xi_{\Lambda+\delta_{ki}}, \quad 1 \le k \le m-1; \tag{24}$$

$$f_k\xi_\Lambda = \sum_{i=1}^{k}\frac{\Pi_{j=1}^{k-1}(l_{k-1,j}-l_{ki})}{\Pi_{j\neq i,j=1}^{k}(l_{kj}-l_{ki})}\xi_{\Lambda-\delta_{ki}}, \quad 1 \le k \le m-1; \tag{25}$$

$$\begin{aligned} e_m\xi_\Lambda &= \sum_{i=1}^{m}\theta_{mi}(-1)^{i-1}(-1)^{\theta_{m1}+\ldots+\theta_{m,i-1}} \\ &\quad\times\frac{\Pi_{1\le j<i}(l_{mj}-l_{mi}-1)}{\Pi_{i<j\le m}(l_{mj}-l_{mi})\Pi_{j\neq i,j=1}^{m}(l_{m+1,j}-l_{mi}-1)}\xi_{\Lambda+\delta_{mi}}, \end{aligned} \tag{26}$$

$$\begin{aligned} f_m\xi_\Lambda &= \sum_{i=1}^{m}(1-\theta_{mi})(-1)^{i-1}(-1)^{\theta_{m1}+\ldots+\theta_{m,i-1}} \\ &\times\frac{(l_{m,i}-l_{m+1,m+1})\Pi_{i<j\le m}(l_{mj}-l_{mi}+1)\Pi_{j=1}^{m-1}(l_{m-1,j}-l_{mi})}{\Pi_{1\le j<i}(l_{mj}-l_{mi})}\xi_{\Lambda-\delta_{mi}}, \end{aligned} \tag{27}$$

$$\begin{aligned} e_k\xi_\Lambda &= \sum_{i=1}^{m}\theta_{ki}(-1)^{\theta_{k1}+\ldots+\theta_{k,i-1}+\theta_{k-1,i+1}+\ldots+\theta_{k-1,m}}(1-\theta_{k-1,i}) \\ &\quad\times\prod_{j\neq i,j=1}^{m}\left(\frac{l_{kj}-l_{ki}-1}{l_{k+1,j}-l_{ki}-1}\right)\xi_{\Lambda+\delta_{ki}} \\ &\quad-\sum_{i=m+1}^{k}\Pi_{j=1}^{m}\left(\frac{(l_{kj}-l_{ki})(l_{kj}-l_{ki}+1)}{(l_{k+1,j}-l_{ki})(l_{k-1,j}-l_{ki}+1)}\right) \\ &\quad\times\frac{\Pi_{j=m+1}^{k+1}(l_{k+1,j}-l_{ki})}{\Pi_{j\neq i,j=m+1}^{k}(l_{kj}-l_{ki})}\xi_{\Lambda+\delta_{ki}}, \quad m+1 \le k \le m+n-1; \end{aligned} \tag{28}$$

(29)
$$f_k\xi_\Lambda = \sum_{i=1}^{m} \theta_{k-1,i}(-1)^{\theta_{k1}+\ldots+\theta_{k,i-1}+\theta_{k-1,i+1}+\ldots+\theta_{k-1,m}}(1-\theta_{ki}) \times \prod_{j\neq i=1}^{m}\left(\frac{l_{kj}-l_{ki}+1}{l_{k-1,j}-l_{ki}+1}\right) \times \frac{\Pi_{j=m+1}^{k+1}(l_{k+1,j}-l_{ki})\Pi_{j=m+1}^{k-1}(l_{k-1,j}-l_{ki}+1)}{\Pi_{j=m+1}^{k}(l_{kj}-l_{ki})(l_{kj}-l_{ki}+1)}\xi_{\Lambda-\delta_{ki}} + \sum_{i=m+1}^{k}\frac{\prod_{j=m+1}^{k-1}(l_{k-1,j}-l_{ki})}{\prod_{j\neq i,j=m+1}^{k}(l_{k,j}-l_{ki})}\xi_{\Lambda-\delta_{ki}} \quad m+1\leq k\leq m+n-1.$$

The arrays $\Lambda\pm\delta_{ki}$ are obtained from Λ by replacing λ_{ki} by $\lambda_{ki}\pm 1$. We assume that $\xi_\Lambda = 0$ if the array Λ does not satisfy the conditions of the theorem.

The formulas in the theorem above can be obtained from [Pal1] by multiplying the basis elements by $\sqrt{|a(\Lambda)|}$, where $a(\Lambda)=\prod_{k=1}^{m+n}a_k(\Lambda)$.

(30)
$$a_k(\Lambda) = \prod_{1\leq i\leq j<k}\frac{(l_{ki}-l_{k-1,j})!}{(l_{k-1,i}-l_{k-1,j})!}\prod_{1\leq i<j\leq k}\frac{(l_{ki}-l_{kj}-1)!}{(l_{k-1,i}-l_{kj}-1)!}, \quad 1\leq k\leq m,$$
$$a_k(\Lambda) = \prod_{1\leq i\leq j\leq m, m+1\leq i\leq j<k}\frac{(l_{k-1,j}-l_{ki})!}{(l_{k-1,j}-l_{k-1,i})!} \times \prod_{1\leq i<j\leq m, m+1\leq i<j\leq k}\frac{(l_{kj}-l_{ki}-1)!}{(l_{kj}-l_{k-1,i}-1)!}, \quad m+1\leq k\leq m+n.$$

Here the function $a!$ is defined by

(31)
$$a! = \begin{cases} 1\cdot 2\cdots a, & \text{for } a\in\mathbb{Z}_{>0},\\ 1, & \text{for } a=0 \text{ or } \pm 1,\\ \dfrac{1}{(a+1)(a+2)\cdots(-1)}, & \text{for } a\in\mathbb{Z}_{<0}.\end{cases}$$

In the following sections we will identify ξ_Λ with the tableau $[l]$ for convenience.

7. Quasi typical modules

7.1. Relation modules for $\mathfrak{gl}(n)$

Let us recall the construction of relation modules for $\mathfrak{gl}(n)$ [FRZ]. Set $\mathfrak{V} := \{(i,j) \mid 1 \le j \le i \le n\}$ and consider the following subsets of $\mathfrak{V} \times \mathfrak{V}$:

$$\mathcal{R}^+ := \{((i,j);(i-1,t)) \mid 2 \le j \le i \le n,\ 1 \le t \le i-1\} \tag{32}$$

$$\mathcal{R}^- := \{((i,j);(i+1,s)) \mid 1 \le j \le i \le n-1,\ 1 \le s \le i+1\} \tag{33}$$

$$\mathcal{R}^0 := \{((n,i);(n,j)) \mid 1 \le i \ne j \le n\} \tag{34}$$

Set $\mathcal{R} := \mathcal{R}^- \cup \mathcal{R}^0 \cup \mathcal{R}^+$. Any subset $\mathcal{C} \subseteq \mathcal{R}$ will be called a *set of relations*. Let $\mathcal{C}^0 = \mathcal{C} \cap \mathcal{R}^0$ and $\mathcal{C}^\pm = \mathcal{C} \cap \mathcal{R}^\pm$.

To any $\mathcal{C} \subseteq \mathcal{R}$ we can associate a directed graph $G(\mathcal{C})$ with the set of vertices $\mathfrak{V}$ as follows: there is an arrow from (i,j) to (r,s) if and only if $((i,j);(r,s)) \in \mathcal{C}$. Given $(i,j),\ (r,s) \in \mathfrak{V}$ we will write $(i,j) \succeq_{\mathcal{C}} (r,s)$ if there exists a path in $G(\mathcal{C})$ starting in (i,j) and ending in (r,s).

Definition 7.1. *Let $\mathcal{C}$ be any set of relation and $[l] = (l_{ij})$, $1 \le i \le n$, $1 \le j \le i$, a Gelfand-Tsetlin tableau for $\mathfrak{gl}(n)$.*

(i) *We will say that $[l]$ satisfies $\mathcal{C}$ if:*
- *$l_{ij} - l_{rs} \in \mathbb{Z}_{\ge 0}$ for any $((i,j);(r,s)) \in \mathcal{C}^+ \cup \mathcal{C}^0$.*
- *$l_{ij} - l_{rs} \in \mathbb{Z}_{>0}$ for any $((i,j);(r,s)) \in \mathcal{C}^-$.*
- *For any $1 \le k \le n-1$ we have, $l_{ki} - l_{kj} \in \mathbb{Z}$ if and only if (k,i) and (k,j) are in the same connected component of $G(\mathcal{C})$.*

(ii) *If $[l]$ satisfies $\mathcal{C}$ then $\mathcal{B}_{\mathcal{C}}([l])$ will be the set of all $[l+z]$, where $z \in \mathbb{Z}^{\frac{n(n+1)}{2}}$ satisfying $\mathcal{C}$ with $z_{ni} = 0$ for $1 \le i \le n$. By $V_{\mathcal{C}}([l])$ we denote the complex vector space spanned by $\mathcal{B}_{\mathcal{C}}([l])$.*

We call $\mathcal{C}$ *noncritical* if $l_{ki} \ne l_{kj}$, $1 \le i < j \le k \le n-1$, for all $[l]$ that satisfies $\mathcal{C}$.

Following [FRZ], Definition 4.18, we call a set of relations $\mathcal{C}$ *reduced* if for every $(k,j) \in \mathfrak{V}(\mathcal{C})$ the following conditions are satisfied:

(i) There exists at most one i such that $((k,j);(k+1,i)) \in \mathcal{C}$;

(ii) There exists at most one i such that $((k+1,i);(k,j)) \in \mathcal{C}$;

(iii) There exists at most one i such that $((k,j);(k-1,i)) \in \mathcal{C}$;

(iv) There exists at most one i such that $((k-1,i);(k,j)) \in \mathcal{C}$;

(v) No relations in the top row follow from other relations.

Let $\mathcal{C}$ be a set of relations. For $i < j$ we say that (k,i) and (k,j) form an adjoining pair if they are in the same indecomposable subset of $\mathcal{C}$ and there is no pair (k,s) such that $(k,i) \succeq_{\mathcal{C}} (k,s) \succeq_{\mathcal{C}} (k,j)$.

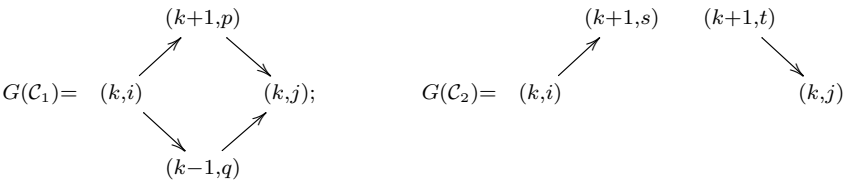
Denote by $\mathfrak{F}$ the set of all indecomposable sets of relations $\mathcal{C}$ satisfying the following condition:

(i) $\mathcal{C}$ is noncritical and reduced;

(ii) $\mathcal{C}$ has no subsets of the form $\{((k,i);(k+1,t)),\ ((k+1,s);(k,j))\}$ with $i < j$ and $s < t$;

(iii) If (n,i) and (n,j), $i \neq j$, are in the same indecomposable subset of $\mathcal{C}$, then $(n,i) \succeq_{\mathcal{C}} (n,j)$ or $(n,j) \succeq_{\mathcal{C}} (n,i)$.

(iv) For every adjoining pair (k,i) and (k,j), $1 \leq k \leq n-1$, there exist p,q such that $\mathcal{C}_1 \subseteq \mathcal{C}$ or, there exist $s < t$ such that $\mathcal{C}_2 \subseteq \mathcal{C}$, where the graphs associated to $\mathcal{C}_1$ and $\mathcal{C}_2$ are as follows



Disconnected unions of sets of $\mathfrak{F}$ are called *admissible*. We have

Theorem 7.2. *[FRZ] Let $\mathcal{C}$ be an admissible set of relations. Then for any $[l^0]$ satisfying $\mathcal{C}$, $V_{\mathcal{C}}([l^0])$ has a $\mathfrak{gl}(n)$-module structure and the action of the generators of $\mathfrak{gl}_n$ on any basis tableau $[l]$ is given by*

$$
\begin{aligned}
E_{k,k+1}[l] &= -\sum_{j=1}^{k} \frac{\prod_i (l_{k+1,i} - l_{k,j})}{\prod_{i \neq j} (l_{k,i} - l_{k,j})} [l + \delta^{kj}], \\
E_{k+1,k}[l] &= \sum_{j=1}^{k} \frac{\prod_i (l_{k-1,i} - l_{k,j})}{\prod_{i \neq j} (l_{k,i} - l_{k,j})} [l - \delta^{kj}], \\
E_{kk}[l] &= \left(\sum_{i=1}^{k} l_{k,i} - \sum_{i=1}^{k-1} l_{k-1,i} + k - 1 \right) [l].
\end{aligned}
\tag{35}
$$

Here $[l \pm \delta^{kj}]$ is the tableau obtained from $[l]$ by adding ± 1 to the (k,j)th entry of $[l]$; if $[l]$ is not in $\mathcal{B}_{\mathcal{C}}([l^0])$ then it is assumed to be zero.

Module $V_{\mathcal{C}}([l^0])$ in Theorem 7.2 is a *relation* $\mathfrak{gl}(n)$-module. We will extend the construction of relation modules to the Lie superalgebra $\mathfrak{gl}(m|n)$ in the next section.

7.2. Relation modules for $\mathfrak{gl}(m|n)$

Definition 7.3. *The set $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$ is called* admissible *if $\mathcal{C}_1$ and $\mathcal{C}_2$ are admissible sets of relations of $\mathfrak{gl}(m)$ and $\mathfrak{gl}(n)$ respectively.*

Definition 7.4. *Let $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$ be admissible.*

(i) *We will say that a tableau $[l]$ of $\mathfrak{gl}(m|n)$ satisfies $\mathcal{C}_1$ if:*
- $l_{ij} - l_{rs} \in \mathbb{Z}_{\geq 0}$ *for any* $((i,j);(r,s)) \in \mathcal{C}_1^+$;
- $l_{ij} - l_{rs} \in \mathbb{Z}_{>0}$ *for any* $((i,j);(r,s)) \in \mathcal{C}_1^- \cup \mathcal{C}_1^0$;
- *For any $1 \leq k \leq n$ we have, $l_{ki} - l_{kj} \in \mathbb{Z}$ if and only if (k,i) and (k,j) are in the same connected component of $G(\mathcal{C}_1)$.*

(ii) *We will say that $[l]$ of $\mathfrak{gl}(m|n)$ satisfies $\mathcal{C}_2$ if:*
- $l_{ij} - l_{rs} \in \mathbb{Z}_{\leq 0}$ *for any* $((i,j);(r,s)) \in \mathcal{C}_2^+ \cup \mathcal{C}_2^0$;
- $l_{ij} - l_{rs} \in \mathbb{Z}_{<0}$ *for any* $((i,j);(r,s)) \in \mathcal{C}_2^-$;
- *For any $m+1 \leq k \leq m+n$ we have, $l_{ki} - l_{kj} \in \mathbb{Z}$ if and only if (k,i) and (k,j) are in the same connected component of $G(\mathcal{C}_2)$.*

(iii) *We say that a tableau $[l]$ of $\mathfrak{gl}(m|n)$ satisfies $\mathcal{C}$ if:*
1) $[l]$ satisfies $\mathcal{C}_1$ and $\mathcal{C}_2$;
2) $l_{ki} - l_{k-1,i} \equiv \theta_{k-1,i}(l) \in \{0,1\}, 1 \leq i \leq m, m+1 \leq k \leq m+n$;
3) $l_{ki} - l_{kj} \neq 0$ for $m+1 \leq k \leq m+n-1$, $1 \leq i \leq m < j \leq k$;
4) If $[l']$ satisfies conditions (1)-(2) and $l'_{ki} = l_{ki} + z_{ki}$, where $z_{ki} \in \mathbb{Z}$, $z_{m+n,i} = 0$ for all i, then $[l']$ satisfies condition (3).

Let $[l^0] = (l^0{}_{ij})$ be a tableau satisfying $\mathcal{C}$ and $\mathcal{B}_{\mathcal{C}}([l^0])$ the set of all possible tableaux $[l] = (l_{ij})$ such that

(1) $l_{ki} = l^0{}_{ki} + z_{ki}$, $z_{ki} \in \mathbb{Z}$, $1 \leq k \leq m+n-1$, $1 \leq i \leq k$;

(2) $[l]$ satisfies $\mathcal{C}$.

Let $V_{\mathcal{C}}([l^0])$ be the vector space spanned by $\mathcal{B}_{\mathcal{C}}([l^0])$.

We are now in the position to state the main result of this section.

Theorem 7.5. *Let $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$ be an admissible set of relations for $\mathfrak{gl}(m|n)$, $[l^0]$ a tableau satisfying $\mathcal{C}$. Then the formulas* (23)-(29) *define a $\mathfrak{gl}(m|n)$-module structure on $V_{\mathcal{C}}([l^0])$.*

The module $V_{\mathcal{C}}([l^0])$ is called *quasi typical* $\mathfrak{gl}(m|n)$-module. Note that any essentially typical module is quasi typical. Also note that quasi typical module need not be typical. The module $V_{\mathcal{C}}([l^0])$ is infinite dimensional in general, it may or may not be highest weight module.

The formulas (23)-(29) define a $\hat{\mathfrak{g}}$-module structure on the space $V_{\mathcal{C}}(l^0)$. Recall that we have a homomorphism $\epsilon : \hat{\mathfrak{g}} \to \mathfrak{gl}(m|n)$ given by $e_i \mapsto E_{i,i+1}$, $f_i \mapsto E_{i+1,i}, 1 \leq i \leq m+n-1$ and $h_j \mapsto E_{jj}, 1 \leq j \leq m+n$. Denote the kernel of the homomorphism ϵ by K.

7.3. Relation removal method

Let $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$. Suppose (k,i) is a vertex involved in the set of relations $\mathcal{C}$, that is (k,i) is the starting or the ending vertex of an arrow in the graph of $\mathcal{C}$. We call a vertex (k,i) in the graph of $\mathcal{C}_1$(resp. $\mathcal{C}_2$) *maximal* if there is no (s,t) such that $((s,t);(k,i)) \in \mathcal{C}_1$(resp. $((k,i);(s,t)) \in \mathcal{C}_2$). A *minimal* pair is defined similarly. Suppose $\mathcal{C}$ is admissible set and (k,i) is maximal or minimal. Denote by $\mathcal{C}_{ki}$ the set of relations obtained from $\mathcal{C}$ by removing all relations that involve (k,i). Following [FRZ] we will say that $\mathcal{C}_{ki} \subsetneq \mathcal{C}$ is obtained from $\mathcal{C}$ by the *relation removal* method, or the RR-method for short.

Theorem 7.6. *Let $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$ be an admissible set of relations for which Theorem 7.5 holds. If $\mathcal{C}'$ is obtained from $\mathcal{C}$ by the RR-method then $\mathcal{C}'$ is admissible and Theorem 7.5 holds for $\mathcal{C}'$.*

Proof. Suppose that $\mathcal{C}'$ is obtained from $\mathcal{C}$ by removing (i,j). Then clearly $\mathcal{C}'$ is admissible. To show that Theorem 7.5 holds for $\mathcal{C}'$ we need to prove that for any generator $g \in K$ and for any tableau $[l']$ satisfying $\mathcal{C}$ one has $g[l'] = 0$.

Consider a tableau $[l]$ satisfying $\mathcal{C}$ and the following conditions. If $i \neq m$ then $l_{rt} = l'_{rt}$ if $(r,t) \neq (i,j)$. If $i = m$ then $l_{rt} = l'_{rt}$ for $(r,t) \neq (k,j), m \leq k \leq m+n$ and $\theta(l) = \theta(l')$ for $m \leq k \leq m+n-1$. Also define $\delta = \delta_{ij}$ if $i \neq m$ and $\delta = \sum_{k=m}^{m+n} \delta_{kj}$ if $i = m$.

Let s be positive (respectively, negative) integer if (i,j) is maximal (respectively minimal) with $|s| >> 0$. Expand $g[l + s\delta]$ and $g[l']$ step by step

(using formulas (23)-(29)). We have

$$g[l+s\delta]=\sum_{z\in A} g_z(l+s\delta)[l+s\delta+z],$$

where A is the set of z such that $[l+s\delta+z]\in\mathcal{B}_{\mathcal{C}}([l])$. Suppose that the tableau $[l+s\delta+z]$ appears in the expansion of $g[l+s\delta]$. Then the tableau $[l'+z]$ appears in the expansion of $g[l']$. Moreover, $[l+s\delta+z]$ belongs to $\mathcal{B}_{\mathcal{C}}([l])$ if and only if $[l'+z]$ belongs to $\mathcal{B}_{\mathcal{C}'}([l'])$. Hence we have

$$g[l']=\sum_{z\in A} g_z(l')[l'+z].$$

Since for which Theorem 7.5 holds for $\mathcal{C}$, one has that $g[l+s\delta]=0$ and the values of rational functions $g_z(l+s\delta)$ are 0 for infinitely many values of s for all $z\in A$. Thus $g_z(l')=0$ for all $z\in A$ and hence Theorem 7.5 holds for $\mathcal{C}'$. □

7.4. Proof of Theorem 7.5

Now we prove Theorem 7.5. The proof follows the idea of the proof of [FRZ], Theorem 4.33 for $\mathfrak{gl}(n)$. We provide the details for the sake of completeness.

Let $\mathcal{C}$ be an admissible set of relations and $[l^0]$ is a tableau satisfying $\mathcal{C}$. Our goal is to show that K vanishes on $[l^0]$.

We say that a tableau $[l]$ satisfies the *generic* condition if

$$l_{ki}-l_{kj}\notin\mathbb{Z} \text{ for } 1\le i\le m, m+1\le j\le k\le m+n. \tag{36}$$

Lemma 7.7. *Let $g\in U(\hat{\mathfrak{g}})$, $[l^0]$ is a tableau satisfying $\mathcal{C}$. If $g[l]=0$ for any tableau $[l]$ satisfying $\mathcal{C}$ and the generic condition then $g[l^0]=0$.*

Proof. We have

$$g[l]=\sum_{z\in A} g_z(l)[l+z],$$

for some finite set of integers A. Here g_z is some rational function and $g_z(l)$ is its evaluation in l. Let 1_n be the tableau such that the (k,j)th entry is 1 for $m+1\le j\le k\le m+n$ and all other entries are 0. There exist infinitely many $x\in\mathbb{C}$ such that $[l^0+x1_n]$ satisfies the generic condition (36), and for

each such x one has

$$g[l^0 + x1_n] = \sum_{z \in A} g_z(l^0 + x1_n)[l + z] = 0$$

by the assumption. Hence $g_z(l^0 + x1_n) = 0$ for infinitely many x. We conclude that $g_z(l^0) = 0$ and the statement follows. □

By Lemma 7.7 it is sufficient to prove the statement of the theorem for $[l]$ with the generic condition (36). Let g be a generator of K. Then

$$g[l] = \sum_z \varphi_z(l)[l + z].$$

Our goal is to verify that $g[l] = 0$. To do so we consider every pair of indices (k, i) involved in $\mathcal{C}$ with $z_{ki} \neq 0$ and construct an admissible set of relations $\mathcal{C}'$ for which Theorem 7.5 holds. Let $[v]$ be a tableau with some variable entries that satisfies $\mathcal{C}'$ (hence $g[v] = 0$). Then $\varphi_z(l)$ is equal to the evaluation of the coefficient $\varphi_z(v)$ of $[v + z]$ in $g[v]$ at $[l]$ which is 0. This proves that $\varphi_z(l) = 0$. Note that for same g and different z one chooses different sets of relations $\mathcal{C}'$ and tableaux $[v]$. In the following we provide the sets of relations $\mathcal{C}'$ and the tableaux $[v]$ for each generator $g \in K$.

Define

$$\mathfrak{V}_m = \{(i, j) | 1 \leq j \leq i \leq m\}, \tag{37}$$

$$\mathfrak{V}_n = \{(i, j) | m + 1 \leq j \leq i \leq m + n\}, \tag{38}$$

$$\mathfrak{V} = \{(i, j) | m + 1 \leq i \leq m + n - 1, 1 \leq j \leq m\}. \tag{39}$$

Let v be the variable tableau such that v_{ab} are variables for $(a, b) \in \mathfrak{V}_m \cup \mathfrak{V}_n$ and $\theta_{ab}(v) = \theta_{ab}(l)$ for $(a, b) \in \mathfrak{V}$.

(i) $[h_i, h_j][l] = 0$. Let $g = [h_i, h_j]$. Then

$$g[l] = \varphi(l)[l] = 0.$$

Then $\varphi(l)$ is the evaluation of the coefficient of $[v]$ in $g[v]$ at $[l]$ which is 0. Thus $[h_i, h_j][l] = 0$.

(ii) $[h_i, e_j][l] = (\delta_{ij} - \delta_{i,j+1})e_j[l]$. Denote $g = [h_i, e_j] - (\delta_{ij} - \delta_{i,j+1})$. Then

$$g[l] = \sum_{s=1}^{j} \varphi_s(l)[l + \delta_{js}].$$

For nonzero vector $[l + \delta_{js}]$, the coefficient $\varphi_s(l)$ is equal to the evaluation of the coefficient of $[v + \delta_{js}]$ in $g[v]$ at $[l]$ which is 0. Therefore, $[h_i, e_j][l] = (\delta_{ij} - \delta_{i,j+1})e_j[l]$.

(iii) $[h_i, f_j][l] = -(\delta_{ij} - \delta_{i,j+1})f_j[l]$. The proof is similar to (ii).

(iv) $[e_i, f_j][l] = 0$ if $i \neq j$. Denote $g = [e_i, f_j]$. Then

$$g[l] = \sum_{s=1}^{i} \sum_{t=1}^{j} \varphi_{s,t}(l)[l + \delta_{is} - \delta_{jt}].$$

(a) Suppose there is no relation between (i, s) and (j, t). Let $[v]$ be the tableau with variable entries.

(b) Suppose there is a relation between (i, s) and (j, t). Then it is $((i, s); (j, t))$ or $((j, t); (i, s))$ and $j = i \pm 1$. We set $\mathcal{C}' = \{((i, s); (j, t))\}$ or $\mathcal{C}' = \{((j, t); (i, s))\}$ respectively. Then C' is admissible. Moreover, applying Theorem 7.6 to the "standard" set of relations (satisfied by all essentially typical tableaux), we obtain that C' satisfies Theorem 7.5. Let $v_{is} = l_{is}, v_{jt} = l_{jt}$ and all other entries of $[v]$ are variables.

In both cases above, the coefficient $\varphi_{s,t}(l)$ by the tableau $[l + \delta_{is} - \delta_{jt}]$ is the evaluation of the coefficient $\varphi_{s,t}(v)$ of $[v + \delta_{is} - \delta_{jt}]$ in $g[v]$ at $[l]$ which is 0. Hence $\varphi_{s,t}(l)$ and $[e_i, f_j][l] = 0$.

(v) $[e_i, f_i][l] = (h_i - h_{i+1})[l]$ for $i \neq m$. If $i < m$, the desired equality follows from [FRZ], Theorem 4.33. Suppose $i > m$. Denote $g = [e_i, f_i] - h_i + h_{i+1}$. Then

$$g[l] = \sum_{s \neq t} \varphi_{s,t}(l)[l + \delta_{is} - \delta_{it}] + \varphi(l)[l].$$

Since there is no relation between (i, s) and (i, t), it can be verified that $\varphi_{s,t}(l) = 0$ using the same argument as in (a) of (iv). Define operators

$\hat{e}_i$ and $\hat{f}_i$ by

$$\hat{e}_i[l] = -\sum_{s=m+1}^{i} \Pi_{t=1}^{m} \left(\frac{l_{it} - l_{is} + 1}{l_{i-1,t} - l_{is} + 1} \right) \frac{\Pi_{t=m+1}^{i+1}(l_{i+1,t} - l_{is})}{\Pi_{t\neq s=m+1}^{k}(l_{it} - l_{is})}[l + \delta_{is}], \tag{40}$$

$$\hat{f}_i[l] = \sum_{s=m+1}^{i} \prod_{t=1}^{m} \left(\frac{l_{it} - l_{is} - 1}{l_{i+1,t} - l_{is} - 1} \right) \frac{\prod_{t=m+1}^{i-1}(l_{i-1,t} - l_{is})}{\prod_{t\neq s=m+1}^{i}(l_{it} - l_{is})}[l - \delta_{is}]. \tag{41}$$

Since the θ is 0 or 1, $\varphi(l)$ is equal to the coefficient of $[l]$ in $(\hat{e}_i\hat{f}_i - \hat{f}_i\hat{e}_i - h_i + h_{i+1})[l]$. Since the set $\mathcal{C}' = (\emptyset, \emptyset)$ is admissible by Theorem 7.6, then the coefficient of $[v]$ in $g[v]$ is 0. Thus the coefficient of $[v]$ in $(\hat{e}_i\hat{f}_i - \hat{f}_i\hat{e}_i - h_i + h_{i+1})[v]$ is 0. Using the argument of [FRZ], Lemma 4.31, we have that the coefficient of $[l]$ in $(\hat{e}_i\hat{f}_i - \hat{f}_i\hat{e}_i - h_i + h_{i+1})[l]$ is the limit of the coefficient of $[v]$ in $(\hat{e}_i\hat{f}_i - \hat{f}_i\hat{e}_i - h_i + h_{i+1})[v]$ by taking v to l. Therefore, $\varphi(l) = 0$.

(vi) $[e_m, f_m][l] = (h_m + h_{m+1})[l]$. The proof is similar to the case (v).

(vii) $[e_i, e_j][l] = [f_i, f_j][l] = 0$ if $|i - j| > 1$. The equality can be obtained by the same argument as in (a) of (iv).

(viii) $[e_m, e_m][l] = [f_m, f_m][l] = 0$. It can be proved using the same argument as in (vi).

Denote $g = [e_m, e_m]$. Then

$$g[l] = \sum_{s\leq t} \varphi_{s,t}(l)[l + \delta_{ms} + \delta_{mt}].$$

If $l_{ms} - l_{mt} > 1$ or $s = t$, then $\varphi_{s,t}(l)$ is equal to the evaluation of the coefficient of $[v + \delta_{is} + \delta_{it}]$ in $g[v]$ at $[l]$ which is 0. If $l_{ms} - l_{mt} = 1$ then without loss of generality we can assume that there exists p such that $((m, s); (m - 1, p)), ((m - 1, p); (m, t)) \in \mathcal{C}$. Thus $l_{m-1,p} = l_{m,s}$ and $[l + \delta_{ms} + \delta_{mt}]$ does not satisfy $\mathcal{C}$. This proves $[e_m, e_m][l] = 0$. Using the same arguments one has that $[f_m, f_m][l] = 0$.

(ix) $[e_i, [e_i, e_{i\pm1}]][l] = [f_i, [f_i, f_{i\pm1}]][l] = 0$, for $i \neq m$. The equalities follow from [FRZ] for $i < m$. We show the equalities for $i > m$. Denote $g = [e_i, [e_i, e_{i\pm1}]]$. Then

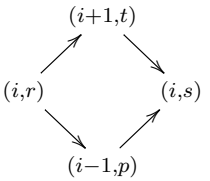
$$g[l] = \sum_{r\leq s} \sum_{t} \varphi_{r,s,t}(l)[l + \delta_{ir} + \delta_{is} + \delta_{i\pm1,t}].$$

(a) Suppose that $r, s, t \geq m+1$.

(1) If there is no relation between (i,r), (i,s) and $(i \pm 1, t)$ then take $\mathcal{C}' = (\emptyset, \emptyset)$ and the variable tableau $[v]$.

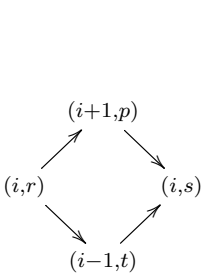
(2) Suppose one of $\{(i,r),(i,s)\}$ is connected to $(i \pm 1, t)$. Let $\mathcal{C}'$ be the set consisting of this relation. Then $\mathcal{C}'$ is admissible and satisfies Theorem 7.5. Without loss of generality we may assume that $\mathcal{C}' = \{((i,r);(i \pm 1,t))\}$. Choose $[v]$ with $v_{ir} = l_{ir}$ and $v_{i\pm1,t} = l_{i\pm1,t}$. Applying the argument as above in both cases (1) and (2) we obtain $g[l] = 0$.

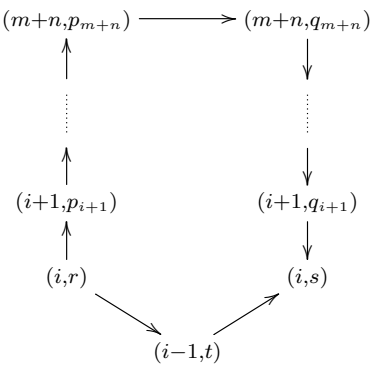
(3) Suppose there are relations between $(i+1,t)$ and both (i,r) and (i,s). Without loss of generality we may assume that $r < s$. Then $\{((i,r);(i+1,t)),((i+1,t);(i,s))\} \subseteq \mathcal{C}$ and there exists p such that $\{((i,r);(i-1,p)),((i-1,p);(i,s))\} \subseteq \mathcal{C}$. Set $\mathcal{C}' = \{((i,r);(i+1,t)),((i+1,t);(i,s)),((i,r);(i-1,p)),((i-1,p);(i,s))\}$. Then $\mathcal{C}'$ is admissible with the associated graph as follows.



Hence $\mathcal{C}'$ satisfies Theorem 7.5. We choose $[v]$ such that $v_{ir} = l_{ir}$, $v_{is} = l_{is}$, $v_{i-1,p} = l_{i-1,p}$, $v_{i+1,t} = l_{i+1,t}$.
The coefficient $\varphi_{r,s,t}(l)$ of the nonzero vector $[l + \delta_{ir} + \delta_{is} + \delta_{i+1,t}]$ is the evaluation at $[l]$ of the coefficient of $[v + \delta_{ir} + \delta_{is} + \delta_{i+1,t}]$ in $g[v]$, which is 0. Hence $\varphi_{r,s,t}(l) = 0$.

(4) Suppose there are relations between $(i-1,t)$ and both (i,r) and (i,s). Without loss of generality we may assume that $r < s$. Then, either there exists p such that $\{((i,r);(i-1,p)),((i-1,p);(i,s))\} \subseteq \mathcal{C}$, or there exist $p_k < q_k$, $i \leq k \leq m+n$ such that $p_i = r, q_i = s$ and $((i_k,p_k);(i_{k+1},p_{k+1})),((i_k,q_k);(i_{k+1},q_{k+1})) \in \mathcal{C}$ for $i+1 \leq k \leq m+n-1$. In the first case we choose $\mathcal{C}' = \{((i,r);(i-1,t)),((i-1,t);(i,s)),((i,r);(i+1,p)),((i+1,p);(i,s))\}$ and tableau $[v]$ such that $v_{ir} = l_{ir}$, $v_{is} = l_{is}$, $v_{i-1,p} = l_{i-1,p}$, $v_{i+1,t} = l_{i+1,t}$. In the second case we choose $\mathcal{C}' = \{((i,r);(i-1,t)),((i-1,t);(i,s)\} \cup \{((m+n,p_{m+n});(m+n,q_{m+n}))\} \cup \{((i_k,p_k);(i_{k+1},p_{k+1}))|i \leq k \leq m+n-1\} \cup \{((i_k,q_{k+1});(i_{k+1},q_k))|i \leq k \leq m+n-1,\}$ and tableau $[v]$ such that $v_{i-1,t} = l_{i-1,t}$, $v_{k,p_k} = l_{i_k,p_k}$, $v_{k,q_k} = l_{k,q_k}$, $i \leq k \leq m+n$. The associated graphs are respectively as follows.





In each of these cases the set $\mathcal{C}'$ is admissible and it satisfies Theorem 7.5. The coefficient $\varphi_{r,s,t}(l)$ of the nonzero vector $[l + \delta_{ir} + \delta_{is} + \delta_{i-1,t}]$ is the evaluation at $[l]$ of the coefficient of $[v + \delta_{ir} + \delta_{is} + \delta_{i-1,t}]$ in $g[v]$ which equals 0. Hence $\varphi_{r,s,t}(l) = 0$.

(b) Suppose that only one number in $\{r, s, t\}$ is $\leq m$. Then there exists at most one relation between (i, r), (i, s) and $(i \pm 1, t)$. By the above argument, $\varphi_{r,s,t}(l) = 0$.

(c) Suppose that two numbers in $\{r, s, t\}$ are $\geq m$. Similarly to (viii), $\varphi_{r,s,t}(\Lambda) = 0$.

The equation $[f_i, [f_i, f_{i\pm1}]][l] = 0$ can be shown by a similar argument.

(x) $[e_m[e_{m\pm1}, [e_m, e_{m\mp1}]]][l] = [f_m[f_{m\pm1}, [f_m, f_{m\mp1}]]][l] = 0$. Denote $g = [e_m[e_{m\pm1}, [e_m, e_{m\mp1}]]]$. Let $\mathcal{C}' = (\emptyset, \emptyset)$ be the empty set of relations and $v = (v_{ij})$ the tableau with variable entries. Applying the Gelfand-Tsetlin formulas to a nonzero $x \in \mathfrak{g}$ one can write $x[v]$ as follows:

$$x[v] = \sum_z a(v)[v + z] + \sum_{z'} a'(v)[v + z'],$$

where $[l + z]$ satisfies $\mathcal{C}$ and $[l + z']$ does not satisfy $\mathcal{C}$. Hence $g[v]$ has the form:

$$g[v] = \sum (\varphi_{s,t,p,q}(v) + \varphi'_{s,t,p,q}(v))[v + \delta_{ms} + \delta_{mt} + \delta_{m+1,p} + \delta_{m-1,q}].$$

Since $\mathcal{C}'$ is admissible, $\varphi_{s,t,p,q}(v) + \varphi'_{s,t,p,q}(v) = 0$. Evaluating at $[l]$ we have

$$g[l] = \sum_{s \leq t} \sum_{p,q} \varphi_{s,t,p,q}(l)[l + \delta_{ms} + \delta_{mt} + \delta_{m+1,p} + \delta_{m-1,q}].$$

If $[l+\delta_{ms}+\delta_{mt}+\delta_{m+1,p}+\delta_{m-1,q}]$ does not satify $\mathcal{C}$, then $\varphi_{s,t,p,q}(l)$ is zero by definition. So we can assume that $[l+\delta_{ms}+\delta_{mt}+\delta_{m+1,p}+\delta_{m-1,q}]$ satisfies $\mathcal{C}$. If $l_{mi}-l_{mj}\notin\mathbb{Z}$ or $l_{mi}-l_{mj}\neq\mathbb{Z}$, then all the tableaux appearing in the expansion of $g[l]$ satisfy $\mathcal{C}$. So we do not have $\varphi'_{s,t,p,q}(v)$ in this case and $\varphi_{s,t,p,q}(l)=0$. Suppose $l_{mi}-l_{mj}=1$. The rational function $\varphi'_{s,t,p,q}$ is regular at $v_{mi}=v_{mj}-1$. Moreover, its evaluation at $[l]$ is zero. Thus $\varphi_{s,t,p,q}(l)=0$.

Using the same argument, we have $[f_m[f_{m\pm1},[f_m,f_{m\mp1}]]][l]=0$.

This completes the proof of Theorem 7.5.

Example 7.8 (Highest weight modules). *Let $\lambda=(\lambda_1,\dots,\lambda_{m+n})$ such that $\lambda_i-\lambda_j\notin\mathbb{Z}$ or $\lambda_i-\lambda_j>i-j$ for $1\leq i<j\leq m$, $m+1\leq i<j\leq m+n$ and $\lambda_s+\lambda_t\notin\mathbb{Z}$ for $1\leq s\leq m<t\leq m+n$. Let $[l^0]$ be the tableau such that*

$$l^0_{ki}=\lambda_{ki}-i+1,(1\leq i\leq m);\quad l^0_{kj}=-\lambda_{kj}+j-2m,(m+1\leq j\leq k),$$

where $\lambda_{ki}=\lambda_i$, $i=1,\dots,m+n$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be maximal sets of relations satisfied by $[l^0]$. Then the irreducible highest weight module $L(\lambda)$ is isomorphic to $V_{\mathcal{C}}([l^0])$.

8. Irreducibility of quasi typical modules

In this section we obtain necessary and sufficient conditions for the irreducibility of quasi typical $\mathfrak{g}$-modules. We have

Proposition 8.1. *The action of $B(t)$ on any basis tableau $[l]$ of $V_{\mathcal{C}}([l^0])$ is given by*

$$B_k(t)[l]=\begin{cases}(1+tl_{k1})\cdots(1+tl_{kk})[l], & \text{if } 1\leq k\leq m,\\ \dfrac{(1+tl_{k1})\cdots(1+tl_{km})}{(1+tl_{k,m+1})\cdots(1+tl_{k,k})}[l], & \text{if } m+1\leq k\leq m+n.\end{cases}\tag{42}$$

Proof. Let $\mathcal{C}$ be an admissible set of relations. If $\tilde{\mathcal{C}}$ is obtained from $\mathcal{C}$ by the RR-method then $B_k(t)$ acts on $V_{\mathcal{C}}([l^0])$ by (46) for any $[l^0]$ if and only if it acts by (46) on $V_{\tilde{\mathcal{C}}}([l^0])$ for any $[l^0]$. This follows by the argument from [FRZ], Lemma 5.1. Since the set $\emptyset$ can be obtained from any admissible set by the RR-method, the statement follows. □

Corollary 8.2. *Let $\sum_{i=1}^r a_i[l_i]\in V_{\mathcal{C}}([l])$ with all a_i non-zero, and $[l_i]$ are mutually distinct. Then $[l_i]\in V_{\mathcal{C}}([l])$ for all $1\leq i\leq r$.*

Proof. It follows from Proposition 8.1 that the Gelfand-Tsetlin subalgebra has different characters on tableaux $[l] \neq [l']$. Then $[l_i] \in V_{\mathcal{C}}([l])$ for all $1 \leq i \leq r$. □

Theorem 8.3. *Let $\mathcal{C}$ be an admissible set of relations and $[l^0]$ a tableau satisfying $\mathcal{C}$. The quasi typical $\mathfrak{g}$-module $V_{\mathcal{C}}([l^0])$ is irreducible if and only if $\mathcal{C}$ is the maximal set of relations satisfied by $[l^0]$ and $l^0_{m+n,i} \neq l^0_{m+n,j}, 1 \leq i \leq m < j \leq m+n$.*

Proof. Suppose $\mathcal{C}$ is the maximal set of relations satisfied by $[l^0]$ and $l^0_{m+n,i} \neq l^0_{m+n,j}, 1 \leq i \leq m < j \leq m+n$. First we show that $V_{\mathcal{C}}([l^0])$ is generated by $[l^0]$. For any tableau $[l] \in V_{\mathcal{C}}([l^0])$ there exist $\{(i_t, j_t, \epsilon_t)\}$, $\epsilon_t = \pm 1$, $1 \leq t \leq s$, such that for any $r \leq s$, $[l^0 + \sum_{t=1}^{r} \epsilon_t \delta_{i_t,j_t}]$ satisfies $\mathcal{C}$ and $[l] = [l^0 + \sum_{t=1}^{s} \epsilon_t \delta_{i_t,j_t}]$.

(a) Suppose $[l^0]$ and $[l] = [l^0 + \delta_{ki}]$ satisfy $\mathcal{C}$.
   (i) If $1 \leq i \leq k \leq m-1$, then $l^0_{k+1,j} \neq l^0_{ki}$ for any $1 \leq j \leq k+1$. Therefore the coefficient of $[l]$ in $E_{k,k+1}[l^0]$ is nonzero by the maximality of $\mathcal{C}$. Hence $[l] \in \mathfrak{g}[l^0]$ by Corollary 8.2.
   (ii) If $1 \leq i \leq m, k = m$, then $\theta_{mi}(l^0) = 1$, i.e. $l^0_{m+1,i} = l^0_{m,i}$. Since $[l]$ satisfies $\mathcal{C}$, we have $l_{mj} \neq l_{mi}$ for any $1 \leq j \leq m$. It implies that $l^0_{mj} - l^0_{mi} - 1 \neq 0$ for any $1 \leq j \leq m$. Therefore, the coefficient of $[l]$ in $E_{k,k+1}[l^0]$ is nonzero. By Corollary 8.2, $[l] \in \mathfrak{g}[l^0]$.
   (iii) If $1 \leq i \leq m, m+1 \leq k \leq m+n-1$, then $\theta_{ki}(l^0) = 1, \theta_{k-1,i}(l^0) = 0$. Since $[l]$ satisfies $\mathcal{C}$, $l^0_{kj} - l^0_{ki} - 1 \neq 0$ for any $j \neq i, 1 \leq j \leq m$. Therefore, the coefficient of $[l]$ in $E_{k+1,k}[l^0]$ is nonzero. By Corollary 8.2, $[l] \in \mathfrak{g}[l^0]$.
   (iv) Suppose that $m+1 \leq i \leq k \leq m+n-1$. Since $\mathcal{C}$ is the maximal set of relations satisfied by $[l^0]$, $l_{ki} \neq l_{k-1,j}$ for any $m+1 \leq j \leq k-1$. Then the coefficient of $[l]$ in $E_{k+1,k}[l^0]$ is nonzero. By Corollary 8.2, $[l] \in \mathfrak{g}[l^0]$.

(b) Suppose $[l^0]$ and $[l] = [l^0 - \delta_{ki}]$ satisfy $\mathcal{C}$. If $1 \leq i \leq m < k \leq m+n-1$, then $\theta_{ki}(l^0) = 0$. Since $[l]$ satisfies $\mathcal{C}$, $l^0_{kj} - l^0_{ki} + 1 \neq 0$ for any $1 \leq j \leq m$.

In the following we show that

$$\frac{\Pi_{j=m+1}^{k+1}(l_{k+1,j} - l_{ki})\Pi_{j=m+1}^{k-1}(l_{k-1,j} - l_{ki} + 1)}{\Pi_{j=m+1}^{k}(l_{kj} - l_{ki})(l_{kj} - l_{ki} + 1)} \neq 0$$

If $l_{k+1,j} = l_{k,i}$ for some j, then there exist tableau $[l']$ such that $l'_{kq} = l_{kq} + z_{kq}$, where $z_{kq} \in \mathbb{Z}$, $z_{m+n,q} = 0$ for all q, $[l']$ satisfies $\mathcal{C}_1$ and $\mathcal{C}_2$, $l'_{kq} - l'_{k-1,q} \equiv \theta_{k-1,q}(l) \in \{0,1\}, 1 \leq q \leq m, m+1 \leq k \leq m+n$, $l_{k+1,j} = l_{k+1,i}$ which contradicts with the definition of tableau satisfying set of relations. Therefore the coefficient of $[l]$ in $E_{k+1,k}[l^0]$ is nonzero. By Corollary 8.2, $[l] \in \mathfrak{g}[l^0]$.

Using the same argument, it can be shown that $[l] \in \mathfrak{g}[l^0]$ in all other cases.

Repeating the argument s times we conclude that $V_{\mathcal{C}}([l^0])$ is generated by $[l^0]$. Suppose W is a nonzero submodule of $V_{\mathcal{C}}([l^0])$. Then W contains a tableau $[l]$ by Corollary 8.2. Using the above argument one can show that $[l]$ generates $V_{\mathcal{C}}([l])$ which is in fact equal to $V_{\mathcal{C}}([l^0])$. This proves the irreducibility of $V_{\mathcal{C}}([l^0])$.

Conversely, suppose $\mathcal{C}$ is not the maximal set of relations satisfied by $[l^0]$. Then $l^0_{k+1,i} - l^0_{k,j} \in \mathbb{Z}$ for some indexes and there is no relation between $(k+1, i)$ and (k, j). So there exist tableaux $[l^1], [l^2] \in V_{\mathcal{C}}([l^0])$ such that $l^1_{k+1,i} - l^1_{k,j} \in \mathbb{Z}_{\geq 0}$ and $l^2_{k,j} - l^2_{k+1,i} \in \mathbb{Z}_{>0}$. By the Gelfand-Tsetlin formulas one has that $[l^2]$ is not in the submodule of $V_{\mathcal{C}}[l^0])$ generated by $[l^1]$ and thus $V_{\mathcal{C}}([l^0])$ is not irreducible.

Suppose that $l_{m+n,i} = l_{m+n,j}$ for some $1 \leq i \leq m < j \leq m+n$. Then there exist tableaux $[l^1], [l^2] \in V_{\mathcal{C}}[l^0])$ such that $\theta_{m+n-1,i}(l^1) = 0$, i.e. $l^1_{m+n-1,i} = l^1_{m+n,j}$ and $\theta_{m+n-1,i}(l^2) = 1$. By the Gelfand-Tsetlin formulas one has that $[l^2]$ is not in the submodule of $V_{\mathcal{C}}[l^0])$ generated by $[l^1]$ and thus $V_{\mathcal{C}}([l^0])$ is not irreducible. □

Theorem 8.4. *$V_{\mathcal{C}}([l^0]) \simeq K(V_{\mathcal{C}}([l^0])^{\mathfrak{g}_1})$ if $V_{\mathcal{C}}([l^0])$ is irreducible.*

Proof. It follows immediately from the Gelfand-Tsetlin formulas that

(43)
$$W = \{[l] \in V_{\mathcal{C}}([l^0]) |\ \theta_{ki}(l) = 0, 1 \leq i \leq m \leq k \leq m+n-1\} \subset V_{\mathcal{C}}([l^0])^{\mathfrak{g}_1}.$$

Suppose there exists $v \notin W$ such that $v \in V_{\mathcal{C}}([l^0])^{\mathfrak{g}_1}$. Since Γ separates the tableaux by Proposition 8.1, we may assume that $v = [l^1]$. For any tableau $[l] \in W$ we have $(E_{11} + \cdots + E_{mm})[l] = (l_{m1} + \cdots + l_{mm})[l]$. Then $(E_{11} + \cdots + E_{mm})[l^1] = (l_{m1} + \cdots + l_{mm} - \alpha)[l^1]$ for some positive integer α. Since the elements of $U(\mathfrak{g}_0)$ commute with $E_{11} + \cdots + E_{mm}$, then all tableaux in $U(\mathfrak{g})[l^1]$ have eigenvalues which are smaller or equal than the eigenvalue of $[l^1]$. This contradicts the irreducibility of $V_{\mathcal{C}}([l^0])$. Thus $W = V_{\mathcal{C}}([l^0])^{\mathfrak{g}_1}$ and $K(V_{\mathcal{C}}([l^0])^{\mathfrak{g}_1}) = K(W) \simeq U(\mathfrak{g}_{-1}) \otimes W$.

Define the linear map $\pi : K(W) \longrightarrow V_{\mathcal{C}}([l^0])$ by

$$\pi(x \otimes [l]) = x[l], \tag{44}$$

$x \in U(\mathfrak{g}_{-1})$ and $[l] \in W$. It is easy to see that π is a $\mathfrak{g}$-module homomorphism. Since the image of π is nonzero and $V_{\mathcal{C}}([l^0])$ is irreducible, then π is surjective.

The algebra $U(\mathfrak{g}_{-1})$ has a basis

$$E_{m+1,1}^{\epsilon_1} E_{m+1,2}^{\epsilon_2} \cdots E_{m+1,m}^{\epsilon_m} E_{m+2,1}^{\epsilon_{m+1}} \cdots E_{m+n,m}^{\epsilon_{mn}}, \quad \epsilon_i = 0 \text{ or } 1, 1 \leq i \leq mn.$$

Suppose $\pi(v) = 0$. Then v can be written as

$$v = \sum_i x_i \otimes w_i,$$

where x_i is a product of E_{kj}, $1 \leq j \leq m < k \leq m+n$ with $x_i \neq x_j$ for $i \neq j$ and w_i is a nonzero vector in W. Without loss of generality we assume that x_1 has the smallest number of factors. Let y be the product of all negative odd root elements E_{kj}'s which are not factors of x_1. Then we have $\pi(yv) = 0$ and $yx = z \otimes w_1$, where z is the product of all negative odd root elements. Thus $zw_1 = 0$ in $V_{\mathcal{C}}([l^0])$.

The module $V_{\mathcal{C}}([l^0])$ is irreducible, thus $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$ is a maximal set of relations satisfied by $[l^0]$ and $W = U(\mathfrak{g}_0)w_1$ is irreducible $\mathfrak{g}_0$-module. Now we will show that $zw_1 \neq 0$ which gives a contradiction. Let $[l']$ be the tableau in $V_{\mathcal{C}}([l^0])$ such that

$$l'_{ki} = l_{ki} - (m+n-k), 1 \leq i \leq m \leq k \leq m+n.$$

Then there exists an element $u \in U(\mathfrak{g})$ such that $uw_1 = [l']$. The element u can be written in the following form:

$$u = \sum_i u_1^{(i)} u_0^{(i)} u_{-1}^{(i)}, \tag{45}$$

where $u_{\pm 1}^{(i)} \in U(\mathfrak{g}_{\pm 1})$, $u_0^{(i)} \in U(\mathfrak{g}_0)$. By the formula (23) we have $(E_{11} + \cdots + E_{mm})[l'] = (l_{m1} + \cdots + l_{mm} - mn)[l']$. Since $u_0^{(i)}$ does not change the eigenvalue of the tableau and $[l']$ has the smallest eigenvalue, then $u_1^{(i)} = a_i$ and $u_{-1}^{(i)} = b_i z$, $a_i, b_i \in \mathbb{C}$ for any i. Thus u has the form $u_0 z, u_0 \in U(\mathfrak{g}_0)$. Therefore, $[l'] = uw_1 = u_0 z w_1$ which implies that zw_1 is nonzero vector in $V_{\mathcal{C}}([l^0])$ which is a contradiction. We conclude that π is injective and $V_{\mathcal{C}}([l^0]) \simeq K(V_{\mathcal{C}}([l^0])^{\mathfrak{g}_1})$. □

9. Quasi covariant modules

In this section we extend our construction of quasi typical modules to include all covariant tensor modules constructed in [SV]. We recall their construction.

Theorem 9.1. *[SV] The set of vectors ξ_Λ parameterized by all the Gelfand-Tsetlin tableaux Λ satisfying the conditions*

1) $\lambda_{m+n,i} \in \mathrm{Z}_{\geq 0}$ *are fixed and* $\lambda_{m+n,j} - \lambda_{m+n,j+1} \in \mathrm{Z}_{\geq 0}, j \neq m, 1 \leq j \leq m+n-1$, $\lambda_{m+n,m} \geq \#\{i : \lambda_{m+n,i} > 0,\ m+1 \leq i \leq m+n\}$;

2) $\lambda_{ki} - \lambda_{k-1,i} \equiv \theta_{k-1,i} \in \{0,1\}, 1 \leq i \leq m; m+1 \leq k \leq m+n$;

3) $\lambda_{km} \geq \#\{i : \lambda_{ki} > 0,\ m+1 \leq i \leq k\}, m+1 \leq k \leq m+n$;

4) if $\lambda_{m+1,m} = 0$, *then* $\theta_{mm} = 0$;

5) $\lambda_{ki} - \lambda_{k,i+1} \in \mathrm{Z}_{\geq 0}, 1 \leq i \leq m-1; m+1 \leq k \leq m+n-1$;

6) $\lambda_{k+1,i} - \lambda_{ki} \in \mathrm{Z}_{\geq 0}$ *and* $\lambda_{k,i} - \lambda_{k+1,i+1} \in \mathrm{Z}_{\geq 0},\ 1 \leq i \leq k \leq m-1$ *or* $m+1 \leq i \leq k \leq m+n-1$,

constitutes a basis of $L(\lambda)$, where $\lambda = (\lambda_1, \lambda_2, \ \ldots, \lambda_{m+n})$ and $\lambda_i = \lambda_{m+n,i}$, $i = 1, \ldots, m+n$. The action of generators of $\mathfrak{gl}(m|n)$ on the irreducible covariant tensor module $L(\lambda)$ is given by the formulas (23)-(29).

Let $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$ be an admissible set of relations. We will say that $\mathcal{C}$ is a *covariant admissible* set of relations if it satisfies the following conditions:

1) $\mathcal{C}_2$ has a subset

$$\mathcal{C}' = \{((i,i);(i+1,i+1))|p \leq i \leq m+n-1\},$$

for a fixed p with $m+1 \leq p \leq m+n$ and there is no $(p-1,i)$ such that $((p,p);(p-1,i)) \in \mathcal{C}_2$ or $((p-1,i);(p,p)) \in \mathcal{C}_2$).

2) If (k,i) is in the component of $\mathcal{C}'$ then (k,j) is in the same component for $i \leq j \leq k$.

Definition 9.2. *A tableau $[l]$ of $\mathfrak{gl}(m|n)$ is called* $\mathcal{C}$-covariant *if:*

1) $[l]$ satisfies $\mathcal{C}_1$ and $\mathcal{C}_2$;

2) $l_{ki} - l_{k-1,i} \equiv \theta_{k-1,i}(l) \in \{0,1\}, 1 \leq i \leq m, m+1 \leq k \leq m+n$;

3) $l_{m+n,n} - l_{m+n,m+n} \in \mathbb{Z}$, *and* $l_{ki} - l_{kj} \notin \mathbb{Z}$ *for* $m+1 \leq k \leq m+n$, $1 \leq i \leq m < j \leq k$ *with* (k,j) *not in the component of* $\mathcal{C}'$.

4) If $l_{m+n,n} - l_{m+n,m+n} \leq 0$ *then for* $m+1 \leq k \leq m+n$

- $l_{km} \geq l_{m+n,m+n} - m - n + p_k$, *if there exists* p_k *such that* p_k *is the smallest number among* $\{m+1, m+2, \ldots, k\}$ *with* $l_{k,p_k} - l_{m+n,m+n} = p_k - m - n$.
- $l_{km} \geq l_{m+n,m+n} - m - n + k + 1$ *if there exists no such* p_k.

5) If $l_{m+1,m} - l_{m+n,m+n} = 1 - n$ *then* $\theta_{mm} = 0$.

Let $[l^0] = (l^0_{ij})$ be a $\mathcal{C}$-covariant tableau and $\mathcal{B}_{\mathcal{C}}([l^0])$ the set of all possible $\mathcal{C}$-covariant tableaux $[l] = (l_{ij})$ satisfying

$$l_{ki} = l^0_{ki} + z_{ki},\ z_{ki} \in \mathbb{Z},\ 1 \leq k \leq m+n-1,\ 1 \leq i \leq k.$$

Denote by $V_{\mathcal{C}}([l^0])$ the vector space spanned by $\mathcal{B}_{\mathcal{C}}([l^0])$. We have the following extension of Theorem 7.5 and Theorem 8.3 for covariant admissible set of relations.

Theorem 9.3. *Let* $\mathcal{C}$ *be a covariant admissible set of relations.*

(i) *The the formulas* (23)-(29) *define a* $\mathfrak{gl}(m|n)$*-module structure on* $V_{\mathcal{C}}([l^0])$ *for any* $\mathcal{C}$*-covariant tableau* $[l^0]$.

(ii) *The action of* $B(t)$ *on any basis tableau* $[l]$ *of* $V_{\mathcal{C}}([l^0])$ *is given by*

$$B_k(t)[l] = \begin{cases} (1+tl_{k1})\cdots(1+tl_{kk})[l], & \text{if } 1 \leq k \leq m, \\ \dfrac{(1+tl_{k1})\cdots(1+tl_{km})}{(1+tl_{k,m+1})\cdots(1+tl_{k,k})}[l], & \text{if } m+1 \leq k \leq m+n. \end{cases} \tag{46}$$

(iii) *The module* $V_{\mathcal{C}}([l^0])$ *is irreducible if and only if* $\mathcal{C}$ *is the maximal set of relations satisfied by* $[l^0]$.

The proof of this theorem is fully analogous to the proof of Theorem 7.5 and Theorem 8.3 so we leave the details out. The module $V_{\mathcal{C}}([l^0])$ in Theorem 9.3 will be called *quasi covariant* $\mathfrak{g}$-module. Note that all covariant tensor finite dimensional modules are quasi covariant. Hence, we have a generalization of the class of finite dimensional covariant tensor modules.

Acknowledgements

V.F. is supported by CNPq (200783/2018-1) and by Fapesp (2018/23690-6). J. Z. was supported by Fapesp (2015/05927-0). V.S. was supported by

NSF grant 1701532. J. Z. acknowledges the hospitality and support of Max Planck Institute for Mathematics in Bonn. All authors are grateful to the International Center for Mathematics of SUSTech for support and hospitality during their visit when the paper was completed.

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Received February 15, 2020
Accepted May 3, 2020