Sharp lower bound for the first eigenvalue of the weighted p-Laplacian II

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Combined with our previous work [14], we prove sharp lower bound estimates for the first nonzero eigenvalue of the weighted p-Laplacian with $1 < p < \infty$ on a compact Bakry-Émery manifold (M^n, g, f), without boundary or with a convex boundary and Neumann boundary condition, satisfying $\mathrm{Ric} + \nabla^2 f \geq \kappa\, g$ for some $\kappa \in \mathbb{R}$.

1. Introduction

The determination of lower bounds for the first nonzero eigenvalue of an elliptic operator is an important issue in both mathematics and physics, since this constant determines the convergence rate of numerical schemes in numerical analysis, describes the energy of a particle in the ground state in quantum mechanics, and determines the decay rate of heat flows in thermodynamics. Given its physical and mathematical significance, sharp lower bounds of the first nonzero eigenvalue of the Laplace-Beltrami operator on a compact Riemannian manifold or the f-Laplacian on a compact Bakry-Émery manifold (without boundary or with a convex boundary and Neumann boundary condition) in terms of geometric data have been established via the efforts of many mathematicians including [5, 7, 8, 10, 24] by the year 2000. Simple alternative proofs via the estimates of the modulus of continuity were found in recent years in [3, 4].

In the last two decades, much attention has been focused on eigenvalue problems of nonlinear operators, especially the p-Laplacian Δ_p and

the weighted p-Laplacian $\Delta_{p,f}$, defined for $1 < p < \infty$ by

$$\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$$

and

$$\Delta_{p,f} u := e^f \mathrm{div}(e^{-f}|\nabla u|^{p-2}\nabla u),$$

respectively. On a compact Riemannian manifold satisfying $\mathrm{Ric} \geq (n-1)\kappa g$ for $\kappa \in \mathbb{R}$, sharp lower bounds for the first nonzero eigenvalue of Δ_p have been obtained in [16] for $\kappa > 0$, [21] for $\kappa = 0$, and [17] for $\kappa < 0$, as well as a sharpened result for $\kappa > 0$. We refer the reader to [14, Theorem 1.3] for a unified statement in terms of eigenvalue comparisons with associated one-dimensional models.

The purpose of this paper is to prove sharp lower bounds for the first nonzero eigenvalue of $\Delta_{p,f}$ on compact Bakry-Émery manifolds in terms of dimension, diameter, and lower bound of the Bakry-Émery Ricci tensor $\mathrm{Ric} + \nabla^2 f$, thus completing our previous work [14] and generalizing the above-mentioned results for compact Riemannian manifolds with Ricci curvature lower bounds.

Recall that a triple (M^n, g, f) consisting of an n-dimensional Riemannian manifold (M^n, g) and a function $f \in C^\infty(M)$ is called a Bakry-Émery manifold if it satisfies

$$\mathrm{Ric} + \nabla^2 f \geq \kappa \, g$$

for some $\kappa \in \mathbb{R}$. Here Ric denotes the Ricci curvature of (M, g) and $\nabla^2 f$ denotes the Hessian of the function f. By taking f to be a constant function, we see that Bakry-Émery manifolds include all Riemannian manifolds with a lower bound on the Ricci curvature. The tensor $\mathrm{Ric} + \nabla^2 f$ is called the Bakry-Émery Ricci tensor and it shares many important properties as the Ricci curvature, see for instance [15] and [22].

The first nonzero eigenvalue of $\Delta_{p,f}$ on a closed Bakry-Émery manifold (M^n, g, f), denoted by $\lambda_{p,f}$, is defined by

$$\lambda_{p,f} = \inf\left\{ \frac{\int_M |\nabla u|^p e^{-f} d\mu}{\int_M |u|^p e^{-f} d\mu} : u \in W^{1,p}(M, e^{-f}d\mu) \setminus \{0\}, \int_M |u|^{p-2}u \, e^{-f} d\mu = 0 \right\}.$$

It was shown in [20] that this infimum is achieved by an eigenfunction $u \in C^{1,\alpha}(M)$ satisfying the Euler-Lagrange equation

$$\Delta_{p,f} u = -\lambda_{p,f}|u|^{p-2}u.$$

In case $\partial M \neq \emptyset$, the Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$ on ∂M is always imposed, where ν is the outward unit normal vector along ∂M. In our recent

work [14], we obtained sharp lower bound estimates for $\lambda_{p,f}$ in terms of p, κ, and the diameter of M, provided that either $1 < p \leq 2$ or $\kappa \leq 0$. In this paper, we take care of the remaining case $p > 2$ and $\kappa > 0$.

Combined with our results in [14], we prove that

Theorem 1.1. *Let (M^n, g, f) be a compact Bakry-Émery manifold (possibly with a C^2 convex boundary) with diameter D and $\mathrm{Ric} + \nabla^2 f \geq \kappa\, g$ for $\kappa \in \mathbb{R}$. For $1 < p < \infty$, let $\lambda_{p,f}$ be the first nonzero eigenvalue of the weighted p-Laplacian $\Delta_{p,f}$ (with Neumann boundary condition if $\partial M \neq \emptyset$). Then we have*

$$(1.1) \qquad \lambda_{p,f} \geq \mu_p(\kappa, D),$$

where $\mu_p(\kappa, D)$ is the first nonzero Neumann eigenvalue of the one-dimensional eigenvalue problem

$$(1.2) \qquad (p-1)|\varphi'|^{p-2}\varphi'' - \kappa\, t\, |\varphi'|^{p-2}\varphi' = -\lambda|\varphi|^{p-2}\varphi$$

on $[-D/2, D/2]$.

When $\kappa = 0$, the ODE (1.2) can be solved explicitly (see [6] or [21]) and we have in particular that

$$\mu_p(0, D) = (p-1)\left(\frac{\pi_p}{D}\right)^p, \text{ where } \pi_p = \frac{2\pi}{p\sin(\pi/p)}.$$

Therefore, we see that Theorem 1.1 reduces to the sharp lower bound of first nonzero eigenvalue of the Laplacian by Zhong and Yang [24] if $p = 2$ and $f \equiv 0$, and to the sharp lower bound of first nonzero eigenvalue of the p-Laplacian by Valtorta [21] if $f \equiv 0$. Moreover, when $\kappa = 0$, the equality in (1.1) is achieved when M is a one-dimensional circle (when M has no boundary) or the line segment $[-D/2, D/2]$ (when M has boundary).

For general $\kappa \in \mathbb{R}$, the $p = 2$ case of Theorem 1.1 was due to Bakry and Qian [5]. Andrews and Ni [4] gave a simple alternative proof using the modulus of continuity approach and they also demonstrated the sharpness in all dimensions by constructing a sequence of Riemannian manifolds collapsing to the interval $[-D/2, D/2]$. We refer the reader to [3, 5, 14, 17] and the references therein for more historical developments and other related results.

In [14], we used the method of modulus of continuity estimates to give a simple proof of Theorem 1.1 for $1 < p \leq 2$ and $\kappa \in \mathbb{R}$. Recent years have

witnessed the great success of this approach and sharp eigenvalues esti-
mates were obtained in [2–4, 9, 11–14, 18, 19, 23]. However, it seems dif-
ficult to handle the $p > 2$ case with this method, as pointed out in the
wonderful survey by Andrews [1, Section 8]. The proof presented below uses
the classical gradient estimates method initiated by Li [8] and Li and Yau
[10], which indeed works for all $1 < p < \infty$ and $\kappa \in \mathbb{R}$. This approach has
been used successfully by various authors to estimate eigenvalues, including
[5, 7, 8, 10, 14, 17, 21, 24].

In order to handle the $\kappa > 0$ case, we have to overcome two difficulties
in this paper. The sharp gradient comparison theorem (see Theorem 3.1),
which is the most important ingredient, was only proved for $\kappa \leq 0$ in [14] us-
ing a two-point maximum principle argument. In the present paper, we use a
Bochner type formula for the weighted p-Laplacian (see Proposition 2.1) to
prove the sharp gradient comparison theorem. Despite its technicality, this
approach works regardless of the sign of κ, and the value of $p \in (1, \infty)$. The
second key estimate that we shall establish in this paper is Proposition 4.2,
which says that $\lambda_{p,f}$ is bounded from below by the first nonzero eigenvalue
of the eigenvalue problem (1.2) on the real line. This bound is trivial when
$\kappa \leq 0$ (the first eigenvalue of the real line is zero in this case), but it is non-
trivial when $\kappa > 0$ and turns out to be a consequence of the sharp gradient
comparison theorem and the compactness of the manifold. Once we have
these two key ingredients, the rest of proof consists of careful analysis of the
one-dimensional models and standard comparison arguments.

The paper is organized as follows. We derive the Bochner formula for
$\Delta_{p,f}$ in Section 2 and use it to prove the sharp gradient comparison theorem
in Section 3. In Section 4, we study the qualitative behavior of solutions
to the ODE (1.2) and construct one-dimensional models for the purpose
of applying the gradient comparison theorem effectively. The proof of The-
orem 1.1 is then given in Section 5. Finally, examples are constructed in
Section 6 to demonstrate the sharpness of Theorem 1.1.

2. Bochner formula for the weighted p-Laplacian

The goal of this section is to establish the Bochner formula for $\Delta_{p,f}$ in
Proposition 2.1, which will play a key role in proving the sharp gradient
comparison theorem in Section 3. This is a slight extension of [21, Section
3], where the author proves the Bochner formula for the p-Laplacian.

Let H_u denote the Hessian of u and set

$$A_u = \frac{H_u(\nabla u, \nabla u)}{|\nabla u|^2}$$

whenever $|\nabla u| \neq 0$. The linearization of $\Delta_{p,f}$ near a function u is given by

$$
\begin{aligned}
P_{u,f}(\eta) &:= \frac{d}{dt}\bigg|_{t=0} \Delta_{p,f}(u + t\eta) \\
&= |\nabla u|^{p-2}\Delta_f \eta + (p-2)|\nabla u|^{p-4}H_\eta(\nabla u, \nabla u) \\
&\quad + (p-2)\Delta_{p,f}u \frac{\langle \nabla u, \nabla \eta \rangle}{|\nabla u|^2} \\
&\quad + 2(p-2)|\nabla u|^{p-4}H_u\left(\nabla u, \nabla \eta - \frac{\nabla u}{|\nabla u|}\left\langle \frac{\nabla u}{|\nabla u|}, \nabla \eta \right\rangle\right)
\end{aligned}
$$

wherever $|\nabla u| \neq 0$. It is easy to see that this operator is strictly elliptic where the gradient of u does not vanish. Let $P_{u,f}^{II}(\eta)$ be defined by

$$P_{u,f}^{II}(\eta) := |\nabla u|^{p-2}\Delta_f \eta + (p-2)|\nabla u|^{p-4}H_\eta(\nabla u, \nabla u).$$

Proposition 2.1 (Bochner formula for $\Delta_{p,f}$). *Let $x \in M$ and $U \subset M$ be an open set containing x. For any function $u \in C^3(U)$ with $\nabla u \neq 0$ on U, we have at x,*

$$
\begin{aligned}
\frac{1}{p}P_{u,f}^{II}(|\nabla u|^p) &= |\nabla u|^{p-2}\langle \nabla \Delta_{p,f}u, \nabla u \rangle - (p-2)|\nabla u|^{p-2}A_u\Delta_{p,f}u \\
&\quad + |\nabla u|^{2(p-2)}\left(|H_u|^2 + p(p-2)A_u^2 + \mathrm{Ric}_f(\nabla u, \nabla u)\right),
\end{aligned}
$$

where $\mathrm{Ric}_f = \mathrm{Ric} + \nabla^2 f$.

Proof. Let $P_u^{II}(\eta) := |\nabla u|^{p-2}\Delta \eta + (p-2)|\nabla u|^{p-4}H_\eta(\nabla u, \nabla u)$ be the second order part of $P_{u,f}(\eta)$. The Bochner formula for the p-Laplacian, shown in [21, Proposition 3.1], states that

$$
\begin{aligned}
\frac{1}{p}P_u^{II}(|\nabla u|^p) &= |\nabla u|^{p-2}\langle \nabla \Delta_p u, \nabla u \rangle - (p-2)|\nabla u|^{p-2}A_u\Delta_p u \\
&\quad + |\nabla u|^{2(p-2)}\left(|H_u|^2 + p(p-2)A_u^2 + \mathrm{Ric}(\nabla u, \nabla u)\right).
\end{aligned}
$$

We then compute, using the identities $\Delta_p u = \Delta_{p,f} u + |\nabla u|^{p-2} \langle \nabla u, \nabla f \rangle$ and $\mathrm{Ric} = \mathrm{Ric}_f - \nabla^2 f$, that

$$
\begin{aligned}
\frac{1}{p} P_u^{II}(|\nabla u|^p) ={}& |\nabla u|^{p-2} \langle \nabla \Delta_{p,f} u, \nabla u \rangle - (p-2)|\nabla u|^{p-2} A_u \Delta_{p,f} u \\
& + |\nabla u|^{2(p-2)} \left(|H_u|^2 + p(p-2) A_u^2 + \mathrm{Ric}_f(\nabla u, \nabla u) \right) \\
& + |\nabla u|^{p-2} \left\langle \nabla(|\nabla u|^{p-2} \langle \nabla u, \nabla f \rangle), \nabla u \right\rangle \\
& - (p-2)|\nabla u|^{2(p-2)} A_u \langle \nabla u, \nabla f \rangle \\
& - |\nabla u|^{2(p-2)} \left(\nabla^2 f(\nabla u, \nabla u) \right) \\
={}& |\nabla u|^{p-2} \langle \nabla \Delta_{p,f} u, \nabla u \rangle - (p-2)|\nabla u|^{p-2} A_u \Delta_{p,f} u \\
& + |\nabla u|^{2(p-2)} \left(|H_u|^2 + p(p-2) A_u^2 + \mathrm{Ric}_f(\nabla u, \nabla u) \right) \\
& + \frac{1}{p}|\nabla u|^{p-2} \langle \nabla |\nabla u|^p, \nabla f \rangle.
\end{aligned}
$$

The desired formula then follows from

$$
\frac{1}{p} P_{u,f}^{II}(|\nabla u|^p) = \frac{1}{p} P_u^{II}(|\nabla u|^p) - \frac{1}{p}|\nabla u|^{p-2} \langle \nabla f, \nabla |\nabla u|^p \rangle.
$$

$\square$

3. The gradient comparison theorem

In this section, we use the Bochner formula for $\Delta_{p,f}$ proved in Proposition 2.1 to derive sharp gradient comparison theorems for eigenfunctions of $\Delta_{p,f}$. We emphasize that the proof works for all $1 < p < \infty$ and $\kappa \in \mathbb{R}$.

Theorem 3.1. *Let (M^n, g, f) be a compact Bakry-Émery manifold (possibly with C^2 convex boundary) satisfying $\mathrm{Ric} + \nabla^2 f \geq \kappa\, g$ for some $\kappa \in \mathbb{R}$. Let u be a solution of*

$$
\text{(3.1)} \qquad\qquad \Delta_{p,f} u = -\lambda |u|^{p-2} u,
$$

normalized so that $-1 = \min\{u\} < 0 < \max\{u\} \leq 1$ (in case $\partial M \neq \emptyset$, we assume that u satisfies the Neumann boundary condition). Suppose $w : [a, b] \to \mathbb{R}$ is a solution of the one-dimensional equation:

$$
\text{(3.2)} \qquad (p-1)|w'|^{p-2} w'' - \kappa\, t |w'|^{p-2} w' = -\lambda |w|^{p-2} w
$$

which is strictly increasing on $[a, b]$, and such that the range of u is contained in $[w(a), w(b)]$. Then we have for all $x \in M$,

$$|\nabla u(x)| \leq w' \left(w^{-1}(u(x)) \right).$$

Proof. **Case 1:** $\partial M = \emptyset$.
We may assume that $[\min\{u\}, \max\{u\}] \subset (w(a), w(b))$, because otherwise we can replace u by αu and then let $\alpha \to 1^-$. Let $w := w_\varepsilon(t)$ be the solution of the initial value problem

$$(3.3) \qquad \begin{cases} (p-1)|w'|^{p-2}w'' - (\kappa - \varepsilon)\, t|w'|^{p-2}w' + \lambda|w|^{p-2}w = 0, \\ w(a) = -1, w'(a) = 0, \end{cases}$$

which is strictly increasing on $[a, b_\varepsilon]$ with $\lim_{\varepsilon \to 0} b_\varepsilon = b$. For $\varepsilon > 0$ sufficiently small, we still have that $[\min\{u\}, \max\{u\}] \subset (w(a), w(b_\varepsilon))$.

Let $c_0 \geq 1$ be the number defined by

$$c_0 = \inf\{c \geq 1 : Z_c(x) = |\nabla u(x)|^p - (cw')^p \big|_{(cw)^{-1}(u(x))} \leq 0 \text{ on } M\}.$$

Clearly c_0 is finite since $Z_c(x)$ approaches negative infinity uniformly as c approaches infinity. By continuity, there exists $x_0 \in M$ such that

$$0 = Z_{c_0}(x_0) = \max_{x \in M} Z_{c_0}(x).$$

For simplicity of notations, we use $t = (c_0 w)^{-1}(u(x))$ as an intermediate variable and write $\varphi(t) = (c_0 w)(t)$. Thus the function $Z_{c_0}(x) = |\nabla u|^p - (\varphi')^p$ attains zero maximum at x_0. Note that $|\nabla u(x_0)| \neq 0$ since φ has positive derivative.

Hereafter, we assume that $u \in C^3(U)$ for some open neighborhood U of x_0. This is certainly the case if $u(x_0) \neq 0$ or if $p \geq 2$, as pointed out in Remark 3.2. In case $1 < p < 2$ and $u(x_0) = 0$, then u has only $C^{2,\alpha}$ regularity near x_0. However, this regularity issue is not an obstacle to the argument, as we will explain in Remark 3.3. Here and below, all the derivatives of φ are evaluated at $t_0 = (c_0 w)^{-1}(u(x_0))$.

The first derivative test implies that $\nabla Z_{c_0}(x_0) = 0$, which produces the following identity at x_0,

$$p|\nabla u|^{p-2}H_u\nabla u = \frac{p}{p-1}\mathcal{L}_p\varphi\nabla u,$$

where $\mathcal{L}_p \varphi := (p-1)(\varphi')^{p-2} \varphi''$ is the one-dimensional p-Laplacian. In particular, we have

$$(3.4) \qquad (p-1)|\nabla u|^{p-2} A_u = \mathcal{L}_p \varphi.$$

Next we calculate and estimate the second derivatives. Using Proposition 2.1, we obtain that at x_0,

$$
\begin{aligned}
\frac{1}{p} P^{II}_{u,f}(|\nabla u|^p) &= |\nabla u|^{p-2} \langle \nabla \Delta_{p,f} u, \nabla u \rangle - (p-2)|\nabla u|^{p-2} A_u \Delta_{p,f} u \\
&\quad + |\nabla u|^{2(p-2)} \left(|H_u|^2 + p(p-2) A_u^2 + \mathrm{Ric}_f(\nabla u, \nabla u) \right) \\
&\geq -\lambda |\nabla u|^{p-2} \langle \nabla \left(|u|^{p-2} u \right), \nabla u \rangle - (p-2)\lambda |u|^{p-2} u |\nabla u|^{p-2} A_u \\
&\quad + |\nabla u|^{2(p-2)} \left((p-1)^2 A_u^2 + \kappa |\nabla u|^2 \right) \\
&= -(p-1)\lambda u^{p-2} |\nabla u|^p - \frac{p-2}{p-1} \lambda |u|^{p-2} u \mathcal{L}_p \varphi \\
(3.5) &\quad + (\mathcal{L}_p \varphi)^2 + \kappa |\nabla u|^{2p-2},
\end{aligned}
$$

where we have used $|H_u|^2 \geq A_u^2$ and $\mathrm{Ric}_f(\nabla u, \nabla u) \geq \kappa |\nabla u|^2$ in the inequality, and (3.4) in getting the last equality.

On the other hand, direct calculation shows

$$
\begin{aligned}
\frac{1}{p} \nabla (\varphi')^p &= \frac{1}{p-1} \mathcal{L}_p \varphi \nabla u, \\
\frac{1}{p} \Delta_f (\varphi')^p &= \frac{1}{p-1} \mathcal{L}_p \varphi \Delta_f u + \frac{1}{p-1} \frac{d}{dt}(\mathcal{L}_p \varphi) \frac{1}{\varphi'} |\nabla u|^2, \\
\frac{1}{p} H_{(\varphi')^p}(\nabla u, \nabla u) &= \frac{1}{p-1} \mathcal{L}_p \varphi A_u |\nabla u|^2 + \frac{1}{p-1} \frac{d}{dt}(\mathcal{L}_p \varphi) \frac{1}{\varphi'} |\nabla u|^4.
\end{aligned}
$$

Putting the above identities together, we obtain that

$$
\begin{aligned}
\frac{1}{p} P^{II}_{u,f}((\varphi')^p) &= \frac{1}{p} |\nabla u|^{p-2} \Delta_f (\varphi')^p + \frac{1}{p}(p-2)|\nabla u|^{p-4} H_{(\varphi')^p}(\nabla u, \nabla u) \\
&= \frac{\mathcal{L}_p \varphi}{p-1} |\nabla u|^{p-2} (\Delta_f u + A_u) + |\nabla u|^p \frac{d}{dt}(\mathcal{L}_p \varphi) \frac{1}{\varphi'} \\
(3.6) &= \frac{-\lambda |u|^{p-2} u}{p-1} \mathcal{L}_p \varphi + |\nabla u|^p \frac{d}{dt}(\mathcal{L}_p \varphi) \frac{1}{\varphi'},
\end{aligned}
$$

where we have used $|\nabla u|^{p-2} (\Delta_f u + A_u) = \Delta_{p,f} u = -\lambda |u|^{p-2} u$ in the last equality.

The second derivative test implies $P_{u,f}^{II}(Z_{\bar{c}}) \leq 0$ at x_0, thus we have by (3.5) and (3.6) that

$$0 \geq \frac{1}{p} P_{u,f}^{II}(Z_{\bar{c}}) = \frac{1}{p} P_{u,f}^{II}(|\nabla u|^p) - \frac{1}{p} P_{u,f}^{II}((\varphi')^p)$$

$$\geq -(p-1)\lambda u^{p-2}|\nabla u|^p - \frac{p-2}{p-1}\lambda|u|^{p-2}u\mathcal{L}_p\varphi$$

$$(3.7) \qquad + (\mathcal{L}_p\varphi)^2 + \kappa|\nabla u|^{2p-2} + \frac{\lambda|u|^{p-2}u}{p-1}\mathcal{L}_p\varphi - |\nabla u|^p\frac{d}{dt}(\mathcal{L}_p\varphi)\frac{1}{\varphi'}.$$

Substituting $|\nabla u(x_0)|^p = (\varphi'(t_0))^p$ and $u(x_0) = \varphi(t_0)$ into (3.7) gives that at t_0,

$$0 \geq -(p-1)\lambda\varphi^{p-2}(\varphi')^p + \frac{(p-2)}{p-1}\lambda|\varphi|^{p-2}\varphi\mathcal{L}_p\varphi$$

$$+ (\mathcal{L}_p\varphi)^2 + \kappa(\varphi')^{2p-2} + \frac{\lambda|\varphi|^{p-2}\varphi}{p-1}\mathcal{L}_p\varphi - (\varphi')^{p-1}\frac{d}{dt}(\mathcal{L}_p\varphi)$$

$$\geq \mathcal{L}_p\varphi\left(\mathcal{L}_p\varphi - (\kappa-\varepsilon)t(\varphi')^{p-1} + \lambda|\varphi|^{p-2}\varphi\right) + \varepsilon(\varphi')^{2p-2}$$

$$- (\varphi')^{p-1}\left(\frac{d}{dt}(\mathcal{L}_p\varphi) + (p-1)\lambda\varphi^{p-2}\varphi'\right.$$

$$(3.8) \qquad \left. - (\kappa-\varepsilon)t\,\mathcal{L}_p\varphi - (\kappa-\varepsilon)(\varphi')^{p-1}\right).$$

Since φ satisfies the ODE

$$\mathcal{L}_p\varphi - (\kappa-\varepsilon)t(\varphi')^{p-1} + \lambda|\varphi|^{p-2}\varphi = 0,$$

it follows that $\mathcal{L}_p\varphi$ satisfies

$$\frac{d}{dt}(\mathcal{L}_p\varphi) - (\kappa-\varepsilon)(\varphi')^{p-1} - (\kappa-\varepsilon)t\,\mathcal{L}_p\varphi + (p-1)\lambda\varphi^{p-2}\varphi' = 0.$$

We then easily get from (3.8) that

$$0 \geq \varepsilon(\varphi')^{2p-2} > 0,$$

which is clearly a contradiction. The desired gradient estimate follows immediately since the solutions w_ε converges in C^1 to the solution w as $\varepsilon \to 0$.

Case 2: $\partial M \neq \emptyset$ and u satisfies the Neumann boundary condition. The proof for Case 1 remains valid as long as x_0 is in the interior of M. If $x_0 \in \partial M$, we follow the argument in [17, Lemma 18] to show that $\nabla Z_{c_0}(x_0) = 0$. Once this is established, it then follows that $P_{u,f}^{II}(Z_{c_0}(x_0)) \leq 0$ and the rest

of the proof proceeds as in Case 1. Thus it suffices to show the following claim.

Claim: The equation $\nabla Z_{c_0}(x_0) = 0$ remains valid even if $x_0 \in \partial M$.

Proof of Claim. Let ν be the outward unit normal vector field of ∂M. Since $Z_{c_0}(x)$ attains its maximum at $x_0 \in \partial M$, we know that all tangential derivatives of $Z_{c_0}(x)$ vanish at x_0 and

$$
\begin{aligned}
0 \leq \langle \nabla Z_{c_0}, \nu \rangle(x_0) &= p|\nabla u|^{p-2} H_u(\nabla u, \nu) - p(\varphi')^{p-2}\varphi''\langle \nabla u, \nu \rangle \\
&= p|\nabla u|^{p-2} H_u(\nabla u, \nu) \\
&= -p|\nabla u|^{p-2} II(\nabla u, \nabla u) \leq 0.
\end{aligned}
$$

Here the last step is because of the convexity of ∂M. Therefore we have $\nabla Z_{c_0}(x_0) = 0$ and the claim is proved. $\qquad\square$

The proof of Theorem 3.1 is complete now. $\qquad\square$

Remark 3.2. *The eigenfunction u is in general not smooth. We have $u \in C^{1,\alpha}(M) \cap W^{1,p}(M)$, and elliptic theory ensures that u is smooth where $\nabla u \neq 0$ and $u \neq 0$. If $\nabla u \neq 0$ and $u(x) = 0$, then $u \in C^{3,\alpha}(U)$ if $p > 2$ and $u \in C^{2,\alpha}(U)$ if $1 < p < 2$, where U is a small neighborhood of x. We refer the reader to [20] for these results.*

Remark 3.3. *If $1 < p < 2$ and $u(x_0) = 0$, we only know that u is $C^{2,\alpha}$ near x_0 and Z is $C^{1,\alpha}$ near x_0. Thus $P_{u,f}^{II}(Z)$ may not be defined since there are two diverging terms in its expression. As we can see in equation (3.7), these terms are*

$$
-(p-1)\lambda|u|^{p-2}|\nabla u|^p \;\; and \;\; -|\nabla u|^p \frac{1}{\varphi'} \frac{d}{dt}(\mathcal{L}_p\varphi).
$$

Since $\nabla u(x_0) \neq 0$, there exists an open set U containing x_0 such that $U \setminus \{u = 0\}$ is open and dense in U. On this set, we see that these two terms exactly cancel each other, and all the other term in $P_{u,f}^{II}(Z_{c_0})$ are well-defined and continuous on U. Thus the formula $P_{u,f}^{II}Z_{c_0} \leq 0$ is valid even in this low regularity setting.

4. One-dimensional models for $\kappa > 0$

In this section, we study the qualitative behavior of the one-dimensional equation

$$
(4.1) \qquad (p-1)|w'|^{p-2}w'' - \kappa\, t|w'|^{p-2}w' + \lambda|w|^{p-2}w = 0.
$$

for $\kappa > 0$. The case $\kappa \leq 0$ was treated in [14].

We recall some basic definitions and properties of p-trigonometric functions and refer the reader to [6, Chapter 1] for more details. For $1 < p < \infty$, let π_p be the positive number defined by

$$\pi_p = \int_{-1}^{1} \frac{ds}{(1 - s^p)^{1/p}} = \frac{2\pi}{p \sin(\pi/p)}.$$

The p-sine function $\sin_p : \mathbb{R} \to [-1, 1]$ is defined implicitly on $[-\pi_p/2, 3\pi_p/2]$ by

$$\begin{cases} t = \int_0^{\sin_p(t)} \frac{ds}{(1 - s^p)^{1/p}} & \text{if } t \in [-\frac{\pi_p}{2}, \frac{\pi_p}{2}], \\ \sin_p(t) = \sin_p(\pi_p - t) & \text{if } t \in [\frac{\pi_p}{2}, \frac{3\pi_p}{2}], \end{cases}$$

and is periodic on $\mathbb{R}$ with period $2\pi_p$. It's easy to see that for $p \neq 2$ this function is smooth around noncritical points, but only $C^{1,\alpha}(\mathbb{R})$ with $\alpha = \min\{p - 1, (p - 1)^{-1}\}$. By defining

$$\cos_p(t) = \frac{d}{dt}\sin_p(t) \text{ and } \tan_p(t) = \frac{\sin_p(t)}{\cos_p(t)},$$

we then have the following generalized trigonometric identities:

$$|\sin_p(t)|^p + |\cos_p(t)|^p = 1,$$
$$\frac{d}{dt}\tan_p(t) = \frac{1}{|\cos_p(t)|^p} = 1 + |\tan_p(t)|^p,$$
$$\frac{d}{dt}\arctan_p(t) = \frac{1}{1 + |t|^p}.$$

Let $\alpha = \left(\frac{\lambda}{p-1}\right)^{1/p}$. We introduce the p-polar coordinates r and θ defined by

(4.2) $$\alpha w = r \sin_p(\theta), \ w' = r \cos_p(\theta),$$

or equivalently,

(4.3) $$r = \left((w')^p + \alpha^p w^p\right)^{\frac{1}{p}}, \ \theta = \arctan_p\left(\frac{\alpha w}{w'}\right).$$

If w is a solution of (4.1) with $w(a) = -1$ and $w'(a) = 0$, then direct calculation shows that θ and r satisfy

(4.4)
$$\begin{cases} \theta' = \alpha - \frac{\kappa t}{p-1} \cos_p^{p-1}(\theta) \sin_p(\theta), \\ \theta(a) = -\frac{\pi_p}{2}, \ (\mathrm{mod}\ \pi_p); \end{cases}$$

(4.5)
$$\begin{cases} \frac{d}{dt} \log r = \frac{\kappa t}{p-1} \cos_p^p(\theta), \\ r(a) = \alpha. \end{cases}$$

Since both $\sin_p(t)$ and $\cos_p^{p-1}(t)$ are Lipschitz functions with Lipschitz constant 1, we can apply Cauchy's theorem to obtain existence, uniqueness, and continuous dependence on the parameters for (4.1), (4.4) and (4.5). Indeed, we have the following proposition.

Proposition 4.1. *For any $a \in \mathbb{R}$, there exists a unique solution w to (4.1) with $w(a) = -1$ and $w'(a) = 0$, defined on $\mathbb{R}$ with $w, (w')^{p-2}w' \in C^1(\mathbb{R})$. Moreover, the solution depends continuously on the parameters in the sense of local uniform convergence of w and $\dot{w}$ in $\mathbb{R}$.*

Let λ_0 be the first nonzero eigenvalue of the eigenvalue problem (4.1) on the real line, i.e.,

(4.6)
$$\lambda_0 = \inf \left\{ \frac{\int_{\mathbb{R}} |\varphi'|^p e^{-\frac{\kappa}{2}s^2} ds}{\int_{\mathbb{R}} |\varphi|^p e^{-\frac{\kappa}{2}s^2} ds} : \varphi \in W^{1,p}(\mathbb{R}, e^{-\frac{\kappa}{2}s^2} ds) \setminus \{0\}, \int_{\mathbb{R}} |\varphi|^{p-2}\varphi e^{-\frac{\kappa}{2}s^2} ds = 0 \right\}.$$

Let's first discuss the special case $p = 2$. In this case, equation (4.1) (when normalizes so that $\kappa = 1$) is the so-called Hermite's differential equation and solutions with polynomial growth are given by Hermitian polynomials. In particular, we have $\lambda_0 = \kappa$ and the corresponding eigenfunction is κt. On the other hand, it is known that the first nonzero eigenvalue of the f-Laplacian is bounded from below by κ if $\mathrm{Ric} + \nabla^2 f \geq \kappa g$, see for example [14, Theorem 1.6]. Thus when $p = 2$, we have $\lambda_{p,f} \geq \lambda_0$ (strict inequality when M is compact) and this inequality plays an important role in the construction of one-dimensional models in [5].

Our first step here is to get the non-sharp bound $\lambda_{p,f} > \lambda_0$ for all $1 < p < \infty$ and $\kappa > 0$. It turns out that this is a consequence of the sharp gradient estimates in Theorem 3.1 and the compactness of M.

Proposition 4.2. *Let (M^n, g, f) be a compact Bakry-Émery manifold (possibly with C^2 convex boundary) satisfying $\mathrm{Ric} + \nabla^2 f \geq \kappa g$ for some $\kappa > 0$. Let $\lambda_{p,f}$ be the first nonzero eigenvalue of $\Delta_{p,f}$ (with Neumann boundary*

condition if $\partial M \neq \emptyset$). Then

$$\lambda_{p,f} > \lambda_0,$$

where λ_0 is defined in (4.6).

We first prove an elementary lemma.

Lemma 4.1. *For any $\lambda \leq \lambda_0$, the ODE*

(4.7) $$(p-1)|w'|^{p-2}w'' - \kappa t|w'|^{p-2}w' + \lambda|w|^{p-2}w = 0$$

admits an odd solution $w : \mathbb{R} \to \mathbb{R}$ satisfying $w'(t) > 0$ for all $t \in \mathbb{R}$.

Proof of Lemma 4.1. By Proposition 4.1, there exists a solution w of (4.7) satisfying $w(0) = 0$ and $w'(0) = 1$. The oddness of w follows from uniqueness of solutions. The condition $w'(t) > 0$ for all $t \in \mathbb{R}$ is a consequence of $\lambda \leq \lambda_0$. Otherwise, we get a solution of (4.7) on some interval $[-a, a]$ with $w'(-a) = w'(a) = 0$ and $w'(t) > 0$ on $(-a, a)$, implying that λ is the first nonzero eigenvalue of (4.7) on $[-a, a]$, which would contradict $\lambda \leq \lambda_0$. $\qquad\square$

Proof of Proposition 4.2. We argue by contradiction. Suppose $\lambda \leq \lambda_0$, then by Lemma 4.1, there exists an odd solution $v : \mathbb{R} \to \mathbb{R}$ of the ODE (4.7) satisfying $v'(t) > 0$ for all $t \in \mathbb{R}$. Since $v(t)$ is strictly increasing, we have that either $v(t)$ approaches infinity as $t \to \infty$ or $\lim_{t\to\infty} v(t) = A \in (0, \infty)$.

Let u be an eigenfunction associated to the eigenvalue $\lambda_{p,f}$ normalized so that $-1 = \min\{u\} < 0 < \max\{u\} \leq 1$. Consider two points x_0 and y_0 such that $u(x_0) = \min_{x\in M} u(x)$, and $u(y_0) = \max_{x\in M} u(x)$. Let $L = d(x_0, y_0)$ and $\gamma : [0, L] \to M$ be a unit speed geodesic joining x_0 and y_0. Define $h(t) = u(\gamma(t))$ and choose $I \subset [0, L]$ such that $h' > 0$ on I and h^{-1} is well-defined in a subset of full measure of $[-1, \max\{u\}]$.

For any number c such that the range of cv contains the interval $[-1, 1]$, we can apply Theorem 3.1 with $w = cv$ on the interval $[w^{-1}(-1), w^{-1}(\max\{u\})]$ to conclude that

$$D \geq \int_0^L dt \geq \int_I dt \geq \int_{-1}^{\max\{u\}} \frac{dy}{h'(h^{-1}(y))} \geq \int_{-1}^{\max\{u\}} \frac{dy}{w'(w^{-1}(y))}$$

$$= \int_{w^{-1}(-1)}^{w^{-1}(\max\{u\})} dt = w^{-1}(\max\{u\}) - w^{-1}(-1) \geq -w^{-1}(-1) = v^{-1}\left(\frac{1}{c}\right).$$

If $\lim_{t\to\infty} v(t) = \infty$, we see that the right hand side goes to infinity by letting $c \to 0$. If $\lim_{t\to\infty} v(t) = A \in (0, \infty)$, the right hand side goes to infinity by

letting c decrease to A^{-1}. Either way, this is a contradiction to the finiteness of the diameter of M. $\qquad\square$

For the purpose of getting sharp eigenvalue estimates, we need to show that for any eigenfunction u of $\Delta_{p,f}$ with eigenvalue $\lambda > \lambda_0$, there exist an interval $[a, b]$ and a solution w of (4.1) such that w is strictly increasing on $[a, b]$ with $w(a) = \min\{u\}$ and $w(b) = \max\{u\}$. As we shall see, this can be achieved by varying the initial data.

The rest of this section is a slight modification of [14, Section 4]. Fix $\lambda > \lambda_0$ and $\kappa > 0$. Let w_a be the solution of the following initial value problem (IVP)

$$(4.8) \qquad \begin{cases} (p-1)|w'|^{p-2}w'' - \kappa\, t|w'|^{p-2}w' + \lambda|w|^{p-2}w = 0, \\ w(a) = -1, w'(a) = 0, \end{cases}$$

with $a \in \mathbb{R}$.

Proposition 4.3. *Fix $\lambda > \lambda_0$ and $\kappa > 0$. There exists a unique $\bar{a} > 0$ such that the solution $w_{-\bar{a}}$ to the IVP (4.8) is odd. In particular, $w_{-\bar{a}}$ restricted to $[-\bar{a}, \bar{a}]$ has nonnegative derivative and has maximum value equal to one.*

Proof. Consider the initial value problem

$$(4.9) \qquad \begin{cases} \dot{\theta} = \alpha - \frac{\kappa t}{p-1}\cos_p^{p-1}(\theta)\sin_p(\theta), \\ \theta(0) = 0. \end{cases}$$

The uniqueness of solutions implies that $\theta(t)$ is an odd function. The fact $\lambda > \lambda_0$ implies there exists $\bar{a} > 0$ such that $\theta(\bar{a}) = \pi_p/2$. It's easily seen that the corresponding solution $r(t)$ to (4.5) is even, regardless of its initial value. The proposition follows by translating obtained information on θ and r back to w. $\qquad\square$

For the solution $w = w_a$ of (4.8), we define

$$b(a) = \inf\{b > a : w'(b) = 0\},$$
$$m(a) = w_a(b(a)),$$
$$\delta(a) = b(a) - a.$$

In other words, $b(a)$ is the first value $b > a$ such that $w'(b) = 0$ with the convention that $b(a) = \infty$ if such a value does not exist. Thus w is strictly increasing on the interval $[a, b(a)]$ and $m(a)$ is the maximum of w on $[a, b(a)]$.

The function $\delta(a)$ measures the length of the interval where $w(a)$ increases from -1 to $m(a)$.

We are concerned with the range of the function $m(a)$ as a varies. Clearly $m(a) > 0$ since the eigenfunction w changes sign. Also, Proposition 4.3 implies $m(-\bar{a}) = 1$. We shall show in the next proposition that $m(a)$ goes to zero as a goes to $-\infty$. It then follows from the Intermediate Value Theorem that the range of $m(a)$ covers $(0, 1]$ when a varies in $(-\infty, -\bar{a}]$.

Proposition 4.4.

$$\lim_{a \to -\infty} m(a) = 0.$$

Proof. The argument is dual to the proof of [14, Proposition 4.3], and we omit it here. □

Proposition 4.5. *We have $\delta(a) \geq \delta(-\bar{a})$ for all $a \leq -\bar{a}$ with strict inequality if $a \neq -\bar{a}$.*

Proof. The argument is dual to the proof of [14, Proposition 4.4], and we omit it here. □

At last, we study $\bar{\delta} := \delta(-\bar{a}) = 2\bar{a}$ as a function of λ, having fixed p and κ. It's easy to see that $\bar{\delta}$ is a strictly decreasing function and so invertible. Thus we can define its inverse $\lambda(\delta)$, which is a continuous and decreasing function. Moreover, it can be characterized in the following equivalent way.

Proposition 4.6. *For fixed $\kappa > 0, 1 < p < \infty$, we have that given $\delta > 0$, λ is the first nonzero Neumann eigenvalue of the one-dimensional problem*

$$(p-1)|w'|^{p-2}w'' - \kappa\,t|w'|^{p-2}w' = -\lambda|w|^{p-2}w$$

on $[-\delta/2, \delta/2]$.

5. Proof of Theorem 1.1

After all the preparations in the previous two sections, we finally prove Theorem 1.1 in this section.

Proof of Theorem 1.1. It suffices to prove the case $\kappa > 0$, as the case $\kappa \leq 0$ was proved in [14]. Let u be an eigenfunction of $\Delta_{p,f}$ associated to the

eigenvalue λ. In view of

$$\int_M |u|^{p-2} u\, e^{-f} = 0,$$

we can normalize u so that $\min\{u\} = -1$ and $\max\{u\} \in (0, 1]$.

By Proposition 4.3 and 4.4, there exist an interval $[a, b]$ and a solution w of (4.8) such that w is strictly increasing on $[a, b]$ with $w(a) = -1 = \min\{u\}$ and $w(b) = \max\{u\} \in (0, 1]$. Moreover, we have $\lambda = \lambda_p([a, b])$, the first nonzero Neumann eigenvalue of the eigenvalue problem $|w'|^{p-2}w'' - \kappa\, t(p-1)|w'|^{p-2}w' + \lambda|w|^{p-2}w = 0$ on $[a, b]$.

Let x and y be such that $u(x) = \min_M u$ and $u(y) = \max_M u$. Consider a unit speed minimizing geodesic $\gamma : [0, d(x, y)] \to M$ joining x and y. Let $h(t) = u(\gamma(t))$ and choose $I \subset [0, d(x, y)]$ such that $h' > 0$ on I and h^{-1} is well-defined in a subset of full measure of $[-1, u_{\max}]$. Then we get, by change of variables and the sharp gradient estimate proved in Theorem 3.1, that

$$\begin{aligned}
D &\geq \int_0^{d(x,y)} dt \geq \int_I dt \geq \int_{-1}^{u_{\max}} \frac{dy}{h'(h^{-1}(y))} \geq \int_{-1}^{w_{\max}} \frac{dy}{w'(w^{-1}(y))} \\
&= \int_a^{b(a)} dt = \delta(a) \geq \delta(\bar{a}),
\end{aligned}$$

where the last inequality is proved in Proposition 4.5. This and Proposition 4.6 yield immediately the desired estimate. □

6. Sharpness

In this section, we show that the lower bound (1.1) given in Theorem 1.1 is sharp for $n \geq 3$ for any κ or $n \geq 2$ for $\kappa \leq 0$. More precisely, for each $\varepsilon > 0$, we construct a Bakry-Émery manifold (M, g, f) with diameter D and satisfying $\lambda_{p,f} < \mu_p(\kappa, D) + \varepsilon$.

For manifolds with boundary, the construction is rather simple. Take a cylinder $rS^{n-1} \times [-D/2, D/2]$ for r sufficiently small, with quadratic potential $f = \frac{\kappa}{2}s^2$. It is easy to see this is a Bakry-Émery manifold with diameter $D(1 + o(r))$ and $\mathrm{Ric} + \nabla^2 f \geq \kappa\, g$. Let w be an eigenfunction associated to the eigenvalue $\mu_p(\kappa, D)$ of the one-dimensional eigenvalue problem (1.2) on $[-D/2, D/2]$. Substituting the test function $\psi(z, s) = w(s)$ into the Rayleigh quotient yields

$$\lambda_{p,f} \leq \frac{\int_M |\nabla \psi|^p e^{-f} d\mu_g}{\int_M |\psi|^p e^{-f} d\mu_g} = \frac{\int_{-D/2}^{D/2} |w'|^p e^{-f} ds}{\int_{-D/2}^{D/2} |w|^p e^{-f} ds} = \mu_p(\kappa, D)$$

It follows that $\lambda_{p,f} \to \mu_p(\kappa, D)$ as $r \to 0$, proving the sharpness of the estimate (1.1) in Theorem 1.1.

To demonstrate the sharpness of (1.1) in the smaller class of manifolds without boundary, we need a more involved construction. The idea is to attach spherical caps to the ends of the above examples. The Bakry-Émery manifolds are constructed exactly the same as in [4]. The Riemannian manifold M, which is approximately a thin cylinder with hemispherical caps attached at each end, is constructed as follows. Let γ be the curve in $\mathbb{R}^2$ with curvature k given as function of arc length for suitably small $r > 0$ and $\delta > 0$ small compared ro r, by

$$(6.1) \qquad k(s) = \begin{cases} \frac{1}{r}, & s \in [0, \frac{\pi r}{2} - \delta], \\ \varphi\left(\frac{s - \frac{\pi r}{2}}{\delta}\right) \frac{1}{r}, & s \in [\frac{\pi r}{2} - \delta, \frac{\pi r}{2} + \delta], \\ 0, & s \in [\frac{\pi r}{2} + \delta, D], \end{cases}$$

and extended to be even under reflection in both $s = 0$ and $s = D/2$. Here φ is a smooth nonincreasing function with $\varphi(s) = 1$ for $s \leq -1$, $\varphi(s) = 0$ for $s \geq 1$, and satisfying $\varphi(s) + \varphi(-s) = 1$. Geometrically, this corresponds to a pair of line segments parallel to the x axis, capped by semicircles of radius r and smoothed at the joins. Let $(x(s), y(s))$ be the corresponding embedding and we choose the point corresponding to $s = 0$ to have $y(0) = 0$ and $y'(0) = 1$. Let (M^n, g) be the hypersurface of rotation in $\mathbb{R}^{n+1}$ given by $\{(x(s), y(s), z) : s \in \mathbb{R}, z \in S^{n-1}\}$.

The function f on M is a function of s only, given by $f'(0) = 0$ (the value of $f(0)$ is immaterial) and

(6.2)
$$f''(s) = \begin{cases} \kappa\left(1 - \frac{D}{\pi r}\right), & s \in [0, \frac{\pi r}{2} - \delta], \\ \kappa\varphi\left(\frac{s - \frac{\pi r}{2}}{\delta}\right)\left(1 - \frac{D}{\pi r}\right) + \kappa\left(1 - \varphi\left(\frac{s - \frac{\pi r}{2}}{\delta}\right)\right), & s \in [\frac{\pi r}{2} - \delta, \frac{\pi r}{2} + \delta], \\ \kappa, & s \in [\frac{\pi r}{2} + \delta, D]. \end{cases}$$

This implies $f'(D/2) = 0$. We also extend f to be even under the reflection in $s = 0$ and $s = D/2$.

With the above choices, let's compute the Bakry-Émery Ricci tensor of (M, g, f). The eigenvalues of the second fundamental form are $k(s)$ (in the s direction) and $\frac{\sqrt{1-(y')^2}}{y}$ in the orthogonal directions. Therefore the Gauss-Codazzi equations imply that the Ricci tensor has eigenvalues $(n-1)k(s)\frac{\sqrt{1-(y')^2}}{y}$ in the s direction, and $k(s)\frac{\sqrt{1-(y')^2}}{y} + (n-1)\frac{1-(y')^2}{y^2}$ in the orthogonal directions. The eigenvalues of the Hessian of f can be calculated

as follows: The curves of fixed z in M are geodesics parametrized by s, so the Hessian in this direction is just f'' as given in (6.2). Since f depends only on s we also have that $\nabla^2 f(\partial_s, e_i) = 0$ for e_i tangent to S^{n-1} and $\nabla^2 f(e_i, e_j) = \frac{y'}{y} f' \delta_{ij}$.

The identities $y(s) = \int_0^s \cos(\theta(\tau)) d\tau$ and $y'(s) = \cos(\theta(s))$, where $\theta(s) = \int_0^s k(\tau) d\tau$ applied to (6.1) yields that as $\delta \to 0$,

$$y(s) = \begin{cases} r \sin(s/r), & s \in [0, \frac{\pi r}{2} - \delta], \\ r(1 + o(\delta)), & s \in [\frac{\pi r}{2} - \delta, D], \end{cases}$$

and

$$y'(s) = \begin{cases} \cos(s/r), & s \in [0, \frac{\pi r}{2} - \delta], \\ o(\delta), & s \in [\frac{\pi r}{2} - \delta, \frac{\pi r}{2} - \delta], \\ 0, & s \in [\frac{\pi r}{2} + \delta, D]. \end{cases}$$

Straightforward calculations then give the following expressions for the Bakry-Émery Ricci tensor $\mathrm{Ric}_f = \mathrm{Ric} + \nabla^2 f$:

$$\mathrm{Ric}_f(\partial_s, \partial_s) = \begin{cases} \kappa + \frac{n-1}{r^2} - \frac{\kappa D}{\pi r}, & s \in [0, \frac{\pi r}{2} - \delta], \\ \kappa + \varphi\left(\frac{2 - \frac{\pi r}{2}}{\delta}\right)\left(\frac{n-1}{r^2}(1 + o(\delta)) - \frac{\kappa D}{\pi r}\right), & s \in [\frac{\pi r}{2} - \delta, \frac{\pi r}{2} - \delta], \\ \kappa, & s \in [\frac{\pi r}{2} + \delta, D], \end{cases}$$

and

$$\mathrm{Ric}_f(e, e) = \begin{cases} \frac{n-1}{r^2} + \frac{\kappa s\left(1 - \frac{D}{\pi r}\right)}{r \tan(s/r)}, & s \in [0, \frac{\pi r}{2} - \delta], \\ \frac{n-2}{r^2} + o(\delta), & s \in [\frac{\pi r}{2} - \delta, \frac{\pi r}{2} - \delta], \\ \frac{n-2}{r^2}(1 + o(\delta)), & s \in [\frac{\pi r}{2} + \delta, D], \end{cases}$$

while $\mathrm{Ric}_f(\partial_s, e) = 0$ for any unit vector e tangent to S^{n-1}. Clearly, we have $\mathrm{Ric}_f \geq \kappa g$ for r and δ small enough for any $\kappa \in \mathbb{R}$ if $n \geq 3$ and for $\kappa \leq 0$ if $n = 2$. Moreover, the diameter of M is $D(1 + o(\delta))$.

Finally, we show that for the Bakry-Émery manifold (M, g, f) constructed above, the first nonzero eigenvalue $\lambda_{p,f}$ of $\Delta_{p,f}$ can be made arbitrarily close to $\mu_p(\kappa, D)$ by choosing r and δ sufficiently small. Let ψ be defined by

$$\psi(z, s) = \begin{cases} w(s - D/2), & s \in [\frac{\pi r}{2} + \delta, D - \left(\frac{\pi r}{2} + \delta\right)], \\ w(D/2 - \frac{\pi r}{2} - \delta), & s \in [0, \frac{\pi r}{2} + \delta] \cup [D - \frac{\pi r}{2} - \delta, D], \end{cases}$$

where w is the solution of

$$(p-1)|w'|^{p-2}w'' - \kappa s|w'|^{p-2}w' + \mu_p(\kappa, D - \pi r - 2\delta)|w|^{p-2}w = 0$$

with $w(0) = 0$, $w'(0) = 1$ and $w'(D/2 - \frac{\pi r}{2} - \delta) = 0$. Plugging the test function ψ into the Rayleigh quotient gives

$$\lambda_{p,f} \leq \frac{\int_M |\nabla \psi|^p e^{-f} d\mu_g}{\int_M |\psi|^p e^{-f} d\mu_g}$$

$$= \frac{\mu_p(\kappa, D - \pi r - \delta) \int_{-\frac{\pi r}{2} - \delta/2}^{D - \frac{\pi r}{2} - \delta/2} |w|^p e^{-f} \, ds}{\int_0^D |w|^p e^{-f} \, ds}$$

$$\leq \mu_p(\kappa, D - \pi r - \delta).$$

It follows that $\lambda_{p,f} \to \mu_p(\kappa, D)$ as r and δ approach zero, thus proving the sharpness of the estimate (1.1) in Theorem 1.1.

Acknowledgments

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