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An approach to the Griffiths conjecture

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The Griffiths conjecture asserts that every ample vector bundle E over a compact complex manifold S admits a hermitian metric with positive curvature in the sense of Griffiths. In this article, we first give a sufficient condition for a positive hermitian metric on $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ to induce a Griffiths positive L^2-metric on the vector bundle E. This result suggests to study the relative Kähler-Ricci flow on $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ for the fibration $\mathbb{P}(E^*) \to S$. We define this flow and prove its convergence.

1. Introduction

Let $E \to S$ be a holomorphic vector bundle of rank r over a compact complex manifold S. The projectivized bundle $\mathcal{X} := \mathbb{P}(E^*)$ whose fiber over $s \in S$ parametrizes hyperplanes of E_s carries the tautological line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ which we also denote by $\mathcal{O}_E(1)$. There is an isomorphism $f_*(\mathcal{O}(1)) \cong E$ where $f : \mathcal{X} \to S$ is the projection map. In [11] Hartshorne defined the ampleness of a vector bundle over a projective manifold: A vector bundle is called ample if the tautological line bundle $\mathcal{O}_E(1)$ is ample over $\mathcal{X}$. On the other hand, Griffiths defined in [10] a notion of positivity for a hermitian holomorphic vector bundle (E, H) by using the curvature of (E, H) which is now called Griffiths positivity. For a line bundle the ampleness of the bundle is equivalent to Griffiths positivity. It is also well-known that a Griffiths positive metric on E induces a positive metric on $\mathcal{O}_E(1)$, hence any Griffiths positive vector bundle is always ample. Griffiths conjectured also the converse, that an ample vector bundle E also carries a Griffiths positive metric. Umemura [18] as well as Campana and Flenner [6] gave an affirmative answer to this question when the base S is a curve. But in general finding a Griffiths positive metric on an ample vector bundle seems to be very difficult and is worth to be investigated.

Another important contribution which points into the direction of this conjecture comes from the general positivity theory for direct images of adjoint bundles. Berndtsson proved in [2] that the bundle $E \otimes \det E$ has always a hermitian metric which is Nakano positive for any given ample vector bundle E. Together with the result of Demailly and Skoda [9] which says that $(E \otimes \det E, H \otimes \det H)$ is Nakano positive if (E, H) is Griffiths positive, this can be seen as a further indication for the Griffiths conjecture. The Nakano positivity of $E \otimes \det E$ follows from Berndtsson's main theorem by the identity $f_*(K_{\mathcal{X}/S} \otimes \mathcal{O}(r+1)) \cong E \otimes \det E$. This article is motivated by his result. In order to study E instead of $E \otimes \det E$ we look at $f_*(\mathcal{O}(1)) \cong E$ which is no longer the direct image of an adjoint bundle (an ample twisted relative canonical bundle).

Now assume that E is ample and choose a positive hermitian metric h on $\mathcal{O}_E(1)$. We define the positive form

$$\omega_{\mathcal{X}} := -\sqrt{-1}\partial\bar{\partial} \log h.$$

This gives Kähler forms

$$\omega_s := \omega_{\mathcal{X}}|_{X_s}$$

on the fibers X_s which induce a hermitian metric on $\det T_{\mathcal{X}/S} = K_{\mathcal{X}/S}^{-1}$ denoted by

$$(-\sqrt{-1}\partial\bar{\partial} \log h)^n,$$

where $n = \dim X_s = r - 1$. We define the L^2-inner product of two sections $u, v \in H^0(X_s, \mathcal{O}_{\mathbb{P}(E_s^*)}(1)) \cong E_s$ on the direct image $f_*(\mathcal{O}(1)) \cong E$ at a point $s \in S$ by

$$\langle u, v \rangle_{L^2}(s) := \int_{X_s} h(u, v) \, \frac{\omega_s^n}{n!}. \tag{1}$$

When the hermitian metric h is induced from a hermitian metric H on E, we already know from [12, Theorem 7.1] that the L^2-metric then gives back the original metric H on E up to a constant factor. In particular we know that if we start with a Griffiths positive metric H on E, it should be possible to prove the Griffiths positivity of the L^2-metric directly by looking at its curvature expression. By analyzing this situation very carefully and using a general curvature formula for direct images due to To and Weng [17], we give a proof of the following result:

Theorem 1. *Suppose E is an ample vector bundle such that $\mathcal{O}_E(1)$ is equipped with a positively curved metric h. Let $\omega_{\mathcal{X}} = -\sqrt{-1}\partial\bar{\partial} \log h$ be the curvature of h and regard the family of volume forms $(\omega_s^n)_{s \in S} = (\omega_{\mathcal{X}}|_{X_s})_{s \in S}^n$*

as a metric on $K^{-1}_{\mathcal{X}/S}$. *Then, if the canonical isomorphism*

$$K^{-1}_{\mathcal{X}/S} \cong \mathcal{O}_E(r) \otimes f^* \det(E)^{-1} \tag{2}$$

becomes an isometry for some hermitian metric on $f^* \det(E)^{-1}$ *that is the inverse of the pullback of a hermitian metric* G *on* $\det E$*, the* L^2*-metric on* $f_*(\mathcal{O}(1)) = E$ *is Griffiths positive.*

The assumption in the statement implies that the curvature form of h defines Kähler-Einstein metrics on the fibers, which are projective spaces. But the Fubini-Study metrics on $\mathbb{P}(E_s^*)$ are in one-to-one correspondence with hermitian structures on E_s^*, hence the theorem actually gives a characterisation of those metrics on $\mathcal{O}_E(1)$ which are induced by metrics on E. For the proof we give in the next section we work on the level of curvature. This gives some insight into the structure of the general curvature formula for L^2-metrics. More precisely, the proof is a study of the conditions needed to insure Griffiths positivity of the L^2-metric on the ample bundle $f_*\mathcal{O}_E(1) = E$.

Theorem 1 gives a link between the Kähler-Einstein problem on projective spaces and the Griffiths conjecture. Therefore we propose in section 3 to study the relative Kähler-Ricci flow on the bundle $\mathcal{O}_E(r)$ instead of $K^{-1}_{\mathcal{X}/S}$. Here we rely on Berman's article [1] and first recall his definition for the normalized relative Kähler-Ricci flow on the level of hermitian metrics for the bundle $K^{-1}_{\mathcal{X}/S}$. By using more recent results for the Kähler-Ricci flow on Fano Kähler-Einstein manifolds, we explain how to extend his result on the convergence of the flow to the case of non-discrete automorphism groups of the fibers. Afterwards we define the relative Kähler-Ricci flow on $\mathcal{O}_E(r)$ and obtain

Theorem 2. *The relative Kähler-Ricci flow on* $\mathcal{O}_E(r)$ *exists and it is smooth over* $\mathcal{X} \times [0, \infty[$. *The flow converges to a hermitian metric* φ_∞ *on* $\mathcal{O}_E(r)$ *that is of class* C^∞ *on each fiber* X_s.

Finally we state the evolution equation for the geodesic curvature function of the evolving metrics on $\mathcal{O}_E(r)$ which completely encodes its positivity.

2. Griffiths positivity of L^2-metrics

In this section we prove Theorem 1. First we introduce the main objects which appear along the computation of the curvature. We cite the formula of To and Weng and prove the Propositions 1 and 2 that allow to simplify the curvature expression. It will also clarify the structure of the geodesic curvature function that finally allows to evaluate the curvature integrals.

We use local holomorphic coordinates (s^k) on the base S and coordinates (z^α) on the fibers and write for the curvature of the positively curved metric h on $\mathcal{O}_E(1)$

$$\omega_{\mathcal{X}} = \sqrt{-1}\left(g_{\alpha\overline{\beta}}\, dz^\alpha \wedge dz^{\overline{\beta}} + h_{k\overline{\beta}}\, ds^k \wedge dz^{\overline{\beta}} + h_{\alpha\bar{l}}\, dz^\alpha \wedge ds^{\bar{l}} + h_{k\bar{l}}\, ds^k \wedge ds^{\bar{l}}\right)$$

Thus the Kähler forms are given by

$$\omega_s := \sqrt{-1}\, g_{\alpha\overline{\beta}}\, dz^\alpha \wedge dz^{\overline{\beta}}$$

and the induced metric on $K^{-1}_{\mathcal{X}/S}$ can be written as

$$\det(g_{\alpha\overline{\beta}}).$$

According to [14] we denote the horizontal lift of a tangent vector $\partial_k = \partial/\partial s^k$ on the base S by v_k. It is given by

$$v_k = \partial_k + a^\alpha_k\, \partial_\alpha$$

where $\partial^\alpha = \partial/\partial z^\alpha$ and

$$a^\alpha_k = -g^{\overline{\beta}\alpha}\, h_{k\overline{\beta}}.$$

We obtain the Kodaira-Spencer forms by

$$A_k := \bar{\partial}(v_k)|_{X_s}$$

and define the *geodesic curvature* in the direction of k, l by

$$c(\varphi)_{k\bar{l}} = \langle v_k, v_l \rangle_{\omega_{\mathcal{X}}}.$$

Here we differ slightly in notation from [14] and use instead the notation from [3] to indicate that $c(\varphi)$ depends on h where locally $h = e^{-\varphi}$. In our local coordinates we have

$$c(\varphi)_{k\bar{l}} = h_{k\bar{l}} + a_k^\alpha h_{\alpha\bar{l}} = h_{k\bar{l}} - a_k^\alpha a_{\bar{l}\alpha}.$$

We observe that the matrix $(c(\varphi)_{k\bar{l}})$ is positive definite iff the hermitian line bundle $(\mathcal{O}_E(1), h)$ is positive which was our assumption.

The main ingredient in the proof of Theorem 1 is a general curvature formula for the direct images of the form p_*L due to To and Weng:

Theorem ([17]). *Let $p : \mathcal{X} \to S$ be a smooth family of n-dimensional compact complex manifolds and $(L, h) \to \mathcal{X}$ be a hermitian holomorphic line bundle. Furthermore assume that $\mathcal{X}$ admits a smooth $(1,1)$-form $\omega_{\mathcal{X}}$ such that its restrictions $\omega_s := \omega_{\mathcal{X}}|_{X_s}$ are Kähler forms on X_s and such that*

$$c_1(L, h) = \frac{k}{2\pi}\omega_{\mathcal{X}}$$

for some $k \in \mathbb{R}$. Assume that p_*L is locally free on S with fiber $(p_*L)_s = H^0(X_s, L_s)$. Then the curvature tensor Θ of the associated L^2-metric (defined as in (1)) on p_*L is given by

$$\begin{aligned}\Theta_{a\bar{b}i\bar{\jmath}}(s) = &-\int_{X_s} \langle G(A^\gamma_{i\overline{\beta}} t_{a;\gamma} dz^{\overline{\beta}}), A^\gamma_{j\overline{\beta}} t_{b;\gamma} dz^{\overline{\beta}}\rangle_h \frac{\omega_s^n}{n!}\\ &+\int_{X_s} (kc(\varphi)_{i\bar{\jmath}} + \Box_{\omega_s} c(\varphi)_{i\bar{\jmath}})\langle t_a, t_b\rangle_h \frac{\omega_s^n}{n!}\end{aligned}$$

for $t_a, t_b \in H^0(X_s, L_s)$. Here G denotes the Green operator.

Here as before $A^\gamma_{i\overline{\beta}}\partial_\gamma dz^{\overline{\beta}}$ are the Kodaira-Spencer forms and $t_{a;\gamma}$ stands for the covariant derivative of the section t_a in the fiber direction $\partial/\partial z^\gamma$ with respect to the Chern connection on $(L_s, h|_{X_s})$. The formula in this general form is of course difficult to deal with. We prove the following two results (cf. [14]) that allow considerable simplifications in our setting:

Proposition 1. *Under the assumption of Theorem 1, the Kodaira-Spencer forms A_i are harmonic, hence zero.*

Proof. We use the symbol ; to denote covariant derivatives with respect to the Chern connection on (X_s, ω_s). On the fiber X_s we have

$$\begin{aligned}
g^{\overline{\delta}\gamma} A_{i\overline{\beta}\overline{\delta};\gamma} &= g^{\overline{\delta}\gamma} a_{i\overline{\delta};\overline{\beta}\gamma} = g^{\overline{\delta}\gamma} a_{i\overline{\delta};\gamma\overline{\beta}} - g^{\overline{\delta}\gamma} a_{i\overline{\tau}} R^{\overline{\tau}}_{\overline{\delta}\ \overline{\beta}\gamma} \\
&= \left(g^{\overline{\delta}\gamma} \partial_\gamma \left(\frac{\partial^2 \log h}{\partial s^i \partial z^{\overline{\delta}}} \right) \right)_{;\overline{\beta}} + a_{i\overline{\tau}} R^{\overline{\tau}}_{\overline{\beta}} \\
&= \left(g^{\overline{\delta}\gamma} \partial_i \left(\frac{\partial^2 \log h}{\partial s^\gamma \partial z^{\overline{\delta}}} \right) \right)_{;\overline{\beta}} + a_{i\overline{\tau}} g^{\overline{\tau}\alpha} R_{\alpha\overline{\beta}} \\
&= \left(-g^{\overline{\delta}\gamma} \partial_i (g_{\gamma\overline{\delta}}) \right)_{;\overline{\beta}} + a_i^\alpha (n+1) g_{\alpha\overline{\beta}} \\
&= -\partial_{\overline{\beta}} \partial_i \log \det(g) + (n+1) a_{i\overline{\beta}} \\
&= -\frac{\partial^2 \log \det(g)}{\partial s^i \partial z^{\overline{\beta}}} + (n+1) \frac{\partial^2 \log h}{\partial s^i \partial z^{\overline{\beta}}} \\
&= -\frac{\partial^2 \log(h^{n+1} \cdot (f^*G)^{-1})}{\partial s^i \partial z^{\overline{\beta}}} + \frac{\partial^2 \log h^{n+1}}{\partial s^i \partial z^{\overline{\beta}}} \\
&= -\frac{\partial^2 \log h^{n+1}}{\partial s^i \partial z^{\overline{\beta}}} + \frac{\partial^2 \log h^{n+1}}{\partial s^i \partial z^{\overline{\beta}}} = 0,
\end{aligned}$$

because by our assumption $(\det T_{\mathcal{X}/S}, \det(g)) \cong (\mathcal{O}(n+1) \otimes f^*(\det E)^{-1}, h^{n+1} \cdot (f^*G)^{-1})$ and the pullback metric f^*G is constant on the fibers. This proves the harmonicity of A_i. The last part follows from the fact that $H^1(\mathbb{P}^n, T_{\mathbb{P}^n}) = 0$. $\square$

Proposition 2. *Under the assumption of Theorem 1, the following equation holds:*

$$(3) \qquad (\Box_{\omega_s} - (n+1)) c(\varphi)_{k\bar{l}} = -f^*(R_G^{\det E})_{k\bar{l}},$$

where $(R_G^{\det E})_{k\bar{l}}$ *is the curvature of* $(\det E, G)$ *in the direction of* ∂_k *and* ∂_l.

Proof. It holds $-\Box_{\omega_s} c(\varphi)_{k\bar{l}} = g^{\overline{\delta}\gamma}(h_{k\bar{l}} - a_k^\sigma a_{\bar{l}\sigma})_{;\gamma\overline{\delta}}$. We compute

$$\begin{aligned}
g^{\overline{\delta}\gamma} h_{k\bar{l};\gamma\overline{\delta}} &= g^{\overline{\delta}\gamma} \partial_k \partial_{\bar{l}} h_{\gamma\overline{\delta}} \\
&= g^{\overline{\delta}\gamma} \partial_k \partial_{\bar{l}} g_{\gamma\overline{\delta}} \\
&= \partial_k (g^{\overline{\delta}\gamma} \partial_{\bar{l}} g_{\gamma\overline{\delta}}) - \partial_k g^{\overline{\delta}\gamma} \partial_{\bar{l}} g_{\gamma\overline{\delta}}
\end{aligned}$$

$$
\begin{aligned}
&= \partial_k(g^{\overline{\delta}\gamma}\partial_{\overline{l}}g_{\gamma\overline{\delta}}) + g^{\overline{\delta}\alpha}g^{\overline{\beta}\gamma}\partial_k g_{\alpha\overline{\beta}}\partial_{\overline{l}}g_{\gamma\overline{\delta}} \\
&= \partial_k\partial_{\overline{l}}\log\det(g_{\gamma\overline{\delta}}) + a_k^{\gamma;\overline{\delta}}a_{\overline{l}\gamma;\overline{\delta}} \\
&= \partial_k\partial_{\overline{l}}\log(h^{n+1}\cdot(f^*G)^{-1}) + a^\sigma_{k;\gamma}a_{\overline{l}\sigma;\overline{\delta}}g^{\overline{\delta}\gamma} \\
&= -(n+1)h_{k\overline{l}} + f^*(R_G^{\det E})_{k\overline{l}} + a^\sigma_{k;\gamma}a_{\overline{l}\sigma;\overline{\delta}}g^{\overline{\delta}\gamma}
\end{aligned}
$$

and

$$
(a_k^\sigma a_{\overline{l}\sigma})_{\gamma\overline{\delta}}g^{\overline{\delta}\gamma} = (a^\sigma_{k;\gamma\overline{\delta}}a_{\overline{l}\sigma} + A^\sigma_{k\overline{\delta}}A_{\overline{l}\sigma\gamma} + a^\sigma_{k;\gamma}a_{\overline{l}\sigma;\overline{\delta}} + a^\sigma_k A_{\overline{l}\sigma\gamma;\overline{\delta}})g^{\overline{\delta}\gamma}.
$$

The two terms involving $A_{\overline{l}}$ vanish because of Proposition 1. For the first we get

$$
a^\sigma_{k;\gamma\overline{\delta}}g^{\overline{\delta}\gamma} = A^\sigma_{k\overline{\delta};\gamma}g^{\overline{\delta}\gamma} + a^\lambda_k R^\sigma_{\lambda\overline{\delta}\gamma}g^{\overline{\delta}\gamma} = 0 - a^\lambda_k R^\sigma_\lambda = -(n+1)a^\sigma_k.
$$

□

We observe that by integrating the equation (3) against the volume form induced by ω_s we get

Corollary 1.

$$
\text{(4)} \qquad \frac{f^*(R_G^{\det E})_{k\overline{l}}}{n!} = \int_{X_s}(n+1)\,c(\varphi)_{k\overline{l}}\,\frac{\omega_s^n}{n!} \quad \textit{if} \quad \int_{X_s}\omega_s^n = 1.
$$

Now we can prove Theorem 1:

Proof. First we can make some reductions: By placing one-dimensional discs in all directions in the base S, we can restrict to the case of a one-dimensional base S. We fix a point $s \in S$ and consider the corresponding Kähler-Einstein form ω_s. Because this is a Fubini-Study form, we can choose a local holomorphic frame $e_1, \ldots, e_{n+1}$ of E^* around s and the corresponding coordinates homogenous coordinates $W_1, \ldots, W_{n+1}$ on $\mathbb{P}(E_s^*)$ such that this form becomes the standard Fubini-Study form on $\mathbb{P}(E_s^*) \cong \mathbb{P}^n$ which we denote by ω_{FS}. At the same time we view the coordinates $W_1, \ldots, W_{n+1}$ as a base of global holomorphic sections of $f_*(\mathcal{O}_E(1)|_{X_s}) \cong E_s$. Because the hermitian metric $h|_{X_s}$ and the standard hermitian metric on $\mathcal{O}_E(1)|_{X_s}$ given by

$$
\frac{1}{|W|^2} := \frac{1}{|W_1|^2 + |W_2|^2 + \ldots + |W_{n+1}|^2}
$$

have the same curvature form on $X_s = \mathbb{P}(E_s^*)$, they differ by a positive constant C on the fixed fiber X_s.

Now we apply the formula of To and Weng to study the curvature of the L^2-metric given by (1) on $f_*\mathcal{O}_E(1) \cong E$. Using Proposition 1 the first term in the curvature formula completely disappears. Using the further reductions we made the second term reads as

$$\Theta_{i\bar{\jmath}s\bar{s}}(s) = \int_{X_s} (c(\varphi) + \Box_{\omega_{FS}} c(\varphi))\, C\, \frac{W_i W_{\bar{\jmath}}}{|W|^2}\, \frac{\omega_{FS}^n}{n!}.$$

We invoke also Proposition 2 to obtain

$$\Theta_{i\bar{\jmath}s\bar{s}}(s) = \int_{X_s} ((n+2)c(\varphi) - f^* R_G^{\det E})\, C\, \frac{W_i W_{\bar{\jmath}}}{|W|^2}\, \frac{\omega_{FS}^n}{n!}.$$

Now we remind ourselves that we are integrating over $\mathbb{P}^n$ with respect to the standard Fubini-Study form. For such integrals we have the following (see [12, Lemma 4.1])

Lemma 1.

$$\int_{\mathbb{P}^n} \frac{W_\alpha W_{\bar{\beta}}}{|W|^2} \frac{\omega_{FS}^n}{n!} = \frac{\delta_{\alpha\bar{\beta}}}{(n+1)!}$$

$$\int_{\mathbb{P}^n} \frac{W_\alpha W_{\bar{\beta}} W_\gamma W_{\bar{\delta}}}{|W|^4} \frac{\omega_{FS}^n}{n!} = \frac{\delta_{\alpha\bar{\beta}}\delta_{\gamma\bar{\delta}} + \delta_{\alpha\bar{\delta}}\delta_{\gamma\bar{\beta}}}{(n+2)!}$$

Thus, using also the identity (4) from the corollary above, we can write

$$\begin{aligned} &\int_{X_s} f^* R_G^{\det E}\, C\, \frac{W_i W_{\bar{\jmath}}}{|W|^2}\, \frac{\omega_{FS}^n}{n!} \\ &= f^* R_G^{\det E} C \int_{X_s} \frac{W_i W_{\bar{\jmath}}}{|W|^2}\, \frac{\omega_{FS}^n}{n!} \\ &= \frac{f^* R_G^{\det E}}{(n+1)!}\, C\, \delta_{i\bar{\jmath}} \\ &= \int_{X_s} C\, \delta_{i\bar{\jmath}}\, c(\varphi)\, \frac{\omega_{FS}^n}{n!} \end{aligned}$$

Hence we can rewrite the expression for the curvature as

$$\Theta_{i\bar{\jmath}s\bar{s}}(s) = C \int_{X_s} \left((n+2)c(\varphi)\, \frac{W_i W_{\bar{\jmath}}}{|W|^2} - \delta_{i\bar{\jmath}}\, c(\varphi)\right) \frac{\omega_{FS}^n}{n!}. \tag{5}$$

From this expression alone it it not yet possible to read off the Griffiths positivity of the direct image metric. We need a further ingredient that explains the structure of the geodesic curvature function $c(\varphi)$ which allows us to evaluate the integral. For this we go back to the elliptic equation (3). We can rewrite it as

$$(\Box_{\omega_{FS}} - (n+1))((n+1)c(\varphi) - f^* R_G^{\det E}) = 0.$$

This means that the function $(n+1)c(\varphi) - f^* R_G^{\det E}$ is an eigenfunction to the smallest positive eigenvalue $n+1$ on the Kähler manifold $(\mathbb{P}^n, \omega_{FS})$. But those functions are known. A generating system of eigenfunctions is given by

$$\left(\phi_{\alpha\overline{\beta}} := (n+1)\frac{W_\beta W_{\overline{\alpha}}}{|W|^2} - \delta_{\overline{\alpha}\beta} \right)_{\alpha,\beta=1}^{n+1}.$$

Hence we can write

$$(6) \qquad \begin{aligned} &(n+1)c(\varphi) - \int_{X_s} (n+1)\, c(\varphi)\, \omega_{FS}^n \\ &\quad = (n+1) \sum_{\alpha,\beta=1}^{n+1} \lambda_{\overline{\alpha}\beta} \frac{W_\beta W_{\overline{\alpha}}}{|W|^2} - \sum_{\alpha,\beta=1}^{n+1} \delta_{\overline{\alpha}\beta} \lambda_{\overline{\alpha}\beta} \end{aligned}$$

for some constants $\lambda_{\overline{\alpha}\beta} \in \mathbb{C}$. Here we assume now that $\mathrm{Vol}(X_s, \omega_{FS}^n) = 1$. Because the system $(\phi_{\alpha\overline{\beta}})_{\alpha\overline{\beta}}$ satisfies the unique relation

$$\sum_{\alpha=1}^{n+1} \phi_{\alpha\overline{\alpha}} = 0,$$

the coefficients $\lambda_{\alpha\overline{\beta}}$ are only determined up to

$$\lambda_{\alpha\overline{\beta}} \mapsto \lambda_{\alpha\overline{\beta}} + C' \delta_{\alpha\overline{\beta}}$$

for a constant C'. Therefore we can choose $\lambda_{\alpha\overline{\beta}}$ so that $\sum_\alpha \lambda_{\alpha\overline{\alpha}} = f^* R_G$ which then implies that

$$\begin{aligned} (n+1)c(\varphi) &= (n+1) \sum_{\alpha,\beta=1}^{n+1} \lambda_{\overline{\alpha}\beta} \frac{W_\beta W_{\overline{\alpha}}}{|W|^2} - \sum_{\alpha+1}^{n+1} \lambda_{\alpha\overline{\alpha}} + f^* R_G \\ &= (n+1) \sum_{\alpha,\beta=1}^{n+1} \lambda_{\overline{\alpha}\beta} \frac{W_\beta W_{\overline{\alpha}}}{|W|^2}. \end{aligned}$$

Plugging this into the equation (5), we can proceed as

$$\begin{aligned}\Theta_{i\bar{j}s\bar{s}}(s) &= C\int_{X_s}((n+2)c(\varphi)\,\frac{W_iW_{\bar{j}}}{|W|^2} - \delta_{i\bar{j}}\,c(\varphi))\,\frac{\omega_{FS}^n}{n!}\\ &= C\int_{X_s}((n+2)\lambda_{\overline{\alpha}\beta}\frac{W_\beta W_{\overline{\alpha}}}{|W|^2}\,\frac{W_iW_{\bar{j}}}{|W|^2} - \delta_{i\bar{j}}\,\lambda_{\overline{\alpha}\beta}\frac{W_\beta W_{\overline{\alpha}}}{|W|^2})\,\frac{\omega_{FS}^n}{n!}\\ &= C(n+2)\lambda_{\overline{\alpha}\beta}\int_{X_s}\frac{W_\beta W_{\overline{\alpha}}W_iW_{\bar{j}}}{|W|^4}\,\frac{\omega_{FS}^n}{n!} - C\delta_{i\bar{j}}\,\lambda_{\overline{\alpha}\beta}\int_{X_s}\frac{W_\beta W_{\overline{\alpha}}}{|W|^2}\,\frac{\omega_{FS}^n}{n!}\\ &= \frac{C\lambda_{\overline{\alpha}\beta}}{(n+1)!}\,((\delta_{\beta\overline{\alpha}}\delta_{i\bar{j}} + \delta_{\beta\bar{j}}\delta_{i\overline{\alpha}}) - \delta_{\beta\overline{\alpha}}\delta_{i\bar{j}})\\ &= \lambda_{i\bar{j}}\frac{C}{(n+1)!}.\end{aligned}$$

Now because the expression

$$\sum_{i,j=1}^{n+1}\Theta_{i\bar{j}s\bar{s}}\frac{W_iW_{\bar{j}}}{|W|^2} = \frac{C}{(n+1)!}c(\varphi)$$

is positive (because $(\mathcal{O}_E(1),h)$ was positive by assumption), we conclude that the L^2-metric on $f_*\mathcal{O}_E(1)\cong E$ is indeed Griffiths positive. This proves Theorem 1. □

3. The relative Kähler-Ricci Flow

3.1. The flow on $K_{\mathcal{X}/S}^{-1}$

First we recall the relative Kähler-Ricci flow for the bundle $K_{\mathcal{X}/S}^{-1}$ and a family of Fano-Kähler-Einstein manifolds X_s as introduced in [1]. As proved in [1] this flow converges in C^∞ to a Kähler-Einstein weight ϕ_∞ on any Fano-Kähler-Einstein manifold X with $H^0(X,T_X)=0$. But using the recent developments concerning the Kähler-Ricci flow on Fano-Kähler-Einstein manifolds, we show that the condition about the automorphism group can be removed.

For this, we consider the absolute case of a Fano-Kähler-Einstein manifold X ($\dim X = n$) with a metric $h=e^{-\phi}$ on K_X^{-1} of positive curvature. We write $\phi\in\mathcal{H}_X$, where $\mathcal{H}_X$ is the set of smooth and positive (weights for) metrics on K_X^{-1}. In this notation $h=e^{-\phi}$ is the pointwise norm squared of a local trivializing section

$$(dz)^{-1} := \partial/\partial z^1\wedge\ldots\wedge\partial/\partial z^n$$

of K_X^{-1}. But more globally, we can view $e^{-\phi}$ as a volume form on X by regarding

$$c_n dz^1 \wedge \ldots \wedge dz^n \wedge d\overline{z^1} \wedge \ldots \wedge d\overline{z^n} e^{-\phi}, \tag{7}$$

where by abuse of notation we write $e^{-\phi}$ for $|(dz)^{-1}|^2 e^{-\varphi}$ (cf. also [5, Sect.2.1]). Here c_n is a unimodular constant that makes the expression positive definite. Therefore, we can define the *canonical measure*

$$\mu_\phi := \frac{e^{-\phi}}{\int_X e^{-\phi}}.$$

We have also the *Monge-Ampère measure* defined as

$$\mathrm{MA}(\phi) := V^{-1}(dd^c\phi)^n,$$

where $V = \int_X (dd^c\phi)^n$. These two measures only depend on the curvature form $\omega = dd^c\phi$ of ϕ and thus we write $\mu_\phi = \mu_\omega$. The Kähler-Einstein condition becomes

$$V^{-1}\omega^n = \mu_\omega.$$

The smooth function

$$u = u_\omega := \log\left(\frac{V^{-1}\omega^n}{\mu_\omega}\right) = \left(\frac{\mathrm{MA}(\phi)}{\mu_\phi}\right) \tag{8}$$

satisfies $dd^c u = \omega - \mathrm{Ric}(\omega)$ and $V^{-1}\int_X e^{-u}\omega^n = 1$ and hence coincides with the normalized Ricci potential.

We consider the (normalized) Kähler-Ricci flow

$$\dot{\omega}_t = \omega_t - \mathrm{Ric}(\omega_t) \tag{9}$$

on the level of Kähler forms starting with some initial Kähler form $\omega_0 \in c_1(X)$ which is the curvature form of a positively curved hermitian metric $h_0 = e^{-\phi_0}$ on K_X^{-1}. This flow has a global solution [7], and Perelman's estimates show that the normalized Ricci potential

$$u_t := \log\left(\frac{V^{-1}\omega_t^n}{\mu_{\omega_t}}\right)$$

is uniformly bounded in C^0. This means that ω_t^n and μ_{ω_t} remain uniformly comparable along the flow. The main result of [13] is the following:

Theorem. *If there exists constants $C, \epsilon > 0$ such that the Mabuchi K-energy $M(\omega_t)$ and the first positive eigenvalue $\lambda_1(\omega_t)$ of the $\bar{\partial}$-Laplacian with respect to ω_t on the space of smooth vector fields satisfy $M(\omega_t) \geq -C$ and $\lambda_1(\omega_t) \geq \epsilon$ along the flow, then ω_t converges at exponential rate in C^∞ topology to a Kähler-Einstein metric.*

As observed in [8], this combines with the main result of [16] to give

Corollary. *If X is Fano and Kähler-Einstein, then the flow of Kähler forms ω_t according to (9) converges at exponential rate to a Kähler-Einstein metric.*

Proof. Since the Mabuchi K-energy is minimized at a Kähler-Einstein metric, it is bounded from below. By the result of Tian and Zhu, there exists a path $g_t \in \mathrm{Aut}^0(X)$ such that $g_t^*\omega_t$ converges to a Kähler-Einstein metric. In particular, $\lambda_1(\omega_t) = \lambda_1(g_t^*\omega_t)$ is bounded away from 0 and we conclude by the previous result. ☐

Now we lift the flow (9) to the level of hermitian metrics on K_X^{-1}. We denote the space of hermitian metrics on K_X^{-1} with positive curvature form by $\mathcal{H}_{K_X^{-1}}$. As introduced in [1], we consider the *normalized flow* on the space $\mathcal{H}_{K_X^{-1}}$ as follows:

$$\dot{\phi}_t = \left(\frac{\mathrm{MA}(\phi_t)}{\mu_{\phi_t}}\right) = \log\left(\frac{V^{-1}(dd^c\phi_t)^n}{e^{-\phi_t}/\int_X e^{-\phi_t}}\right), \quad \phi_0 = \phi. \tag{10}$$

Using the above corollary we can prove

Theorem 3. *The flow (10) exists and it is smooth over $X \times [0, \infty[$. Furthermore, ϕ_t converges in C^∞ to a smooth Kähler-Einstein potential ϕ_∞.*

Proof. The first part is due to [7]. As already observed in [1], the fact that the two measures $\mathrm{MA}(\phi_t)$ and μ_{ϕ_t} are both probability measures gives that the time derivative $\dot{\phi}_t$ coincides with the normalized Ricci potential u_t defined by (8). But now since ω_t converges exponentially fast to a Kähler-Einstein metric ω_∞, $\dot{\phi}_t = u_t$ converges exponentially fast to $u_\infty = 0$. But since the operator

$$\omega \mapsto u_\omega = \left(\frac{V^{-1}\omega^n}{\mu_\omega}\right)$$

is Lipschitz on bounded sets with respect to the C^∞ topology, the exponential convergence can be integrated to imply that $t \mapsto \phi_t$ is Cauchy as $t \to \infty$.

Hence ϕ_t converges to a hermitian metric ϕ_∞ which gives a Kähler-Einstein potential for ω_∞. □

For the relative case of a family $f : \mathcal{X} \to S$ of Fano-Kähler-Einstein manifolds, we define the flow fiberwise as in (10). This time we are looking for solutions in $\mathcal{H}_{K^{-1}_{\mathcal{X}/S}}$, the space of hermitian metrics on $K^{-1}_{\mathcal{X}/S}$ that has positive curvature along the fibers, starting with a metric $\phi = \phi_0 \in \mathcal{H}_{K^{-1}_{\mathcal{X}/S}}$. Using the result form the absolute case, we get

Theorem 4. *(= Theorem 2) The fiberwise Kähler-Ricci flow for a family $f : \mathcal{X} \to S$ of Fano-Kähler-Einstein manifolds as defined in (10) exists and it is smooth over $\mathcal{X} \times [0, \infty[$. Moreover, it converges to a hermitian metric ϕ_∞ on $K^{-1}_{\mathcal{X}/S}$ that is of class C^∞ on each fiber X_s.*

Proof. The first part is already contained in [1, Th. 4.1]. The second part is just the convergence statement in Theorem 3. □

Remark 1. *Under the assumption that the fibers X_s of the family $\mathcal{X} \to S$ all satisfy the condition $H^0(X_s, T_{X_s}) = 0$, the convergence is locally uniform with respect to the base, see [4, Th.4.1]. We expect this to be true as well in our situation. Moreover, in the case $H^0(X_s, T_{X_s}) = 0$ one obtains that the limit metric ϕ_∞ is of class C^∞ globally on $\mathcal{X}$. This is due to the fact that we have uniqueness of the Kähler-Einstein metric in this case. In our more general setting, the regularity for the limit metric in the horizontal direction of the fibration is unclear.*

3.2. The flow on $\mathcal{O}_E(r)$ - Definition and convergence

In this section, we work with an arbitrary vector bundle $E \to S$, not necessarily ample. We would like to run the flow on the bundle $\mathcal{O}_E(r)$ instead of $K^{-1}_{\mathcal{X}/S}$. Therefore, we first recall that the isomorphism

$$K^{-1}_{\mathcal{X}/S} \cong \mathcal{O}_E(r) \otimes f^* \det(E)^{-1} \tag{11}$$

is canonical, because it comes from the natural relative Euler sequence

$$0 \to \Omega_{\mathcal{X}/S} \to f^*E \otimes \mathcal{O}_E(-1) \to \mathcal{O} \to 0.$$

Furthermore, the line bundle

$$f_*(K_{\mathcal{X}/S} \otimes \mathcal{O}_E(r)) \cong \det E \tag{12}$$

is fiberwise trivial and hence we have that

$$H^0(X_s, K_{X_s} \otimes \mathcal{O}_{\mathbb{P}(E_s^*)}(r)) \cong \det E_s$$

and it is one-dimensional. Now we start with a hermitian metric $h = e^{-\varphi}$ on $\mathcal{O}(r)$ with fiberwise positive curvature. A local holomorphic (trivializing) section u of $\det E$ can be represented under the isomorphism (12) by a holomorphic section of $K_{\mathcal{X}/S} \otimes \mathcal{O}(r)$ on the total space (after shrinking S), which are fiberwise holomorphic n-forms with values in $\mathcal{O}(r)$ (see also [15]). On the other hand, the pullback section f^*u can locally on $\mathcal{X}$ be written as (by abuse of notation we write again u for f^*u)

$$u = dz \otimes s$$

with local trivializing sections $dz := dz_1 \wedge \ldots \wedge dz_n$ of $K_{\mathcal{X}/S}$ and s of $\mathcal{O}_E(r)$ using the isomorphism

$$f^*(\det E) \cong K_{\mathcal{X}/S} \otimes \mathcal{O}_E(r).$$

Of course, both descriptions are compatible with each other. This means that dz and s are only local, but $dz \otimes s$ is the local description of the global form u. We write (see again [5, Sect. 2.1] for this notation)

$$|s|^2_\varphi := |s|^2 e^{-\varphi}$$

for the pointwise norm squared and

$$|u|^2 e^{-\varphi} := c_n dz \wedge d\overline{z}\, |s|^2_\varphi,$$

which gives volume forms on the fibers X_s. The associated norm of $u_s := u|_{X_s} \in H^0(X_s, K_{X_s} \otimes \mathcal{O}_{\mathbb{P}(E^*)}(r)) \cong \det E_s$ is given by the L^2-norm

$$||u_s||^2_\varphi := \int_{X_s} |u_s|^2 e^{-\varphi} := \int_{X_s} c_n |s|^2 e^{-\varphi} dz \wedge d\overline{z}.$$

Now we write $|f^*u_s|^2 e^{-\psi} := ||u_s||^2_\varphi$, which gives a metric $e^{-\psi}$ on $f^*\det E$; the pullback of the L^2-metric on $\det E$. Hence

$$e^{-\phi} := e^{-\varphi} e^{\psi}$$

is a metric on $-K_{\mathcal{X}/S}$ corresponding to the trivializing section $(dz)^{-1}$. This is a relative volume form on $\mathcal{X}$, in other words

$$\mu_\varphi := |u_s|^2 e^{-\varphi} / \int_{X_s} |u_s|^2 e^{-\varphi}$$

is a volume form on each fiber X_s, which does not depend on the choice of the local trivializing section u. Obviously, we have

Proposition 3. *μ_φ is a probability measure.*

Thus we define the flow of hermitian metrics φ_t on $\mathcal{O}_E(r)$, which are all positive along the fibers X_s, by the fiberwise flow

$$\dot{\varphi}_t = \log\left(\frac{\mathrm{MA}(\varphi_t)}{\mu_{\varphi_t}}\right), \quad \varphi_0 = \varphi. \tag{13}$$

Theorem 5. *The relative Kähler-Ricci flow on $\mathcal{O}_E(r)$ as defined in (13) exists and it is smooth over $\mathcal{X} \times [0, \infty[$. The flow converges to a hermitian metric φ_∞ on $\mathcal{O}_E(r)$ that is of class C^∞ on each fiber X_s.*

Proof. We have $f^*u = s \otimes dz$, where u is a local section of $\det E$ on S. As above, we write for short $u = s \otimes dz$, where u is constant on the fibers X_s. This gives $(dz)^{-1} \otimes u = s$, which just means that we can identify local sections of $K^{-1}_{\mathcal{X}/S}$ and $\mathcal{O}(r)$ by means of the local trivializing of $f^*(\det E)$ given by u. In particular, this gives an identification of $K^{-1}_{X_s}$ and $\mathcal{O}_{\mathbb{P}(E^*_s)}(r)$ on the fibers. Hence we can write

$$\begin{aligned}
\frac{|u|^2 e^{-\varphi}}{\int_{X_s} |u|^2 e^{-\varphi}} &= \frac{c_n dz \wedge d\bar{z}\, |s|^2 e^{-\varphi}}{\int_{X_s} c_n dz \wedge d\bar{z}\, |s|^2 e^{-\varphi}} \\
&= \frac{c_n |u|^2 dz \wedge d\bar{z}\, |(dz)^{-1}|^2 e^{-\varphi}}{\int_{X_s} c_n |u|^2 dz \wedge d\bar{z}\, |(dz)^{-1}|^2 e^{-\varphi}} \\
&= \frac{c_n dz \wedge d\bar{z}\, |(dz)^{-1}|^2 e^{-\varphi}}{\int_{X_s} c_n dz \wedge d\bar{z}\, |(dz)^{-1}|^2 e^{-\varphi}},
\end{aligned}$$

because the section u was constant and non-vanishing on the fibers. By using the notation from (7), we can write the last expression simply as

$$\frac{e^{-\varphi}}{\int_{X_s} e^{-\varphi}}.$$

Hence fiberwise, there is now difference between the flow 10 and 13. The difference only occurs globally, where the denominator of the canonical measure in 10 is interpreted as a metric on the trivial line bundle on $\mathcal{X}$, where in the case of 13 the corresponding normalization gives a metric on $f^* \det E$ which is only trivial fiberwise. But this does not affect the analysis of the fiberwise flow, hence we obtain Theorem 5 from Theorem 4. $\square$

Remark 2. *As an observation we remark that the hermitian metric on $K_{\mathcal{X}/S}^{-1}$ given by the Monge-Ampère measure $(dd^c\varphi_t)^n/n!$ splits in the limit into the product metric*

$$e^{-\varphi_\infty} \cdot \left(\int |u_s|^2 e^{-\varphi_\infty} \right)^{-1},$$

which is of course what we expect in the Kähler-Einstein limit.

Remark 3. *In the particular case where $\det E$ is trivial, the bundle $K_{\mathcal{X}/S}^{-1}$ is canonically isomorphic to $\mathcal{O}_E(r)$ and the flow (13) coincides with the normalized flow in [1].*

Remark 4. *By writing the metric $e^{-\phi}$ on $K_{\mathcal{X}/S}^{-1}$ as a product metric $e^{-\varphi}e^{\psi}$ on $\mathcal{O}(r) \otimes (f^* \det E)^{-1}$ we get*

$$c_n dz \wedge d\overline{z} \, |(dz)^{-1}|^2 e^{-\phi} = \frac{c_n dz \wedge d\overline{z} \, |s|^2 e^{-\varphi}}{|u|^2 e^{-\psi}}.$$

Therefore, we are actually forced to set

$$|u|^2 e^{-\psi} = \int_{X_s} c_n dz \wedge d\overline{z} \, |s|^2 e^{-\varphi},$$

i.e. we have to choose the pullback of the L^2-metric for the metric on $f^* \det E$ in order to normalize the volume form $|u|^2 e^{-\varphi}$, which is in turn necessary for the convergence.

3.3. The evolution equation for the geodesic curvature function

Now we turn back to the Griffiths problem and start with a positive hermitian metric h on $\mathcal{O}_E(1)$. Then we run the flow (13) on $\mathcal{O}_E(r)$ for the positive initial metric $h^{\otimes r} = e^{-\varphi_0}$. In order to apply Theorem 1 to the limit metric, we are left to prove that the positivity of the geodesic curvature function $c(\varphi_t)$ is preserved under the flow (13). (Note that $e^{-\varphi_t}$ is now the metric on $\mathcal{O}_E(r)$.) Hence we need to study how it evolves along the flow. For this we can again assume that our base manifold S is a one-dimensional disc. By using the computation from [1, Theorem 3.3, Theorem 4.7], we get the following evolution equation for $c(\varphi)$:

Proposition 4. *On a fiber X_s we have*

$$\frac{\partial c(\varphi_t)}{\partial t} = -\Box_{\omega_t} c(\varphi_t) + rc(\varphi_t) + |A_s^{\varphi_t}|^2_{\omega_t} + \partial_s \partial_{\overline{s}} \log \int_{X_s} |u_s|^2 e^{-\varphi_t}. \tag{14}$$

Note that in the limit we get indeed equation (3). For the last term on the right hand side, which is minus the curvature of

$$\int |u_s|^2 e^{-\varphi_t},$$

we can apply for instance the curvature formula from [5, Theorem 3.1]. This result says that the above L^2-metric on $f_*(K_{\mathcal{X}/S} \otimes \mathcal{O}(r)) = \det E$ is positive if $(\mathcal{O}(r), h^{\otimes r})$ is positive. But because the latter is equivalent to the positivity of the function $c(\varphi_t)$, this means that the expression

$$\partial_s \partial_{\overline{s}} \log \int_{X_s} |u_s|^2 e^{-\varphi_t}$$

always gives a negative contribution on the right hand side in this case, which is an obstruction to apply the usual maximum principles for the heat equation. On the other hand, this term has to be there, otherwise $c(\varphi_\infty)$ could never be positive, compare [1, Remark 4.11.]). But the question whether the positivity of $c(\varphi_0)$ is preserved or not remains open.

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