

Poitou–Tate sequence for complex of tori over p -adic function fields

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We complete the picture of local and global arithmetic duality theorems for short complexes of finite Galois modules and tori over p -adic function fields. In view of the duality theorems, we deduce a 12-term Poitou–Tate exact sequence which relates global Galois cohomology groups to restricted topological products of local Galois cohomology groups.

Introduction

Recently, there have been several developments concerning the arithmetic of linear groups over fields of arithmetic type having cohomological dimension strictly larger than 2 (for example, the function field K of a smooth projective geometrically integral curve defined over some finite extension of \mathbb{Q}_p). In [CTPS12], Colliot-Thélène, Parimala and Suresh investigated the Hasse principle for certain varieties over such K . Later, Hu [Hu14] obtained results on the Hasse principle for simply connected groups while Harari and Szamuely [HS16] found cohomological obstructions to the Hasse principle for quasi-split reductive groups over p -adic function fields. On the other hand, Harari, Scheiderer and Szamuely [HSS15, HS16] studied systematically obstructions to the Hasse principle and weak approximation for tori over p -adic function fields. All these developments motivate investigations concerning the arithmetic of reductive linear algebraic groups over such fields K .

This paper is some sort of complement to [Tia21] where the author established some arithmetic duality results and obtained obstructions to weak approximation for connected reductive algebraic groups over K . In the present paper, we give a full picture of arithmetic duality results for a short complex of tori and deduce a 12-term Poitou–Tate style exact sequence (in particular, we obtain such an exact sequence for groups of multiplicative type over K).

Let us state the main results of this article. Let X be a smooth projective geometrically integral curve over a p -adic field k and let $K = k(X)$ be the

function field of X . Suppose that $\rho : T_1 \rightarrow T_2$ is a morphism of K -tori and let $C = [T_1 \rightarrow T_2]$ be the associated complex concentrated in degree -1 and 0 . For a K -torus P , its dual P' is the torus whose module of characters is the module of cocharacters of P . Let T'_1 and T'_2 be the respective dual tori of T_1 and T_2 , and put $C' = [T'_2 \rightarrow T'_1]$ for the dual complex concentrated in degree -1 and 0 . We put $\text{III}^i(C) := \text{Ker}(\mathbb{H}^i(K, C) \rightarrow \prod_{v \in X^{(1)}} \mathbb{H}^i(K_v, C))$ to be the Tate–Shafarevich group of C , where $X^{(1)}$ denotes the set of closed points on X . The first main result is

Theorem. *The following is a functorial perfect pairing of finite groups for $0 \leq i \leq 2$:*

$$\text{III}^i(C) \times \text{III}^{2-i}(C') \rightarrow \mathbb{Q}/\mathbb{Z}.$$

So far, the theorem was known when $i = 1$ by [Tia21, Theorem 1.18]. Thus the above result provides a generalization and complement to *loc. cit.* into all possible degrees (see Remark 3.4 for details). In the context of higher dimensional local fields, similar results of Izquierdo [Izq16, pp. 80, Théorème 4.17] are highly relevant where he obtained perfect pairings between quotients of Tate–Shafarevich groups by their maximal divisible subgroups. In the case of number fields, Demarche [Dem11a] established perfect pairings $\text{III}^i(C) \times \text{III}^{2-i}(\widehat{C}) \rightarrow \mathbb{Q}/\mathbb{Z}$ of finite groups when either $\text{Ker } \rho$ is finite or ρ is surjective. Here $\widehat{C} = [\widehat{T}_2 \rightarrow \widehat{T}_1]$ with \widehat{T}_i the respective the module of characters associated to T_i . Actually, we would like to consider the complex $C' = [T'_2 \rightarrow T'_1]$ instead of $\widehat{C} = [\widehat{T}_2 \rightarrow \widehat{T}_1]$ for the following reasons:

- If we consider a connected reductive group G and the universal covering $G^{\text{sc}} \rightarrow G^{\text{ss}}$ of its derived subgroup, we may obtain an associated short complex $C = [T^{\text{sc}} \rightarrow T]$ where T is a maximal torus of G and T^{sc} is the inverse image of T in G^{sc} . Then [Tia21, Theorem 2.4(2)] tells us that the kernel of the map $\text{III}^1_{\omega}(C')^D \rightarrow \text{III}^1(C)$ (which is induced by the global duality $\text{III}^1(C) \simeq \text{III}^1(C')^D$) provides a defect of weak approximation for G . Here $\text{III}^1_{\omega}(C')$ denotes the subgroup of elements in $\mathbb{H}^1(K, C')$ that are zero in all but finitely many $\mathbb{H}^1(K_v, C')$.
- Let M be a group of multiplicative type over K and embed it into a short exact sequence $0 \rightarrow M \rightarrow T_1 \rightarrow T_2 \rightarrow 0$ with T_i being K -tori. Thus $M[1]$ is quasi-isomorphic to $C = [T_1 \rightarrow T_2]$. Actually it is too greedy to expect that there is an algebraic group (which is of finite type) playing the role of the dual of M . Instead, the natural "dual" of M should be $C' = [T'_2 \rightarrow T'_1]$ because it does not lose the information on the torsion part of \widehat{M} . The same phenomenon already arose when

one tries to handle the dual of a semi-abelian variety. As pointed out in [HS05], the dual of a semi-abelian variety is a so-called 1-motive.

In order to define the maps in the Poitou–Tate sequence (1) below, we shall need an auxiliary global duality result. For a closed point $v \in X^{(1)}$, we let \mathcal{O}_v be the ring of integers in K_v . We fix a non-empty open subset X_0 of X such that T_1 and T_2 extend to X_0 -tori \mathcal{T}_1 and \mathcal{T}_2 respectively, and we put $\mathcal{C} = [\mathcal{T}_1 \rightarrow \mathcal{T}_2]$. Let $\mathbb{P}^i(K, C)$ be the restricted topological product of $\mathbb{H}^i(K_v, C)$ with respect to the subgroups $\mathbb{H}^i(\mathcal{O}_v, C)$ (we will show that $\mathbb{H}^i(\mathcal{O}_v, C)$ is a subgroup of $\mathbb{H}^i(K_v, C)$ in the sequel). For an abelian group A , we denote $A_\wedge = \varprojlim_n A/n$. Let $\mathbb{III}_\wedge^0(C) := \text{Ker}(\mathbb{H}^0(K, C)_\wedge \rightarrow \mathbb{P}^0(K, C)_\wedge)$. We shall see that $\mathbb{III}_\wedge^0(C) \simeq \mathbb{III}^0(C)$ if $\text{Ker } \rho$ is finite in view of the following result.

Theorem. *Suppose that $\text{Ker } \rho$ is finite. The following is a perfect pairing of finite groups:*

$$\mathbb{III}_\wedge^0(C) \times \mathbb{III}^2(C') \rightarrow \mathbb{Q}/\mathbb{Z}.$$

The proof of the theorem is analogous to that of [Dem11a, Proposition 5.10] in the number field context. This result is more complicated because we have to handle the inverse limits $\mathbb{H}^0(K, C)_\wedge$ and $\mathbb{P}^0(K, C)_\wedge$, and so it is not an immediate consequence of Artin–Verdier duality and local duality. The idea is to describe $\mathbb{III}_\wedge^0(C)$ and $\mathbb{III}^2(C')$ by various limits. Thus a crucial problem is to define respective transition maps and now the finiteness of $\text{Ker } \rho$ plays a role. Finally, note that if $\text{coker } \rho$ is trivial, then the kernel of $\rho' : T'_2 \rightarrow T'_1$ is finite. Thus we obtain a perfect pairing $\mathbb{III}^2(C) \times \mathbb{III}_\wedge^0(C') \rightarrow \mathbb{Q}/\mathbb{Z}$ of finite groups as well.

The classical Poitou–Tate sequence is a 9-term exact sequence which relates global Galois cohomology with restricted ramification of a finite Galois module over a global field (for example, see [Har17, Théorème 17.13] and also [Čes15] for a generalization). In [HSS15], Harari, Scheiderer and Szamuely constructed a 12-term Poitou–Tate style exact sequence for finite Galois modules and a 9-term one for tori in the p -adic function field context. Now we arrive at the main result of the present article:

Theorem. *Suppose either $\text{Ker } \rho$ is finite or $\text{Coker } \rho$ is trivial. Then the following is a functorial exact sequence of topological abelian groups*

$$(1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{H}^{-1}(K, C)_\wedge & \longrightarrow & \mathbb{P}^{-1}(K, C)_\wedge & \longrightarrow & \mathbb{H}^2(K, C')^D \\ & & \underbrace{\hspace{10em}} & & & & \\ & & \longrightarrow & \mathbb{H}^0(K, C)_\wedge & \longrightarrow & \mathbb{P}^0(K, C)_\wedge & \longrightarrow & \mathbb{H}^1(K, C')^D \\ & & \underbrace{\hspace{10em}} & & & & \\ & & \longrightarrow & \mathbb{H}^1(K, C) & \longrightarrow & \mathbb{P}^1(K, C)_{\text{tors}} & \longrightarrow & (\mathbb{H}^0(K, C')_\wedge)^D \\ & & \underbrace{\hspace{10em}} & & & & \\ & & \longrightarrow & \mathbb{H}^2(K, C) & \longrightarrow & \mathbb{P}^2(K, C)_{\text{tors}} & \longrightarrow & (\mathbb{H}^{-1}(K, C')_\wedge)^D \longrightarrow 0 \end{array}$$

where $\mathbb{P}^i(K, C)_{\text{tors}}$ denotes the torsion subgroup of the group $\mathbb{P}^i(K, C)$ for $i = 1, 2$.

Actually, the exactness of the first and the last row in diagram (1) hold without any assumption on C . The finiteness of the kernel $\text{Ker } \rho$ or the surjectivity of ρ plays an essential role in the proof of the exactness of the middle two rows. Moreover, as we have seen the finiteness of $\text{Ker } \rho$ provides perfect pairings between Tate–Shafarevich groups which enable us to connect the desired exact sequence. Finally, let us have a glimpse at consequences (see Example 4.13 for details) of the Poitou–Tate sequence given above.

- If we take $C = [0 \rightarrow P]$ to be a single torus, then we obtain [HSS15, Theorem 2.9]. Note that here we have used the fact that $H^3(K, P) = 0$ (see Remark 1.3 below).
- For a connected reductive group G and a maximal torus T in G , let $T^{\text{sc}} \subset G^{\text{sc}}$ be as above. Then $\text{Ker}(T^{\text{sc}} \rightarrow T)$ is finite and we deduce a Poitou–Tate sequence (see Example 4.13(2)) for the reductive group G .
- If ρ is surjective, then the complex $C[-1]$ is quasi-isomorphic to $\text{Ker } \rho$ (which is a group of multiplicative type). Thus we obtain Poitou–Tate style exact sequences for groups of multiplicative type.

Let us close the introduction by potential applications of various Poitou–Tate style exact sequences. Harari and Izquierdo set forth how to use a Poitou–Tate sequence [HI18, Théorème 4.6] to define a defect to strong approximation using a divisible quotient group in [HI18, Théorème 4.8] when the base field is the function field of a smooth projective curve defined over an algebraically closed field of characteristic zero. Besides, Demarche

[Dem11b, Théorème 2.9] studied a defect to strong approximation with the help of another Poitou–Tate sequence. Hopefully our Poitou–Tate sequence gives an interpretation of a defect to strong approximation for connected linear groups as well over p -adic function fields.

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Notations and conventions

Function fields. Throughout this article, K will be the function field of a smooth proper and geometrically integral curve X over a p -adic field. Note that each closed point $v \in X^{(1)}$ defines a discrete valuation of K . We write $\mathcal{O}_{X,v}$ for the local ring at v and $\kappa(v)$ for its residue field. Moreover, K_v (resp. K_v^h) will be the completion (resp. Henselization) of K with respect to v and \mathcal{O}_v (resp. \mathcal{O}_v^h) will be the ring of integers in K_v (resp. K_v^h).

Abelian groups. Let A be an abelian group. We shall denote by ${}_nA$ (resp. $A\{\ell\}$) for the n -torsion subgroup (resp. ℓ -primary subgroup with ℓ a prime) of A . Moreover, let A_{tors} be the torsion subgroup of A . So $A_{\text{tors}} = \varinjlim_n {}_nA$ is the direct limit of n -torsion subgroups of A . We write A^\wedge for the profinite completion of A (that is, the inverse limit of its finite quotients), $A_\wedge := \varprojlim_n A/nA$ and $A^{(\ell)} := \varprojlim_n A/\ell^n$ for the ℓ -adic completion with ℓ a prime number. A torsion abelian group A is of cofinite type if ${}_nA$ is finite for each $n \geq 1$. If A is ℓ -primary torsion of cofinite type, then $A/\text{Div } A \simeq A^{(\ell)}$ where the former group is the quotient of A by its maximal divisible subgroup. For a topological abelian group A , we write $A^D := \text{Hom}_{\text{cont}}(A, \mathbb{Q}/\mathbb{Z})$ for the group of continuous homomorphisms .

Motivic complexes. Let L be a field. For a smooth L -variety Y , we denote the étale motivic complex over Y by the complex of sheaves $\mathbb{Z}(i) := z^i(-, \bullet)[-2i]$ on the small étale site of Y , where $z^i(Y, \bullet)$ is Bloch’s cycle complex [Blo86]. For example, we have quasi-isomorphisms $\mathbb{Z}(0) \simeq \mathbb{Z}$ and $\mathbb{Z}(1) \simeq \mathbb{G}_m[-1]$ by [Blo86, Corollary 6.4]. We write $A(i) := A \otimes^{\mathbf{L}} \mathbb{Z}(i)$ for any abelian group A . Finally, if n is an integer invertible in L , then [GL01, Theorem 1.5] gives a quasi-isomorphism $\mathbb{Z}/n\mathbb{Z}(i) \simeq \mu_n^{\otimes i}$ where μ_n is concentrated in degree 0. We shall write $\mathbb{Q}/\mathbb{Z}(i) := \varinjlim_n \mu_n^{\otimes i}$ for the direct limit of the sheaves $\mu_n^{\otimes i}$ for all $n \geq 1$.

Tori and short complex of tori. Let L be a field of characteristic zero and let \bar{L} be a fixed algebraic closure of L . We write \widehat{T} or $X^*(T)$ (resp. \check{T} or $X_*(T)$) for the character module (resp. cocharacter module) of an L -torus T . These are finitely generated free abelian groups endowed with a $\text{Gal}(\bar{L}|L)$ -action, and moreover \check{T} is the \mathbb{Z} -linear dual of \widehat{T} . The dual torus T' of T is the torus with character group \check{T} , that is, $\widehat{T'} = \check{T}$.

Let $C = [T_1 \xrightarrow{\rho} T_2]$ be a short complex of L -tori concentrated in degree -1 and 0 . We always write $M = \text{Ker } \rho, T = \text{Coker } \rho$ and $C' = [T'_2 \rightarrow T'_1]$ (again it is concentrated in degree -1 and 0). Thus M is a group of multiplicative type and T is a torus. Let $X_0 \subset X$ be a non-empty open subset such that T_1 and T_2 extend to X_0 -tori \mathcal{T}_1 and \mathcal{T}_2 respectively (in the sense of [SGA3II, Exposé IX, Définition 1.3]). We similarly write $\mathcal{M} = \text{Ker}(\mathcal{T}_1 \rightarrow \mathcal{T}_2)$ and $\mathcal{T} = \text{Coker}(\mathcal{T}_1 \rightarrow \mathcal{T}_2)$ over X_0 . So over X_0 we obtain complexes $\mathcal{C} = [\mathcal{T}_1 \rightarrow \mathcal{T}_2]$ and $\mathcal{C}' = [\mathcal{T}'_2 \rightarrow \mathcal{T}'_1]$. Finally, U will always be a suitably sufficiently small non-empty open subset of X_0 .

For the short complex $\mathcal{C} = [\mathcal{T}_1 \rightarrow \mathcal{T}_2]$ over X_0 , we put, for $n \geq 1$, $T_{\mathbb{Z}/n}(\mathcal{C}) := H^0(\mathcal{C}[-1] \otimes^{\mathbf{L}} \mathbb{Z}/n)$ which is an fppf sheaf of abelian groups. Recall [Dem11a, Lemme 2.3] that the sheaf $T_{\mathbb{Z}/n}(\mathcal{C})$ is represented by a finite group scheme of multiplicative type over X_0 , and it fits into a distinguished triangle ${}_n\mathcal{M}[2] \rightarrow \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow T_{\mathbb{Z}/n}(\mathcal{C})[1] \rightarrow {}_n\mathcal{M}[3]$.

Cohomologies. Unless otherwise stated, all cohomologies are understood with respect to the étale topology. Let $j_0 : X_0 \rightarrow X$ be the open immersion. We denote $\mathbb{H}_c^i(X_0, \mathcal{C}) := \mathbb{H}^i(X, j_{0!}\mathcal{C})$ for the compact support cohomology.

Pairings. There is a canonical pairing $C \otimes^{\mathbf{L}} C' \rightarrow \mathbb{Z}(2)[3]$ (see [Izq16, pp. 69, Lemme 4.3]) over K which extends to a pairing $\mathcal{C} \otimes^{\mathbf{L}} \mathcal{C}' \rightarrow \mathbb{Z}(2)[3]$ over X_0 (both pairings are in the bounded derived category of étale sheaves). By [HS16, Lemma 1.1 and Lemma 2.1] respectively, we have identifications $\mathbb{H}_c^5(U, \mathbb{Z}(2)) \simeq \mathbb{Q}/\mathbb{Z}$ and $\mathbb{H}^4(K_v, \mathbb{Z}(2)) \simeq \mathbb{Q}/\mathbb{Z}$. In particular, there are canonical pairings $\mathbb{H}^i(U, \mathcal{C}) \times \mathbb{H}_c^{2-i}(U, \mathcal{C}') \rightarrow \mathbb{Q}/\mathbb{Z}$ for any non-empty open subset $U \subset X_0$ and $\mathbb{H}^i(K_v, \mathcal{C}) \times \mathbb{H}^{1-i}(K_v, \mathcal{C}') \rightarrow \mathbb{Q}/\mathbb{Z}$. We shall also consider pairings in the finite level. More precisely, we may identify $\mathbb{Z}/n \otimes^{\mathbf{L}} \mathbb{Z}/n$ with the short complex $[\mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}]$ concentrated in degree $-2, -1$ and 0 , where the first arrow is given by $x \mapsto (nx, -nx)$ and the second one is given by $(x_1, x_2) \mapsto n(x_1 + x_2)$. In this point of view, we may define a pairing $\mathbb{Z}/n \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow \mathbb{Z}[1]$ by sending $(x_1, x_2) \in \mathbb{Z} \oplus \mathbb{Z}$ to $x_1 + x_2$. Subsequently, we obtain a canonical pairing $(C \otimes^{\mathbf{L}} \mathbb{Z}/n) \otimes^{\mathbf{L}} (C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Z}(2)[4]$ which induces canonical pairings $\mathbb{H}^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{H}_c^{1-i}(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}$ and $\mathbb{H}^i(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{H}^{-i}(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}$.

Triangles. We shall use the following distinguished triangles frequently in the sequel. By definition of C , we obtain a distinguished triangle $T_1 \rightarrow T_2 \rightarrow C \rightarrow T_1[1]$ and similarly ${}_nT_1 \rightarrow {}_nT_2 \rightarrow C \otimes^{\mathbf{L}} \mathbb{Z}/n[-1] \rightarrow {}_nT_1[1]$. By definition of M and T , we have a distinguished triangle $M[1] \rightarrow C \rightarrow T \rightarrow M[2]$. There is a distinguished triangle $C \rightarrow C \rightarrow C \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow C[1]$ obtained from the Kummer sequences for tori. Finally, we have a distinguished triangle ${}_nM[2] \rightarrow C \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow T_{\mathbb{Z}/n}(C)[1] \rightarrow {}_nM[3]$ by [Dem11a, Lemme 2.3], where $T_{\mathbb{Z}/n}(C) := H^0(C[-1] \otimes^{\mathbf{L}} \mathbb{Z}/n)$.

1. Preliminaries on injectivity properties

We begin with the following lemma considering the canonical map

$$\mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n).$$

Lemma 1.1. *For each $i \in \mathbb{Z}$ and $v \in X_0^{(1)}$, the homomorphism*

$$\mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$$

induced by the inclusion $\mathcal{O}_v \subset K_v$ is injective. Therefore we may identify $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ as a subgroup of $\mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$.

Proof. For $i = -2$, we observe that $\mathbb{H}^{-2}(\mathcal{O}_v, \mathcal{C}) = 0$ from the distinguished triangle $\mathcal{T}_1 \rightarrow \mathcal{T}_2 \rightarrow \mathcal{C} \rightarrow \mathcal{T}_1[1]$. Since \mathcal{T}_1 is affine (hence separated), $\mathcal{T}_1(\mathcal{O}_v) \rightarrow \mathcal{T}_1(K_v)$ is injective. Thus the homomorphism $\mathbb{H}^{-1}(\mathcal{O}_v, \mathcal{C}) \rightarrow \mathbb{H}^{-1}(K_v, C)$ is injective by dévissage thanks to the distinguished triangle $\mathcal{T}_1 \rightarrow \mathcal{T}_2 \rightarrow \mathcal{C} \rightarrow \mathcal{T}_1[1]$.

Now we suppose $i \geq -1$. Let \overline{K}_v be a separable closure of K_v and let K_v^{nr} be the maximal unramified extension of K_v . According to [Mil06, II, Proposition 1.1(b)], we obtain $H^i(\mathcal{O}_v, {}_n\mathcal{P}) \simeq H^i(\kappa(v), {}_n\mathcal{P})$ for $i \geq 0$ and $\mathcal{P} = \mathcal{T}_1, \mathcal{T}_2$. It follows that for $i \geq -1$, the canonical map $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^i(\kappa(v), \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is an isomorphism thanks to the distinguished triangle ${}_n\mathcal{T}_1[1] \rightarrow {}_n\mathcal{T}_2[1] \rightarrow \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow {}_n\mathcal{T}_1[2]$. Note that $\mathbb{H}^i(\kappa(v), \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is isomorphic to $\mathbb{H}^i(\text{Gal}(K_v^{\text{nr}}|K_v), \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ by ramification theory. Choose an extension of v to \overline{K}_v and let I_v be the corresponding inertia group. But the short exact sequence $1 \rightarrow I_v \rightarrow \text{Gal}(\overline{K}_v|K_v) \rightarrow \text{Gal}(K_v^{\text{nr}}|K_v) \rightarrow 1$ admits a section [Ser65, II, Appendix, §2], consequently $H^i(K_v^{\text{nr}}|K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow H^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ admits a retraction, hence is injective. \square

We have similar results for the complex C . Namely, the canonical map $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C}) \rightarrow \mathbb{H}^i(K_v, C)$ is injective.

Lemma 1.2. *The homomorphism $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C}) \rightarrow \mathbb{H}^i(K_v, \mathcal{C})$ induced by the canonical morphism $\text{Spec } K_v \rightarrow \text{Spec } \mathcal{O}_v$ is injective for each $i \in \mathbb{Z}$ and $v \in X_0^{(1)}$.*

Proof. First of all, note that $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C}) = 0$ for $i \leq -2$ and $i \geq 3$.

- (1) $i = -1$. This is already proved in the first paragraph of the previous proof.
- (2) $i = 0$. We consider the distinguished triangle $M[1] \rightarrow C \rightarrow T \rightarrow M[2]$. By dévissage, it will be sufficient to show $H^1(\mathcal{O}_v, \mathcal{M}) \rightarrow H^1(K_v, M)$ is injective. We may realize \mathcal{M} as an extension $1 \rightarrow \mathcal{P} \rightarrow \mathcal{M} \rightarrow \mathcal{F} \rightarrow 1$ of a finite group scheme \mathcal{F} by a torus \mathcal{P} over \mathcal{O}_v (since \mathcal{M} is isotrivial by [SGA3II, Chapitre X, Proposition 5.16]). Recall that $H^1(\mathcal{O}_v, \mathcal{P}) \rightarrow H^1(K_v, P)$ and $H^1(\mathcal{O}_v, \mathcal{F}) \rightarrow H^1(K_v, F)$ are injective (see [HSS15, Proposition 1.2 and 1.3]), and that $H^0(\mathcal{O}_v, \mathcal{F}) = H^0(K_v, F)$ since \mathcal{F} is a finite group scheme. It follows that $H^1(\mathcal{O}_v, \mathcal{M}) \rightarrow H^1(K_v, M)$ is injective by dévissage.
- (3) $i = 1$. Since we have $\mathbb{H}^1(\mathcal{O}_v^h, \mathcal{C}) \simeq \mathbb{H}^1(\kappa(v), \mathcal{C}) \simeq \mathbb{H}^1(\mathcal{O}_v, \mathcal{C})$ and $\mathbb{H}^1(K_v^h, \mathcal{C}) \simeq \mathbb{H}^1(K_v, \mathcal{C})$ (thanks to [HS16, Corollary 3.2] and the distinguished triangle $T_1 \rightarrow T_2 \rightarrow C \rightarrow T_1[1]$), the validity of this case is ensured by [Tia21, Corollary 1.16].
- (4) $i = 2$. The following commutative diagram is obtained from the respective Kummer sequences

$$\begin{array}{ccc}
 \varinjlim_n \mathbb{H}^1(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}^2(\mathcal{O}_v, \mathcal{C}) \\
 \downarrow & & \downarrow \\
 \varinjlim_n \mathbb{H}^1(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & H^2(K_v, \mathcal{C}).
 \end{array}$$

Since the groups $\mathbb{H}^1(K_v, \mathcal{C})$ and $\mathbb{H}^1(\kappa(v), \mathcal{C})$ are torsion, we observe that the horizontal arrows are isomorphisms. Since the left vertical arrow is injective by Lemma 1.1, so is the right one by diagram chasing. □

Remark 1.3. To proceed, let us first briefly explain the degrees under consideration.

- (1) Let P be a K -torus. The groups $H^i(K, P)$ and $H^i(K_v, P)$ vanish for $i \geq 3$. See Lemma 3.1(3) below for details. Subsequently, $\mathbb{H}^i(K, C) = 0$ for $i \geq 3$ thanks to the distinguished triangle $T_1 \rightarrow T_2 \rightarrow C \rightarrow T_1[1]$.
- (2) The group $H^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) = 0$ for $i \leq -3$ or $i \geq 3$. This is a direct consequence of dévissage thanks to the distinguished triangle $C \rightarrow C \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow C[1]$.
- (3) The groups $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ vanish for $i \geq 2$ or $i \leq -3$. Indeed, we consider over \mathcal{O}_v the distinguished triangle ${}_n\mathcal{T}_1[1] \rightarrow {}_n\mathcal{T}_2[1] \rightarrow \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow {}_n\mathcal{T}_1[2]$. Therefore it will be sufficient to show $H^i(\mathcal{O}_v, {}_n\mathcal{P}) = 0$ for $i \geq 3$ and $i \leq -1$, and for any \mathcal{O}_v -torus \mathcal{P} by dévissage. Finally, we have $H^i(\mathcal{O}_v, {}_n\mathcal{P}) = H^i(\kappa(v), {}_n\mathcal{P})$ by [Mil06, II, Proposition 1.1(b)] and the latter group vanishes for $i \geq 3$ for cohomological dimension reasons (see [Ser65, Chapitre II, §5.3]), and $H^i(\mathcal{O}_v, {}_n\mathcal{P}) = 0$ for $i \leq -1$ by construction.

We denote by $\mathbb{P}^i(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) := \prod' \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ the restricted topological product of the finite discrete groups $\mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ with respect to the subgroups $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$. Note that the only non-trivial degrees are $-2 \leq i \leq 2$ by Remark 1.3. Since $\mathbb{P}^i(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is a direct limit of the compact groups $\prod_{v \in U} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \prod_{v \notin U} \mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ over all $U \subset X_0$, it is locally compact. By Remark 1.3 we obtain

$$\mathbb{P}^{-2}(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) = \prod_{v \in X^{(1)}} \mathbb{H}^{-2}(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$$

and

$$\mathbb{P}^2(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) = \bigoplus_{v \in X^{(1)}} \mathbb{H}^2(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n).$$

Therefore $\mathbb{P}^{-2}(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is profinite, and $\mathbb{P}^2(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is discrete (it is a direct sum of finite groups).

Similarly, we let $\mathbb{P}^i(K, C)$ be the restricted topological product of the groups $\mathbb{H}^i(K_v, C)$ with respect to the subgroups $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C})$ (see Lemma 1.2) for $v \in X_0^{(1)}$ and $-1 \leq i \leq 2$.

We close this section by the following injectivity between quotient groups.

Lemma 1.4. *For $n \geq 1$ and $i \geq -1$, the canonical homomorphism $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C})/n \rightarrow \mathbb{H}^i(K_v, C)/n$ induced by the inclusion $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C}) \rightarrow \mathbb{H}^i(K_v, C)$ is injective as well for $v \in X_0^{(1)}$. In particular, the homomorphism*

$\mathbb{P}^i(K, C)/n \rightarrow \prod_{v \in X^{(1)}} \mathbb{H}^i(K_v, C)/n$ induced by the inclusion $\mathbb{P}^i(K, C) \subset \prod_{v \in X^{(1)}} \mathbb{H}^i(K_v, C)$ is injective for $i \geq -1$. Moreover, the image is the restricted topological product of $\mathbb{H}^i(K_v, C)/n$ with respect to the subgroups $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C})/n$.

Proof. Thanks to the distinguished triangle $C \rightarrow C \rightarrow C \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow C[1]$, it suffices to show that $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is injective for each $v \in X_0^{(1)}$ which is ensured by Lemma 1.1. Let $(x_v) \in \mathbb{P}^i(K, C)/n$ and let $(\tilde{x}_v) \in \mathbb{P}^i(K, C)$ be a family of lifts of $x_v \in \mathbb{H}^i(K_v, C)/n$ in $\mathbb{H}^i(K_v, C)$. So $\tilde{x}_v \in \mathbb{H}^i(\mathcal{O}_v, \mathcal{C})$ for all but finitely many v . In particular, its image in $\mathbb{H}^i(K_v, C)/n$ lies in the subgroup $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C})/n$. \square

2. Arithmetic dualities in finite level

We first develop some arithmetic duality results and a 15-term Poitou–Tate exact sequence concerning the complexes $C \otimes^{\mathbf{L}} \mathbb{Z}/n$ and $C' \otimes^{\mathbf{L}} \mathbb{Z}/n$ for any $n \geq 1$.

2.1. Local dualities

The following local arithmetic duality is a special case of [Izq16, pp. 73, Proposition 4.7]. We quote it here and we briefly recall the idea of the proof.

Proposition 2.1. *The following pairing is a functorial perfect pairing of finite groups for $i \in \mathbb{Z}$*

$$(2) \quad \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{H}^{-i}(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. Recall [HS16, pp. 6, pairing (10)] that the following is a perfect pairing¹ of finite groups for $j \in \mathbb{Z}$ and $i = 1, 2$

$$H^j(K_v, {}_nT_i) \times H^{3-j}(K_v, {}_nT'_i) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Therefore the distinguished triangles ${}_nT_1[1] \rightarrow {}_nT_2[1] \rightarrow C \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow {}_nT_1[2]$ and ${}_nT'_2[1] \rightarrow {}_nT'_1[1] \rightarrow C' \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow {}_nT'_2[2]$ yield an isomorphism

$$\mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^{-i}(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D$$

¹Let P be a torus over a field L of characteristic zero. Then we have canonical isomorphisms ${}_nP(\bar{L}) \simeq \text{Hom}(\widehat{P}, \mu_n(\bar{L}))$ and ${}_nP'(\bar{L}) \simeq \text{Hom}(\check{P}, \mu_n(\bar{L}))$. Moreover, taking the identity element $\text{id} \in \widehat{P} \otimes \check{P} \simeq \text{End}(\widehat{P})$ into account, we obtain a pairing ${}_nP \otimes^{\mathbf{L}} {}_nP' \rightarrow \mu_n^{\otimes 2}$.

by dévissage (see [Izq16, pp. 73, Proposition 4.7] for details). □

We shall need the following additional result on respective annihilators of local dualities.

Proposition 2.2. *For $-2 \leq i \leq 2$, the annihilator of $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is $\mathbb{H}^{-i}(\mathcal{O}_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ under the perfect pairing*

$$\mathbb{H}^i(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{H}^{-i}(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. The distinguished triangle ${}_n\mathcal{T}_1[1] \rightarrow {}_n\mathcal{T}_2[1] \rightarrow \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow {}_n\mathcal{T}_1[2]$ over \mathcal{O}_v yields a commutative diagram

$$\begin{array}{ccccccccc} H^{i+1}(\mathcal{O}_v, {}_n\mathcal{T}_1) & \longrightarrow & H^{i+1}(\mathcal{O}_v, {}_n\mathcal{T}_2) & \longrightarrow & \mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & H^{i+2}(\mathcal{O}_v, {}_n\mathcal{T}_1) & \longrightarrow & H^{i+2}(\mathcal{O}_v, {}_n\mathcal{T}_2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow * \\ H^{i+1}(K_v, {}_n\mathcal{T}_1) & \longrightarrow & H^{i+1}(K_v, {}_n\mathcal{T}_2) & \longrightarrow & \mathbb{H}^i(K_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & H^{i+2}(K_v, {}_n\mathcal{T}_1) & \longrightarrow & H^{i+2}(K_v, {}_n\mathcal{T}_2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{2-i}(\mathcal{O}_v, {}_n\mathcal{T}'_1)^D & \longrightarrow & H^{2-i}(\mathcal{O}_v, {}_n\mathcal{T}'_2)^D & \longrightarrow & \mathbb{H}^{-i}(\mathcal{O}_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & \longrightarrow & H^{1-i}(\mathcal{O}_v, {}_n\mathcal{T}'_1)^D & \longrightarrow & H^{1-i}(\mathcal{O}_v, {}_n\mathcal{T}'_2)^D \end{array}$$

of finite groups with exact rows. For cohomological dimension reasons, the following pairing

$$\mathbb{H}^i(\mathcal{O}_v, {}_n\mathcal{P}) \times \mathbb{H}^{3-i}(\mathcal{O}_v, {}_n\mathcal{P}') \rightarrow \mathbb{H}^3(\mathcal{O}_v, \mathbb{Q}/\mathbb{Z}(2)) \simeq \mathbb{H}^3(\kappa(v), \mathbb{Q}/\mathbb{Z}(2))$$

is trivial for any X_0 -torus \mathcal{P} , and similarly

$$\mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{H}^{-i}(\mathcal{O}_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}$$

is also trivial. Thus the columns in the diagram are complexes. Note that it will be sufficient to consider $-2 \leq i \leq 0$ by symmetry and that the arrow $*$ is an isomorphism for $i = -2$ and is injective for $i = -1, 0$ (see [HSS15, Proposition 1.2] and its proof). In the sequel, we show the exactness of the middle column case by case.

- (1) $i = -2$. In this case, we have $H^{i+2}(\mathcal{O}_v, {}_n\mathcal{T}_j) \simeq H^{i+2}(K_v, {}_n\mathcal{T}_j)$ and $H^{i+1}(\mathcal{O}_v, {}_n\mathcal{T}_j) = H^{i+1}(K_v, {}_n\mathcal{T}_j) = 0$ for $j = 1, 2$. Thus exactness of the middle column follows from a diagram chase.
- (2) $i = -1$. The right two columns of the diagram above are exact by [HSS15, Proposition 1.2]. Moreover, we have an isomorphism $H^{i+1}(\mathcal{O}_v, {}_n\mathcal{T}_2) \simeq H^{i+1}(K_v, {}_n\mathcal{T}_2)$. Now a diagram chase yields the exactness of the middle column.

(3) $i = 0$. Note that $H^{i+1}(K_{v,n}T_1) \rightarrow H^{2-i}(\mathcal{O}_{v,n}T_1)^D$ is surjective because $H^{2-i}(\mathcal{O}_{v,n}T_1) \rightarrow H^{2-i}(K_{v,n}T_1)$ is an inclusion (see [HSS15, Proposition 1.2]) of finite groups. The exactness of the middle column follows from a diagram chase. \square

Corollary 2.3. *For each $i \in \mathbb{Z}$, the following pairing of locally compact topological groups induced by the local dualities is perfect*

$$\mathbb{P}^i(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{P}^{-i}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. This is an immediate consequence of Proposition 2.1 and Proposition 2.2. \square

2.2. Global dualities

We begin with an Artin–Verdier style duality result which plays a role in the proof of the global duality

$$\mathbb{H}^i(C \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{H}^{1-i}(C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}$$

for $-1 \leq i \leq 2$. We quote the following proposition [Izq16, pp. 70, I.4.4] for convenience and completeness.

Proposition 2.4 (Artin–Verdier duality). *Let $U \subset X_0$ be a non-empty open subset. For $i \in \mathbb{Z}$, the following is a perfect pairing between finite groups*

$$\mathbb{H}^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{H}_c^{1-i}(U, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Another input for the proof of global duality is the key exact sequence (3) below. We need the following results to assure its exactness.

Lemma 2.5. *There are canonical isomorphisms $\mathbb{H}^i(K_v^h, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ for all $i \geq -1$.*

Proof. Let F be a finite étale commutative group scheme over K . Note that F is locally constant in the étale topology and that K_v^h and K_v have the same absolute Galois group, therefore $H^i(K_v^h, F) \simeq H^i(K_v, F)$ for any $i \in \mathbb{Z}$. Now the result follows thanks to the distinguished triangle ${}_nT_1 \rightarrow {}_nT_2 \rightarrow C \otimes^{\mathbf{L}} \mathbb{Z}/n[-1] \rightarrow {}_nT_1[1]$ by dévissage. \square

The following proposition is proved in [Tia21, Proposition 1.12]. Since it frequently plays a role in the demonstrations later, we quote it here for

convenience of the reader. Although loc. cit. only deals with the case $\mathcal{A} = \mathcal{C}$, the same argument works for $\mathcal{A} = \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n$ as well.

Proposition 2.6. *Let $U \subset X_0$ be a non-empty open subset. Let \mathcal{A} be either \mathcal{C} or $\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n$.*

- (1) *Let $V \subset U$ be a further non-empty open subset. We have an exact sequence*

$$\cdots \rightarrow \mathbb{H}_c^i(V, \mathcal{A}) \rightarrow \mathbb{H}_c^i(U, \mathcal{A}) \rightarrow \bigoplus_{v \in U \setminus V} \mathbb{H}^i(\kappa(v), i_v^* \mathcal{A}) \rightarrow \mathbb{H}_c^{i+1}(V, \mathcal{A}) \rightarrow \cdots$$

where $i_v : \text{Spec } \kappa(v) \rightarrow U$ is the closed immersion.

- (2) *We have an exact sequence for $i \geq 1$ if $\mathcal{A} = \mathcal{C}$, and for $i \geq -1$ if $\mathcal{A} = \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n$:*

$$\cdots \rightarrow \mathbb{H}_c^i(U, \mathcal{A}) \rightarrow \mathbb{H}^i(U, \mathcal{A}) \rightarrow \bigoplus_{v \notin U} \mathbb{H}^i(K_v^h, \mathcal{A}) \rightarrow \mathbb{H}_c^{i+1}(U, \mathcal{A}) \rightarrow \cdots$$

where by abuse of notation we write \mathcal{A} for the pull-back of \mathcal{A} by the natural morphism $\text{Spec } K_v^h \rightarrow U$.

- (3) *We have an exact sequence for $i \geq 1$ if $\mathcal{A} = \mathcal{C}$, and for $i \geq -1$ if $\mathcal{A} = \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n$:*

$$\cdots \rightarrow \mathbb{H}_c^i(U, \mathcal{A}) \rightarrow \mathbb{H}^i(U, \mathcal{A}) \rightarrow \bigoplus_{v \notin U} \mathbb{H}^i(K_v, \mathcal{A}) \rightarrow \mathbb{H}_c^{i+1}(U, \mathcal{A}) \rightarrow \cdots$$

- (4) (Three Arrows Lemma). *Let $V \subset U$ be a further non-empty open subset. We have a commutative diagram*

$$\begin{array}{ccc} \mathbb{H}_c^i(V, \mathcal{A}) & \longrightarrow & \mathbb{H}_c^i(U, \mathcal{A}) \\ \downarrow & & \downarrow \\ \mathbb{H}^i(V, \mathcal{A}) & \longleftarrow & \mathbb{H}^i(U, \mathcal{A}). \end{array}$$

Put $\mathbb{D}_K^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) := \text{Im}(\mathbb{H}_c^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^i(K, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n))$. Now we arrive at the key exact sequence for the proof of global duality between the respective Tate–Shafarevich groups of $\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n$ and $\mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n$.

Proposition 2.7. *The following is an exact sequence for $-1 \leq i \leq 1$*

$$(3) \quad \bigoplus_{v \in X^{(1)}} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}_c^{i+1}(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{D}_K^{i+1}(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow 0.$$

Proof. We can construct a map

$$\bigoplus_{v \in X^{(1)}} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}_c^{i+1}(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$$

in a similar way as [HS16, pp. 11]. Let us recall the construction for the convenience of the readers. Suppose that $\alpha \in \bigoplus_{v \in X^{(1)}} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ lies in $\bigoplus_{v \notin V} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ for some non-empty open subset V of U . By Proposition 2.6(3), we can send α to $\mathbb{H}_c^{i+1}(V, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ and hence to $\mathbb{H}_c^{i+1}(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ by covariant functoriality of $\mathbb{H}_c^{i+1}(-, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$. The following commutative diagram for $W \subset V$

$$\begin{array}{ccc} \bigoplus_{v \notin W} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}_c^{i+1}(W, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \\ \uparrow & & \downarrow \\ \bigoplus_{v \notin V} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}_c^{i+1}(V, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \end{array}$$

shows that the construction does not depend on the choice of V . Finally, the sequence (3) is a complex by Proposition 2.6(3) and the square in diagram (4) below commutes by the same argument as in the proof of [HS16, Proposition 4.2].

Conversely, take $\alpha \in \text{Ker}(\mathbb{H}_c^{i+1}(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{D}_K^{i+1}(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n))$. Let $V \subset U$ be a non-empty open subset. We consider the following diagram for $-1 \leq i \leq 1$:

$$(4) \quad \begin{array}{ccccc} \mathbb{H}_c^{i+1}(V, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}_c^{i+1}(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \bigoplus_{v \in U \setminus V} \mathbb{H}^{i+1}(\kappa(v), C \otimes^{\mathbf{L}} \mathbb{Z}/n) \\ & & \downarrow & & \downarrow \\ & & \mathbb{H}^{i+1}(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \bigoplus_{v \in U \setminus V} \mathbb{H}^{i+1}(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n). \end{array}$$

The upper row is exact by Proposition 2.6(1). The left vertical arrow is just the composition

$$\mathbb{H}_c^{i+1}(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^{i+1}(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^{i+1}(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n).$$

The right vertical arrow is given by the composition

$$\mathbb{H}^{i+1}(\kappa(v), C \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^{i+1}(\mathcal{O}_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^{i+1}(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n).$$

By Lemma 1.1, the right vertical arrow in diagram (4) is injective.

Finally, thanks to the exactness of the upper row in diagram (4), α comes from an element $\beta \in \mathbb{H}_c^{i+1}(V, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ by diagram chasing. Since β goes to zero in $\mathbb{H}^{i+1}(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$, we may choose V sufficiently small such that β already maps to zero in $\mathbb{H}^{i+1}(V, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$. Now the proof is completed by Proposition 2.6(3). \square

Let

$$\mathbb{H}^i(C \otimes^{\mathbf{L}} \mathbb{Z}/n) := \text{Ker} \left(\mathbb{H}^i(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \prod_{v \in X^{(1)}} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \right).$$

Now we construct a perfect pairing $\mathbb{H}^i(C \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{H}^{1-i}(C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}$ of finite groups for $i = -1, 0$.

Theorem 2.8. *The following is a perfect pairing of finite groups for each $i \in \mathbb{Z}$:*

$$\mathbb{H}^i(C \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{H}^{1-i}(C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. Thanks to the distinguished triangle ${}_nT_1[1] \rightarrow {}_nT_2[1] \rightarrow C \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow {}_nT_1[2]$, we see that $\mathbb{H}^{-2}(C \otimes^{\mathbf{L}} \mathbb{Z}/n) = 0$ by the injectivity of ${}_nT_1(K) \rightarrow {}_nT_1(K_v)$. For $i \leq -3$ and $i \geq 3$, we have $H^i(K, {}_nT_j) = 0$ for $i \geq 4$ and $j = 1, 2$ by Remark 1.3(1) and the Kummer sequences. Now it follows that $\mathbb{H}^i(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) = 0$ for $i \leq -3$ and $i \geq 3$ by the above distinguished triangle and dévissage. In particular, $\mathbb{H}^i(C \otimes^{\mathbf{L}} \mathbb{Z}/n) = 0$ for $i \leq -3$ and $i \geq 3$. Thus it will be sufficient to consider the cases $-1 \leq i \leq 2$.

Clearly by symmetry it will be sufficient to consider $i = -1, 0$. We define $D_{\text{sh}}^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ to be the kernel of the last arrow of the upper row in the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & D_{\text{sh}}^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \prod_{v \in X^{(1)}} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{D}_K^{1-i}(U, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & \longrightarrow & \mathbb{H}_c^{1-i}(U, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & \longrightarrow & \left(\bigoplus_{v \in X^{(1)}} \mathbb{H}^{-i}(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \right)^D. \end{array}$$

The middle vertical arrow is an isomorphism by Proposition 2.4 and that for the right one by Proposition 2.1. It follows that the left vertical arrow is

an isomorphism as well. Since the group $\mathbb{H}_c^{1-i}(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is finite and the functor $\mathbb{H}_c^{1-i}(-, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is covariant, $\{\mathbb{D}_K^{1-i}(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)\}_{U \subset X_0}$ forms a decreasing family of finite abelian groups. Hence there exists a non-empty open subset $U_0 \subset X_0$ such that $\mathbb{D}_K^{1-i}(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) = \mathbb{D}_K^{1-i}(U_0, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ for all non-empty open subset $U \subset U_0$ (here we used Proposition 2.6(3) implicitly). We deduce that $\mathbb{D}_K^{1-i}(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^{1-i}(\mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ for all $U \subset U_0$ by Proposition 2.6. Now passing to the direct limit over all U of the isomorphism $D_{\text{sh}}^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^{1-i}(\mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D$ yields $\mathbb{H}^i(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \varinjlim_n D_{\text{sh}}^i(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^{1-i}(\mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D$. \square

2.3. The Poitou–Tate sequence

Lemma 2.9. *Let \mathcal{A} be either \mathcal{C} or $\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n$ over $U \subset X_0$ and let A be its generic fibre. For $V \subset U$, if $\alpha \in \mathbb{H}^i(V, \mathcal{A})$ is such that $\alpha_v \in \mathbb{H}^i(K_v, A)$ belongs to $\mathbb{H}^i(\mathcal{O}_v, \mathcal{A})$ for all $v \in U \setminus V$, then $\alpha \in \text{Im}(\mathbb{H}^i(U, \mathcal{A}) \rightarrow \mathbb{H}^i(V, \mathcal{A}))$ for $i \in \mathbb{Z}$.*

Proof. The localization sequences [Fu11, Proposition 5.6.11] for the respective pairs of open immersions $V \subset U$ and $\text{Spec } K_v \subset \text{Spec } \mathcal{O}_v$ (actually here we do the same argument as loc. cit. by replacing injective resolutions by injective Cartan–Eilenberg resolutions) together with [Mil80, pp. 93, 1.28] induce the following commutative diagram with exact rows²

$$\begin{array}{ccccc}
 \mathbb{H}^i(U, \mathcal{A}) & \longrightarrow & \mathbb{H}^i(V, \mathcal{A}) & \longrightarrow & \bigoplus_{v \in U \setminus V} \mathbb{H}_v^{i+1}(\mathcal{O}_v^h, \mathcal{A}) \\
 \downarrow & & \downarrow & & \downarrow \\
 \bigoplus_{v \in U \setminus V} \mathbb{H}^i(\mathcal{O}_v, \mathcal{A}) & \longrightarrow & \bigoplus_{v \in U \setminus V} \mathbb{H}^i(K_v, A) & \longrightarrow & \bigoplus_{v \in U \setminus V} \mathbb{H}_v^{i+1}(\mathcal{O}_v, \mathcal{A}).
 \end{array}$$

By [DH18, Lemma 2.6] the right vertical map is an isomorphism, so a diagram chasing yields the desired result. \square

²Here we have used the fact that $\text{Cone}(\mathbf{R}\Gamma_{U \setminus V} \rightarrow \mathbf{R}\Gamma_U)$ agrees with $\mathbf{R}\Gamma_V$. Indeed, it suffices to show that Cone preserves triangles and satisfies the desired universal property. It preserves triangles since it forms a functor on the derived category and it satisfies the desired universal property because it agrees with $\mathbf{R}\Gamma_V$ for sheaves by [Mil80, pp. 93, 1.28]. Therefore we can pass from a single sheaf to a short complex.

Lemma 2.10. *There are exact sequences for $n \geq 1$ and $-2 \leq i \leq 2$:*

$$\mathbb{H}^i(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{P}^i(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^{-i}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D.$$

Proof. For $V \subset U \subset X_0$ and $-1 \leq i \leq 2$, we obtain an exact sequence

$$\mathbb{H}^i(V, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \prod_{v \notin V} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}_c^{i+1}(V, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$$

by Proposition 2.6(3). Subsequently the following is an exact sequence by Lemma 2.9

$$\begin{aligned} \mathbb{H}^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) &\rightarrow \prod_{v \notin U} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \prod_{v \in U \setminus V} \mathbb{H}^i(\mathcal{O}_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \\ &\rightarrow \mathbb{H}_c^{i+1}(V, C \otimes^{\mathbf{L}} \mathbb{Z}/n). \end{aligned}$$

By Artin–Verdier duality 2.4, we obtain an isomorphism $\mathbb{H}_c^{i+1}(V, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^{-i}(V, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D$. Taking inverse limit over V then yields an exact sequence

$$\begin{aligned} \mathbb{H}^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) &\rightarrow \prod_{v \notin U} \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \prod_{v \in U} \mathbb{H}^i(\mathcal{O}_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \\ &\rightarrow \mathbb{H}^{-i}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D. \end{aligned}$$

Now we conclude the desired exact sequence by taking direct limit over U .

In particular, we obtain an exact sequence $\mathbb{H}^2(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{P}^2(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^{-2}(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)^D$ by applying the case $i = 2$ to C' . It follows that there are exact sequences

$$\mathbb{H}^{-2}(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{P}^{-2}(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^2(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D$$

by dualizing the above sequence of discrete abelian groups (recall that double dual of a finite abelian group is itself). □

To close this section, we summarize all the above arithmetic dualities into a 15-term exact sequence as follows.

Theorem 2.11. *The following is a 15-term exact sequence for $n \geq 1$*

$$\begin{aligned}
 (5) \quad & 0 \longrightarrow \mathbb{H}^{-2}(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{P}^{-2}(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{H}^2(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D \\
 & \longrightarrow \mathbb{H}^{-1}(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{P}^{-1}(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{H}^1(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D \\
 & \longrightarrow \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D \\
 & \longrightarrow \mathbb{H}^1(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{P}^1(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{H}^{-1}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D \\
 & \longrightarrow \mathbb{H}^2(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{P}^2(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{H}^{-2}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D \longrightarrow 0
 \end{aligned}$$

Proof. We have seen that $\mathbb{H}^{-2}(C \otimes^{\mathbf{L}} \mathbb{Z}/n) = 0$ in Theorem 2.8, and it follows that the first arrow is injective. The surjectivity of the last arrow follows by dualizing the injective map $\mathbb{H}^{-2}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{P}^{-2}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$. The exactness at each $\mathbb{P}^i(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ for $-2 \leq i \leq 2$ is proved in Lemma 2.10.

Next, we show that the map $\mathbb{H}^i(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{P}^i(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ has discrete image for $-1 \leq i \leq 2$. Since $\mathbb{P}^2(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ itself is discrete, there is nothing to do. For $i = 0, \pm 1$, suppose that $\alpha \in \text{Im}(\mathbb{H}^i(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{P}^i(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n))$ lies in $\prod_{v \notin U} \mathbb{H}^i(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \prod_{v \in U} \mathbb{H}^i(\mathcal{O}_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ for some $U \subset X_0$. Then α comes from the finite group $\mathbb{H}^i(U, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ by Lemma 2.9. Now dualizing the exact sequences $0 \rightarrow \mathbb{H}^i(C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^i(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{P}^i(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ (together with Corollary 2.3 and Theorem 2.8) yields the exactness at all the remaining terms. \square

3. Results for the complex C

3.1. Global duality: preliminaries

In this subsection, we establish an Artin–Verdier style duality and some local duality results. We begin with a list of properties of abelian groups under consideration. Recall that $C = [T_1 \xrightarrow{\rho} T_2]$.

Lemma 3.1. *Let P be a K -torus that extends to a U_0 -torus \mathcal{P} for some sufficiently small non-empty open subset U_0 of X . Let U be a non-empty open subset of U_0 . Let L be either K or K_v .*

- (1) *The torsion groups $\mathbb{H}^1(U, \mathcal{C})_{\text{tors}}$ and $\mathbb{H}_c^1(U, \mathcal{C})_{\text{tors}}$ are of cofinite type.*
- (2) *For $i \geq 2$, the groups $\mathbb{H}^i(U, \mathcal{C})$ and $\mathbb{H}_c^i(U, \mathcal{C})$ are torsion of cofinite type.*

- (3) *The group $H^1(K, P)$ has finite exponent and the group $H^1(K_v, P)$ is finite. Moreover, $H^i(L, P) = 0$ for $i \geq 3$. Finally, the groups $\mathbb{H}^i(P)$ are finite for each $i \geq 0$.*
- (4) *Let Φ be a group of multiplicative type over K . Then the groups $H^1(L, \Phi)$ and $H^3(L, \Phi)$ have finite exponents.*
- (5) *Suppose that $M := \text{Ker } \rho$ is finite. Then the groups $\mathbb{H}^{-1}(K_v, C)$ have a common finite exponent for all $v \in X^{(1)}$. Moreover, the groups $\mathbb{H}^{-1}(K, C)$ and $\mathbb{H}^1(K, C)$ are torsion of finite exponent.*
- (6) *Suppose that $T := \text{coker } \rho$ is trivial. The groups $\mathbb{H}^0(K, C)$ and $\mathbb{H}^2(K, C)$ are torsion of finite exponent.*

Proof. For (1) and (2), see [Tia21, Lemma 1.1].

- (3) First of all, we observe for $L = K$ or K_v that $H^1(L, P)$ has finite exponent by Hilbert’s Theorem 90 and a restriction-corestriction argument. The group $H^3(K, P)$ is the direct limit of the groups $H^3(V, \mathcal{P})$ for $V \subset U_0$, but by [SvH03, Corollary 4.10] $H^3(V, \mathcal{P}) = 0$ for V sufficiently small and so $H^3(K, P) = 0$. For the group $H^3(K_v, P)$, we deduce that $H^3(K_v, P) \simeq \varinjlim_n H^3(K_v, {}_n P)$ from the Kummer sequence. Thus it suffices to show that $\varprojlim_n H^0(K_v, {}_n P') = 0$ by [HS16, (10)]. Note that $(K_v^\times)_{\text{tors}} = (\kappa(v)^\times)_{\text{tors}}$ is finite, so $H^0(K_v, P')_{\text{tors}}$ is finite as well by a restriction-corestriction argument. As a consequence, $H^0(K_v, {}_n P')$ has a common finite exponent for each n and it follows that the limit $\varprojlim_n H^0(K_v, {}_n P') = 0$ vanishes. For cohomological dimension reasons, we have $H^i(L, {}_n P) = 0$ for $i \geq 4$. Subsequently, we see that $H^i(L, P) \simeq \varinjlim_n H^i(L, {}_n P) = 0$ for $i \geq 4$ where the first isomorphism follows from the Kummer sequence $0 \rightarrow H^{i-1}(L, P)/n \rightarrow H^i(L, {}_n P) \rightarrow {}_n H^i(L, P) \rightarrow 0$. The remaining claims are proved in [Tia21, Lemma 1.1].
- (4) Embed Φ into a short exact sequence $0 \rightarrow P \rightarrow \Phi \rightarrow F \rightarrow 0$ where P is an L -torus and F is a finite étale commutative group scheme. Thus we obtain an exact sequence $H^i(L, P) \rightarrow H^i(L, \Phi) \rightarrow H^i(L, F)$ for $i \geq 1$. By dévissage, it follows that $H^i(L, \Phi)$ has finite exponent by (3) for $i = 1, 3$.
- (5) Note that we have an isomorphism $\mathbb{H}^{-1}(K_v, C) \simeq \mathbb{H}^0(K_v, M)$ thanks to the distinguished triangle $M[1] \rightarrow C \rightarrow T \rightarrow M[2]$. Since M is finite by assumption, $\mathbb{H}^{-1}(K_v, C)$ has a common finite exponent for each $v \in X^{(1)}$. The group $\mathbb{H}^{-1}(K, C)$ has finite exponent for the same reason.

Thanks to the exact sequence $H^2(K, M) \rightarrow \mathbb{H}^1(K, C) \rightarrow H^1(K, T)$, we deduce that $\mathbb{H}^1(K, C)$ has finite exponent by dévissage using (3).

(6) In this case, the short complex C is quasi-isomorphic to $M[1]$. Thus the desired results follow from (4). □

Remark 3.2. Note that the finiteness of $\text{Ker } \rho$ is equivalent to the finiteness of $\text{Coker}(\widehat{T}_2 \rightarrow \widehat{T}_1)$, and hence it is equivalent to the injectivity of $\widetilde{T}_1 \rightarrow \widetilde{T}_2$. Therefore the finiteness of $\text{Ker } \rho$ amounts to saying that $\rho' : T'_2 \rightarrow T'_1$ is surjective, and vice versa. By Lemma 3.1(5,6), we see that

- If $\text{Ker } \rho$ is finite, then $\mathbb{H}^{-1}(K, C)$, $\mathbb{H}^0(K, C')$, $\mathbb{H}^1(K, C)$ and $\mathbb{H}^2(K, C')$ are torsion of finite exponent.
- If $\text{Coker } \rho$ is trivial, then $\mathbb{H}^{-1}(K, C')$, $\mathbb{H}^0(K, C)$, $\mathbb{H}^1(K, C')$ and $\mathbb{H}^2(K, C)$ are torsion of finite exponent.

3.2. Global duality

The goal of this section is to establish global duality results. We begin with the finiteness of $\mathbb{I}\mathbb{I}\mathbb{I}^0(C)$ and $\mathbb{I}\mathbb{I}\mathbb{I}^2(C)$ (and that of $\mathbb{I}\mathbb{I}\mathbb{I}^0(C')$ and $\mathbb{I}\mathbb{I}\mathbb{I}^2(C')$ by symmetry).

Lemma 3.3. *The groups $\mathbb{I}\mathbb{I}\mathbb{I}^0(C)$ and $\mathbb{I}\mathbb{I}\mathbb{I}^2(C)$ are finite.*

Proof. Let L be K or K_v for $v \in X^{(1)}$. We consider the distinguished triangle

$$(6) \quad M[1] \rightarrow C \rightarrow T \rightarrow M[2]$$

over L . By Lemma 3.1(4), the groups $H^1(L, M)$ and $H^3(L, M)$ have finite exponents.

- The distinguished triangle (6) yields exact sequences $H^3(L, M) \rightarrow \mathbb{H}^2(L, C) \rightarrow H^2(L, T) \rightarrow H^4(L, M)$. Note that $\mathbb{I}\mathbb{I}\mathbb{I}^2(T)$ is finite by Lemma 3.1(3). In particular, $\mathbb{I}\mathbb{I}\mathbb{I}^2(C)$ has finite exponent by dévissage and it remains to show that $\mathbb{I}\mathbb{I}\mathbb{I}^2(C)$ is of cofinite type. Since $\mathbb{H}^2(K, C)$ is the direct limit of $\mathbb{H}^2(U, C)$, each $\alpha \in \mathbb{I}\mathbb{I}\mathbb{I}^2(C)$ comes from some $\mathbb{H}^2(U, C)$ with U a non-empty open subset of X_0 . In particular, α lies in the image of $\mathbb{H}^2_c(U, C)$ by Proposition 2.6(3). We conclude that α comes from $\mathbb{H}^2_c(X_0, C)$ by the Three Arrows Lemma and hence $\mathbb{I}\mathbb{I}\mathbb{I}^2(C)$ is a subquotient of $\mathbb{H}^2_c(X_0, C)$ (which is torsion of cofinite type by Lemma 3.1(2)). As a consequence, $\mathbb{I}\mathbb{I}\mathbb{I}^2(C)$ is of cofinite type. Therefore $\mathbb{I}\mathbb{I}\mathbb{I}^2(C)$ is finite.

- The exact sequence $0 \rightarrow H^1(K, M) \rightarrow \mathbb{H}^0(K, C) \rightarrow H^0(K, T)$ obtained from (6) yields an isomorphism $\mathbb{III}^1(M) \simeq \mathbb{III}^0(C)$ as $\mathbb{III}^0(T) = 0$. Since we may embed M into a short exact sequence $0 \rightarrow M \rightarrow P_1 \rightarrow P_2 \rightarrow 0$ with P_1 and P_2 being K -tori, there is a quasi-isomorphism $M[1] \simeq [P_1 \rightarrow P_2]$. Subsequently, an analogous argument as above implies that $\mathbb{III}^1(M) \subset \text{Im}(H_c^1(X_0, \mathcal{M}) \rightarrow H^1(K, M))$. But this map factors through $H_c^1(X_0, \mathcal{M})/N \rightarrow H^1(K, M)$ because $H^1(K, M)$ has finite exponent for some positive integer N by Lemma 3.1(4). Finally, $H_c^1(X_0, \mathcal{M})/N$ injects into the finite group $H_c^1(X_0, \mathcal{M} \otimes^{\mathbf{L}} \mathbb{Z}/N)$ (we have seen its finiteness in Proposition 2.4) thanks to the distinguished triangle $M \rightarrow M \rightarrow M \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow M[1]$, so it is finite as well. Hence $\mathbb{III}^1(M) \simeq \mathbb{III}^0(C)$ is contained in this finite image which completes the proof. \square

Remark 3.4. In fact, all non-trivial Tate–Shafarevich groups of the complex C are finite:

- $\mathbb{III}^1(C)$ is a finite group. This fact is more complicated because the group $\mathbb{H}_c^1(U, C)$ needs not be torsion. See [Tia21, Proposition 1.14] for a proof.
- $\mathbb{III}^i(C) = 0$ for $i \leq -1$ and $i \geq 3$ by Remark 1.3(1) thanks to the distinguished triangle $T_1 \rightarrow T_2 \rightarrow C \rightarrow T_1[1]$.

Now we prove a first global duality result between the finite groups $\mathbb{III}^0(C)$ and $\mathbb{III}^2(C')$. Let $\mathbb{D}_K^i(U, C) := \text{Im}(\mathbb{H}_c^i(U, C) \rightarrow \mathbb{H}^i(K, C))$. The first input is the following exact sequence constructed as [Tia21, Proposition 1.17]

$$(7) \quad \bigoplus_{v \in X^{(1)}} \mathbb{H}^1(K_v, C) \rightarrow \mathbb{H}_c^2(U, C) \rightarrow \mathbb{D}_K^2(U, C) \rightarrow 0.$$

In our situation, the exactness is guaranteed by Lemma 1.2 below instead of [Tia21, Corollary 1.16] there.

Now we arrive at:

Theorem 3.5. *The following is a perfect pairing of finite groups:*

$$\mathbb{III}^0(C') \times \mathbb{III}^2(C) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. We first identify $\mathbb{III}^2(C)$ with the image of $\mathbb{H}_c^2(U, C)$ in $\mathbb{H}^2(K, C)$ for U sufficiently small. Since $\mathbb{H}_c^2(U, C)$ is torsion of cofinite type by Lemma 3.1, so is $\mathbb{D}_K^2(U, C)$. Hence the decreasing family $\{\mathbb{D}_K^2(U, C)\{\ell\}\}_{U \subset X_0}$ of ℓ -primary torsion groups must be stable by [HS16, Lemma 3.7]. Let us say

$\mathbb{D}_K^2(U, \mathcal{C})\{\ell\} = \mathbb{D}_K^2(U_0, \mathcal{C})\{\ell\}$ for some open subset $U_0 \subset X_0$ and for each non-empty open subset $U \subset U_0$. Letting U run through all non-empty open subsets of U_0 , we conclude $\mathbb{D}_K^2(U_0, \mathcal{C})\{\ell\} = \text{III}^2(C)\{\ell\}$ by Proposition 2.6(3).

Let $\mathbb{D}_{\text{sh}}^0(U, \mathcal{C}')$ be the kernel of $\mathbb{H}^0(U, \mathcal{C}') \rightarrow \prod_{v \in X^{(1)}} \mathbb{H}^0(K_v, \mathcal{C}')$. According to [Tia21, Proposition 1.2], there is a pairing

$$\mathbb{H}^i(U, \mathcal{C})\{\ell\} \times \mathbb{H}_c^{2-i}(U, \mathcal{C}')^{(\ell)} \rightarrow \mathbb{Q}/\mathbb{Z}$$

with divisible left kernel. We consider now the following commutative diagram with exact rows (the lower row is exact by the exactness of (7) and [Tia21, Lemma 1.7])

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{D}_{\text{sh}}^0(U, \mathcal{C}')\{\ell\} & \longrightarrow & \mathbb{H}^0(U, \mathcal{C}')\{\ell\} & \longrightarrow & \left(\prod_{v \in X^{(1)}} \mathbb{H}^0(K_v, \mathcal{C}') \right)\{\ell\} \\ & & \downarrow \Phi_U & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (\mathbb{D}_K^2(U, \mathcal{C})^{(\ell)})^D & \longrightarrow & (\mathbb{H}_c^2(U, \mathcal{C})^{(\ell)})^D & \longrightarrow & \left(\bigoplus_{v \in X^{(1)}} \mathbb{H}^1(K_v, \mathcal{C}) \right)^{(\ell)D} \end{array}$$

with the left vertical arrow Φ_U obtained from the commutativity of the right square. Moreover, the right vertical arrow is an isomorphism by local duality [Tia21, Remark 1.7] and the middle one is surjective with divisible kernel by [Tia21, Proposition 1.2] (the surjectivity is ensured by the proof of loc. cit.). It follows that the left vertical arrow is surjective and $\text{Ker } \Phi_U$ is divisible. In particular, $\varinjlim_U \text{Ker } \Phi_U$ is also divisible. But $\varinjlim_U \mathbb{D}_{\text{sh}}^0(U, \mathcal{C}')\{\ell\} \simeq \text{III}^0(C')\{\ell\}$ is finite, therefore it does not contain a non-trivial divisible subgroup, i.e. $\varinjlim_U \text{Ker } \Phi_U = 0$ is trivial. Consequently, we obtain the following identifications:

$$\text{III}^0(C')\{\ell\} \simeq \varinjlim_U \mathbb{D}_{\text{sh}}^0(U, \mathcal{C}')\{\ell\} \simeq \varinjlim_U (\mathbb{D}_K^2(U, \mathcal{C})^{(\ell)})^D.$$

Recall that $\text{III}^2(C)$ is finite and $\mathbb{D}_K^2(U, \mathcal{C})\{\ell\} = \text{III}^2(C)\{\ell\}$ for U sufficiently small. So $\mathbb{D}_K^2(U, \mathcal{C})^{(\ell)} \simeq \text{III}^2(C)\{\ell\}^{(\ell)} \simeq \text{III}^2(C)\{\ell\}$, and it follows that $\text{III}^0(C')\{\ell\} \rightarrow \text{III}^2(C)\{\ell\}^D$ is an isomorphism. Since $\text{III}^0(C')$ and $\text{III}^2(C)$ are finite, they are finite direct sums of ℓ -primary parts and therefore $\text{III}^0(C') \times \text{III}^2(C) \rightarrow \mathbb{Q}/\mathbb{Z}$ is a perfect pairing. \square

Remark 3.6. Taking [Tia21, Theorem 1.18] into account, we see that the pairing $\text{III}^i(C) \times \text{III}^{2-i}(C') \rightarrow \mathbb{Q}/\mathbb{Z}$ is perfect for each integer i .

To connect the first two rows in the Poitou–Tate sequence (1), we shall need an additional global duality concerning inverse limits.

Theorem 3.7. *Put $\text{III}_\wedge^0(C) := \text{Ker}(\mathbb{H}^0(K, C)_\wedge \rightarrow \mathbb{P}^0(K, C)_\wedge)$. If $\text{Ker } \rho$ is finite, then the following is a perfect pairing of finite groups*

$$\text{III}_\wedge^0(C) \times \text{III}^2(C') \rightarrow \mathbb{Q}/\mathbb{Z}.$$

The rest of this section is devoted to the proof of Theorem 3.7 which is analogous to that of [Dem09, pp. 86-88]. We proceed by reducing the question into various limits in finite level.

Lemma 3.8. *Let $C = [T_1 \xrightarrow{\rho} T_2]$ (here $\text{Ker } \rho$ is not necessarily finite). The natural map is an isomorphism:*

$$\text{III}_\wedge^0(C) \rightarrow \varprojlim_n \text{III}^0(C \otimes^{\mathbf{L}} \mathbb{Z}/n).$$

Proof. Consider the Kummer exact sequences

$$0 \rightarrow \mathbb{H}^0(K_v, C)/n \rightarrow \mathbb{H}^0(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{H}^1(K_v, C) \rightarrow 0$$

for all $v \in X^{(1)}$, and

$$0 \rightarrow \mathbb{H}^0(\mathcal{O}_v, \mathcal{C})/n \rightarrow \mathbb{H}^0(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{H}^1(\mathcal{O}_v, \mathcal{C}) \rightarrow 0$$

for all $v \in X_0^{(1)}$. Moreover, the complex $0 \rightarrow \mathbb{P}^0(K, C)/n \rightarrow \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{P}^1(K, C) \rightarrow 0$ is an exact sequence by Lemma 1.4. Therefore there is a commutative diagram with exact rows by taking inverse limit over all n in the respective Kummer sequences

$$(8) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{H}^0(K, C)_\wedge & \longrightarrow & \varprojlim_n \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \Phi_K \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{P}^0(K, C)_\wedge & \longrightarrow & \varprojlim_n \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \Phi_\Pi \longrightarrow 0. \end{array}$$

where $\Phi_K \subset \varprojlim_n {}_n\mathbb{H}^1(K, C)$ and $\Phi_\Pi \subset \varprojlim_n {}_n\mathbb{P}^1(K, C)$ (here the inverse limit may not be right exact because the involved groups are infinite). Recall that $\text{III}^1(C)$ is finite (Remark 3.4). As a consequence, the kernel of the right

vertical arrow is contained in $\varprojlim_n \mathbb{H}^1(C) = 0$. Therefore there are isomorphisms

$$\begin{aligned} \mathbb{H}^0_\wedge(C) &\simeq \text{Ker} \left(\varprojlim_n \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \varprojlim_n \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \right) \\ &\simeq \varprojlim_n \mathbb{H}^0(C \otimes^{\mathbf{L}} \mathbb{Z}/n) \end{aligned}$$

by the snake lemma, as required. □

The following lemmas tell us that $\varprojlim_n \mathbb{H}^0(C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is an inverse limit of subgroups of $\mathbb{H}^0(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$.

Lemma 3.9. *Let \mathcal{F} be a finite étale commutative group scheme over X_0 . Then $\varprojlim_n H^i(X_0, {}_n\mathcal{F}) = 0$ for $i \geq 0$.*

Proof. Since \mathcal{F} is finite, $\mathcal{F} = {}_N\mathcal{F}$ for some positive integer N . Take $(x_n) \in \varprojlim_n H^i(X_0, {}_n\mathcal{F})$. Then $x_n = Nx_{Nn}$ for each positive integer n and it follows that $x_n = 0$, i.e. $\varprojlim_n H^i(X_0, {}_n\mathcal{F}) = 0$. □

Lemma 3.10. *Suppose that $\text{Ker } \rho$ is finite. Then $\varprojlim_n \mathbb{H}^0(X_0, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \varprojlim_n \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is injective.*

Proof. By Lemma 3.9, we have $\varprojlim_n H^i(X_0, {}_n\mathcal{M}) = 0$. Consider the distinguished triangle ${}_nM[2] \rightarrow C \otimes^{\mathbf{L}} \mathbb{Z}/n[1] \rightarrow T_{\mathbb{Z}/n}(C)[1] \rightarrow {}_nM[3]$. Thus

$$\varprojlim_n \mathbb{H}^0(X_0, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \varprojlim_n \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$$

is injective by dévissage (indeed, by the same argument of [Dem11a, Proposition 5.3(2)] we see that $H^1(X_0, T_{\mathbb{Z}/n}(C)) \rightarrow H^1(K, T_{\mathbb{Z}/n}(C))$ is injective). □

We put $\mathbb{D}^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) := \text{Im} (\mathbb{H}^i_c(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n))$. So there are inclusions

$$\varprojlim_n \mathbb{D}^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \subset \varprojlim_n \mathbb{H}^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \subset \varprojlim_n \mathbb{H}^i(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n).$$

If $V \subset U$ is an open subset, then $\varprojlim_n \mathbb{D}^i(V, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \subset \varprojlim_n \mathbb{D}^i(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$. In this way, we can take the inverse limit $\varprojlim_U \varprojlim_n \mathbb{D}^0(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ over all U .

Lemma 3.11. *Suppose that $\text{Ker } \rho$ is finite. The following map is an isomorphism*

$$\varprojlim_U \varprojlim_n \mathbb{D}^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \varprojlim_n \mathbb{H}^0(C \otimes^{\mathbf{L}} \mathbb{Z}/n)$$

with transition maps given by covariant functoriality of the functor $\mathbb{H}_c^0(-, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$.

Proof. Since

$$\varprojlim_n \mathbb{H}^0(X_0, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \varprojlim_n \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$$

is injective, we can take $\bigcap_{U \subset X_0} \varprojlim_n \mathbb{D}^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ in the inverse limit $\varprojlim_n \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$. It follows from the definition of \varprojlim_U that the intersection over $U \subset X_0$ coincides with inverse limit. By Proposition 2.6(1) and (3), we conclude that $\bigcap_{U \subset X_0} \varprojlim_n \mathbb{D}^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) = \varprojlim_n \mathbb{H}^0(C \otimes^{\mathbf{L}} \mathbb{Z}/n)$. \square

Next we describe $\mathbb{H}^2(C')$. Again we write L for K or K_v and consider the Kummer sequence $0 \rightarrow \mathbb{H}^1(L, C')/n \rightarrow \mathbb{H}^1(L, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{H}^2(L, C') \rightarrow 0$. Since $\mathbb{H}^i(L, C')$ is torsion for $i \geq 1$, taking the direct limit over all n yields an isomorphism $\mathbb{H}^1(L, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^2(L, C')$. In particular, we obtain $\mathbb{H}^1(\varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^2(C')$. Put

$$\begin{aligned} \mathbb{D}_{\text{sh}}^1(U, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n) &:= \text{Ker} \left(\mathbb{H}^1(U, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \right. \\ &\left. \rightarrow \prod_{v \in X^{(1)}} \varinjlim_n H^1(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \right). \end{aligned}$$

If $V \subset U$ is a smaller open subset, then there is a homomorphism $\mathbb{D}_{\text{sh}}^1(U, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{D}_{\text{sh}}^1(V, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ induced by the restriction maps $\mathbb{H}^1(U, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^1(V, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$. In particular, the direct limit $\varinjlim_U \mathbb{D}_{\text{sh}}^1(U, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ makes sense and is isomorphic to $\mathbb{H}^1(\varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ by construction. Consequently, we reduce Theorem 3.7 to showing that $\varprojlim_n \mathbb{D}^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{D}_{\text{sh}}^1(U, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}$ is a perfect pairing.

We shall need the following compatibility between local duality and Artin–Verdier duality.

Lemma 3.12. *Let U be a sufficiently small non-empty open subset of X_0 . The following is a commutative diagram*

$$(9) \quad \begin{array}{ccc} \bigoplus_{v \in X^{(1)}} \mathbb{H}^{-1}(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \times & \prod_{v \in X^{(1)}} \mathbb{H}^1(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{Q}/\mathbb{Z} \\ \downarrow & & \uparrow \parallel \\ \mathbb{H}_c^0(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \times & \mathbb{H}^1(U, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{Q}/\mathbb{Z} \end{array}$$

where the left vertical arrow is constructed analogous to the first arrow of (7), and the middle one is the composition

$$\mathbb{H}^1(U, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^1(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \prod \mathbb{H}^1(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n).$$

Proof. The proof is essentially the same as that of [CTH15, Proposition 4.3(f)]. We first observe by the same argument as *loc. cit.* that it will be sufficient to show the commutativity of diagram (9) when $v \notin U$. Since the duality pairings

$$\mathbb{H}^1(U, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{H}_c^0(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}$$

and

$$\mathbb{H}^1(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{H}^{-1}(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}$$

are induced by the pairings $\text{Ext}_U^1(C \otimes^{\mathbf{L}} \mathbb{Z}/n, \mu_n^{\otimes 2}) \times \mathbb{H}_c^0(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}$ and $\text{Ext}_{K_v}^1(C \otimes^{\mathbf{L}} \mathbb{Z}/n, \mu_n^{\otimes 2}) \times \mathbb{H}^{-1}(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}$ respectively by [Mil80, V, Proposition 1.20], it suffices to show the following diagram

$$\begin{array}{ccc} \text{Hom}_{K_v}(C \otimes^{\mathbf{L}} \mathbb{Z}/n, \mu_n^{\otimes 2}[1]) & \times & \mathbb{H}^{-1}(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{Q}/\mathbb{Z} \\ \uparrow & & \downarrow \parallel \\ \text{Hom}_U(C \otimes^{\mathbf{L}} \mathbb{Z}/n, \mu_n^{\otimes 2}[1]) & \times & \mathbb{H}_c^0(U, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \longrightarrow \mathbb{Q}/\mathbb{Z}, \end{array}$$

where the Hom are in the sense of respective derived categories. Take $\alpha_U \in \text{Hom}_U(C \otimes^{\mathbf{L}} \mathbb{Z}/n, \mu_n^{\otimes 2}[1])$ and let α_v be its image in $\text{Hom}_{K_v}(C \otimes^{\mathbf{L}} \mathbb{Z}/n, \mu_n^{\otimes 2}[1])$. Let $j_U : U \rightarrow X$ and $j_v : \text{Spec } K_v \rightarrow \text{Spec } \mathcal{O}_v$ be the respective open immersions. Recall that there is an isomorphism

$$\mathbb{H}^{-1}(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}_v^0(\mathcal{O}_v, j_{v!}(C \otimes^{\mathbf{L}} \mathbb{Z}/n))$$

(see the proof of [CTH15, Proposition 4.3(c)]). In view of the commutative diagram

$$\begin{CD}
 \mathbb{H}_v^0(\mathcal{O}_v, j_{v!}(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)) @<\simeq<< \mathbb{H}_v^0(X, j_{U!}(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)) @>> \mathbb{H}^0(X, j_{U!}(\mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)) \\
 @V(\alpha_v)_*VV @V(\alpha_v)_*VV @V(\alpha_v)_*VV \\
 \mathbb{H}_v^1(\mathcal{O}_v, j_{v!}(\mu_n^{\otimes 2})) @<\simeq<< \mathbb{H}_v^1(X, j_{U!}(\mu_n^{\otimes 2})) @>> \mathbb{H}^1(X, j_{U!}(\mu_n^{\otimes 2})),
 \end{CD}$$

we conclude the desired commutativity of diagram (9). □

Lemma 3.13. *Suppose that $M := \text{Ker } \rho$ is finite. Then the canonical map*

$$\mathbb{H}^1(U, \varinjlim_n \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \prod \varinjlim_n H^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$$

factors through $\bigoplus \varinjlim_n H^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \subset \prod \varinjlim_n H^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$.

Proof. Indeed, the finiteness of $\text{Ker } \rho$ is equivalent to the surjectivity of $\rho' : T'_2 \rightarrow T'_1$. Therefore we obtain a quasi-isomorphism $(\text{Ker } \rho')[1] \simeq \mathcal{C}'$. But $\text{Ker } \rho'$ is a group of multiplicative type, we may extend it to X_0 and embed it into a short exact sequence $0 \rightarrow \mathcal{P} \rightarrow \text{Ker } \rho' \rightarrow \mathcal{F} \rightarrow 0$ over \mathcal{O}_v for $v \in X_0^{(1)}$ where \mathcal{P} is a torus and \mathcal{F} is a finite étale commutative group scheme. Note that the groups $H^3(\mathcal{O}_v, \mathcal{P}) \simeq H^3(\kappa(v), \mathcal{P})$ and $H^3(\mathcal{O}_v, \mathcal{F}) \simeq H^3(\kappa(v), \mathcal{F})$ vanish, so we conclude that $H^3(\mathcal{O}_v, \text{Ker } \rho') = 0$. According to the distinguished triangle $\mathcal{C}' \rightarrow \mathcal{C}' \rightarrow \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow \mathcal{C}'[1]$, we have an identification

$$\mathbb{H}^2(\mathcal{O}_v, \mathcal{C}') \simeq \varinjlim_n \mathbb{H}^1(\mathcal{O}_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n).$$

But $\mathbb{H}^2(\mathcal{O}_v, \mathcal{C}') \simeq H^3(\mathcal{O}_v, \text{Ker } \rho') = 0$, it follows that $\varinjlim_n \mathbb{H}^1(\mathcal{O}_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) = 0$. Hence the image of $\varinjlim_n \mathbb{H}^1(U, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ in $\prod_v \varinjlim_n \mathbb{H}^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ lies in the subgroup $\bigoplus_v \varinjlim_n \mathbb{H}^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$. □

Since direct limits commute with direct sums, we obtain an exact sequence

$$\begin{aligned}
 0 \rightarrow \mathbb{D}_{\text{sh}}^1(U, \varinjlim_n \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) &\rightarrow \mathbb{H}^1(U, \varinjlim_n \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n) \\
 &\rightarrow \varinjlim_n \prod_v H^1(K_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)
 \end{aligned}$$

via

$$\begin{aligned} \bigoplus_v \varinjlim_n H^1(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) &\simeq \varinjlim_n \bigoplus_v H^1(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \\ &\subset \varinjlim_n \prod_v H^1(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n). \end{aligned}$$

Lemma 3.14. *There is a perfect pairing of abelian groups*

$$\varinjlim_n \mathbb{D}^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) \times \mathbb{D}_{\text{sh}}^1(U, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. We consider the following diagram with exact rows

$$(10) \quad \begin{array}{ccccc} 0 & \longrightarrow & \mathbb{D}_{\text{sh}}^1(U, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}^1(U, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \varinjlim_n \prod_v H^1(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \\ & & \uparrow & & \uparrow \simeq & & \uparrow * \\ 0 & \longrightarrow & \varinjlim_n (\mathbb{D}^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n))^D & \longrightarrow & \varinjlim_n (\mathbb{H}_c^0(U, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n))^D & \longrightarrow & \varinjlim_n (\bigoplus \mathbb{H}^{-1}(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n))^D \end{array}$$

where the lower row is exact by Proposition 2.7. Note that the arrow $*$ is constructed as the composition

$$\begin{aligned} \varinjlim_n \left(\bigoplus_v \mathbb{H}^{-1}(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \right)^D &\simeq \varinjlim_n \prod_v \left(\mathbb{H}^{-1}(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)^D \right) \\ &\simeq \varinjlim_n \prod_v \mathbb{H}^1(K_v, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \end{aligned}$$

where the last isomorphism follows from local dualities. The square in diagram (10) commutes by Lemma 3.12. Now a diagram chase shows that the left vertical arrow is an isomorphism. □

4. Poitou–Tate sequences

We begin with the topologies on $\mathbb{H}^i(K, C)$ and $\mathbb{P}^i(K, C)$.

- For each i , the groups $\mathbb{H}^i(K, C)$ are endowed with the discrete topology. The groups $\mathbb{H}^i(K, C)_\wedge$ are endowed with the subspace topology of the product $\prod_n \mathbb{H}^i(K, C)/n$. Its topology is not profinite since each component $\mathbb{H}^i(K, C)/n$ is not necessarily a finite group in general.
- For $i = -1, 0$, we give $\mathbb{P}^i(K, C)$ the restricted product topology. Moreover, the group $\mathbb{P}^i(K, C)_\wedge$ is equipped with the subspace topology of the product $\prod_n \mathbb{P}^i(K, C)/n$.

- For $i = 1, 2$, The group $\mathbb{P}^i(K, C)_{\text{tors}}$ is endowed with the direct limit topology. More precisely, ${}_n\mathbb{P}^i(K, C)$ is equipped with the restricted product topology with respect to the discrete topology on each ${}_n\mathbb{H}^i(K_v, C)$, and their direct limit $\mathbb{P}^i(K, C)_{\text{tors}}$ is equipped with the corresponding direct limit topology.

Now we arrive the main result of this paper.

Theorem 4.1. *Suppose either $\text{Ker } \rho$ is finite or $\text{Coker } \rho$ is trivial. Then the following is a 12-term functorial exact sequence of topological abelian groups*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{H}^{-1}(K, C)_{\wedge} & \longrightarrow & \mathbb{P}^{-1}(K, C)_{\wedge} & \longrightarrow & \mathbb{H}^2(K, C')^D \\
 & & \underbrace{\hspace{10em}} & & & & \\
 & & \longrightarrow & \mathbb{H}^0(K, C)_{\wedge} & \longrightarrow & \mathbb{P}^0(K, C)_{\wedge} & \longrightarrow & \mathbb{H}^1(K, C')^D \\
 & & \underbrace{\hspace{10em}} & & & & \\
 & & \longrightarrow & \mathbb{H}^1(K, C) & \longrightarrow & \mathbb{P}^1(K, C)_{\text{tors}} & \longrightarrow & (\mathbb{H}^0(K, C')_{\wedge})^D \\
 & & \underbrace{\hspace{10em}} & & & & \\
 & & \longrightarrow & \mathbb{H}^2(K, C) & \longrightarrow & \mathbb{P}^2(K, C)_{\text{tors}} & \longrightarrow & (\mathbb{H}^{-1}(K, C')_{\wedge})^D \longrightarrow 0
 \end{array}$$

where the map $H^1(K, C')^D \rightarrow H^1(K, C)$ is induced by the global duality pairing $\text{III}^1(C) \times \text{III}^1(C') \rightarrow \mathbb{Q}/\mathbb{Z}$ (see [Tia21, Theorem 1.18]).

The proof of the theorem consists of several steps. We first establish perfect pairings between the restricted topological products for any short complex C . Subsequently, we deduce the exactness of the first and the last rows again for any C . Finally, we deal with the more complicated exact sequence in the middle of the diagram with either $\text{Ker } \rho$ being finite or $\text{Coker } \rho$ being trivial.

Step 1: dualities between restricted topological products.

We proceed as in the finite level to obtain pairings between $\mathbb{P}^i(K, C)_{\wedge}$ and $\mathbb{P}^{1-i}(K, C')_{\text{tors}}$ for $i = -1, 0$. Recall Lemma 1.4 that $\mathbb{H}^i(\mathcal{O}_v, C)/n \rightarrow \mathbb{H}^i(K_v, C)/n$ is injective for each $v \in X_0^{(1)}$. Therefore we are allowed to identify $\mathbb{H}^i(\mathcal{O}_v, C)_{\wedge}$ with a subgroup of $\mathbb{H}^i(K_v, C)_{\wedge}$ for $v \in X_0^{(1)}$ by the left exactness of inverse limits. In this step, all the conclusions are valid without any assumption on $\text{Ker } \rho$ and $\text{Coker } \rho$.

Proposition 4.2. *For $i = -1, 0$, the annihilator of $\mathbb{H}^{1-i}(\mathcal{O}_v, \mathcal{C}')$ is $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C})_\wedge$ under the perfect pairing (see [Tia21, Remark 1.5])*

$$\mathbb{H}^i(K_v, C)_\wedge \times \mathbb{H}^{1-i}(K_v, C') \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. We consider the following commutative diagram with exact rows for $i = -1, 0$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{H}^i(\mathcal{O}_v, \mathcal{C})/n & \longrightarrow & \mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}^{i+1}(\mathcal{O}_v, \mathcal{C}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{H}^i(K_v, C)/n & \longrightarrow & \mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}^{i+1}(K_v, C). \end{array}$$

Take $t \in \mathbb{H}^i(K_v, C)/n$ such that t is orthogonal to ${}_n\mathbb{H}^{1-i}(\mathcal{O}_v, \mathcal{C}')$. Then the image s of t in $\mathbb{H}^i(K_v, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ is orthogonal to $\mathbb{H}^{-i}(\mathcal{O}_v, \mathcal{C}' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ and it follows that $s \in \mathbb{H}^i(\mathcal{O}_v, \mathcal{C} \otimes^{\mathbf{L}} \mathbb{Z}/n)$ by Proposition 2.2. The right vertical arrow is injective by Lemma 1.2, thus a diagram chase shows that t lies in $\mathbb{H}^i(\mathcal{O}_v, \mathcal{C})/n$. □

Corollary 4.3. *For $i = -1, 0$, there are isomorphisms $\mathbb{P}^i(K, C)_\wedge \simeq (\mathbb{P}^{1-i}(K, C')_{\text{tors}})^D$ of locally compact groups.*

Proof. This is an immediate consequence of Lemma 1.4 and Proposition 4.2. □

Step 2: exactness of the first and the last rows of diagram (1).

In this step, all the conclusions are valid without any assumption on $\text{Ker } \rho$ and $\text{Coker } \rho$.

Proposition 4.4. *The following is an exact sequence of locally compact groups*

$$(11) \quad 0 \rightarrow \text{III}^2(C) \rightarrow \mathbb{H}^2(K, C) \rightarrow \mathbb{P}^2(K, C)_{\text{tors}} \rightarrow (\mathbb{H}^{-1}(K, C')_\wedge)^D \rightarrow 0.$$

Proof. We consider the following commutative diagram with exact rows and exact middle column by Theorem 2.11:

$$(12) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{H}^1(K, C)/n & \longrightarrow & \mathbb{H}^1(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & {}_n\mathbb{H}^2(K, C) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{P}^1(K, C)/n & \longrightarrow & \mathbb{P}^1(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & {}_n\mathbb{P}^2(K, C) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & ({}_n\mathbb{H}^0(K, C'))^D & \longrightarrow & \mathbb{H}^{-1}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & \longrightarrow & (\mathbb{H}^{-1}(K, C')/n)^D \longrightarrow 0. \end{array}$$

Taking direct limit over all n of the last two columns in diagram (12) yields the commutative diagram

$$(13) \quad \begin{array}{ccccccc} \varinjlim_n \mathbb{H}^1(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \varinjlim_n \mathbb{P}^1(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & (\varprojlim_n \mathbb{H}^{-1}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n))^D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}^2(K, C) & \longrightarrow & \mathbb{P}^2(K, C)_{\text{tors}} & \longrightarrow & (\mathbb{H}^{-1}(K, C')_{\wedge})^D & & \end{array}$$

Since $\mathbb{H}^2(K, C)$ is torsion and $\mathbb{H}^3(K, C) = 0$ (thanks to Remark 1.3 and the distinguished triangle $T_1 \rightarrow T_2 \rightarrow C \rightarrow T_1[1]$), taking direct limit of the Kummer sequence

$$0 \rightarrow \mathbb{H}^2(K, C)/n \rightarrow \mathbb{H}^2(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{H}^3(K, C) \rightarrow 0.$$

yields $\varinjlim_n \mathbb{H}^2(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) = 0$. Taking direct limit in Theorem 2.11 yields the exactness of the upper row. The left vertical arrow in (13) is an isomorphism since $\mathbb{H}^1(K, C) \otimes \mathbb{Q}/\mathbb{Z} = 0$ and the middle one is surjective by the exactness of \varinjlim_n . If the right vertical arrow is an isomorphism, then a diagram chase yields the exact sequence (11).

So it remains to show $\varprojlim_n \mathbb{H}^{-1}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \simeq \mathbb{H}^{-1}(K, C')_{\wedge}$. We see that the vanishing $\varprojlim_n {}_n\mathbb{H}^0(K, C') = 0$ is enough because of the Kummer sequence

$$0 \rightarrow \mathbb{H}^{-1}(K, C')/n \rightarrow \mathbb{H}^{-1}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{H}^0(K, C') \rightarrow 0,$$

and therefore we reduce to show $\mathbb{H}^0(K, C')_{\text{tors}}$ has finite exponent. Indeed, since $H^0(K, \mathbb{G}_m)_{\text{tors}} = (K^\times)_{\text{tors}} = (k^\times)_{\text{tors}}$ is finite, we see that $H^0(K, P)_{\text{tors}}$ has finite exponent for a K -torus P by a restriction-corestriction argument. The distinguished triangle $\text{Ker } \rho'[1] \rightarrow C' \rightarrow \text{Coker } \rho' \rightarrow \text{Ker } \rho'[2]$ yields an exact sequence $0 \rightarrow H^1(K, \text{Ker } \rho')_{\text{tors}} \rightarrow \mathbb{H}^0(K, C')_{\text{tors}} \rightarrow \mathbb{H}^0(K, \text{Coker } \rho')_{\text{tors}}$.

By Lemma 3.1(4), $H^1(K, \text{Ker } \rho')$ has finite exponent, therefore so does $\mathbb{H}^0(K, C')_{\text{tors}}$ by dévissage. \square

We thank the referee for pointing out to us the following lemma.

Lemma 4.5. *Let $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_4$ be an exact sequence of Hausdorff, second countable and locally compact topological abelian groups with continuous maps. Then the map $A_2 \rightarrow A_3$ is strict. In particular, we have an exact sequence $A_3^D \rightarrow A_2^D \rightarrow A_1^D$.*

Proof. We show that $A_2/\text{Im } A_1 \rightarrow A_3$ induces a homeomorphism onto a closed subgroup. Its image is closed since it equals the closed subgroup $\text{Ker}(A_3 \rightarrow A_4)$ of A_3 . It induces a homeomorphism onto its image by [Bou74, Chapitre IX, §5, Proposition 6]. Now if an element of A_2^D goes to zero in A_1^D , then it becomes an element of $(A_2/\text{Im } A_1)^D$. But $A_2/\text{Im } A_1 \rightarrow A_3$ is a homeomorphism onto a closed subgroup of A_3 , we conclude that $A_3^D \rightarrow A_2^D \rightarrow A_1^D$ is exact. \square

Corollary 4.6. *The following is an exact sequence of locally compact groups*

$$0 \rightarrow \mathbb{H}^{-1}(K, C)_\wedge \rightarrow \mathbb{P}^{-1}(K, C)_\wedge \rightarrow \mathbb{H}^2(K, C')^D \rightarrow \mathbb{H}^2(C')^D \rightarrow 0.$$

Proof. Applying Proposition 4.4 to C' yields an exact sequence

$$(14) \quad \mathbb{H}^2(K, C') \rightarrow \mathbb{P}^2(K, C')_{\text{tors}} \rightarrow (\mathbb{H}^{-1}(K, C)_\wedge)^D \rightarrow 0.$$

It follows that the desired sequence is exact at the first three terms by dualizing the sequence (14) and applying Corollary 4.3 and Lemma 4.5. Applying Lemma 4.5 to the exact sequence $0 \rightarrow \mathbb{H}^2(C') \rightarrow \mathbb{H}^2(K, C') \rightarrow \mathbb{P}^2(K, C')_{\text{tors}}$ yields the desired exactness at the last three terms. \square

Step 3.1: Exactness of the second and the third rows of diagram (1): finite kernel case

We will systematically assume that $M := \text{Ker } \rho$ is finite from Proposition 4.7 to Proposition 4.11.

Proposition 4.7. *The following is an exact sequence of locally compact groups*

$$\varprojlim_n \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \varprojlim_n \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow \mathbb{H}^0(K, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D.$$

Proof. We consider the following commutative diagram³

(15)

$$\begin{array}{ccccccc}
 H^0(K, T_{\mathbb{Z}/n}(C)) & \xrightarrow{\Phi_n^K} & H^2(K, {}_nM) & \longrightarrow & \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & H^1(K, T_{\mathbb{Z}/n}(C)) \xrightarrow{\Psi_n^K} H^3(K, {}_nM) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \mathbb{P}^0(K, T_{\mathbb{Z}/n}(C)) & \xrightarrow{\Phi_n^{\Pi}} & \mathbb{P}^2(K, {}_nM) & \longrightarrow & \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{P}^1(K, T_{\mathbb{Z}/n}(C)) \xrightarrow{\Psi_n^{\Pi}} \mathbb{P}^3(K, {}_nM) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H^3(K, T_{\mathbb{Z}/n}(C)')^D & \rightarrow & H^1(K, ({}_nM)')^D & \rightarrow & \mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & \rightarrow & H^2(K, T_{\mathbb{Z}/n}(C)')^D \rightarrow H^0(K, ({}_nM)')^D
 \end{array}$$

where the upper two rows are exact since they are induced by the distinguished triangle ${}_nM[2] \rightarrow C \otimes^{\mathbf{L}} \mathbb{Z}/n \rightarrow T_{\mathbb{Z}/n}(C)[1] \rightarrow {}_nM[3]$, and the columns except the middle one are exact by [HSS15, Theorem 2.3]. For a finite étale commutative group scheme F , recall that $F' := \text{Hom}(F, \mathbb{Q}/\mathbb{Z}(2))$. Since $\text{Hom}(-, \mathbb{Q}/\mathbb{Z}(2))$ is an exact functor, $\mathbf{R} \text{Hom}(-, \mathbb{Q}/\mathbb{Z}(2))$ may be computed on a complex by applying $\text{Hom}(-, \mathbb{Q}/\mathbb{Z}(2))$. Therefore we may dualize the map $T_{\mathbb{Z}/n}(C) \rightarrow {}_nM[2]$ in the derived category to obtain a map ${}_nM'[-2] \rightarrow T_{\mathbb{Z}/n}(C)'$. The product

$$\mathbb{P}^0(K, T_{\mathbb{Z}/n}(C)) = \prod_{v \in X^{(1)}} \mathbb{H}^0(K_v, T_{\mathbb{Z}/n}(C))$$

is compact since $H^0(K_v, T_{\mathbb{Z}/n}(C))$ is finite. Now taking inverse limit of the middle three columns of diagram (15) over all n yield the following commutative diagram in which the upper two rows are exact

(16)

$$\begin{array}{ccccccc}
 \varprojlim_n H^2(K, {}_nM) & \longrightarrow & \varprojlim_n H^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \varprojlim_n \text{Ker } \Psi_n^K & \longrightarrow & \varprojlim_n^1 \text{Coker } \Phi_n^K \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \varprojlim_n \mathbb{P}^2(K, {}_nM) & \longrightarrow & \varprojlim_n \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \varprojlim_n \text{Ker } \Psi_n^{\Pi} & \longrightarrow & \varprojlim_n^1 \text{Coker } \Phi_n^{\Pi} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \varprojlim_n H^1(K, ({}_nM)')^D & \rightarrow & \varprojlim_n H^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & \rightarrow & \varprojlim_n H^2(K, T_{\mathbb{Z}/n}(C)')^D & &
 \end{array}$$

³Recall that $H^3(\mathcal{O}_v, {}_n\mathcal{M}) = H^3(\kappa(v), {}_n\mathcal{M}) = 0$ for cohomological dimension reasons. Thus we have $\mathbb{P}^3(K, {}_nM) = \bigoplus \mathbb{H}^3(K_v, {}_nM)$.

Moreover, the third column of the above diagram (16) also fits into the commutative diagram with exact rows and columns

(17)

$$\begin{array}{ccccc}
 0 & \longrightarrow & \varprojlim_n \text{Ker } \Psi_n^K & \longrightarrow & \varprojlim_n H^1(K, T_{\mathbb{Z}/n}(C)) & \longrightarrow & \varprojlim_n \text{Im } \Psi_n^K \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \varprojlim_n \text{Ker } \Psi_n^\Pi & \longrightarrow & \varprojlim_n \mathbb{P}^1(K, T_{\mathbb{Z}/n}(C)) & \longrightarrow & \varprojlim_n \text{Im } \Psi_n^\Pi \\
 & & & & \downarrow & & \\
 & & & & \varprojlim_n H^2(K, T_{\mathbb{Z}/n}(C))^D & &
 \end{array}$$

with $\varprojlim_n \text{Im } \Psi_n^K \subset \varprojlim_n H^3(K, {}_nM) = 0$ being zero (here we have used the fact that M is finite). Therefore $\varprojlim_n \text{Ker } \Psi_n^K \rightarrow \varprojlim_n H^1(K, T_{\mathbb{Z}/n}(C))$ is surjective. Note that the middle column is exact because \varprojlim_n^1 of the finite groups $\text{Ker}(H^1(K, T_{\mathbb{Z}/n}(C)) \rightarrow \mathbb{P}^1(K, T_{\mathbb{Z}/n}(C)))$ vanishes according to [Jen72, Théorème 7.3] (the finiteness of the kernel follows from [HS16, Theorem 4.4]).

Take $\alpha \in \varprojlim_n \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n)$ such that it goes to zero in

$$\varprojlim_n \mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D.$$

By functoriality, $\beta := \text{Im } \alpha \in \varprojlim_n \mathbb{P}^1(K, T_{\mathbb{Z}/n}(C))$ goes to zero in

$$\varprojlim_n H^2(K, T_{\mathbb{Z}/n}(C))^D.$$

Thus β comes from $\gamma \in \varprojlim_n H^1(K, T_{\mathbb{Z}/n}(C))$ by the exactness of the middle column in diagram (17). Since $\varprojlim_n \text{Ker } \Psi_n^K \rightarrow \varprojlim_n H^1(K, T_{\mathbb{Z}/n}(C))$ is surjective, γ comes from some $\gamma' \in \varprojlim_n \text{Ker } \Psi_n^K$. But

$$\varprojlim_n^1 \text{Coker } \Phi_n^K \rightarrow \varprojlim_n^1 \text{Coker } \Phi_n^\Pi$$

is injective by Lemma 4.8 below, so γ' comes from $\tau \in \varprojlim_n H^0(K, T_{\mathbb{Z}/n}(C))$. Since α and $\text{Im } \tau$ have the same image in $\varprojlim_n \text{Ker } \Psi_n^\Pi$ by construction, the exactness of the second row of diagram (16) implies that α and $\text{Im } \tau$ differs from an element in $\varprojlim_n \mathbb{P}^2(K, {}_nM)$. Recall that $\text{Ker } \rho$ is finite, thus $\varprojlim_n \mathbb{P}^2(K, {}_nM) = 0$ and α comes from $\varprojlim_n H^0(K, T_{\mathbb{Z}/n}(C))$. \square

Lemma 4.8. *The homomorphism $\varprojlim_n^1 \text{Coker } \Phi_n^K \rightarrow \varprojlim_n^1 \text{Coker } \Phi_n^\Pi$ in diagram (16) is an isomorphism.*

Proof. Since $H^0(L, T_{\mathbb{Z}/n}(C))$ is finite for $L = K, K_v, \mathbb{P}^0(K, T_{\mathbb{Z}/n}(C)) = \prod_{v \in X^{(1)}} H^0(K_v, T_{\mathbb{Z}/n}(C))$ is compact. Thus we obtain $\varprojlim_n^1 H^0(K, T_{\mathbb{Z}/n}(C)) = 0$ and $\varprojlim_n^1 \mathbb{P}^0(K, T_{\mathbb{Z}/n}(C)) = 0$ by [Jen72, Théorème 7.3]. Moreover, the image $\text{Im } \Phi_n^K$ of $H^0(K, T_{\mathbb{Z}/n}(C))$ in $H^2(K, {}_nM)$ is finite, so $\varprojlim_n^1 H^2(K, {}_nM) \simeq \varprojlim_n^1 \text{Coker } \Phi_n^K$ by the short exact sequence $0 \rightarrow \text{Im } \Phi_n^K \rightarrow H^2(K, {}_nM) \rightarrow \text{Coker } \Phi_n^K \rightarrow 0$. Similarly, we obtain $\varprojlim_n^1 \mathbb{P}^2(K, {}_nM) \simeq \varprojlim_n^1 \text{Coker } \Phi_n^\Pi$.

Let I_n denote the image of $H^2(K, {}_nM) \rightarrow \mathbb{P}^2(K, {}_nM)$. So

$$0 \rightarrow \text{III}^2({}_nM) \rightarrow H^2(K, {}_nM) \rightarrow I_n \rightarrow 0$$

is an exact sequence. The finiteness of $\text{III}^2({}_nM)$ (by [HS16, Theorem 4.4]) yields an isomorphism $\varprojlim_n^1 H^2(K, {}_nM) \simeq \varprojlim_n^1 I_n$. Moreover, the cokernel of $I_n \rightarrow \mathbb{P}^2(K, {}_nM)$ is a subgroup of the group $H^1(K, ({}_nM)')^D$ by diagram (15). The finiteness of M yields the vanishing of $\varprojlim_n (H^1(K, ({}_nM)')^D) \simeq H^1(K, \varinjlim_n ({}_nM)')^D$ and thus $\varprojlim_n \text{Coker}(I_n \rightarrow \mathbb{P}^2(K, {}_nM)) = 0$ by the left exactness of inverse limits. Moreover, the finiteness of M implies that there is an integer (say e) such that ${}_{ne}M \rightarrow {}_nM$ the multiplication by e is trivial for each integer n . In particular, the system $\{\text{Coker}(I_n \rightarrow \mathbb{P}^2(K, {}_nM))\}$ satisfies the Mittag–Leffler condition, hence $\varprojlim_n^1 \text{Coker}(I_n \rightarrow \mathbb{P}^2(K, {}_nM)) = 0$ by [Wei94, Proposition 3.5.7]. We now conclude that $\varprojlim_n^1 I_n \rightarrow \varprojlim_n^1 \mathbb{P}^2(K, {}_nM)$ is an isomorphism. Therefore we have $\varprojlim_n^1 \text{Coker } \Phi_n^K \simeq \varprojlim_n^1 I_n \simeq \varprojlim_n^1 \text{Coker } \Phi_n^\Pi$. \square

Corollary 4.9. *The following is an exact sequence of locally compact groups*

$$0 \rightarrow \text{III}_\wedge^0(C) \rightarrow \mathbb{H}^0(K, C)_\wedge \rightarrow \mathbb{P}^0(K, C)_\wedge \rightarrow \mathbb{H}^1(K, C')^D \rightarrow \text{III}^1(C')^D \rightarrow 0.$$

Proof. We consider again the diagram (8) with rows and the middle column being exact (by Proposition 4.7):

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{H}^0(K, C)_\wedge & \longrightarrow & \varprojlim_n \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \Phi_K \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{P}^0(K, C)_\wedge & \longrightarrow & \varprojlim_n \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \Phi_\Pi \longrightarrow 0. \\ & & \downarrow & & \downarrow & & \\ & & \mathbb{H}^1(K, C')^D & \longrightarrow & \mathbb{H}^0(K, \varinjlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & & \end{array}$$

Since $\mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{H}^1(K, C')$ is a surjective map between discrete groups, there is an injective map $({}_n\mathbb{H}^1(K, C'))^D \rightarrow (\mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n))^D$ and hence an injection $\mathbb{H}^1(K, C') \rightarrow \mathbb{H}^0(K, \varprojlim_n C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$. So the left vertical column is a complex. But the map $\Phi_K \rightarrow \Phi_\Pi$ is injective (see the proof of Lemma 3.8), another diagram chase then tells us the left column is exact.

To show the exactness of the last three terms, we consider the exact sequence $0 \rightarrow \mathbb{I}\mathbb{I}^1(C') \rightarrow \mathbb{H}^1(K, C') \rightarrow \mathbb{P}^1(K, C')_{\text{tors}}$. Now we obtain the desired exactness by Lemma 4.5. \square

Remark 4.10. By Lemma 3.1, the groups $\mathbb{H}^{-1}(K, C)$ and $\mathbb{P}^{-1}(K, C)$ are torsion of finite exponent, therefore they are isomorphic to respective completions $\mathbb{H}^{-1}(K, C)_\wedge$ and $\mathbb{P}^{-1}(K, C)_\wedge$. Note that $\mathbb{H}^1(K, C)$ and $\mathbb{P}^1(K, C) \subset \prod \mathbb{H}^1(K_v, C)$ have finite exponents according to Lemma 3.1. Consequently, we obtain $\mathbb{H}^1(K, C) \simeq \mathbb{H}^1(K, C)_\wedge$ and $\mathbb{P}^1(K, C)_{\text{tors}} = \mathbb{P}^1(K, C) \simeq \mathbb{P}^1(K, C)_\wedge$. Moreover, the finiteness of M implies that $\rho' : T'_2 \rightarrow T'_1$ is surjective and hence $\mathbb{H}^0(K, C')$ has finite exponent by Lemma 3.1. Summing up, the subscripts "tors" and " \wedge " in the first and the third rows of (1) are superfluous.

Proposition 4.11. *The following sequence of locally compact groups is exact*

$$0 \rightarrow \mathbb{I}\mathbb{I}^1(C) \rightarrow \mathbb{H}^1(K, C) \rightarrow \mathbb{P}^1(K, C)_{\text{tors}} \rightarrow (\mathbb{H}^0(K, C')_\wedge)^D \rightarrow \mathbb{H}^2(K, C).$$

Proof. Taking inverse limit in diagram (12) yields a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{H}^1(K, C) & \longrightarrow & \varprojlim_n \mathbb{H}^1(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \Psi_K \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{P}^1(K, C) & \longrightarrow & \varprojlim_n \mathbb{P}^1(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \Psi_\Pi \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{H}^0(K, C')^D & \longrightarrow & \varprojlim_n \mathbb{H}^{-1}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & & \end{array}$$

We observe that $\Psi_K \rightarrow \Psi_\Pi$ is injective because its kernel is contained in $\varprojlim_n {}_n\mathbb{I}\mathbb{I}^2(C) = 0$ (recall that $\mathbb{I}\mathbb{I}^2(C)$ is finite). The exactness of the middle column follows from the vanishing $\varprojlim_n^1 \mathbb{I}\mathbb{I}^1(C \otimes^{\mathbf{L}} \mathbb{Z}/n) = 0$ (by [Jen72, Théorème 7.3]). Thus a diagram chase yields the exactness of the left column.

Now we verify the exactness of $\mathbb{P}^1(K, C) \rightarrow \mathbb{H}^0(K, C')^D \rightarrow \mathbb{H}^2(K, C)$. Consider the commutative diagram with vertical arrows obtained from respective Kummer sequences:

$$(18) \quad \begin{array}{ccccc} \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & \longrightarrow & \mathbb{H}^1(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \\ \downarrow & & \downarrow & & \downarrow \\ {}_n\mathbb{P}^1(K, C) & \longrightarrow & (\mathbb{H}^0(K, C')/n)^D & \xrightarrow{*} & {}_n\mathbb{H}^2(K, C), \end{array}$$

where the upper row is exact by Theorem 2.11 and the arrow $*$ is the composite $(\mathbb{H}^0(K, C')/n)^D \rightarrow (\mathbb{H}^0(C')/n)^D \simeq {}_n\mathbb{H}^2(C) \rightarrow {}_n\mathbb{H}^2(K, C)$. Note that the squares in (18) commute by construction. Finally, the middle vertical arrow is surjective because $\mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ and $\mathbb{H}^0(K, C')/n$ are discrete. Passing to the direct limit over all n , the right vertical arrow in diagram (18) becomes an isomorphism as $\mathbb{H}^1(K, C) \otimes \mathbb{Q}/\mathbb{Z} = 0$. Now a diagram chasing implies the exactness of $\mathbb{P}^1(K, C) \rightarrow \mathbb{H}^0(K, C')^D \rightarrow \mathbb{H}^2(K, C)$. \square

Step 3.2: Exactness of the second and the third rows of diagram (1): the surjective case

Suppose that $\rho : T_1 \rightarrow T_2$ is surjective. Thus $\widehat{T}_2 \rightarrow \widehat{T}_1$ is injective and $\widetilde{T}_1 \rightarrow \widetilde{T}_2$ has finite cokernel, i.e. $\rho' : T'_2 \rightarrow T'_1$ has finite kernel. Therefore $\mathbb{H}^{-1}(K, C')$, $\mathbb{H}^0(K, C)$, $\mathbb{H}^1(K, C')$ and $\mathbb{H}^2(K, C)$ are torsion groups having finite exponent by Lemma 3.1. It follows that $\mathbb{H}^0(K, C)_\wedge = \mathbb{H}^0(K, C)$, $\mathbb{H}^{-1}(K, C')_\wedge = \mathbb{H}^{-1}(K, C')$. Moreover, we have $\mathbb{P}^0(K, C)_\wedge = \mathbb{P}^0(K, C)$ and $\mathbb{P}^2(K, C)_{\text{tors}} = \mathbb{P}^2(K, C)$ thanks to the distinguished triangle $M[1] \rightarrow C \rightarrow T \rightarrow M[2]$. We conclude that the subscripts of the second and the last rows in diagram (1) are superfluous. After Corollary 4.6 and Proposition 4.4, it remains to show the following proposition.

Proposition 4.12. *Suppose that $\rho : T_1 \rightarrow T_2$ is surjective. Then the sequence*

$$\begin{aligned} 0 \rightarrow \mathbb{H}^0(C) \rightarrow \mathbb{H}^0(K, C) \rightarrow \mathbb{P}^0(K, C) \rightarrow \mathbb{H}^1(K, C')^D \\ \rightarrow \mathbb{H}^1(K, C) \rightarrow \mathbb{P}^1(K, C)_{\text{tors}} \rightarrow (\mathbb{H}^0(K, C')_\wedge)^D \rightarrow \mathbb{H}^2(K, C) \end{aligned}$$

is exact.

Proof.

- We show the exactness of $\mathbb{H}^1(K, C) \rightarrow \mathbb{P}^1(K, C)_{\text{tors}} \rightarrow (\mathbb{H}^0(K, C')_{\wedge})^D$. Since $\mathbb{H}^1(K, C')$ has finite exponent, $\varprojlim_n \mathbb{H}^1(K, C') = 0$ and hence $\mathbb{H}^0(K, C')_{\wedge} \simeq \varprojlim_n \mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)$ by the Kummer sequence $0 \rightarrow \mathbb{H}^0(K, C')/n \rightarrow \mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n) \rightarrow {}_n\mathbb{H}^1(K, C') \rightarrow 0$. Consider the commutative diagram

$$\begin{array}{ccccc} \varinjlim_n \mathbb{H}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \varinjlim_n \mathbb{P}^0(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & (\varprojlim_n \mathbb{H}^0(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n))^D \\ \downarrow & & \downarrow & & \downarrow \simeq \\ \mathbb{H}^1(K, C) & \longrightarrow & \mathbb{P}^1(K, C)_{\text{tors}} & \longrightarrow & (\mathbb{H}^0(K, C')_{\wedge})^D. \end{array}$$

Now the exactness of the lower row follows from that of the upper row (thanks to Theorem 2.11) by diagram chasing.

- We consider the following commutative diagram (see diagram (18)) for $i = -1, 0$ in which the top row is exact by Theorem 2.11:

$$(19) \quad \begin{array}{ccccc} \mathbb{P}^i(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) & \longrightarrow & \mathbb{H}^{-i}(K, C' \otimes^{\mathbf{L}} \mathbb{Z}/n)^D & \longrightarrow & \mathbb{H}^{i+1}(K, C \otimes^{\mathbf{L}} \mathbb{Z}/n) \\ \downarrow & & \downarrow & & \downarrow \\ {}_n\mathbb{P}^{i+1}(K, C) & \longrightarrow & (\mathbb{H}^{-i}(K, C')/n)^D & \xrightarrow{*} & {}_n\mathbb{H}^{i+2}(K, C) \end{array}$$

where the arrow $*$ is constructed in the same way as (18). Note that $\mathbb{H}^0(K, C) \simeq H^1(K, M)$ and $\mathbb{H}^1(K, C)$ are torsion. Thus

$$\mathbb{H}^i(K, C) \otimes \mathbb{Q}/\mathbb{Z} = 0 \quad \text{for } i = 0, 1.$$

It follows that the right vertical arrow becomes an isomorphism after taking direct limit in diagram (19). Therefore we get the exactness of

$$\mathbb{P}^0(K, C) \rightarrow \mathbb{H}^1(K, C')^D \rightarrow \mathbb{H}^1(K, C)$$

and

$$\mathbb{P}^1(K, C)_{\text{tors}} \rightarrow (\mathbb{H}^0(K, C')_{\wedge})^D \rightarrow \mathbb{H}^2(K, C)$$

by diagram chasing.

- According to the previous points, we have an exact sequence $\mathbb{H}^1(K, C') \rightarrow \mathbb{P}^1(K, C')_{\text{tors}} \rightarrow (\mathbb{H}^0(K, C')_{\wedge})^D \rightarrow \mathbb{H}^2(K, C')$. Dualizing it yields an exact sequence $\mathbb{H}^0(K, C)_{\wedge} \rightarrow \mathbb{P}^0(K, C)_{\wedge} \rightarrow \mathbb{H}^1(K, C')^D$ by Lemma 4.5. But the groups $\mathbb{H}^0(K, C)$ and $\mathbb{P}^0(K, C)$ have finite exponents, thus we obtain an exact sequence $\mathbb{H}^0(K, C) \rightarrow \mathbb{P}^0(K, C) \rightarrow \mathbb{H}^1(K, C')^D$. \square

Example 4.13.

- (1) Let P be a K -torus that extends to an X_0 -torus \mathcal{P} . We consider the special case that $C = [0 \rightarrow P]$ and $C' = P'[1]$. By definition, $\mathbb{H}^{-1}(L, P) = 0$ for $L = K$ or K_v and hence the first two terms in diagram (1) vanish automatically. The third term vanishes by Lemma 3.1. Moreover, $\mathbb{P}^1(K, P)$ has finite exponent by Hilbert’s Theorem 90, so it is torsion. Finally, $H^1(K, P')$ has finite exponent, thus the canonical map $H^1(K, P') \rightarrow H^1(K, P')_\wedge$ is an isomorphism. The remaining 9 terms in diagram (1) read as

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^0(K, P)_\wedge & \longrightarrow & \mathbb{P}^0(K, P)_\wedge & \longrightarrow & H^2(K, P')^D \\
 & & \underbrace{\hspace{10em}} & & & & \\
 & & \longrightarrow & H^1(K, P) & \longrightarrow & \mathbb{P}^1(K, P) & \longrightarrow & H^1(K, P')^D \\
 & & \underbrace{\hspace{10em}} & & & & \\
 & & \longrightarrow & H^2(K, P) & \longrightarrow & \mathbb{P}^2(K, P)_{\text{tors}} & \longrightarrow & (H^0(K, P')_\wedge)^D \longrightarrow 0
 \end{array}$$

which is the Poitou–Tate exact sequence for tori [HSS15, Theorem 2.9].

- (2) Let M be a group of multiplicative type over K . We may embed it into a short exact sequence $0 \rightarrow M \rightarrow T_1 \rightarrow T_2 \rightarrow 0$ with T_1 and T_2 being K -tori. In particular, there is a quasi-isomorphism $M[1] \simeq C_M$ with $C_M = [T_1 \xrightarrow{\rho} T_2]$. In this case, $\text{Coker } \rho$ is trivial and we obtain a Poitou–Tate sequence for short complexes C_M and thus for groups of multiplicative type.
- (3) Let G be a connected reductive group over K . Let G^{sc} be the universal covering of the derived subgroup G^{ss} of G . Let $\rho : G^{\text{sc}} \rightarrow G^{\text{ss}} \rightarrow G$ be the composite. Let T be a maximal torus of G and let $T^{\text{sc}} := \rho^{-1}(T)$ be the inverse image of T in G^{sc} . Thus T^{sc} is a maximal torus of G^{sc} . Following [Bor98], we write $H_{\text{ab}}^i(K, G) = \mathbb{H}^i(K, C)$ and $\mathbb{P}_{\text{ab}}^i(K, G) = \mathbb{P}^i(K, C)$ with $C = [T^{\text{sc}} \rightarrow T]$ for the abelianized Galois cohomologies. So the Poitou–Tate sequence (1) yields an exact sequence for the abelianization of Galois cohomology of G as follows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_{\text{ab}}^{-1}(K, G) & \longrightarrow & \mathbb{P}_{\text{ab}}^{-1}(K, G) & \longrightarrow & \mathbb{H}^2(K, C')^D \\
 & & \underbrace{\hspace{10em}} & & & & \\
 & & \longrightarrow & H_{\text{ab}}^0(K, G)_\wedge & \longrightarrow & \mathbb{P}_{\text{ab}}^0(K, G)_\wedge & \longrightarrow & \mathbb{H}^1(K, C')^D \\
 & & \underbrace{\hspace{10em}} & & & & \\
 & & \longrightarrow & H_{\text{ab}}^1(K, G) & \longrightarrow & \mathbb{P}_{\text{ab}}^1(K, G) & \longrightarrow & \mathbb{H}^0(K, C')^D \\
 & & \underbrace{\hspace{10em}} & & & & \\
 & & \longrightarrow & H_{\text{ab}}^2(K, G) & \longrightarrow & \mathbb{P}_{\text{ab}}^2(K, G)_{\text{tors}} & \longrightarrow & (\mathbb{H}^{-1}(K, C')_\wedge)^D \longrightarrow 0.
 \end{array}$$

Hopefully it will give a defect to strong approximation, which is analogous to the number field case [Dem11b].

Finally, we relate $\mathbb{H}^i(K, C')$ in Example 4.13(3) with a simpler cohomology group. Recall that the algebraic fundamental group $\pi_1^{\text{alg}}(\overline{G})$ (see [Bor98, §1] and [CT08, §6] for more information) of a connected reductive group G is $\pi_1^{\text{alg}}(\overline{G}) := X_*(T)/\rho_*X_*(T^{\text{sc}})$ where $\rho_* : X_*(T^{\text{sc}}) \rightarrow X_*(T)$ is induced by $\rho : T^{\text{sc}} \rightarrow T$.

Corollary 4.14. *Let G be a connected reductive group. Let $C = [T^{\text{sc}} \xrightarrow{\rho} T]$ be as above. Let G^* be the group of multiplicative type such that $X^*(G^*) = \pi_1^{\text{alg}}(\overline{G})$. Then the following is an exact sequence*

$$0 \rightarrow \text{III}^1(C) \rightarrow \mathbb{H}^1(K, C) \rightarrow \mathbb{P}^1(K, C) \rightarrow H^1(K, G^*)^D.$$

Proof. Let $T^{\text{ss}} := T \cap G^{\text{ss}}$ and let $G^{\text{tor}} := T/T^{\text{ss}}$. Thus there is a short exact sequence of short complexes

$$(20) \quad 0 \rightarrow [(G^{\text{tor}})' \rightarrow 0] \rightarrow [T' \rightarrow (T^{\text{sc}})'] \rightarrow [(T^{\text{ss}})' \rightarrow (T^{\text{sc}})'] \rightarrow 0.$$

By [CT08, Proposition 6.4], there is a short exact sequence of abelian groups

$$0 \rightarrow (\text{Ker } \rho)(-1) \rightarrow \pi_1^{\text{alg}}(\overline{G}) \rightarrow X_*(G^{\text{tor}}) \rightarrow 0.$$

Here $(\text{Ker } \rho)(-1) := \text{Hom}_{\mathbb{Z}}(X^*(\text{Ker } \rho), \mathbb{Q}/\mathbb{Z})$ is the module of characters of $(\text{Ker } \rho)' := \text{Hom}(\text{Ker } \rho, \mathbb{Q}/\mathbb{Z}(2))$. Thus there is an exact sequence of groups of multiplicative type

$$(21) \quad 0 \rightarrow (G^{\text{tor}})' \rightarrow G^* \rightarrow (\text{Ker } \rho)' \rightarrow 0.$$

Since $T^{\text{sc}} \rightarrow T^{\text{ss}}$ is an isogeny with kernel $\text{Ker } \rho$, its dual isogeny $(T^{\text{ss}})' \rightarrow (T^{\text{sc}})'$ has kernel $(\text{Ker } \rho)'$, i.e. there is a quasi-isomorphic $[(T^{\text{ss}})' \rightarrow (T^{\text{sc}})'] \simeq (\text{Ker } \rho)'[1]$. By definition there is an exact sequence $X_*(T^{\text{sc}}) \xrightarrow{\rho_*} X_*(T) \rightarrow \pi_1^{\text{alg}}(\overline{G}) \rightarrow 0$, so there is a corresponding exact sequence $0 \rightarrow G^* \rightarrow T' \rightarrow (T^{\text{sc}})'$ of groups of multiplicative type. In particular, we obtain a morphism of short complexes $G^*[1] \rightarrow C'$. Summing up, there is a commutative diagram of short complexes with exact rows obtained from (21) and (20):

$$\begin{array}{ccccccc} 0 & \longrightarrow & (G^{\text{tor}})'[1] & \longrightarrow & G^*[1] & \longrightarrow & (\text{Ker } \rho)'[1] \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (G^{\text{tor}})'[1] & \longrightarrow & C' & \longrightarrow & [(T^{\text{ss}})' \rightarrow (T^{\text{sc}})'] \longrightarrow 0. \end{array}$$

Since the right vertical arrow is a quasi-isomorphism, so is the middle one as is seen by taking cohomology and applying the 5-lemma. Thus $H^{i+1}(K, G^*) \simeq \mathbb{H}^i(K, C')$ and the desired sequence follows from Example 4.13(3). \square

Remark 4.15. Corollary 4.14 gives an abelianized version of the Kottwitz–Borovoi sequence [Bor98, Theorem 5.16] over the p -adic function field K . Hopefully over such K there is an exact sequence $1 \rightarrow \text{III}^1(G) \rightarrow H^1(K, G) \rightarrow \mathbb{P}^1(K, G) \rightarrow H^1(K, G^*)^D$ of pointed sets for connected linear groups which can be used to give an obstruction to weak approximation for homogenous spaces under some linear group with stabilizer G .

References

- [Blo86] Spencer Bloch, *Algebraic cycles and higher K-theory*, Adv. in Math. **61** (1986), no. 3, 267–304, DOI 10.1016/0001-8708(86)90081-2.
- [Bor98] Mikhaïl Borovoi, *Abelian Galois cohomology of reductive groups*, Mem. Amer. Math. Soc. **132** (1998), no. 626, viii+50.
- [Bou74] Nicolas Bourbaki, *Éléments de mathématique. Topologie générale. Chapitres 5 à 10*, Hermann, Paris, 1974 (French).
- [Čes15] Kęstutis Česnavičius, *Poitou-Tate without restrictions on the order*, Math. Res. Lett. **22** (2015), no. 6, 1621–1666, DOI 10.4310/MRL.2015.v22.n6.a5.
- [CT08] Jean-Louis Colliot-Thélène, *Résolutions flasques des groupes linéaires connexes*, J. Reine Angew. Math. **618** (2008), 77–133, DOI 10.1515/CRELLE.2008.034 (French).
- [CTH15] Jean-Louis Colliot-Thélène and David Harari, *Dualité et principe local-global pour les tores sur une courbe au-dessus de $\mathbb{C}((t))$* , Proc. Lond. Math. Soc. (3) **110** (2015), no. 6, 1475–1516, DOI 10.1112/plms/pdv019 (French, with English and French summaries).
- [CTPS12] Jean-Louis Colliot-Thélène, Raman Parimala, and Venapally Suresh, *Patching and local-global principles for homogeneous spaces over function fields of p -adic curves*, Comment. Math. Helv. **87** (2012), no. 4, 1011–1033, DOI 10.4171/CMH/276.
- [Dem09] Cyril Demarche, *Méthodes cohomologiques pour l'étude des points rationnels sur les espaces homogènes*, Thèse de doctorat de l'Université Paris-Sud, 2009 (French, with English and French summaries).
- [Dem11a] ———, *Suites de Poitou-Tate pour les complexes de tores à deux termes*, Int. Math. Res. Not. IMRN **1** (2011), 135–174, DOI 10.1093/imrn/rnq060 (French, with English and French summaries).
- [Dem11b] ———, *Le défaut d'approximation forte dans les groupes linéaires connexes*, Proc. Lond. Math. Soc. (3) **102** (2011), no. 3, 563–597, DOI 10.1112/plms/pdq033 (French, with English and French summaries).

- [Fu11] Lei Fu, *Étale cohomology theory*, Nankai Tracts in Mathematics, vol. 13, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.
- [GL01] Thomas Geisser and Marc Levine, *The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky*, J. Reine Angew. Math. **530** (2001), 55–103, DOI 10.1515/crll.2001.006.
- [Har17] David Harari, *Cohomologie galoisienne et théorie du corps de classes*, Savoirs Actuels (Les Ulis). [Current Scholarship (Les Ulis)], EDP Sciences, Les Ulis; CNRS Éditions, Paris, 2017 (French).
- [DH18] Cyril Demarche and David Harari, *Artin-Mazur-Milne duality for fppf cohomology* (2018), available at <https://arxiv.org/abs/1804.03941>.
- [HI18] David Harari and Diego Izquierdo, *L'espace adélique d'un tore sur un corps de fonctions*, To appear in Ann. Inst. Fourier (Grenoble) (2018) (French).
- [HSS15] David Harari, Claus Scheiderer, and Tamás Szamuely, *Weak approximation for tori over p -adic function fields*, Int. Math. Res. Not. IMRN **10** (2015), 2751–2783, DOI 10.1093/imrn/rnu019.
- [HS05] David Harari and Tamás Szamuely, *Arithmetic duality theorems for 1-motives*, J. Reine Angew. Math. **578** (2005), 93–128, DOI 10.1515/crll.2005.2005.578.93.
- [HS16] ———, *Local-global questions for tori over p -adic function fields*, J. Algebraic Geom. **25** (2016), no. 3, 571–605, DOI 10.1090/jag/661.
- [Hu14] Yong Hu, *Hasse principle for simply connected groups over function fields of surfaces*, J. Ramanujan Math. Soc. **29** (2014), no. 2, 155–199.
- [Izq16] Diego Izquierdo, *Dualité et principe local-global sur les corps de fonctions*, Thèse de doctorat de l'Université Paris-Saclay préparée à l'Université Paris-Sud, 2016 (French, with English and French summaries).
- [Jen72] Christian Jensen, *Les foncteurs dérivés de \varprojlim et leurs applications en théorie des modules*, Lecture Notes in Mathematics, Vol. 254, Springer-Verlag, Berlin-New York, 1972.
- [Mil80] James Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980.
- [Mil06] ———, *Arithmetic duality theorems*, 2nd ed., BookSurge, LLC, Charleston, SC, 2006.
- [SvH03] Claus Scheiderer and Joost van Hamel, *Cohomology of tori over p -adic curves*, Math. Ann. **326** (2003), no. 1, 155–183, DOI 10.1007/s00208-003-0416-y.
- [Ser65] Jean-Pierre Serre, *Cohomologie galoisienne*, With a contribution by Jean-Louis Verdier. Lecture Notes in Mathematics, No. 5. Troisième édition, vol. 1965, Springer-Verlag, Berlin-New York, 1965 (French).
- [Tia21] Yisheng Tian, *Obstructions to weak approximation for reductive groups over p -adic function fields*, J. Number Theory **220** (2021), 128–162, DOI 10.1016/j.jnt.2020.08.008.

- [Wei94] Charles Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.
- [SGA3II] Michael Artin, Alexander Grothendieck, and Michel Raynaud, *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 152, Springer-Verlag, Berlin-New York, 1970 (French).

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