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Open Torelli locus and complex ball quotients

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We study the problem of non-existence of totally geodesic complex ball quotients in the open Torelli locus in a moduli space of principally polarized Abelian varieties using analytic techniques.

1. Introduction

1.1 Let $\mathcal{M}_g$ be the moduli space or stack of Riemann surfaces of genus $g \geqslant 2$. Let $\overline{\mathcal{M}}_g$ be the Deligne-Mumford compactification of $\mathcal{M}_g$. Let $\mathcal{A}_g$ be the moduli space of principally polarized Abelian varieties of complex dimension g. We know that $\mathcal{A}_g = \mathcal{S}_g/Sp(2g,\mathbb{Z})$ is the quotient of the Siegel Upper Half Space $\mathcal{S}_g$ of genus g. Let $\overline{\mathcal{A}}_g$ be the Bailey-Borel compactification of $\mathcal{A}_g$. Associating a smooth Riemann surface represented by a point in $\mathcal{M}_g$ to its Jacobian, we obtain the Torelli map $j_g : \mathcal{M}_g \to \mathcal{A}_g$. The Torelli map extends to $j_g : \overline{\mathcal{M}}_g \to \overline{\mathcal{A}}_g$. The image $T_g^o := j_g(\mathcal{M}_g)$ is called the open Torelli locus of $\mathcal{A}_g$. It is well-known that the Torelli map j_g is injective on $\mathcal{M}_g$. As a mapping between stacks, the mapping $t_g|_{\mathcal{M}_g}$ is known to be an immersion apart from the hyperelliptic locus, which is denoted by H_g. H_g is the set of points in $\mathcal{M}_g$ parametrizing hyperelliptic curves of genus g.

It is a natural problem to study $\mathcal{M}_g$, j_g and to characterize the Torelli locus in $\mathcal{A}_g$. There are many interesting directions and approaches to the problems. Our motivation comes from the following conjecture in the literature.

Conjecture 1. *(Oort [O]) Let T_g^o be the open Torelli locus in the Siegel modular variety $\mathcal{A}_g$. Then for g sufficiently large, the intersection of T_g^o with any Shimura variety $M \subset \mathcal{A}_g$ of strictly positive dimension is not Zariski dense in M.*

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The problem is related to a conjecture of Coleman [C] that the cardinality of CM points on $\mathcal{M}_g$ cannot be infinite if g is sufficiently large. Shimura varieties are arithmetic locally Hermitian symmetric spaces. Hence we may consider a geometrically slightly more general question of whether there exists a locally Hermitian symmetric space in $\mathcal{A}_g$ with a Zariski open set in T_g^o. In such case, the lattice Γ involved in the complex rank one case, namely complex balls $B_{\mathbb{C}}^n = PU(n,1)/P(U(n)\times U(1))$, may not be arithmetic.

Conjecture 1 is open, but there are quite a few interesting partial results. First of all there is the result of Hain [Ha] that for M locally Hermitian symmetric of rank at least 2, it cannot happen that $M \subset T_g(\overline{\mathcal{M}}_g) - (t_g(H_g) \cup t_g(\overline{\mathcal{M}}_g \backslash \mathcal{M}_g))$, cf. also de Jong-Zhang [dJZ] for precise formulation and some results. On the other hand, there is the result of de Jong-Noot [dJN] that there are examples of Shimura curves in $\mathcal{M}_g$ for $g = 4$ and 6. A more systematic and complete treatment in this direction can be found in Moonen [Moo]. To the knowledge of the author, not much is known for complex ball quotients of dimension $n \geqslant 2$, apart from some restrictions in terms of Higgs bundles given in Chen-Lu-Tan-Zuo [CLTZ]. A possible reason is that on the one hand rigidity results in general are not strong enough for super-rigidity properties in the rank one complex cases, and on the other hand the problem is not concrete enough to be handled by geometric techniques developed for specific Riemann surfaces.

1.2 The goal of this paper is to provide a method which is applicable to locally Hermitian symmetric spaces and in particular to all complex ball quotients of dimension $n \geqslant 2$. It is a weaker statement but is in support of Conjecture 1.

Theorem 1. *The set $T_g^o - j_g(H_g) \subset \mathcal{A}_g$ for $g \geqslant 2$ does not contain any complex hyperbolic complex ball quotient, compact or non-compact with finite volume, of complex dimension at least 2 as a totally geodesic complex suborbifold of $\mathcal{A}_g$.*

Remarks. The method of proof applies immediately to other locally Hermitian symmetric spaces of complex dimension at least 2. We refer the readers to Section **3.5** and **4.4** for more details. However, the end result for $\mathrm{rank}_{\mathbb{R}} M \geqslant 2$ followed already from the results of [Ha], where the proof is completely different.

Combining Theorem 1, the results of [Ha], the results of [Mö] and the very recent result of [AN], we have now a rather complete picture for Shimura varieties on the complement of the hyperelliptic locus in the open Torelli locus.

Theorem 2. *Let $g \geqslant 2$. The space $T_g^o - j_g(H_g) \subset \mathcal{A}_g$ in the Siegel modular variety $\mathcal{A}_g$ does not contain any Shimura subvariety of $\mathcal{A}_g$, except when M is the Torelli image of a Riemann surface with genus $g = 3, 4$. There is only one curve for each of $g = 3, 4$, with universal families given by $y^4 = x(x-1)(x-t)$ and $y^6 = x(x-1)(x-t)$ respectively.*

2. Preliminaries and rigidity

2.1 The approach we take is complex analytic, trying to compare various Kobayashi metrics making use of Schwarz Lemma and results in rigidity. The reader may refer to **3.1** for a brief summary of facts needed about Kobayashi metric.

As mentioned in the introduction, we may consider either M as a smooth submanifold of $\mathcal{A}_g$, or a suborbifold. In the latter case the Kobayashi metrics are considered to be in orbifold sense as follows. Recall that all the singularities of $\mathcal{A}_g$ are orbifold singularities since $\mathcal{A}_g$ is a quotient of the Siegel Upper Half Space $\mathcal{S}_g$ by a discrete group and $\mathcal{S}_g$ is smooth. By an orbifold embedding (resp. mapping) $f : M \to \mathcal{A}_g$, we mean that there is a finite covering $\pi : \mathcal{A}'_g \to \mathcal{A}_g$ so that $\mathcal{A}'_g$ is smooth and there is an embedding (resp. mapping) $f' : M' \to \mathcal{A}'$ for which M' is smooth and $f \circ \pi = \pi \circ f'$. Since Kobayashi metric is invariant under a biholomorphism and in particular invariant under a local holomorphic covering map, the argument throughout the article would be independent of the local uniformization taken at each orbifold singularity.

Hence the Kobayashi metric studied throughout the article is in orbifold sense as explained.

2.2 The proof of Theorem 1 makes use of the following result in [A].

Proposition 1. *([A] Theorem 1.1) Let $\widetilde{M} = B_{\mathbb{C}}^N$ be the complex unit ball of dimension $N \geqslant 2$. There is no holomorphic embedding of $\widetilde{M}$ into $\mathcal{T}_g$ which is isometric with respect to the Kobayashi metrics on $\widetilde{M}$ and $\mathcal{T}_g$.*

3. Totally geodesic embeddings from complex balls to Siegel upper spaces

3.1 Our approach relies on basic properties concerning totally geodesic embeddings of complex balls in the Siegel Upper Half Space. The purpose of this section is to explain results in this direction relevant to our purpose. For this purpose, it is more convenient to consider the bounded model of $\mathcal{S}_g$, namely, the classical bound domain III_g. The main technical result of this paper is Proposition 3 stated in **3.4**.

Basic properties of classical domains can be found in [He], [Mok1] and [S]. Since we are going to use the results of Satake, we follow closely the exposition in [S]. First let us explain briefly classical domains of type $I_{p,q}$ and III_k according to **1.2, 1.3** of [S], of which the terminology is to be used in later parts of this section.

(i). $I_{p,q}$: Consider V a vector space over $\mathbb{C}$ equipped with a non-degenerate Hermitian form F of signature (p,q) with $p > q > 0$. $I_{p,q} = \mathcal{D}(V,F) =: \mathcal{D}$ is the space of q dimensional complex subspace V_- of V so that $F|_{V_-}$ is negative definite. Let V_+ be the orthogonal complement of V_- in V, so that $F|_{V_+}$ is positive definite. A point in $\mathcal{D}$ is determined by V_-, or the pair (V_+, V_-). Let z_0 be a fixed point $\mathcal{D}$, determined by $(V_+^{(0)}, V_-^{(0)})$ and for convenience can be chosen to be the origin. Let $(e_1, \cdots, e_p)$ and $(e_{p+1}, \cdots, e_{p+q})$ be orthonormal basis of $V_+^{(0)}$ and $V_-^{(0)}$ respectively, so that together they form a basis of V. A point $z \in \mathcal{D}$ is now determined by (V_+, V_-) with V_- spanned by the basis

$$\sum_{i=1}^{p} e_i z_{ij} + e_{p+j}, \ 1 \leqslant j \leqslant q, \tag{1}$$

where the (p,q) matrix $Z = (z_{ij})$ satisfies $I_q -{}^t\,\overline{Z}Z > 0$. Denote by $M_{p,q}$ the space of all $p \times q$ matrices with entries in $\mathbb{C}$. Identifying $\mathcal{D}$ with $\{Z \in M_{p,q} : I_q - \overline{Z}^t Z > 0\}$, we realize $I_{p,q}$ as a bounded domain in $\mathbb{C}^{pq}$.

The complex ball $B^n_{\mathbb{C}}$ in $\mathbb{C}^n$ is just $I_{n,1}$.

(ii). III_k: Consider $V_{\mathbb{R}}$ a vector space over $\mathbb{R}$ of dimension $2k$ equipped with a non-degenerate alternating bilinear form A. Let $V = V_{\mathbb{C}}$ be the complexification of $V_{\mathbb{R}}$. A extends naturally to V. The Hermitian form defined by

$$F(x,y) := iA(\bar{x}, y) \tag{2}$$

has signature (k,k) on V. $III_k = \mathcal{D}(V_{\mathbb{R}}, A)$, or simply, $\mathcal{D}$, is the space of all complex structures I on $V_{\mathbb{R}}$ so that the bilinear form $A(x, Iy)$ is symmetric and positive definite. Let $W = \{x \in V | Ix = ix\}$ so that $V = W + \overline{W}$. It follows that

$$A|_W = 0, \quad F|_W > 0 \tag{3}$$

and $\overline{W}$ is the orthogonal complement of W in V with respect to F. Hence $I \in \mathcal{D}$ is determined by W or the pair $(W, \overline{W})$ satisfying (3). Fix a point $z_o \in \mathcal{D}$ corresponding to W^o. Let $(e_1, \cdots, e_p)$ be an orthonormal basis of W^o with respect to F and let

$$e_{k+i} = \overline{e}_i, \ 1 \leqslant i \leqslant k. \tag{4}$$

As described in $I_{p,q}$ with $p = q = k$ in the description of $W \in \mathcal{D}$, it follows that W is described by a $k \times k$ symmetric complex matrices Z with $I_k - \overline{Z}^t Z > 0$. Hence $\mathcal{D}$ is identified with the bounded domain $\{Z \in M_{k,k} : Z = Z^t, I_k - \overline{Z}^t Z > 0\} \subset \mathbb{C}^{n(n+1)/2}$.

The Siegel Upper Half Space $\mathcal{S}_g$ is biholomorphic to III_g.

3.2 Here we recall briefly the classification of holomorphic totally geodesic embeddings of a Hermitian symmetric domain into another. In general, the classification of holomorphic totally geodesic embedding of a Hermitian symmetric space $N_1 = G_1/K_1$ into another Hermitian symmetric space $N_2 = G_2/K_2$ with respect to the Bergman metrics has been given by Satake [S] and Ihara [I], where G_i is a semi-simple Lie group and K_i a maximal compact subgroup for $i = 1, 2$. Since the manifolds involved are symmetric, the classification of totally geodesic embeddings is reduced to the classification of injective Lie algebra homomorphisms $\rho : \mathfrak{g}_1 \to \mathfrak{g}_2$ for the corresponding Lie groups. The invariant complex structure on N_i is given by an element $H_{oi} \in K_i$. The totally geodesic embedding is holomorphic if the condition (H_1), namely, $\rho \circ \mathrm{ad}(H_{o1}) = \mathrm{ad}(H_{o2}) \circ \rho$ is satisfied. The condition (H_1) is a consequence of the condition (H_2), namely, $\rho(H_{o1}) = H_{o2}$. This is explained on page 427 of [S]. Some explanation in terms of root system and Dynkin diagrams by Ihara in [I]. We refer the reader to [S] for any unexplained notation and terminology.

3.3 We consider now the specific situation of classification of holomorphic totally geodesic embeddings of $I_{N,1}$ into III_g. Let 0 be the origin in $B^N_{\mathbb{C}}$, which may be assumed to be mapped to the origin 0 of $\mathcal{S}_g$ realized as the bounded domain III_g as above, since the spaces involved are homogeneous. The classification is described in Theorem 1, **3.2** and the Table on page 460 of [S]. From the description there, any holomorphic embedding of $I_{N,1}$ to

III_g is a direct sum of a number of compositions of the following types of totally geodesic mappings.

Type 1, standard embeddings: This includes embedding $i_{1a} : I_{N,1} \to I_{p,q}$, for $N \leqslant p$ and $i_{1b} : III_k \to III_l$, for $k < l$, given by the standard representation or standard embedding.

Type 2, connecting embeddings: The embedding $i_2 = \iota_{p,q} : I_{p,q} \to III_{p+q}$ is given in page 432-433 of [S],

$$I_{p,q} \ni Z \longmapsto \begin{pmatrix} 0 & {}^tZ \\ Z & 0 \end{pmatrix} \in III_g \cong \mathcal{S}_{p+q}.$$

Type 3, absolutely irreducible embeddings: The embedding $i_3 : I_{p,1} \to I_{r,s}$, $r = \binom{p}{m}$, $s = \binom{p}{m-1}$, and $i_3 : I_{p,1} \to I_r$ in the case of $r = s$, which happens when $p \equiv 1 \pmod 4$ and $m = \frac{p+1}{2}$. This corresponds to the skew-symmetric tensor representations of degree m as explained in **3.2** and the Table on page 460 of [S].

The construction is described in page 448 of [S]. The representation is given by $\rho = \Lambda_m$ of $G \cong SU(p,1)$, corresponding to skew-symmetric tensors of degree m. In terms of a fundamental system of roots of the Lie algebra involved, the highest weight λ_ρ of ρ is given by

$$\lambda_\rho = (\overbrace{1, 1, \cdots, 1}^{\text{m}}, 0, \cdots 0)$$

with $1 \leqslant m \leqslant p$. Recall the setting given in **(3.1)(i)** in the case of $I_{p,1}$. Let $(e_i)_{i=1,\cdots,p+1}$ be an orthonormal basis of V, a basis of the exterior algebra Λ^m is given by $e_{i_1\cdots i_m} = e_{i_1} \wedge \cdots \wedge e_{i_m}$, $i_1 < i_2 < \cdots < i_m$. As in **(3.1)(i)**, there is a Hermitian form F on V, which induces $F^{(m)}$ on Λ^m given by

$$F^{(m)}(x_1 \wedge \cdots \wedge x_m, y_1 \wedge \cdots \wedge y_m) = \det(F(x_i, y_j))$$

for $x_i, y_j \in V$ and is invariant under $\rho(\mathfrak{g}_{su_{p,1}})$. Since

$$F^{(m)}(e_{i_1\cdots i_m}, e_{i_1\cdots i_m}) = \begin{cases} 1 & i_m < p+1, \\ -1 & i_m = p+1, \end{cases}$$

the Hermitian form $F^{(m)}$ has signature (r, s) with $r = \binom{p}{m}$, $s = \binom{p}{m-1}$.

The totally geodesic isometry of the symmetric domains $f : I_{p,1} \to I_{r,s}$ described here is given in page 448 of [S] by

(5)

$$\begin{array}{ccccccccc} \mathcal{D}(V,F) & \ni & (V_+,V_-) & \longmapsto & (\Lambda_m(V_+)\otimes 1, \Lambda_{m-1}(V_+)\otimes V_-) & \in & \mathcal{D}(\Lambda_m(V),F^{(m)}). \\ \| & & \| & & \| & & \| \\ I_{p,1} & \ni & z & \longmapsto & z' & \in & I_{r,s} \end{array}$$

In the case of representation in $III_r \subset I_{r,r}$, this corresponds to the above discussion with $r = s$ and hence $m = \frac{p+1}{2}$ and $p \equiv 1 \pmod 2$. In such case, we can define a Bilinear form B on $\Lambda_m(V) \times \Lambda_m(V)$ by

$$x \wedge y = B(x,y)e_{1\cdots p+1}$$

which satisfies

$$B(y,x) = (-1)^{m^2} B(x,y).$$

Hence if $m \equiv 1 \pmod 2$ or $p \equiv 1 \pmod 4$, the bilinear form $B(x,y)$ is an alternating bilinear form. Furthermore, there is a semi-linear transformation σ on $\Lambda_m(V)$ so that

$$F^{(m)}(x,y) = iB(x^\sigma, y),$$

where σ satisfying $\sigma^2 = 1$ is explicitly written in [S], page 449, as follows. Let $M = (i_1, \cdots, i_m)$ be an oriented subset of $(1, 2, \cdots, p+1)$ and M^c the complement. Then

$$e_M^\sigma = a(M)e_{M^c},\ a(M) = -i\epsilon(M^c, M)\eta(M), \tag{6}$$

where $\epsilon(M^c, M) = \pm 1$ is the signature of the permutation of (M^c, M) with respect to $(1, 2, \cdots, p+1)$ and $\eta(M) = -1$ (resp. 1) if $p+1 \in M$ (resp. $p+1 \notin M$). Hence $a(M) = \pm i$. σ serves and complex conjugate as in **3.1(ii)**. In this case, the bilinear form B serves as the bilinear form A as needed in equation (2) in **3.1(ii)** for the definition of III_r. The totally geodesic isometry of the symmetric domains $f : I_{p,1} \to III_r$ is given by (5) with $r = s$.

We summarize the result of [S], which is relevant to us, from Theorem 1 and the Table on page 460 in [S]. As explained in **3.2**, totally geodesic holomorphic embedding corresponds to condition (H_1), which by (a) below reduces the problem to representations satisfying condition (H_2). The Table on page 460 in [S] summarizes the representations satisfying condition (H_2) determined in Section 3 of [S].

Proposition 2. *(Satake)*
(a). Let ρ be a representation of $\mathfrak{g} = \mathfrak{su}(\mathfrak{m}, \mathfrak{1})$ into III_g satisfying condition (H_1). Then there exists absolutely irreducible representations ρ_i $(1 \leqslant i \leqslant r_1)$

of $\mathfrak{g}$ *into* $(III)_{p_i}$ $(p_i > 0)$ *satisfying* (H_2) *and absolutely irreducible representations* ρ_i $(r_1 \leqslant i \leqslant r_1 + r_2)$ *of* $\mathfrak{g}$ *into* $(I)_{p_i,q_i}$ $(p_i, q_i \geqslant 0, p_i + q_i > 0)$ *satisfying* (H_2) *such that* ρ *is* k *equivalent to the direct sum of representations* $\sum_{i=1}^{r_1} \rho_i + \sum_{i=r_1+1}^{r_1+r_2} \iota_{p_i,q_i} \circ \rho_i$ *up to a trivial representation, where* $\sum_{i=1}^{r_1} p_i + \sum_{i=r_1+1}^{r_2} (p_i + q_i) \leqslant g$.

(b). An absolutely irreducible representation ρ *of* $\mathfrak{g}$ *into* $I_{r,s}$ *or* III_r *satisfying* (H_2) *corresponds to embeddings of type* i_3 *described earlier.*

We refer the reader to the original source [S] for any unexplained terminology. In particular, the notion of k-equivariant and direct sum are described in §1 and §2 of [S]. The result of Satake applies to any semi-simple Lie algebra $\mathfrak{g}$ of Hermitian type.

Let us now describe the mapping given in (5) more carefully, which is to be used later.

Lemma 1. *In terms of standard coordinates, the totally geodesic mapping* $f : B_n \to I_{r,s} \subset \mathbb{C}^N$ *or* $III_r \subset \mathbb{C}^N$ *for* $r = s$ *with* $f(0) = 0$ *and* $z' = f(z)$ *as described above is linear in* z*, with image given by the intersection* $f(B_n)$ *with a subspace of* $\mathbb{C}^N$ *of appropriate dimension.*

Proof. We remark that the standard coordinates as used in [S] in the description above are also the Harish-Chandra coordinates.

Consider first $f : I_{p,1} \to I_{r,s}$ as given by (5). In terms of (1) for $I_{p,1}$, the point z in (5) corresponds to V_+ being spanned by $\sum_{i=1}^{p} e_i z_i + e_{p+1}$. Since it is a holomorphic totally geodesic embedding, the mapping is equivariant with respect to the action of G and in particular invariant under the action of the isotropy group $K = S(U(p) \times U(1))$. In particular, it suffices for us to investigate the image $f(z)$ for $z = (z_1, 0, \cdots, 0)$ with $|z_1|^2 < 1$. Again, we use $(e_1, \cdots, e_p)$ and (e_{p+1}) to denote some orthonormal basis of $V_+^{(0)}$ and $V_-^{(0)}$ respectively. With z as described, an orthonormal basis of V_- and V_+ at z are e'_{p+1} and $(e'_1, e_2, \cdots, e_p)$ respectively, where

$$(7) \qquad e'_{p+1} = \frac{1}{\sqrt{1 + |z|^2}}(z_1 e_1 + e_{p+1}), \; e'_1 = \frac{1}{\sqrt{1 + |z|^2}}(e_1 - \overline{z}_1 e_{p+1}).$$

To describe z' in the image of f in (5), we need to investigate $\Lambda_{m-1}(V_+) \otimes V_-$ where $F^{(m)}$ is negative definite, and express them in terms of base vectors of $\Lambda_m^{(0)} = \Lambda_m(V_+^{(0)}) \oplus \Lambda_{m-1}(V_+^{(0)}) \otimes V_-^{(0)}$. From definition, $\Lambda_{m-1}(V_+^{(0)}) \otimes V_-^{(0)}$

is generated by a basis consists of the following two types of elements,
(i) $e_{i_1\cdots i_{m-1}} \wedge e'_{p+1}$, where $1 < i_1 < \cdots < i_{m-1} \leqslant p$,
(ii) $e'_1 \wedge e_{i_1\cdots i_{m-2}} \wedge e'_{p+1}$, where $1 < i_1 < \cdots < i_{m-2} \leqslant p$.

From the formula of e'_{p+1} in (7), we compute in case (i) that

$$e_{i_1\cdots i_{m-1}} \wedge e'_{p+1} = \frac{1}{\sqrt{1+|z|^2}}((-1)^{m-1}z_1 e_{1i_1\cdots i_{m-1}} + e_{i_1\cdots i_{m-1}(p+1)}) \tag{8}$$

which is proportional to

$$(-1)^{m-1}z_1 e_{1i_1\cdots i_{m-1}} + e_{i_1\cdots i_{m-1}(p+1)}. \tag{9}$$

Notice that we need the coefficient of $e_{i_1\cdots i_{m-1}(p+1)}$ to be 1 in the format of (1).

In case (ii), (7) gives

$$e'_1 \wedge e'_{p+1} = e_1 \wedge e_{p+1}$$

and hence

$$e'_1 \wedge e_{i_1\cdots i_{m-2}} \wedge e'_{p+1} = e_{1i_1\cdots i_{m-2}(p+1)}. \tag{10}$$

Let $u = \binom{p-1}{m-1}$. It follows from the above explicit computation that for $f : I_{p,1} \to I_{r,s}$ as given in (5), the coordinates of $z' = f(z)$ are given by

$$\begin{aligned} &z = (z_1, 0, \cdots, 0) \mapsto f(z) = z' \\ &\text{with } z'_{ij} = \begin{cases} (-1)^{m-1}z_1, & 1 \leqslant i \leqslant t, 1 \leqslant j \leqslant u \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{11}$$

Similar constructions apply to $f : I_{p,1} \to III_r \subset I_{r,r}$ corresponding (5) with $r = s$. We recall that the coordinates z' in III_r in the image of f is determined according to (1) in (3.1) with respect to a corresponding basis of vectors in $(\Lambda_m(V_+) \otimes 1, \Lambda_{m-1}(V_+) \otimes V_-)$. Moreover, the choice of the base vectors is given by (4), choosing e_{r+i} to be the complex conjugate of e_i. In our case of $\Lambda_m^{(0)}$, the complex conjugate is given by σ in the setting of (5) and $e^\sigma_{I_m} = a(I_m)e_{I^c_m}$ in terms of earlier notations, here $I_m = i_1 \cdots i_m$ is an index set. Hence $z' = (z'_{ij})_{1\leqslant i,j\leqslant r}$ are determined by having a basis of

$\Lambda_{m-1}(V_+)\otimes V_-$ of the form

$$\begin{aligned}(12)\qquad &\sum_{i=1}^{r} e^{\sigma}_{I_i(p+1)}z'_{ij}+e_{I_j(p+1)}\\ &\qquad=\sum_{i=1}^{r} a(I_i(p+1))e_{(I_i(p+1))^c}z'_{ij}+e_{I_j(p+1)},\ 1\leqslant j\leqslant r\end{aligned}$$

where $a(I_i(p+1))=\pm 1$. From the expressions in (9) and (10), we see that (11) still applies in the sense that z'_{ij} are either 0 or $(-1)^m z_1$. This computation shows that in terms of the standard coordinates of the bounded domains in $\mathbb{C}^N$ as described in **1.3, 1.4** of [S] or Chapter 4 §2 of [Mok 1] for the classical domains that the image is the intersection of a line in $\mathbb{C}^n$ with $I_{r,s}$ or I_r. As mentioned earlier, this works for any complex direction obtained under the action of the isotropy group at 0 on the domain. Since the mapping f is a totally geodesic mapping and is hence equivariant under the action of the isotropy group, the lemma follows. □

3.4 We now state the main result of this section.

Proposition 3. *Let $i: B^N_{\mathbb{C}}\to III_g$ be a totally geodesic embedding. Then there exists a holomorphic map $p: III_g\to B^N_{\mathbb{C}}$ so that $p\circ i$ is the identity mapping on $B^N_{\mathbb{C}}$.*

Proof. Again, we let 0 be the origin in $B^N_{\mathbb{C}}$, which may be assumed to be mapped to the origin 0 of III_g, a bounded domain realization of $\mathcal{S}_g$, as discussed earlier. To streamline the presentation, let us consider each simple type of presentations i_1, i_2, i_3 in details before the general case described in Proposition 2.

Type 1: This includes embedding is given by $i_{1a}: I_{N,1}\to I_{p,q}$, for $N\leqslant p$ or $i_{1b}: III_k\to III_l$, for $k<l$, given by the standard embedding into the corresponding upper left hand corner of the image.

The classical domain $I_{p,q}$ is given as a symmetric space G/K with $G=SU(p,q)$ and $K=S(U(p)\times U(q))$, where K is the isotopy group at 0. For a holomorphic totally geodesic embedding $i_{1a}: B^N_{\mathbb{C}}\cong I_{N,1}\to I_{p,q}$, conjugating by K if necessary, we may assume that $\frac{\partial}{\partial z_{11}}\in (i_{1a})_*(T_{B^N_{\mathbb{C}}})$. Now we observe that $\frac{\partial}{\partial z_{ij}}$ with $i>1$ and $j>1$ cannot lie in $(i_{1a})_*(T_{B^N_{\mathbb{C}}})$, for otherwise the image $i_{1a}(B^N_{\mathbb{C}})$ as a symmetric space would have real rank at least 2 considering the tangent vectors $\frac{\partial}{\partial z_{11}}$ and $\frac{\partial}{\partial z_{ij}}$, contradicting the fact that

$I_{1,N}$ has real rank 1. It follows that $i_{1a}(B_{\mathbb{C}}^N)$ has to lie in one of the following two subspaces of $I_{p,q}$

$$\begin{aligned} & I_{1,q} = \{z = [z_{ij}] \in I_{p,q} | z_{ij} = 0 \text{ for } i \geqslant 2\} \\ \text{or } & I_{p,1} = \{z = [z_{ij}] \in I_{p,q} | z_{ij} = 0 \text{ for } j \geqslant 2\}. \end{aligned}$$

In either case, $N \leqslant \max(p, q) = q$.

Suppose $i_{1a}(B_{\mathbb{C}}^N) \subset I_{1,q}$. Conjugating by some elements in K if necessary, we may assume that mapping i_{1a} is given by

$$i_{1a}(z_1, \cdots, z_N) = \begin{bmatrix} z_1 & \cdots & z_N & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Consider the holomorphic projection map $p_{1a} : I_{p,q} \to B_{\mathbb{C}}^N$ given by

$$p_{1a}(\begin{bmatrix} z_{11} & \cdots & z_{1N} & z_{N+1} & \cdots & z_{1q} \\ \vdots & & & & & \vdots \\ z_{p1} & \cdots & z_{pN} & z_{N+1} & \cdots & z_{pq} \end{bmatrix}) = \begin{bmatrix} z_{11} & \cdots & z_{1N} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Denote by Y the matrix in the domain and Z the matrix in the image. As the $p \times q$ matrix Y satisfies $I -{}^t\overline{Y}Y > 0$ from definition, it follows that $\sum_{i=1}^N |z_i|^2 \leqslant \sum_{i=1}^p |z_i|^2 < 1$ and hence the image of p_{1a} lies in $B_{\mathbb{C}}^N$. Furthermore, it follows from definition that $p_{1a} \circ i_{1a} = 1_{B_{\mathbb{C}}^N}$. Hence p_{1a} gives us the retraction that we need.

For $i_{1b} : III_k \to III_l$, for $k < l$, given by the standard embedding into the corresponding upper left hand corner of the image. Denote by $p_{1b} : III_l \to III_k$ the projection onto the upper left hand corner

$$p_{1b}([z_{ij}]_{i,j=1,\cdots,l}) \quad = \quad [z_{ij}]_{i,j=1,\cdots,k}.$$

Let $Y = [z_{ij}]_{i,j=1,\cdots,l}$ and $U = [z_{ij}]_{i,j=1,\cdots,k}$. The fact that Y is symmetric implies that U is symmetric. Now $I_g - \overline{Y}Y > 0$ implies that $I_{g'} - \overline{U}U > 0$ as each column vector of U is part of a column vector of Y. Hence the image lies in III_k.

It is clear that $p_{1b} \circ i_{1b}|_{III_{g'}}$ is the identity map and hence p_{1b} is a projection.

Type 2: $i_2 = \iota_{p,q} : I_{p,q} \to III_{p+q}$ with $\iota_{p,q}(Z) = \begin{pmatrix} 0 & {}^tZ \\ Z & 0 \end{pmatrix}$. Define $i_2 : III_{p+q} \to I_{p,q}$ the projection

$$III_{p+q} \ni Y = \begin{pmatrix} W_1 & {}^tZ \\ Z & W_2 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & {}^tZ \\ Z & 0 \end{pmatrix} \longmapsto Z \in I_{p,q}.$$

Here Y is symmetric. From the fact that $I_{p+q} - \overline{Y}Y > 0$, it follows that $I_q -^t \overline{W}W -^t \overline{Z}Z > 0$ and hence $I_q -^t \overline{Z}Z > 0$. Hence the image of i_2 is really in $I_{p,q}$. It follows from definition that $p_2 \circ i_2 = 1_{I_{p,q}}$ is the identity map.

Type 3: $i_3 : I_{p,1} \to I_{r,s}$ or III_r corresponding to the skew-symmetric tensor representations of degree m. Since the second case can be considered as a special case of the first case, it suffices for us to consider the case of $I_{r,s}$ being the image. From Lemma 1 in **3.3**, the image of i_3 is a linear subspace $R := \mathrm{Im}(i_3)$ passing through the origin in III_r. We claim that the argument of Lemma 1 shows that there is a projection $p_3 : I_{r,s} \to f(I_{p,1})$ so that $p_3 \circ i_3|_{I_{p,1}}$ is identity on $I_{p,1}$. We actually take p_3 to be the orthogonal projection of $I_{r,s}$ to the complex linear subspace P of $\mathbb{C}^N$ containing $f(I_{p,1})$ as a subdomain. To prove the claim, since the domain involved is a convex domain in $\mathbb{C}^N$, it suffices for us to show that $p_3(I_{r,s}) = f(I_{p,1})$. Clearly $p_3(I_{r,s})$ contains $f(I_{p,1})$. This in turn follows if we can prove the corresponding statement for the projection of $I_{r,s}$ to $f(B^1_{\mathbb{C}})$ for a geodesic $B^1_{\mathbb{C}} \subset B^p_{\mathbb{C}} \cong I_{p,1}$ through the origin, since f is equivariant with respect to the action of the isotropy groups at 0. Hence it suffices for us to show that the projection p_4 of $I_{r,s}$ to the line in $\mathbb{C}^N$ containing $f(B^1_{\mathbb{C}})$ is actually $f(B^1_{\mathbb{C}})$, where $B^1_{\mathbb{C}} = \{(z_1, 0, \ldots, 0) : |z_1| < 1\}$ as studied in the proof of Lemma 1.

Recall that $r = \binom{p}{m}$, $s = \binom{p}{m-1}$ and $u = \binom{p-1}{m-1}$. From (11) in the proof of Lemma 1, the image of f is

$$f(B^1_{\mathbb{C}}) = \left\{ \begin{pmatrix} z_1 I_u & 0_{u,a} \\ 0_{b,u} & 0_{b,a} \end{pmatrix},\ (z_1, 0, \ldots, 0) \in B^1_{\mathbb{C}} \right\} \tag{13}$$

where $b = \binom{p-1}{m}$, $a = \binom{p-1}{m-2}$, I_t is the identity matrix of size t and $0_{c,d}$ is the zero matrix of size $c \times d$. Clearly

$$f(B^1_{\mathbb{C}}) \subset I_{r,r} \cong \begin{pmatrix} I_{r,r} & 0_{u,a} \\ 0_{b,u} & 0_{b,a} \end{pmatrix},$$

where $I_{r.r}$ is a bounded symmetric domain of type I.

Similar to the construction of I_{1a} in the standard embedding earlier, it is clear that there is a projection $q_1 : I_{r,s} \to I_{u,u}$ as explained above by taking zeros in non-relevant entries. Hence it suffices for us to show that there is a retraction $q_2 : I_{u,u} \to \{(z_1 I_u) : |z_1| < 1\}$. Let $w = (w_{ij}) \in I_{u,u}$. By definition, it satisfies $I_q - \bar{w}^t w > 0$ and hence $|w_{ij}| < 1$ for each i, j. It suffices for us to define

$$q_2(w) = (\frac{1}{u^2}\sum_{i,j=1}^{u} w_{ij})I_u.$$

Clearly $|\frac{1}{u^2}\sum_{i,j=1}^{u} w_{ij}| < 1$ as $|w_{ij}| < 1$. Furthermore $q_2(\zeta I_u) = \zeta I_u$ for $\zeta \in \mathbb{C}$, $|\zeta| < 1$, and hence $q_2|_{\mathrm{Im}(f)}$ is the identity map. Now it suffices for us to let $p_4 = q_2 \circ q_1$.

This concludes the discussions on embeddings of the simple types. We now combine the results from the above discussions with those from Proposition 2. According to Proposition 2a, the representation ρ involved is of form

$$\sum_{i=1}^{r_1} \rho_i + \sum_{i=r_1+1}^{r_1+r_2} \iota_{p_i,q_i} \circ \rho_i.$$

Suppose $(r_1, r_2) = (1, 0)$ or $(0, 1)$, that is the representation is irreducible. Then the projection occurs from composition of projections corresponding to Type 1, 2, 3 embeddings respectively, making use of Proposition 2a.

Consider now the general case. Note that each of the factors above corresponds to image lying in some Type III classical domain III_{s_i} for $1 \leqslant i \leqslant r_1$ or $III_{p_j+q_j}$ for $r_1 + 1 \leqslant j \leqslant r_1 + r_2$. As explained in §2 of [S], this corresponds to diagonal blocks of square matrices in III_g, where $\sum_{i=1}^{r_1} s_i + \sum_{j=r_1+1}^{r_1+r_2}(p_i + q_i) \leqslant g$. For simplicity of notation, let us just define $s_j = p_j + q_j$ for $r_1 + 1 \leqslant j \leqslant r_1 + r_2$. Hence we have

$$i : I_{p,1} \xrightarrow{i_a} \begin{pmatrix} III_{s_1} & 0 & & \cdots \\ 0 & III_{s_2} & & \cdots \\ \vdots & \vdots & \cdots & \end{pmatrix} \xhookrightarrow{i_b} I_g.$$

The projection

$$p_b : I_g \longrightarrow \begin{pmatrix} III_{s_1} & 0 & & \cdots \\ 0 & III_{s_2} & & \cdots \\ \vdots & \vdots & \cdots & \end{pmatrix}$$

so that $p_b \circ i_b$ is identity is constructed exactly as in those Type i_1 embeddings discussed earlier. For each $III_{s_i}, 1 \leqslant i \leqslant r_1 + r_2$, there is a projection

$p_{s_i}: III_{s_i} \to I_{p,1}$ so that $p_{s_i} \circ i_1$ is the identity map according to earlier discussions. The projection

$$p_a : \begin{pmatrix} III_{s_1} & 0 & & \cdots \\ 0 & III_{s_2} & & \cdots \\ \vdots & \vdots & \cdots & \end{pmatrix} \longrightarrow I_{p,1}$$

is then defined by identifying $I_{p,1}$ with $i_a(I_{p,1})$ and letting

$$p_a = \frac{1}{r_1 + r_2} p_{s_1} + \cdots + \frac{1}{r_1 + r_2} p_{s_{r_1+r_2}},$$

where addition is given in terms of the coordinate functions of the standard realization, which is the Harish-Chandra coordinates for I_g. Clearly from construction, $p_a \circ i_a$ is the identity. We may then define $p = p_a \circ p_b$. It follows from construction that $p \circ i$ is the identity. □

3.5 Though not really needed for this article, we mention that the argument of Proposition 3 can be applied to other pairs of Hermitian symmetric spaces of non-compact type, following case-by-case checking as done above for classical domains using the results of [S], see also [I]. In a more inspiring way, Ngaiming Mok [Mok3] has shown us a conceptual proof of such a result for all Hermitian symmetric spaces of non-compact type, including those containing factors of exceptional types without using classification results. This was done in terms of the Lie triple system and Harish-Chandra embedding for all pairs of Hermitian symmetric spaces of non-compact type.

4. Proofs of the main results

4.1 Denote by $g_{V,K}$ the Kobayashi (pseudo-)metric of a variety V, which is the positive semi-definite Finsler metric defined as

$$\sqrt{g_{V,K}}(x,v) = \inf\{\frac{1}{r} | \exists f : \Delta_r \to V \text{ holomorphic}, f(0) = x, f'(0) = v\},$$

where Δ_r is the disk of radius r in $\mathbb{C}$ centered at the origin. It follows from definition that the metric on a manifold is the same as on its universal covering from the lifting properties of a map from the unit disk. Note for orbifolds, we are considering orbifold maps and orbifold uniformization as explained in **2.1**.

From Ahlfors Schwarz Lemma, it follows easily that the Kobayashi metric on a complex ball is precisely the same as the Poincaré metric which is the same as the Bergman metric. The reader may consult Proposition 3 of [Y1] and the references quoted there for various forms of Schwarz Lemma. Since we are considering quotients of bounded domains, it follows from Schwarz Lemma that the Kobayashi metric is positive definite in this paper. Note that the Teichmüller space $\mathcal{T}_g$ can be realized as a bounded domain in $\mathbb{C}^{3g-3}$ from Bers Embedding.

Furthermore, it follows immediately from the definition that $g_{V,K}$ has decreasing properties in the following sense. Let $F : M \to N$ be a holomorphic mapping. Then $g_{N,K}(F_* v) \leqslant g_{M,K}(v)$ for all $v \in T_x M$. Again, the Kobayashi metric may be degenerate in general, but in our case it is always non-degenerate. This follows from the fact that the manifolds involved by uniformized by bounded domains in $\mathbb{C}^n$ for some $n > 0$ and the earlier discussions.

We have the following consequence of the decreasing property of the Kobayashi metric.

Lemma 2. *Suppose $M = B^N_{\mathbb{C}}/\Gamma$ is a totally geodesic subvariety of $\mathcal{A}_g$. Then $g_{M,K} = (j_g^{-1})^* g_{T_g^o,K}|_M = g_{\mathcal{A}_g,K}|_M$.*

Proof. The inclusion map $i : M \to \mathcal{A}_g$ is a holomorphic embedding. Now we have the holomorphic mappings

$$M \to \mathcal{M}_g^o \to \mathcal{A}_g. \tag{14}$$

The first holomorphic map in (14) comes from our assumption that $M \subset T_g^o$ and the fact that $j_g^{-1}|_{T_g^o - j_g(H_g)}$ is a holomorphic map. Here we used the fact that j_g is an injective holomorphic map and is an immersion on $\mathcal{M}_g - H_g$. The second holomorphic mapping in (14) follows from Torelli mapping. The Torelli mapping is holomorphic by definition. Since M is a complex submanifold of $T_g^o \subset \mathcal{A}_g$, it follows from definition of the Kobayashi metric in terms of extremal functions that

$$g_{M,K} \geqslant (j_g^{-1})^* g_{T_g^o,K}|_M \geqslant g_{\mathcal{A}_g,K}|_M. \tag{15}$$

On the other hand, the Kobayashi metric on a manifold $g_{V,K}$ is the same as its lift $g_{\widetilde{V},K}$ to the universal covering $\widetilde{V}$ of V. Hence in terms of a totally geodesic B^N in Siegel $\mathcal{S}_g$, we need to compare $g_{B^N,K}$ and $g_{\mathcal{S}_g,K}|_{B^N}$. It follows from Proposition 3 that there is a holomorphic map $p : \mathcal{S}_g \to \widetilde{M}$ so that $p \circ i$ is identity. Hence for $x \in \widetilde{M} \subset \mathcal{S}_g$ and $w \in T_x\widetilde{M} \subset T_x\mathcal{S}_g$, a holomorphic curve $f : \Delta_r \to \mathcal{S}_g$ holomorphic with $f(0) = x$, $f'(0) = w$ gives rise to a holomorphic map $p \circ f : \Delta_r \to \widetilde{M}$ holomorphic with $p \circ f(0) = x$ and $(p \circ f)'(0) = w$. It follows from the decreasing property of the Kobayashi metric that $g_{\mathcal{S}_g,K}|_{B^N} \geqslant g_{B^N,K}$, which is equivalent to $g_{\mathcal{A}_g,K}|_M \geqslant g_{M,K}$ after descending to M from the universal covering as discussed earlier.

Combining the above two paragraphs, we conclude that $g_{\mathcal{S}_g,K}|_{B^N} = g_{B^N,K}$ and that the two inequalities in (15) can be replaced by equalities. □

4.2 Theorem 1 now follows by putting the earlier arguments together.

Proof of Theorem 1. Assume for the sake of proof by contradiction that there exists $M = B^N_{\mathbb{C}}/\Gamma$ so that M is a totally geodesic subvariety of $\mathcal{A}_g$ with $N > 1$, and $M \subset T^o_g - j_g(H_g) = j_g(\mathcal{M}^o_g)$, where $\mathcal{M}^o_g = \mathcal{M}_g - H_g$.

From Lemma 2, $g_{M,K} = (j_g^{-1})^* g_{T^o_g,K}|_M = g_{\mathcal{A}_g,K}|_M$. In particular, $(j_g^{-1})^* g_{\mathcal{M}^o_g,K}|_M = g_{M,K}$. This however contradicts Proposition 1. □

4.3 We remark that the argument in the proof of Theorem 1 can be applied to study the non-existence of locally Hermitian symmetric space in $T^o_g - j_g(H_g)$ as a totally geodesic complex suborbifold for $g \geqslant 2$, a result proved earlier in [Ha]. For this purpose, we observe that an analogue of Proposition 1 is true for $\widetilde{M}$ being any Hermitian symmetric space as given in [A]. Together with the remarks given in **3.5** and the results of [Mok3], the other parts of the proof can be applied. See also [Y2] for some other arguments.

4.4 *Proof of Theorem 2.* From the results of [Ha], we know that any symmetric variety M in $T^o_g - j_g(H_g) \subset \mathcal{A}_g$ has to be of real rank 1 as a locally symmetric space. Alternatively, to make the article more self-contained, this follows from **3.5, 4.3** and the proof of Theorem 1. Since M is Hermitian symmetric, we know that M has to be a complex ball quotient. Theorem 1 implies that the complex dimension of M is 1 and hence M is a hyperbolic Riemann surface. From the discussions above, such a Riemann surface M has to be a Shimura-Teichmüller in the terminology of [Mö], since the Kobayashi metric, which is well-known to be the same as the Teichmüller metric on Teichmüller spaces, is the same as the natural hyperbolic metric on M as it is a totally geodesic curve in $T^o_g - j_g(H_g)$. In such a case, Möller proved in [Mö] that such a Riemann surface does not exist for genus $g \neq 3, 4, 5$, and the only examples for $g = 3, 4$ are given in the statement of Theorem 2. Very recently, it was proved by Aulicino and Norton in [AN] that there is no example in $g = 5$. Theorem 2 follows. □

Theorem 2 gives a necessary and sufficient condition for the existence of a locally Hermitian symmetric space in $T_g^o - j_g(H_g)$.

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