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Some criteria for uniform K-stability

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We prove some criteria for uniform K-stability of log Fano pairs. In particular, we show that uniform K-stability is equivalent to β-invariant having a positive lower bound. Then we study the relation between optimal destabilization conjecture and the conjectural equivalence between uniform K-stability and K-stability in twisted setting.

1. Introduction

K-stability is an important concept introduced in [25] (and later algebraically reformulated in [15]) to test whether there is a Kähler-Einstein metric on a projective Fano manifold (see in particular [11–13, 26]). However, it's difficult to check K-stability of a Fano manifold and various equivalent but simpler criteria have been introduced in terms of special test configurations [24], valuations and filtrations [18, 23] and stability thresholds (or δ-invariants) [5, 19].

In this note, we give some more criteria for uniform K-stability from these perspectives. We note here that the concept of uniform K-stability is introduced by [10, 14]. Since uniform K-stability has certain openness property, i.e. it is preserved after small perturbation of the boundary divisor (see [17]), we first have the following criterion (note that the direction (1) $\Rightarrow$ (2) has been known by [17]).

Theorem 1.1 (=Theorem 3.1). *Let (X, Δ) be a log Fano pair. The following are equivalent:*

1) *(X, Δ) is uniformly K-stable.*
2) *There exists a $\epsilon > 0$ such that $(X, \Delta + \epsilon D)$ is K-semistable for any $D \in |-K_X - \Delta|_{\mathbb{R}}$.*

Our next criterion gives a way to test uniform K-stability using only β-invariant (see Section 2 for related definitions):

Theorem 1.2. *Let (X,Δ) be a log Fano pair. The following are equivalent:*

1) *(X,Δ) is uniformly K-stable.*
2) *There exists $\epsilon > 0$ such that $\beta_{X,\Delta}(E) \geq \epsilon$ for any divisor E over X.*
3) *There exists $\epsilon > 0$ such that $\beta_{X,\Delta}(E) \geq \epsilon$ for any dreamy divisor E over X.*
4) *There exists $\epsilon > 0$ such that $\beta_{X,\Delta}(E) \geq \epsilon$ for any weakly special divisor E over X.*

It is well expected that K-stability is equivalent to uniform K-stability. This is known to be true in the smooth case by [1] and the solution of the Yau-Tian-Donaldson conjecture. In general, the statement is equivalent to the existence of divisorial valuation computing δ-invariant when $\delta(X,\Delta) = 1$ (see Section 5 or [9]). In [8], an algebraic theory of twisted K-stability (first introduced by [14]) is developed to study $\mathbb{Q}$-Fano varieties that are not uniformly K-stable. We introduce the concept of twisted uniform K-stability and similarly expect it to be equivalent to twisted K-stability. We then explore the relation between this equivalence and the existence of divisorial valuation computing δ-invariant when $\delta(X,\Delta) < 1$. In particular, we prove:

Theorem 1.3 (=Theorem 4.4). *Let (X,Δ) be a log Fano pair with $\delta(X,\Delta) \leq 1$, then for any $0 < \mu < \delta(X,\Delta)$, (X,Δ) is μ-twisted uniformly K-stable. Besides, (X,Δ) is $\delta(X,\Delta)$-twisted K-semistable but not $\delta(X,\Delta)$-twisted uniformly K-stable.*

This is a refinement of the twisted valuative criterion established in [8]. Using this result, we establishes the equivalence between the existence of divisorial δ-minimizer and the conjecture "K-stable = Uniformly K-stable" in the twisted setting.

Theorem 1.4 (=Theorem 5.4). *Let (X,Δ) be a log Fano pair with $\delta(X,\Delta) \leq 1$. The following are equivalent:*

1) *$\delta(X,\Delta)$ is computed by a divisorial valuation.*
2) *For any $0 < \mu \leq 1$, μ-twisted K-stable is equivalent to μ-twisted uniformly K-stable.*

The paper is organized as follows. In Section 2, we recall the notion and some preliminaries that will be used later. In Section 3, we prove the criteria for uniform K-stability, i.e. Theorems 1.1 and 1.2. In Section 4, we introduce the concept of twisted K-stability and twisted uniform K-stability and prove Theorem 1.3. In Section 5, we prove Theorem 1.4.

2. Preliminaries

We work over $\mathbb{C}$. We refer to [21, 22] for the definition of singularities of pairs. A projective normal variety X is called $\mathbb{Q}$-Fano if $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor and X admits klt singularities. A pair (X, Δ) is called log Fano if $-K_X - \Delta$ is an ample $\mathbb{Q}$-Cartier divisor and (X, Δ) is klt. The $\mathbb{R}$-linear system of an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor L is defined to be $|L|_{\mathbb{R}} = \{D \geq 0 \,|\, D \sim_{\mathbb{R}} L\}$. Similar one can define the $\mathbb{Q}$-linear system $|L|_{\mathbb{Q}}$ of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor.

2.1. Test configurations

Let (X, Δ) be a log Fano pair. A test configuration $(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L})$ of $(X, \Delta; -K_X - \Delta)$ consists of the following data:

1) A projective morphism $\pi : \mathcal{X} \to \mathbb{A}^1$ and an effective $\mathbb{Q}$-divisor Δ_{tc} on $\mathcal{X}$.
2) A relatively ample $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$.
3) A $\mathbb{C}^*$-action on $(\mathcal{X}, \Delta_{\mathrm{tc}}; r\mathcal{L})$ for some sufficiently divisible integer r such that $(\mathcal{X}^*, \Delta_{\mathrm{tc}}^*; r\mathcal{L}|_{\mathcal{X}^*})$ is $\mathbb{C}^*$-equivariantly isomorphic to $(X, \Delta; -r(K_X + \Delta)) \times (\mathbb{A}^1 \setminus 0)$ via the projection π, where $\mathcal{X}^* = \mathcal{X} \setminus \mathcal{X}_0$ and $\Delta_{\mathrm{tc}}^* = \Delta_{\mathrm{tc}}|_{\mathcal{X}^*}$.

Unless otherwise specified, all test configurations considered in this note are assumed to be normal, i.e. $\mathcal{X}$ is normal in the above definition. One can glue $(\mathcal{X}, \Delta_{\mathrm{tc}})$ and $(X, \Delta) \times (\mathbb{P}^1 \setminus 0)$ along their common open subset $(X, \Delta) \times (\mathbb{A}^1 \setminus 0)$ to get a natural compactification $(\overline{\mathcal{X}}, \overline{\Delta}_{\mathrm{tc}}; \overline{\mathcal{L}})$. A test configuration $(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L})$ is called special (resp. weakly special) if $(\mathcal{X}, \Delta_{\mathrm{tc}} + \mathcal{X}_0)$ is *plt* (resp. *lc*) and $\mathcal{L} \sim_{\mathbb{Q}} -K_{\mathcal{X}/\mathbb{A}^1} - \Delta_{\mathrm{tc}}$.

A test configuration $(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L})$ is trivial if $\mathcal{X}$ is $\mathbb{C}^*$-equivariantly isomorphic to $X \times \mathbb{A}^1$. It is said to be of product type if it's induced by a diagonal $\mathbb{C}^*$-action on $(X, \Delta) \times \mathbb{A}^1$ given by a one parameter subgroup of $\mathrm{Aut}(X, \Delta)$.

2.2. K-stability

Given a test configuration $(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L})$ of an n-dimensional log Fano pair (X, Δ), its *generalized Futaki invariant* is defined as follows:

$$\mathrm{Fut}(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L}) := \frac{n\overline{\mathcal{L}}^{n+1}}{(n+1)(-K_X-\Delta)^n} + \frac{\overline{\mathcal{L}}^n \cdot (K_{\overline{\mathcal{X}}/\mathbb{P}^1} + \overline{\Delta}_{\mathrm{tc}})}{(-K_X-\Delta)^n}.$$

We say (X, Δ) is *K-semistable* if $\mathrm{Fut}(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L}) \geq 0$ for any normal test configuration $(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L})$. We say (X, Δ) is *K-stable* if $\mathrm{Fut}(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L}) > 0$ for any non-trivial normal test configuration $(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L})$. We say (X, Δ) is *K-polystable* if it is K-semistable and $\mathrm{Fut}(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L}) > 0$ for any non product type normal test configuration $(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L})$.

To define uniform K-stability, we introduce the J-functional of a test configuration $(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L})$ as follows [10, 18]:

$$J(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L}) := \frac{\Pi^*(-K_{X\times\mathbb{P}^1/\mathbb{P}^1} - \Delta_{\mathbb{P}^1})^n \cdot \Theta^*\overline{\mathcal{L}}}{(-K_X-\Delta)^n} - \frac{\overline{\mathcal{L}}^{n+1}}{(n+1)(-K_X-\Delta)^n},$$

where $\Pi : \mathcal{Z} \to X \times \mathbb{P}^1$ and $\Theta : \mathcal{Z} \to \mathcal{X}$ denote the normalization of the graph of $X \times \mathbb{P}^1 \dashrightarrow \mathcal{X}$.

We say (X, Δ) is *uniformly K-stable* if there is a positive number $0 < \epsilon < 1$ such that $\mathrm{Fut}(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L}) \geq \epsilon J(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L})$ for any normal test configuration.

2.3. Dreamy divisor and special divisor

In this subsection, we introduce two kinds of divisors which will appear frequently later.

Let (X, Δ) be a log Fano pair. We say E is a divisor over X if there is a normal birational model $\sigma : Y \to X$ such that E is a prime divisor on Y. Note that E induces a valuation ord_E of the function field $\mathbb{C}(X)$; we define divisorial valuations over X as valuations of the form $c \cdot \mathrm{ord}_E$ where $c > 0$ and E is a divisor over X.

Definition 2.1 ([18]). We say that E is a dreamy divisor or ord_E is a dreamy valuation over X if $\bigoplus_{i,j\in\mathbb{N}} H^0(X, -ir\sigma^*(K_X+\Delta) - jE)$ is finitely generated, where r is a positive integer such that $-r(K_X+\Delta)$ is Cartier.

Definition 2.2. We say that E is a (weakly) special divisor or ord_E is a (weakly) special valuation over X if it's induced by a non-trivial (weakly) special test configuration $(\mathcal{X}, \Delta_{\rm tc}; \mathcal{L})$, i.e. $\mathcal{X}_0$ is irreducible (this is automatic if the test configuration is special) and ord_E is proportional to the restriction of $\mathrm{ord}_{\mathcal{X}_0}$ (since $\mathrm{ord}_{\mathcal{X}_0}$ is a divisorial valuation on the function field $\mathbb{C}(\mathcal{X}) = \mathbb{C}(X \times \mathbb{A}^1)$, we just restrict the valuation to $\mathbb{C}(X)$ to get a divisorial valuation over X; see [10]).

We have the following characterization of dreamy divisors (see [18, Theorem 5.1] and [16, Lemma 3.8]).

Lemma 2.3. *If E is a dreamy divisor over X, then there is a test configuration $(\mathcal{X}, \Delta_{\rm tc}; \mathcal{L})$ whose central fiber is integral such that ord_E is proportional to the restriction of $\mathrm{ord}_{\mathcal{X}_0}$. Conversely, if $(\mathcal{X}, \Delta_{\rm tc}; \mathcal{L})$ is a test configuration whose central fiber is integral, then the restriction of $\mathrm{ord}_{\mathcal{X}_0}$ is a dreamy valuation over X.*

Remark 2.4. If $(\mathcal{X}, \Delta_{\rm tc}; \mathcal{L})$ is a test configuration whose central fiber is integral, then $\mathcal{L} \sim_{\mathbb{Q}} -K_{\mathcal{X}/\mathbb{A}^1} - \Delta_{\rm tc}$.

2.4. Various invariants

In this subsection, we recall the β-invariants and δ-invariants of log Fano pairs.

Definition 2.5 ([18]). Let (X, Δ) be a log Fano pair and E a divisor over X. Pick a log resolution $\pi\colon Y \to X$ such that E is a divisor on Y. The β-invariant of E (or ord_E) is defined as:

$$\beta_{X,\Delta}(E) := A_{X,\Delta}(E) - S_{X,\Delta}(E)$$

where $A_{X,\Delta}(E) := 1 + \mathrm{ord}_E(K_Y - \pi^*(K_X + \Delta))$ is the log discrepancy of E with respect to (X, Δ) and

$$S_{X,\Delta}(E) := \frac{1}{(-K_X - \Delta)^n} \int_0^\infty \mathrm{vol}(-K_X - \Delta - xE)\mathrm{d}x$$

where $\mathrm{vol}(-K_X - \Delta - xE) := \mathrm{vol}(-\pi^*(K_X + \Delta) - xE)$. Note that the above definition differs from Fujita's original definition by a multiple. We also let $T_{X,\Delta}(E)$ be the pseudo-effective threshold of $-E$ with respect to

$-(K_X+\Delta)$, i.e.

$$T_{X,\Delta}(E)=\sup\{x\in\mathbb{R}^+\mid \operatorname{vol}(-K_X-\Delta-xE)>0\}=\sup_{D\in|-K_X-\Delta|_{\mathbb{Q}}}\operatorname{ord}_E(D).$$

Finally we let $j_{X,\Delta}(E)=T_{X,\Delta}(E)-S_{X,\Delta}(E)$.

Remark 2.6. We have the following relation between $S_{X,\Delta}(E)$ and $T_{X,\Delta}(E)$ (see e.g. [5, Lemma 2.6]):

$$\frac{1}{n+1}T_{X,\Delta}(E)\le S_{X,\Delta}(E)\le \frac{n}{n+1}T_{X,\Delta}(E).$$

It then follows that

$$\frac{1}{n}S_{X,\Delta}(E)\le j_{X,\Delta}(E)\le nS_{X,\Delta}(E).$$

The β-invariant has a close relation to K-stability, as discovered in [18] and [23] (see also [9] for part of the statement):

Theorem 2.7. *Let (X,Δ) be a log Fano pair. The following are equivalent:*

1) (X,Δ) *is K-semistable (resp. K-stable, uniformly K-stable).*
2) $\beta_{X,\Delta}(E)\ge 0$ *(resp. >0, $\ge \epsilon j_{X,\Delta}(E)$ for some fixed $\epsilon>0$) for any divisorial valuation* ord_E *over X.*
3) $\beta_{X,\Delta}(E)\ge 0$ *(resp. >0, $\ge \epsilon j_{X,\Delta}(E)$ for some fixed $\epsilon>0$) for any dreamy divisorial valuation* ord_E *over X.*
4) $\beta_{X,\Delta}(E)\ge 0$ *(resp. >0, $\ge \epsilon j_{X,\Delta}(E)$ for some fixed $\epsilon>0$) for any special divisorial valuation* ord_E *over X.*

The following δ-invariant is introduced by [19] to characterize K-stability.

Definition 2.8. Let (X,Δ) be a log Fano pair. Let $m>0$ be an integer such that $-m(K_X+\Delta)$ is Cartier and $N_m:=h^0(X,-m(K_X+\Delta))\neq 0$. An m-basis type divisor of (X,Δ) is defined to be a $\mathbb{Q}$-divisor D_m of the form

$$D_m=\frac{\{s_1=0\}+\cdots+\{s_{N_m}=0\}}{mN_m}$$

where $s_1,\cdots,s_{N_m}$ a basis of $H^0(X,-m(K_X+\Delta))$. We set

$$\delta_m(X,\Delta):=\sup\big\{a\in\mathbb{R}^+|(X,\Delta+aD_m)\ \text{is lc for any } m\text{-basis type divisor } D_m\big\},$$

and

$$\delta(X,\Delta) := \limsup_m \delta_m(X,\Delta).$$

By [5], the above limsup is in fact a limit and we have

$$\delta(X,\Delta) = \inf_E \frac{A_{X,\Delta}(E)}{S_{X,\Delta}(E)},$$

where the infimum runs over all divisors E over X.

We have the following K-stability criterion in terms of δ-invariants ([5, 19]):

Theorem 2.9. *Let (X,Δ) be a log Fano pair. Then*

1) *(X,Δ) is K-semistable if and only if $\delta(X,\Delta) \geq 1$.*
2) *(X,Δ) is uniformly K-stable if and only if $\delta(X,\Delta) > 1$.*

3. Criteria for K-stability

In this section, we will establish several criteria for uniform K-stability.

Theorem 3.1. *Suppose (X,Δ) is a log Fano pair, then the following two are equivalent:*

1) *(X,Δ) is uniformly K-stable.*
2) *There exists some $\epsilon > 0$ such that $(X,\Delta+\epsilon D)$ is K-semistable for any $D \in |-K_X-\Delta|_{\mathbb{R}}$.*

Proof. For any divisorial valuation ord_E over X,

$$\beta_{X,\Delta+\epsilon D}(E) = A_{X,\Delta}(E) - \epsilon \cdot \mathrm{ord}_E(D) - (1-\epsilon)S_{X,\Delta}(E).$$

The above equality holds since we have following,

$$\begin{aligned}S_{X,\Delta+\epsilon D}(E) =&\frac{1}{\mathrm{vol}(-(1-\epsilon)(K_X+\Delta))}\int_0^\infty \mathrm{vol}(-(1-\epsilon)(K_X+\Delta)-tE)\mathrm{dt}\\ =&\frac{1}{\mathrm{vol}(-(K_X+\Delta))}\int_0^\infty \mathrm{vol}(-(K_X+\Delta)-\frac{t}{1-\epsilon}E)\mathrm{dt}\\ =&\frac{1-\epsilon}{\mathrm{vol}(-(K_X+\Delta))}\int_0^\infty \mathrm{vol}(-(K_X+\Delta)-\frac{t}{1-\epsilon}E)\mathrm{d}(\frac{t}{1-\epsilon})\\ =&(1-\epsilon)S_{X,\Delta}(E).\end{aligned}$$

Assume (2) holds, then $\beta_{X,\Delta+\epsilon D}(E) \geq 0$ for all $D \in |-K_X-\Delta|_{\mathbb{R}}$ and all divisors E over X. Taking the supremum over D we have

$$A_{X,\Delta}(E)-\epsilon T_{X,\Delta}(E)-(1-\epsilon)S_{X,\Delta}(E) \geq 0,$$

i.e.

$$A_{X,\Delta}(E)-S_{X,\Delta}(E) \geq \epsilon\cdot(T_{X,\Delta}(E)-S_{X,\Delta}(E)) = \epsilon\cdot j_{X,\Delta}(E),$$

which implies (1).

Conversely, suppose (X,Δ) is uniformly K-stable, then by Theorem 2.7, there exists some μ with $0<\mu<1$, such that $\frac{A_{X,\Delta}(E)}{S_{X,\Delta}(E)} \geq 1+\mu$ for any divisorial valuation ord_E over X. By Remark 2.6, we can choose a $0<\epsilon<1$ such that

$$(1+\mu)S_{X,\Delta}(E) \geq \epsilon\cdot T_{X,\Delta}(E)+(1-\epsilon)S_{X,\Delta}(E),$$

say $\epsilon=\frac{\mu}{n+1}$. Thus

$$\begin{aligned}\beta_{X,\Delta+\epsilon\cdot D}(E) &= A_{X,\Delta}(E)-\epsilon\cdot \mathrm{ord}_E(D)-(1-\epsilon)S_{X,\Delta}(E)\\ &\geq A_{X,\Delta}(E)-\epsilon\cdot T_{X,\Delta}(E)-(1-\epsilon)S_{X,\Delta}(E) \geq 0\end{aligned}$$

for any $D \in |-K_X-\Delta|_{\mathbb{R}}$. So, $(X,\Delta+\epsilon D)$ is K-semistable by Theorem 2.7(1). □

Inspired by the above Theorem 3.1, we can define a new invariant for a log Fano pair (X,Δ), the *uniformity* of (X,Δ), which characterizes how uniformly K-stable (X,Δ) is.

Definition 3.2. Suppose (X,Δ) is a given K-semistable log Fano pair with $\alpha(X,\Delta) \leq 1$. The uniformity of (X,Δ) is defined as follows:

$$u(X,\Delta) := \sup\left\{a \in \mathbb{R}_{\geqslant 0} | (X,\Delta+aD) \text{ is K-semistable}, \forall D \in |-K_X-\Delta|_{\mathbb{R}}\right\}.$$

We can give a precise characterization for $u(X,\Delta)$.

Proposition 3.3. *Let (X,Δ) be a K-semistable log Fano pair with $\alpha(X,\Delta) \leq 1$, then*

$$u(X,\Delta) = \inf_E \frac{\beta_{X,\Delta}(E)}{j_{X,\Delta}(E)},$$

where E runs through all divisors over X.

Proof. Suppose a is a nonnegative real number such that $(X,\Delta+aD)$ is K-semistable for any $D \in |-K_X-\Delta|_{\mathbb{R}}$, then we have

$$\beta_{X,\Delta+aD}(E) = A_{X,\Delta}(E) - \mathrm{ord}_E(aD) - (1-a)S_{X,\Delta}(E) \geq 0,$$

$\forall D \in |-K_X-\Delta|_{\mathbb{R}}$ and $\forall E$ over X. This is equivalent to

$$A_{X,\Delta}(E) - S_{X,\Delta}(E) \geq a(T_{X,\Delta}(E) - S_{X,\Delta}(E))$$

for any E over X, i.e.

$$a \leq \inf_E \frac{\beta_{X,\Delta}(E)}{j_{X,\Delta}(E)}.$$

□

By Theorem 3.1, we have the following corollary:

Corollary 3.4. *Suppose (X,Δ) is a K-semistable log Fano pair with $\alpha(X,\Delta) \leq 1$. Then*

1) *(X,Δ) is uniformly K-stable if and only if $u(X,\Delta) > 0$.*
2) *$\delta(X,\Delta) = 1$ if and only if $u(X,\Delta) = 0$.*

Remark 3.5. By Remark 2.6 we have the following relation between $u(X,\Delta)$ and $\delta(X,\Delta)-1$:

$$\frac{1}{n}(\delta(X,\Delta)-1) \leq u(X,\Delta) \leq n(\delta(X,\Delta)-1).$$

Theorem 2.7, Theorem 2.9 and Theorem 3.1 give three characterizations of uniform K-stability. We now give another criterion using only β-invariant.

Theorem 3.6. *Let (X,Δ) be a log Fano pair. The following three are equivalent:*

1) *(X,Δ) is uniformly K-stable.*

2) *There exists $\epsilon > 0$ such that $\beta_{X,\Delta}(E) \geq \epsilon$ for any divisor E over X.*

3) *There exists $\epsilon > 0$ such that $\beta_{X,\Delta}(E) \geq \epsilon$ for any dreamy divisor E over X.*

Proof. (1) $\Rightarrow$ (2): If (X,Δ) is uniformly K-stable, then there exists some $\delta > 1$ such that $A_{X,\Delta}(E) \geq \delta \cdot S_{X,\Delta}(E)$ for all divisors E over X. Since (X,Δ) is log Fano, we have $A_{X,\Delta}(E) \geq \frac{1}{r}$ where r is an integer such that $r(K_X+\Delta)$ is Cartier. Thus

$$\beta_{X,\Delta}(E) = A_{X,\Delta}(E) - S_{X,\Delta}(E) \geq (1-\delta^{-1})A_{X,\Delta}(E) \geq \frac{1-\delta^{-1}}{r}$$

for any divisor E over X and we may simply take $\epsilon = \frac{1-\delta^{-1}}{r} > 0$.

(2) $\Rightarrow$ (1): Suppose that $\beta_{X,\Delta}(E) \geq \epsilon > 0$ for all divisors over X. By [5, Corollary 3.6], there exists a sequence c_m $(m = 1, 2, \cdots)$ of numbers depending only on (X,Δ) such that $\lim_{m\to\infty} c_m = 1$ and $c_m \cdot \mathrm{ord}_E(D_m) \leq S_{X,\Delta}(E)$ for any $m \in \mathbb{N}$, any divisor E over X and all m-basis type divisor $D_m \sim_{\mathbb{Q}} -(K_X+\Delta)$. It follows that

$$\begin{aligned} A_{X,\Delta+c_mD_m}(E) &= A_{X,\Delta}(E) - c_m \cdot \mathrm{ord}_E(D_m) \\ &\geq A_{X,\Delta}(E) - S_{X,\Delta}(E) = \beta_{X,\Delta}(E) \geq \epsilon \end{aligned}$$

for all m, E and D_m as above. In other words, the pair $(X, \Delta + c_m D_m)$ is ϵ-lc. By [2, Theorem 1.6] (applied to the pair $(X, B = \Delta + c_m D_m)$, $M = D_m$ and the very ample divisor $A = -r(K_X+\Delta)$ for some sufficiently large and divisible r), there exists some $t > 0$ depending only on (X,Δ) such that $\mathrm{lct}(X, B; D_m) \geq t$ for all m and D_m. Hence $(X, \Delta + (c_m+t)D_m)$ is lc for all $m \in \mathbb{N}$ and all m-basis type divisor D_m, which implies $\delta_m(X,\Delta) \geq c_m + t$. Letting $m \to \infty$ we see that $\delta(X,\Delta) \geq 1 + t > 1$ and therefore (X,Δ) is uniformly K-stable.

(3) $\Leftrightarrow$ (2): One direction is obvious. For the other direction, note that by Theorem 2.7, (3) implies that (X,Δ) is K-semistable, hence it suffices to show that if (X,Δ) is a K-semistable log Fano pair, then any divisor E over X for which $\beta_{X,\Delta}(E) < 1$ is dreamy. This is proved in Lemma 3.7. □

Lemma 3.7. *Let (X,Δ) be a K-semistable log Fano pair and E a divisor over X. Suppose that $\beta_{X,\Delta}(E) < 1$. Then E is dreamy.*

Proof. By [5, Lemma 3.5 and Corollary 3.6], there exists m-basis type divisors $D_m \sim_{\mathbb{Q}} -(K_X+\Delta)$ $(m \in \mathbb{N})$ such that $\mathrm{ord}_E(D_m) \to S_{X,\Delta}(E)$ $(m \to$

∞). Let $\lambda_m = \min\{\delta_m(X,\Delta),1\}$. Since (X,Δ) is K-semistable, we have $\lim_{m\to\infty}\lambda_m = 1$ and $(X,\Delta+\lambda_m D_m)$ is lc for all $m\in\mathbb{N}$. Then as

$$A_{X,\Delta+\lambda_m D_m}(E) = A_{X,\Delta}(E) - \lambda_m \mathrm{ord}_E(D_m) \to \beta_{X,\Delta}(E) < 1 \ (m\to\infty),$$

we see that $A_{X,\Delta+\lambda_m D_m}(E) < 1$ for $m \gg 0$. Let $D = (\lambda_m - \epsilon)D_m + (1 + \epsilon - \lambda_m)H$ where $m \gg 0$, $0 < \epsilon \ll 1$ and $H \in |-K_X-\Delta|_{\mathbb{Q}}$ is general. Then $(X,\Delta+D)$ is klt, $K_X+\Delta+D \sim_{\mathbb{Q}} 0$ and we still have $A_{X,\Delta+D}(E) < 1$. By [4, Corollary 1.4.3], there is an extraction $\sigma : Y \to X$ from a projective normal variety Y which only extracts the divisor E. For $0<\epsilon\ll 1$, we then have

$$K_Y + \tilde{\Delta} + (1-\epsilon)\tilde{D} + \mu E = \sigma^*(K_X+\Delta+(1-\epsilon)D),$$

where $\tilde{\Delta}$ and $\tilde{D}$ are birational transformations of Δ and D respectively and

$$\mu = 1 - A_{X,\Delta+D}(E) - \epsilon\cdot \mathrm{ord}_E(D) > 0.$$

Since $(X,\Delta+(1-\epsilon)D)$ is log Fano, its crepant pullback $(Y,\tilde{\Delta}+(1-\epsilon)\tilde{D}+\mu E)$ is of Fano type, i.e. the pair admits klt singularities and $-(K_Y+\tilde{\Delta}+(1-\epsilon)\tilde{D}+\mu E)$ is big and nef. Therefore, E is dreamy by [4, Corollary 1.3.1]. □

In general, there are many dreamy divisors over a log Fano pair. We now show that those with small β-invariants are weakly special. In particular, combining with Theorem 3.6, this completes the proof of Theorem 1.2.

Theorem 3.8. *Let (X,Δ) be a K-semistable log Fano pair. Then there exists some $0<\epsilon_0\ll 1$ such that any dreamy divisor E over X with $\beta_{X,\Delta}(E) < \epsilon_0$ induces a weakly special test configuration of (X,Δ) with integral central fiber.*

Proof. Let $\mathfrak{R}\subset[0,1]$ be a finite set of rational numbers containing 1 and all the coefficients of Δ. Choose $\epsilon_0\in\mathbb{Q}\cap(0,1)$ such that a pair $(Y,B+G)$ (where G is a reduced $\mathbb{Q}$-Cartier divisor and $\dim Y \le \dim X + 1$) is lc as long as $(Y,B+(1-\epsilon_0)G)$ is lc and the coefficients of B belongs to $\mathfrak{R}$. Such ϵ_0 exists by the ACC of log canonical threshold [20, Theorem 1.1]. Suppose E is a divisor over X with $\beta_{X,\Delta}(E)<\epsilon_0$, then similar to the proof of Lemma 3.7 we can find a $D\in|-K_X-\Delta|_{\mathbb{Q}}$ such that $(X,\Delta+D)$ is klt and $A_{X,\Delta+D}(E)<\epsilon_0$. By [4, Corollary 1.4.3], one can extract E on a birational model of X, say $\mu : Y \to X$ and

$$K_Y + \widetilde{D} + \widetilde{\Delta} + cE = \mu^*(K_X + \Delta + D),$$

where $\widetilde{D}$ and $\widetilde{\Delta}$ are strict transformations of D and Δ respectively and $1 - \epsilon_0 < c < 1$. Consider the pair $(X_{\mathbb{A}^1}, \Delta_{\mathbb{A}^1} + D_{\mathbb{A}^1} + X_0)$ (where $X_{\mathbb{A}^1} = X \times \mathbb{A}^1$, etc. and $X_0 = X \times \{0\}$) which is a plt pair. Then there is an induced morphism $\mu_{\mathbb{A}^1} : Y_{\mathbb{A}^1} \to X_{\mathbb{A}^1}$. Let v be a quasi-monomial valuation over $X_{\mathbb{A}^1}$ with weight $(1,1)$ along the divisors X_0 and $E_{\mathbb{A}^1}$. It's clear that v is a divisorial valuation over $X_{\mathbb{A}^1}$ whose center is contained in X_0. Denote by $\mathcal{E}$ the corresponding divisor over $X_{\mathbb{A}^1}$, then $A_{X_{\mathbb{A}^1}, \Delta_{\mathbb{A}^1} + D_{\mathbb{A}^1} + X_0}(\mathcal{E}) = A_{X, \Delta + D}(E) < \epsilon_0 < 1$, hence by [4, Corollary 1.4.3] we can extract $\mathcal{E}$ on a projective birational model $\pi : \mathcal{Y} \to X_{\mathbb{A}^1}$ of $X_{\mathbb{A}^1}$. We have

(1)
$$K_{\mathcal{Y}} + \pi_*^{-1}\Delta_{\mathbb{A}^1} + \pi_*^{-1}D_{\mathbb{A}^1} + \widetilde{X}_0 + c\mathcal{E} = \pi^*(K_{X_{\mathbb{A}^1}} + \Delta_{\mathbb{A}^1} + D_{\mathbb{A}^1} + X_0) \sim_{\mathbb{Q}} 0,$$

where $\widetilde{X}_0$ is the strict transformation of X_0 and $c > 1 - \epsilon_0 > 0$.

Let $0 < \epsilon \ll 1$. Then it is easy to check that $(X_{\mathbb{A}^1}, \Gamma_\epsilon := \Delta_{\mathbb{A}^1} + (1 - \epsilon)(D_{\mathbb{A}^1} + X_0))$ is klt, $-(K_{X_{\mathbb{A}^1}} + \Gamma_\epsilon)$ is big and nef over $\mathbb{A}^1$ and $A_{X_{\mathbb{A}^1}, \Gamma_\epsilon}(\mathcal{E}) > 0$. It follows that $\mathcal{Y}$ is of Fano type over $\mathbb{A}^1$ as in the proof of Lemma 3.7. Run the $\widetilde{X}_0$-MMP/$\mathbb{A}^1$ on $\mathcal{Y}$, we get a minimal model $\mathcal{Y} \dashrightarrow \mathcal{Y}'$ where the birational transform of $\widetilde{X}_0$ is nef over $\mathbb{A}^1$. By [24]*Lemma 1, this implies that the MMP contracts at least one component of $\mathcal{Y}_0$. Since all $\widetilde{X}_0$-negative curves are contained in the birational transform of $\widetilde{X}_0$, $\widetilde{X}_0$ is the contracted component. Let $\Delta'_{\mathbb{A}^1}$, $D'_{\mathbb{A}^1}$ and $\mathcal{E}'$ be the pushforward of $\pi_*^{-1}\Delta_{\mathbb{A}^1}$, $\pi_*^{-1}D_{\mathbb{A}^1}$ and $\mathcal{E}$ on $\mathcal{Y}'$ respectively, then we know $\mathcal{Y}' \to \mathbb{A}^1$ has an integral central fiber $\mathcal{Y}'_0$ and the restriction of $\mathrm{ord}_{\mathcal{Y}'_0}$ is exactly ord_E. We next run the $-(K_{\mathcal{Y}'} + \Delta'_{\mathbb{A}^1})$-MMP/$\mathbb{A}^1$ to get an ample model of $(\mathcal{Y}', \Delta'_{\mathbb{A}^1}) \to \mathbb{A}^1$ with respect to $-(K_{\mathcal{Y}'} + \Delta'_{\mathbb{A}^1})$, denoted by $(\mathcal{X}, \Delta_{\mathrm{tc}}) \to \mathbb{A}^1$, where Δ_{tc} is the pushforward of $\Delta'_{\mathbb{A}^1}$. It's clear that $(\mathcal{X}, \Delta_{\mathrm{tc}}) \to \mathbb{A}^1$ is a test configuration of (X, Δ) such that the central fiber $\mathcal{X}_0$ is integral and the restriction of $\mathrm{ord}_{\mathcal{X}_0}$ is ord_E. By (1), $(\mathcal{Y}, \pi_*^{-1}\Delta_{\mathbb{A}^1} + \pi_*^{-1}D_{\mathbb{A}^1} + \tilde{X}_0 + c\mathcal{E})$ is an lc log Calabi-Yau pair and the same is true for its birational contractions (being log Calabi-Yau is preserved under birational contractions and since birational contractions between log CY pairs are crepant, they also preserve log canonicity). In particular, $(\mathcal{Y}', \Delta'_{\mathbb{A}^1} D'_{\mathbb{A}^1} + c\mathcal{E}')$ is lc and the same holds for its strict transform on $\mathcal{X}$. It follows that $(\mathcal{X}, \Delta_{\mathrm{tc}} + c\mathcal{X}_0)$ is an lc pair. As $c > 1 - \epsilon_0$, we see that $(\mathcal{X}, \Delta_{\mathrm{tc}} + \mathcal{X}_0)$ is lc by our choice of ϵ_0. □

Remark 3.9. The above theorem says the following two statements are equivalent:

1) (X,Δ) is uniformly K-stable.
2) There is a $\epsilon>0$ such that $\beta_{X,\Delta}(E)\geq\epsilon$ for any weakly special divisor E over X.

Compared with Theorems 2.7 and 2.9, one would expect that for uniform K-stability it's sufficient to check $\beta(E)\geq\epsilon$ for all special divisors E over X, although this doesn't seem to follow from our current proof.

It's expected that uniformly K-stable and K-stable are the same for any given log Fano pair. One direction is clear. Assume (X,Δ) is K-stable, to confirm uniform K-stability, it suffices to show that there exists some $\epsilon>0$ such that $\beta_{X,\Delta}(E)>\epsilon$ for any weakly special divisor E over X. Our next result (inspired by the recent work [27]) shows that it suffices to consider those E that are bounded in some sense (note that a more general version that applies to all weakly special divisor is independently proved in [7, Theorem A.2] using a somewhat different method):

Theorem 3.10. *Let (X,Δ) be a K-semistable log Fano pair. Then there exist $\epsilon_0>0$ and some positive integer N depending only on (X,Δ) such that if E is a divisor over X with $\beta_{X,\Delta}(E)<\epsilon_0$, then we can find some $G\in\frac{1}{N}|-N(K_X+\Delta)|$ such that $(X,\Delta+G)$ is lc and E is an lc place of $(X,\Delta+G)$.*

Proof. Let ϵ_0 be as in the proof of Theorem 3.8. As in the proof of Theorem 3.8, we can find a $D\in|-K_X-\Delta|_{\mathbb{Q}}$ such that $(X,\Delta+D)$ is klt and $A_{X,\Delta+D}(E)<\epsilon_0$. In addition, we can extract E to be a divisor on a projective birational model of X, say $\mu:Y\to X$ and

$$K_Y+\widetilde{\Delta}+\widetilde{D}+cE=\mu^*(K_X+\Delta+D),$$

where $\widetilde{\Delta}$ and $\widetilde{D}$ are the strict transformations and $1-\epsilon_0<c<1$. Note that Y is of Fano type as before, hence we can run MMP for $-(K_Y+\widetilde{\Delta}+E)$. Suppose the MMP ends with a Mori fiber space $Y\dashrightarrow Y'\to T$ and write $\widetilde{\Delta}'$ and E' for the pushforward of $\widetilde{\Delta}$ and E on Y', then we know $(K_{Y'}+\widetilde{\Delta}'+E')|_F$ is ample where F is the general fiber of $Y'\to T$. As $(Y,\widetilde{\Delta}+\widetilde{D}+cE)$ is a klt log Calabi-Yau pair, so is $(Y',\widetilde{\Delta}'+\widetilde{D}'+cE')$. Since $\rho(Y'/T)=1$, it follows that $(K_{Y'}+\widetilde{\Delta}'+cE')|_F$ is anti-nef and $(Y',\widetilde{\Delta}'+cE')$ is klt, thus the cone over $(F,(\widetilde{\Delta}'+cE')|_F)$ is lc and hence so is the cone over $(F,(\widetilde{\Delta}'+E')|_F)$ by the choice of ϵ_0. But by [21]*Lemma 3.1, this contradicts the fact that $(K_{Y'}+\widetilde{\Delta}'+E')|_F=K_F+(\widetilde{\Delta}'+E')|_F$ is ample (see also [20, Theorem 1.5]). So the MMP above produces a minimal model $Y \dashrightarrow Y'$ and $-(K_{Y'} + \widetilde{\Delta}' + E')$ is nef.

Now by the boundedness of complement [3, Theorem 1.7], there exists some integer $N > 0$ depending only on the dimension and the set $\mathfrak{R}$ of coefficients of Δ such that if $(Y', \widetilde{\Delta}' + E')$ is an lc pair of dimension n with coefficients in $\mathfrak{R}$, Y' is of Fano type and $-(K_{Y'} + \widetilde{\Delta}' + E')$ is nef, then there exists some effective divisor $G' \in \frac{1}{N}|-N(K_{Y'} + \widetilde{\Delta}' + E')|$ such that $(Y', \widetilde{\Delta}' + E' + G')$ is lc. It follows that E is a lc place of the lc pair $(X, \Delta + G)$ where $G \in \frac{1}{N}|-N(K_X + \Delta)|$ is the pushforward of G' to X. □

It is therefore very natural to ask the following question:

Question 3.11. *Given a set S of lc log Calabi-Yau pairs $(X, \Delta + D)$ such that (X, Δ) is log Fano. Let S' be the set of lc log Calabi-Yau pairs that can be realized as weakly special degenerations of pairs in S (i.e. integral central fibers of weakly special test configurations of pairs $(X, \Delta + D)$ in S). Assume that S is bounded. Is S' bounded?*

In particular, a positive answer to this question will lead to a proof that K-stability is equivalent to uniform K-stability (since the Futaki invariants have a bounded denominator in a bounded family). We don't know any proof or counterexample to the above question.

Remark 3.12. Theorem 3.10 also gives an approximation for $\delta(X, \Delta) = 1$ using lc places of bounded lc complements, i.e. if $\delta(X, \Delta) = 1$, then $\delta(X, \Delta) = \inf_E \frac{A_{X,\Delta}(E)}{S_{X,\Delta}(E)}$, where E is a lc place of $(X, \Delta + G)$ for some lc N-complement G of (X, Δ). See [7, Corollary 3.6] for a more general statement when $\delta(X, \Delta) \le 1$.

4. Twisted setting

In this section, we will define K-stability in the twisted setting. To make it simple, we leave out the boundary as it doesn't play essential roles. X always denotes a $\mathbb{Q}$-Fano variety with $\delta(X) \le 1$. We first recall the definition of twisted K-stability [8, 14].

Definition 4.1. Let $(\mathcal{X}, \mathcal{L})$ be a given normal test configuration of X, $0 < \mu \le 1$, then μ-twisted generalized Futaki invariant is defined to be

$$\mathrm{Fut}_{1-\mu}(\mathcal{X}, \mathcal{L}) := \sup_{D \in |-K_X|_{\mathbb{Q}}} \mathrm{Fut}(\mathcal{X}, (1-\mu)\mathcal{D}; \mu\mathcal{L})$$

where $\mathcal{D}$ is closure of $D \times (\mathbb{A}^1 \setminus 0)$ in $\mathcal{X}$.

Definition 4.2. 1) We say X is μ-twisted K-semistable if $\mathrm{Fut}_{1-\mu}(\mathcal{X}, \mathcal{L}) \geq 0$ for every normal test configuration $(\mathcal{X}, \mathcal{L})$.

2) We say X is μ-twisted K-stable if $\mathrm{Fut}_{1-\mu}(\mathcal{X}, \mathcal{L}) > 0$ for every non-trivial normal test configuration $(\mathcal{X}, \mathcal{L})$.

3) We say X is μ-twisted uniformly K-stable if there exists a positive real number $\epsilon > 0$ such that $\mathrm{Fut}_{1-\mu}(\mathcal{X}, \mathcal{L}) \geq \epsilon J(\mathcal{X}, \mathcal{L})$ for every normal test configuration $(\mathcal{X}, \mathcal{L})$.

In the above definition, one should check all normal test configurations to test twisted K-stability. However, by a special test configuration theory in twisted setting that has been established in [8, Theorem 1.6] which is parallel to [24], we have following theorem:

Theorem 4.3. *To test μ-twisted K-semistability (resp. K-stability and uniform K-stability), it suffices to check all special test configurations.*

Proof. In [8, Theorem 1.6], we only establish the parallel twisted special test configuration theory for K-semistability and K-stability. However, for uniform K-stability, we take j-invariant into account just as [18, Section 3] and the proof is all the same. ☐

While X may not be K-semistable, it can still be K-stable in the twisted sense [8]. The following result is a refinement of the twisted valuative criterion established in [8, Theorem 1.5].

Theorem 4.4. *Let X be a $\mathbb{Q}$-Fano variety with $\delta(X) \leq 1$, then X is μ-twisted uniformly K-stable for $0 < \mu < \delta(X)$, and X is μ-twisted K-semistable but not μ-twisted uniformly K-stable for $\mu = \delta(X)$.*

Proof. For $\mu < \delta(X)$, by [6, Theorem C], there is a $D \in |-K_X|_{\mathbb{Q}}$ such that $(X, (1-\mu)D)$ is uniformly K-stable. Thus there is a positive real number $0 < \epsilon < 1$ such that

$$\mathrm{Fut}(\mathcal{X}, (1-\mu)\mathcal{D}; \mathcal{L}) \geq \epsilon J(\mathcal{X}, \mathcal{L})$$

for any normal test configuration, so one has

$$\mathrm{Fut}_{1-\mu}(\mathcal{X}, \mathcal{L}) \geq \epsilon J(\mathcal{X}, \mathcal{L})$$

For $\mu = \delta(X)$, we can choose a sequence of special test configurations $(\mathcal{X}_i, \mathcal{L}_i)$ such that $\lim_i \frac{A(v_{\mathcal{X}_{i,0}})}{S(v_{\mathcal{X}_{i,0}})} = \delta(X)$ [8, Theorem 4.3]. We aim to prove that X is not δ-twisted uniformly K-stable where $\delta = \delta(X)$. If not, there is a positive real number $0 < \epsilon < 1$ such that

$$\mathrm{Fut}_{1-\delta}(\mathcal{X}_i, \mathcal{L}_i) \geq \epsilon J(\mathcal{X}_i, \mathcal{L}_i).$$

One can choose a general $D \in |-K_X|_{\mathbb{Q}}$ such that

$$\mathrm{Fut}_{1-\delta}(\mathcal{X}_i, \mathcal{L}_i) = \mathrm{Fut}(\mathcal{X}_i, (1-\delta)\mathcal{D}; \mathcal{L}_i)$$

for any i, where D doesn't contain any center of $v_{\mathcal{X}_{i,0}}$ [8, Theorem 3.7]. Thus one obtain

$$\mathrm{Fut}_{1-\delta}(\mathcal{X}_i, \mathcal{L}_i) = A(v_{\mathcal{X}_{i,0}}) - \delta S(v_{\mathcal{X}_{i,0}}) \geq \epsilon J(\mathcal{X}_i, \mathcal{L}_i) = \epsilon(T(v_{\mathcal{X}_{i,0}}) - S(v_{\mathcal{X}_{i,0}})),$$

which contradicts $\lim_i \frac{A(v_{\mathcal{X}_{i,0}})}{S(v_{\mathcal{X}_{i,0}})} = \delta(X)$. □

Remark 4.5. By [10]*Proposition 7.8 and Remark 7.12, the J-functional (as a norm on test configurations) is equivalent to the minimum norm as in [14]. Thus we can replace the J-functional by the minimum norm in the definition of twisted uniform K-stability. Under this equivalent definition, the above statement follows almost immediately from [8, Proposition 3.4 and Theorem 3.7]. Note that X is μ-twisted K-semistable (resp. uniformly K-stable) if and only if

$$\inf_{(\mathcal{X},\mathcal{L})} \frac{\mathrm{Fut}_{1-\mu}(\mathcal{X}, \mathcal{L})}{||\mathcal{X}||_m} \geq 0, (\text{resp.} > 0).$$

Using that

$$\mathrm{Fut}_{1-\mu}(\mathcal{X}, \mathcal{L}) = \mathrm{Fut}(\mathcal{X}, \mathcal{L}) + (1-\mu)||\mathcal{X}||_m,$$

we get X is μ-twisted K-semistable (resp. uniformly K-stable) if and only if

$$\inf_{(\mathcal{X},\mathcal{L})} \frac{\mathrm{Fut}(\mathcal{X}, \mathcal{L})}{||\mathcal{X}||_m} \geq \mu - 1, (\text{resp.} > \mu - 1).$$

Since

$$\inf_{(\mathcal{X},\mathcal{L})} \frac{\mathrm{Fut}(\mathcal{X}, \mathcal{L})}{||\mathcal{X}||_m} = \delta(X) - 1,$$

the result follows.

5. Optimal Destabilization Conjecture

It has long been expected that uniform K-stability is equivalent to K-stability. In [9], they reduced the problem to the existence of divisorial δ-minimizer for $\delta(X) = 1$, that is, the divisorial valuation computing δ-invariant. The algebraic twisted K-stability theory has been established to study K-unstable Fano varieties [8], then the case $\delta < 1$ can be studied in parallel to the case $\delta = 1$. In this section, we will explain the relation between the following two conjectures.

Conjecture 5.1. (Optimal Destabilization Conjecture) *Let X be a $\mathbb{Q}$-Fano variety with $\delta(X) \leq 1$, then there exists a divisor E over X computing $\delta(X)$, i.e. $\frac{A(E)}{S(E)} = \delta(X)$.*

Conjecture 5.2. *Let X be a $\mathbb{Q}$-Fano variety with $\delta(X) \leq 1$, and $0 < \mu \leq 1$, then X is μ-twisted K-stable is equivalent to that X is μ-twisted uniformly K-stable.*

By Theorem 4.4, we first have the following lemma as a direct corollary:

Lemma 5.3. *Conjecture 5.2 is equivalent to that X is not $\delta(X)$-twisted K-stable.*

The above two conjectures are equivalent by following result:

Theorem 5.4. *Conjecture 5.1 is equivalent to Conjecture 5.2.*

Proof. We first assume Conjecture 5.1, i.e. there is a divisor E computing $\delta = \delta(X)$, then by [8, Theorem 1.1], E is a dreamy divisor over X which naturally induces a non-trivial test configuration $(\mathcal{X}, \mathcal{L})$ such that $\mathrm{Fut}_{1-\delta}(\mathcal{X}, \mathcal{L}) = 0$, thus X is not δ-twisted K-stable. Conversely, assume X is not δ-twisted K-stable, then there exists a non-trivial test configuration $(\mathcal{X}, \mathcal{L})$ such that $\mathrm{Fut}_{1-\delta}(\mathcal{X}, \mathcal{L}) = 0$. By [8, Theorem 3.9], it must be a special test configuration whose central fiber induces a divisorial valuation computing $\delta(X)$. □

We can also translate optimal destabilization conjecture into vanishing of δ-twisted generalized Futaki invariant [8].

Theorem 5.5. *Suppose X is a klt Fano variety with $\delta(X) \leq 1$. If there is a divisor E over X computing $\delta(X)$, i.e $\frac{A(E)}{S(E)} = \delta(X)$, then there is a test configuration $(\mathcal{X}, \mathcal{L})$ such that $\mathrm{Fut}_{1-\delta}(\mathcal{X}, \mathcal{L}) = 0$. Conversely, if there is a* *test configuration* $(\mathcal{X},\mathcal{L})$ *such that* $\mathrm{Fut}_{1-\delta}(\mathcal{X},\mathcal{L})=0$*, then there is a divisor* E *over* X *computing* $\delta(X)$.

Proof. Suppose there is a divisor E computing $\delta(X)$, then E is a dreamy divisor which naturally induces a test configuration whose δ-twisted generalized Futaki is zero, by [8, Theorem 1.1]. Conversely, if there is a test configuration whose δ-twisted generalized Futaki is zero, then it must be a special test configuration whose central fiber induces a divisorial valuation computing $\delta(X)$, by [8, Theorem 4.6]. □

Remark 5.6. The first two conjectures in this section for $\delta(X)=1$ correspond to the following two conjectures (also see [9]):

1) (Optimal Destabilization Conjecture for $\delta=1$) Suppose X is a $\mathbb{Q}$-Fano variety with $\delta(X)=1$, then there is a divisorial valuation ord_E computing $\delta(X)$, i.e. $\delta(X)=\frac{A(E)}{S(E)}=1$.
2) For Fano varieties, uniform K-stability is equivalent to K-stability.

By Theorem 5.4, we know they are also equivalent.

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