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Representing smooth 4-manifolds as loops in the pants complex

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We show that every smooth, orientable, closed, connected 4-manifold can be represented by a loop in the pants complex. We use this representation, together with the fact that the pants complex is simply connected, to provide an elementary proof that such 4-manifolds are smoothly cobordant to a connected sum of complex projective planes, with either orientation. We also use this association to give information about the structure of the pants complex. Namely, given a loop in the pants complex, L, which bounds a disk, D, we show that the signature of the 4-manifold associated to L gives a lower bound on the number of triangles in D.

1. Overview

Simplicial complexes associated to curves on a surface play a central role in 2- and 3-manifold topology, particularly in the study of mapping class groups and Heegaard splittings. Recently, Kirby and Thompson [10] pushed these techniques into dimension four, assigning a loop in the cut complex to a trisected 4-manifold. The aim of this paper is to, in some sense, reverse this. In particular, given a loop in the pants complex, L, we show how to uniquely build a closed smooth 4-manifold $\mathcal{X}^4_C(L)$. Our main theorem is that all such manifolds arise in this fashion.

Theorem 2. *For every closed, smooth, orientable 4-manifold X^4, there exists a closed loop L in $\mathcal{P}(\Sigma)$ so that X is diffeomorphic to $\mathcal{X}^4_C(L)$.*

In their proof of the finite presentability of the mapping class group [8], Hatcher and Thurston sketch a proof that the pants complex is simply connected. This result was later fully fleshed out in work of Hatcher [7]. As our main theorem associates a loop to any 4-manifold, it is natural to ask what the disk it bounds represents. Viewing 4-manifolds from this perspective yields a natural proof of the following theorem, originally due to Pontrjagin and Rohlin [15], which is our main application.

Theorem 4. *Every smooth, oriented, closed manifold is cobordant to $\coprod_m \mathbb{C}P^2 \coprod_n \overline{\mathbb{C}P}^2$.*

Our proof here follows along the lines of recent work of Gay [6], in which the author proves the same theorem by associating a loop of smooth functions on a surface to a 4-manifold. The similarity in these arguments suggests that the pants complex of a surface Σ is a good discrete model for the space of smooth functions on Σ. It is worth noting that the simple-connectivity of the pants complex was originally proved using properties of generic smooth functions on surfaces. Nevertheless, there now exist multiple proofs of the simple-connectivity of the pants complex which rely on different techniques [2] [3] which give rise to alternative paths to the theorem, some of which (after using [18]) are quite elementary.

We also use our correspondence to gain insight into the structure of the pants complex. In particular, given a loop L in the pants complex, we define an invariant $\sigma(L)$, which is the signature of the 4-manifold associated to L. This may be calculated using information only of the 1-skeleton of the pants complex, but contains information about possible disks that this loop can bound. In particular, we obtain the following proposition, where the 3S-triangles are a particular type of 2-cell in the pants complex (see Figure 3 and the next section):

Proposition 1. *Let L be a loop in the pants complex with $\sigma(L) = n$, then any disk bounded by L must contain at least n 3S-triangles.*

2. The pants complex

We briefly discuss the pants complex of a surface following [7]. Let Σ be a connected closed orientable surface of genus greater than or equal to 2. A **pants decomposition** of Σ is a set of $3g-3$ simple closed curves on Σ such that cutting Σ along these curves results in a disjoint union of $2g-2$ 3-punctured spheres (pairs of pants). Two pants decompositions of Σ are considered the same if the curves are isotopic. We will be considering the 2-complex $\mathcal{P}(\Sigma)$, called the pants complex of Σ, whose vertices correspond to isotopy classes of pants decompositions of Σ.

There are two types of edges in $\mathcal{P}(\Sigma)$: S-edges ("S" for stabilization) and A-edges ("A" for associative). Note that if one removes neighbourhoods of all but one of the curves in a pants decomposition, one is left with $2g-3$ pants and either a 4-punctured sphere or a once punctured torus. Two pants decompositions P_1 and P_2 are connected by an S-move if all but one of the curves in P_1 are the same as in P_2 and the curves that differ intersect each other in exactly one point on a once punctured torus component. Two pants decompositions P_1 and P_2 are connected by an A-edge if all but one of the curves in P_1 are the same as in P_2 and the curves that differ intersect each other in exactly two points on a 4-punctured sphere. See Figure 1.

If Σ is instead a torus, then we will see that the correct analog of a pants decomposition is just an isotopy class of an essential curve. The complex $\mathcal{P}(\Sigma)$ for a torus, whose 1-skeleton is known as the Farey graph, has isotopy classes of a essential curves as vertices and S-edges with the same definition as above, but no analog of A-edges.

There are five different types of 2-cells in $\mathcal{P}(\Sigma)$ which are glued to topological configurations. These configurations are shown in Figures 2-6. With this choice of 2-cells, the following theorem holds.

Theorem 1. *(Hatcher [7]) $\mathcal{P}(\Sigma)$ is connected and simply-connected.*

If P is a pants decomposition for Σ, we may obtain a 3-dimensional handlebody with boundary Σ, by taking $\Sigma \times I$, attaching 2-handles to the curves in P, and then capping off the remaining $2g-2$ sphere components with 3-handles. We denote this handlebody by $\mathcal{H}(P)$, We say two handlebodies, H_1 and H_2, with $\partial H_1 = \partial H_2 = \Sigma$ are equal if the identity map on the boundary extends to a homeomorphism from H_1 to H_2. Note that there are inequivalent pants decompositions that produce equal handlebodies, namely, any two pants decompositions related by A-moves define the same handlebody.

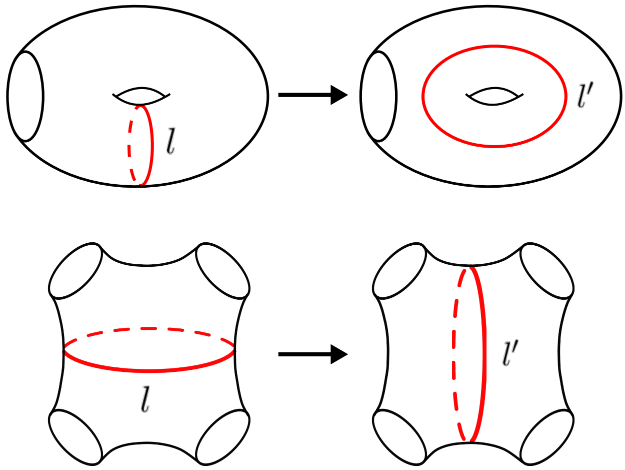

Figure 1: Top: An S-move in the pants complex. Bottom: and A-move in the pants complex.

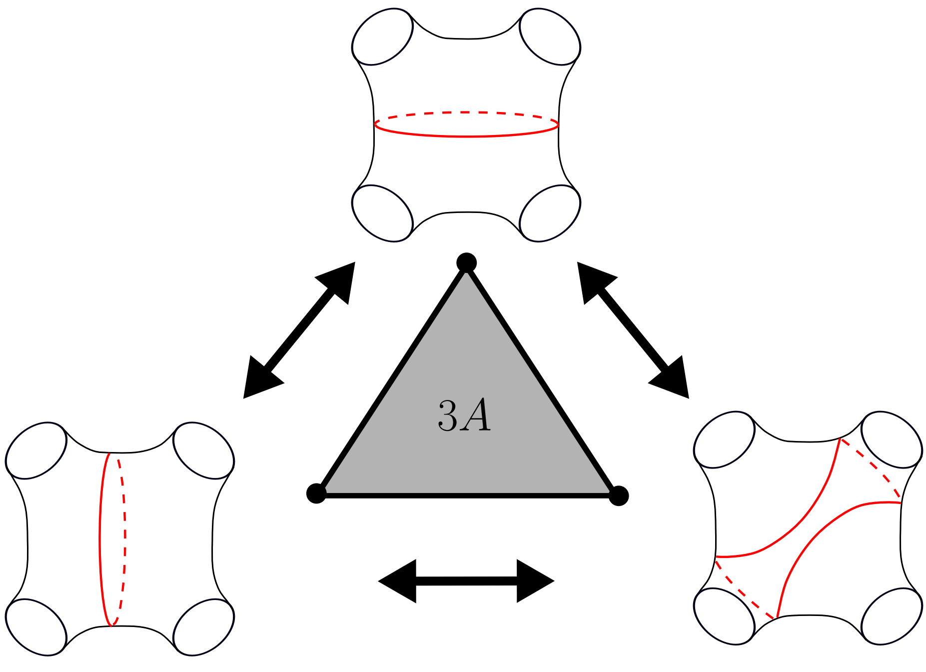

Figure 2: A 3A cycle which is filled in by a triangle.

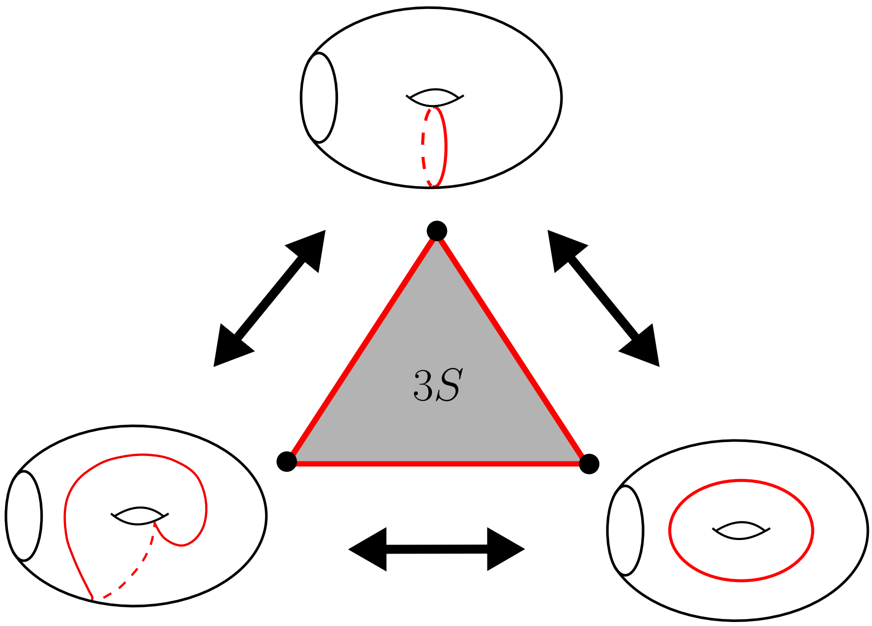

Figure 3: A 3S cycle which is filled in by a triangle.

Given a handlebody H with $\partial H = \Sigma$, the **handlebody set** of H in $\mathcal{P}(\Sigma)$ is the set of pants decompositions P of Σ such that $\mathcal{H}(P) = H$. By our previous discussion, two pants decompositions related by an A-move lie in the same handlebody set. The following result originally proved by Luo shows that handlebody sets are in fact connected by A-moves.

Lemma 1. *(Corollary 1 of [12]) Given a handlebody H with $\partial H = \Sigma$, and two pants decompositions P_1 and P_2 in the handlebody set of H, there exists a path in $\mathcal{P}(\Sigma)$ consisting of exclusively A-edges between P_1 and P_2.*

A **cut system** on a genus g closed orientable surface Σ is a set $\alpha = \{\alpha_1, ..., \alpha_g\}$ of pairwise disjoint, non-separating, essential curves on Σ. At times it will be more natural to consider cut systems instead of pants decompositions. The following two lemmas will allow us to pass between these decompositions freely.

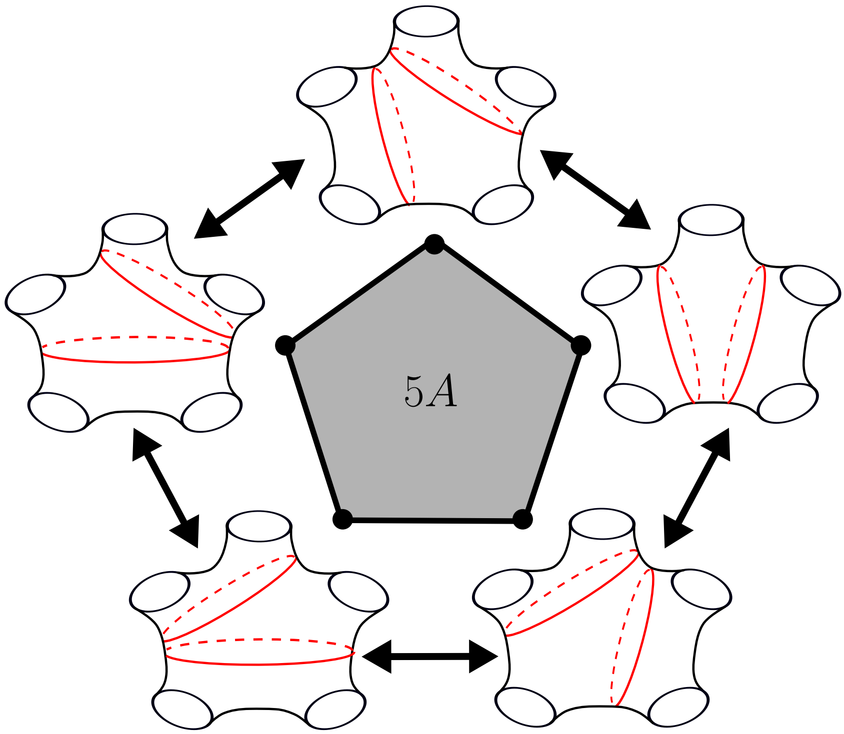

Figure 4: A 5A cycle which is filled in by a pentagon.

Lemma 2. *(Lemma 5 of [9]) Any pants decomposition for Σ contains at least g non-separating curves. Any choice of g mutually non-separating curves in a pants decomposition give a cut system of Σ.*

Given a pants decomposition, choosing a cut system via the previous lemma gives sufficient data to determine a handlebody. This is often enough information for the topological constructions in this paper. To return to the simplicial constructions of the pants complex we will need to complete cut systems or, more generally, any set of curves to a pants decompositions. The following lemma, which may proved by induction on the genus and an Euler characteristic argument, will allow us to do so.

Lemma 3. *Let H be a genus g handlebody with boundary Σ. Let $k < 3g - 3$ and $\{c_1, ..., c_k\}$ be a set of non-isotopic simple closed curves on Σ such that $c_1, ..., c_k$ all bound disjoint properly embedded disks in H. Then there*

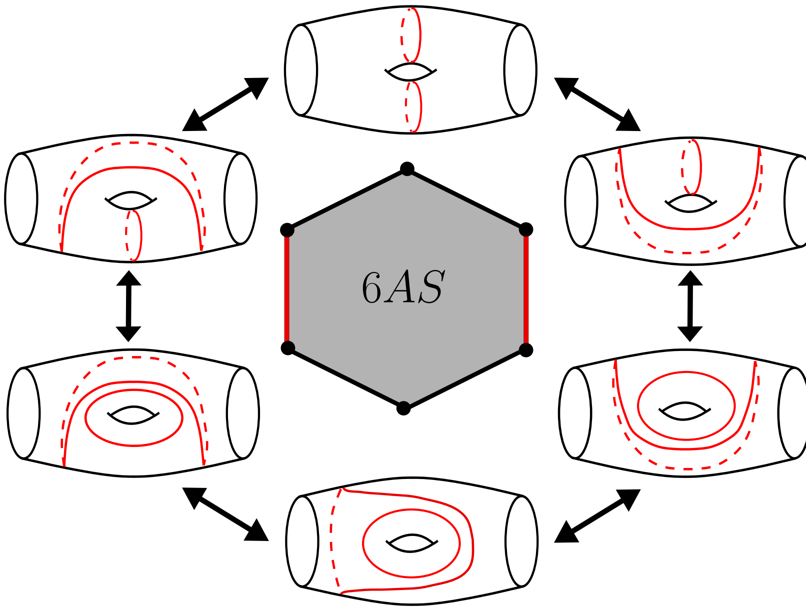

Figure 5: A 6AS cycle which is filled in by a hexagon. A-moves are labeled by black curves while S-moves are labeled by red curves.

exist some additional simple closed curves on Σ, $\{c_{k+1}, ..., c_{3g-3}\}$, *that bound disjoint properly embedded disks in* H, *so that* $P = \{c_1, ..., c_k, c_{k+1}, ..., c_{3g-3}\}$ *is a pants decomposition for* Σ *with* $\mathcal{H}(P) = H$.

3. The 1-skeleton of the pants complex

Our goal in this section is to construct a unique oriented 4-manifold from a walk in the pants complex. Given two vertices P_1 and P_2 in the pants complex, we obtain a Heegaard splitting of a closed orientable 3-manifold $\mathcal{M}^3(P_1, P_2) = \mathcal{H}(P_1) \cup_\Sigma \mathcal{H}(P_2)$. Further, if we consider an ordering, as above with P_1 first then P_2, and if Σ is oriented, then we can orient $\mathcal{H}(P_1)$ to agree with Σ and $\mathcal{H}(P_2)$ to disagree with Σ, we then obtain an orientation of $\mathcal{M}^3(P_1, P_2)$, and we will henceforth assume that it carries this orientation. Note that changing the orientation of Σ, or the order of P_1 or P_2, will change the orientation of the resulting 3-manifold.

Given an oriented edge e between two vertices P_1 and P_2 in the pants complex, we will define a compact orientable 4-manifold $\mathcal{X}^4(W)$ with

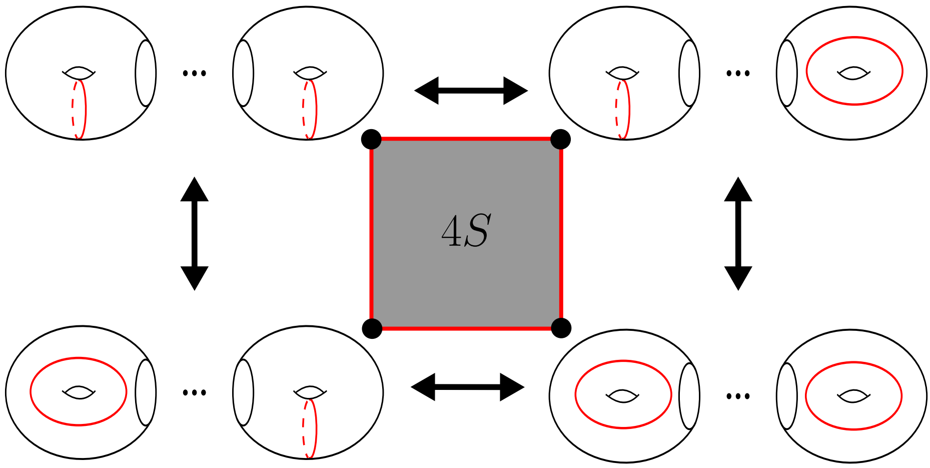

Figure 6: A 4S cycle of S-moves which is filled in by a square. There are also corresponding squares for disjointly supported A-moves and disjointly supported A- and S- moves. We will denote these by 4S-squares, 4A-squares, and 4AS-squares, respectively.

$\partial \mathcal{X}^4(e) = \mathcal{M}^3(P_1, P_2)$. We first need to determine the manifold $\mathcal{M}^3(P_1, P_2)$. By inspecting Heegaard diagrams of the manifolds involved we may determine the 3-manifolds associated to vertices connected by an edge. In particular, we obtain the following lemma.

Lemma 4. *Let P_1 and P_2 be two pants decompositions of a surface Σ of genus g. If P_1 and P_2 are connected by an A-edge, then $\mathcal{M}^3(P_1, P_2) = \sharp^g(S^1 \times S^2)$. If P_1 and P_2 are connected by an S-edge, then $\mathcal{M}^3(P_1, P_2) = \sharp^{g-1}(S^1 \times S^2)$.*

Proof. Note that if Q is a set of disjoint curves on Σ with the property that when 2-handles are attached to Q, only spherical components remain, and P is a set of disjoint curves on Σ that contains Q, then the handlebody bounding Σ given by attaching 2-handles to Q is equal to the handlebody bounding Σ given by attaching 2-handles to P.

First, consider the case where P_1 and P_2 are connected by an A-edge. To see this, let Q be the set of curves where P_1 and P_2 agree - namely, all of the curves not involved in the A-move, and consider the 3-manifold obtained by attaching 2-handles to Q. Then both of the curves involved in the A-move, after attaching these 2-handles, are now on a spherical boundary component and therefore $\mathcal{H}(P_1)$ and $\mathcal{H}(P_2)$ are both equal to the handlebody obtained by attaching 2-handles to Q. Therefore $\mathcal{H}(P_1) = \mathcal{H}(P_2)$ and thus $\mathcal{M}^3(P_1, P_2) = \sharp^g(S^1 \times S^2)$.

Now consider the case where P_1 and P_2 are connected by an S-edge. Let c denote the curve that bounds the torus where the S-move is supported. Let c_1 and c_2 denote the curves in the S-move in P_1 and P_2, respectively. Since c_1 and c_2 intersect in a single point, the Heegaard splitting of $\mathcal{M}^3(P_1, P_2)$ is stabilized. Destabilizing this Heegaard splitting, we obtain a 3-manifold that is described by the Heegaard splitting where the surface Σ has been surgered along c, the torus component containing c_1 and c_2 has been deleted, and the respective handlebodies are described by the corresponding remaining curves in P_1 and P_2. Since these remaining curves are equal, the resulting destabilization is $\sharp^{g-1}(S^1 \times S^2)$ and therefore $\mathcal{M}^3(P_1, P_2) = \sharp^{g-1}(S^1 \times S^2)$. $\square$

Having determined the 3-manifold associated to an edge, we next define a unique oriented 4-manifold filling. If $\mathcal{X}^4(e)$ is an A-edge, then we fill the resulting $\sharp^g(S^1 \times S^2)$ with $\natural^g(S^1 \times D^3)$. Similarly, if the edge is an S-edge, we fill the resulting $\sharp^{g-1}(S^1 \times S^2)$ with $\natural^{g-1}(S^1 \times D^3)$, see Figure 7. By a theorem of Laudenbach and Poenaru [11], these fillings are unique.

Given an oriented walk, W, of arbitrary length, we construct $\mathcal{X}^4(W)$ by first constructing all of the 4-manifolds associated to each of the edges in W, and then gluing together each successive pair of edges along the common 3-dimensional handlebody via the identity map along their shared handlebody (see Figure 8). As a convention, if W is just a single vertex P then we just take the filling of $\mathcal{H}(P) \cup_\Sigma \mathcal{H}(P)$ by $\natural^g(S^1 \times B^3)$. Note that with these orientations $\partial\mathcal{X}^4(W) = \mathcal{M}^3(\partial W)$.

We now address orientations for the 4-manifold. Assume from now on that we have a fixed orientation on Σ and our walk W has a fixed orientation. We will obtain an orientation on $\mathcal{X}^4(W)$ as follows: for a single oriented edge from a vertex P_1 to a vertex P_2, orient $\mathcal{H}(P_1)$ with the orientation that induces the orientation on Σ and orient $\mathcal{H}(P_2)$ with the orientation that induces the opposite orientation on Σ. Then we can glue $\mathcal{H}(P_1)$ to $-\mathcal{H}(P_2)$ via the identity map on Σ and obtain an orientation on $\mathcal{M}^3(P_1, P_2)$. This, in turn, induces an orientation on $\mathcal{X}^4(W)$. If W consists of multiple edges, then we can orient $\mathcal{X}^4(W)$ by orienting the wedges as above, and since each non-end vertex of W has an edge coming in and an edge going out, the resulting identity maps between the handlebodies will be orientation-reversing and we therefore obtain an orientation on all of $\mathcal{X}^4(W)$. Note

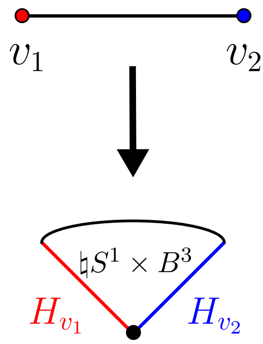

Figure 7: An edge in $P(\Sigma)$ corresponds to a connected sum of copies $S^1 \times S^2$ which may be filled in uniquely with the appropriate boundary sum of copies of $S^1 \times D^3$.

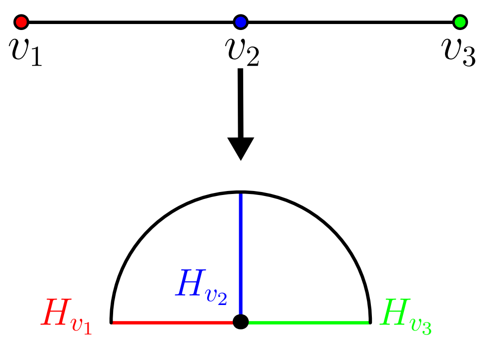

Figure 8: A path in $P(\Sigma)$ gives rise to the manifold obtained by gluing together the wedges corresponding to edges along the handlebodies of the shared vertices

that, either switching the orientation of Σ, or switching the direction of W will change the orientation on $\mathcal{X}^4(W)$.

We next seek to obtain a handle description for 4-manifolds given by a walk in the pants complex. We first need the following lemma.

Lemma 5. *Let H be a handlebody and let $\gamma \subset \partial H$ be a curve such that, for some properly embedded disk $D \subset H$, $|\gamma \cap D| = 1$. Then the result of pushing γ into H, and doing surgery on γ is again a handlebody. Moreover, if we do surgery on γ using the surface framing, then γ bounds a disk in the surgered handlebody.*

Proof. One way to prove that a 3-manifold is a handlebody is to find a collection of disjoint properly embedded disks which cuts the manifold into balls. Let $D_1, ..., D_g$ be a collection of such disks for H with $D_1 = D$. By sliding all of the other disks over D_1, we may arrange so that none of these disks intersect γ. We can then still cut along $D_2, ..., D_g$ in the surgered manifold. We then only need to analyze what is happening in the solid torus $H \backslash \{D_2, ...D_g\}$ containing the curve γ where the surgery is occurring along with its dual disk, D_1.

Since $|\gamma \cap D_1| = 1$, and γ is isotopic into the boundary, γ is isotopic to the core curve $S^1 \times \{0\} \subset S^1 \times D^2$. But any surgery on the core curve in a solid torus results again in a solid torus - one way to see this is that the part of the solid torus that is not affected by the surgery is just a collar neighborhood of the boundary. If we give the surgery curve the surface framing, then there is a disk D_1', disjoint from $D_2, ..., D_g$, with $\partial D_1' = \gamma$, formed by taking the surgery disk for the push-in of γ and extending it to the boundary by adding the annulus coming from the push-in process. We therefore conclude that $D_1', D_2, ..., D_g$ is a collection of properly embedded disks which cut the surgered manifold into a ball, and hence it is a handlebody. □

The following will used repeatedly to identify the 4-manifolds corresponding to paths and loops in the pants complex. The proof is similar to the proof of Lemma 13 of [5].

Lemma 6. *Let W be an oriented walk in $\mathcal{P}(\Sigma)$ starting at P_1. The following process produces a handle decomposition of $\mathcal{X}^4(W)$. Start with $\mathcal{H}(P_1) \times I$; these are the 0- and 1-handles. For every (directed) S-edge in W, we take the new curve in the latter vertex, push it into $\mathcal{H}(P_1) \times \{0\} \subset \mathcal{H}(P_1) \times I$, and give this curve the surface framing from $\Sigma = \partial \mathcal{H}(P_1) \times \{0\}$. The curves that are seen later in the walk along W are not pushed as far inside of $\mathcal{H}(P_1) \times \{0\}$ as earlier curves. These framed curves are the attaching curves for the 2-handles. There are no 3-handles or 4-handles.*

Proof. We start in the case where W is just a single edge. In the case where W is an A-edge, then $\mathcal{X}^4(W) = \natural^g(S^1 \times D^3)$ and indeed this is the manifold that we obtain from our handlebody description, since no 2-handles are added.

In the case where W is an S-edge, $\mathcal{X}^4(W) = \natural^{g-1}(S^1 \times D^3)$ and so we must verify that the attaching sphere of the 2-handle that we are adding intersects the belt sphere of one of the 1-handles in exactly one point, ensuring that the handles cancel to give the desired result. Using Lemma 2, we can choose g mutually nonseparating curves in P_1 on Σ that form a cut system for $\mathcal{H}(P_1)$. The belt spheres of the 1-handles are exactly these g nonseparating curves in Σ together with the disks on both sides of $\gamma \subset \sharp^g(S^1 \times S^2) = \partial(S^1 \times D^3)$. By the definition of an S-move, and the convention for attaching a 2-handle stated in the lemma, we see that the attaching circle for the 2-handle intersects this belt sphere in exactly one point.

Now consider the case of a general walk $W = (P_1, P_2, ...P_{n+1})$. Since A-moves do not affect the resulting 4-manifold, we proceed by induction on the number of S-moves that W contains and we assume that (P_n, P_{n+1}) is an S-edge. The base case was just discussed. Let $W' = (P_1, ..., P_n)$. By the inductive hypothesis, the 4-manifold $\mathcal{X}^4(W')$ has a handlebody diagram as described in the statement of the lemma. Let $H = \mathcal{H}(P_n)$. Attach one end of $H \times I$ to this H in the boundary to obtain a space that (after rounding corners) is still just $\mathcal{X}^4(W')$. Now attach a 2-handle along the new curve in P_{n+1} framed by Σ to the free end of $H \times I$ as in the statement of the lemma.

We now verify that the union of $H \times I$ and this new 2-handle is indeed $\natural^{g-1}(S^1 \times D^3)$. Using Lemma 2, we obtain a cut system for H containing the curve that is changed by the S-move, such that the curves in the cut system bound disjoint disks in H. These disks considered in both ends of $H \times I$ together with the curves cross I form a set of belt spheres for the genus g 4-dimensional handlebody $H \times I$ and the attaching circle for the 2-handle intersects exactly the disk bounding the curve corresponding to the S-move, and in exactly one point, thus verifying that we have $\natural^{g-1}(S^1 \times D^3)$ as desired. Furthermore, by Lemma 5, when we look at the two handlebodies in the boundary of $H \times I$ together with this 2-handle, we have exactly $\mathcal{H}(P_n)$ and $\mathcal{H}(P_{n+1})$. Therefore, what we have attached is a 4-dimensional filling of the desired handlebodies by $\natural^{g-1}(S^1 \times D^3)$, which, by [11], can only be done in one way. $\square$

Since the mapping class group acts transitively on the set of handlebodies with a given boundary, we may apply a mapping class, and insert or delete A-moves, to assume that a walk starts in a pants decomposition which contains the cut system shown in Figure 9. We may then use Lemma 6, to obtain a Kirby diagram for the manifold $X^4(W)$. Namely, the cut system for the handlebody shown in Figure 9 becomes dotted circles, representing the 1-handles and these are inside "at the center" of the shown handlebody. S-moves give rise to 2-handles as in the previous lemma; the 2-handles that arise from the S-moves earlier in the walk are pushed further "inside" of the surface in Figure 9.

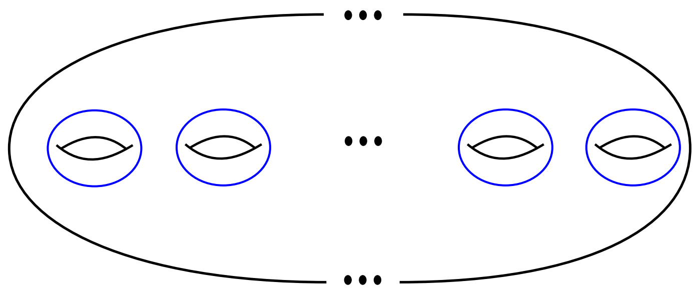

Figure 9: A cut system for the standard handlebody. These curves become the 1-handles in the dotted circle notation for 1-handles in a Kirby diagram.

The following result uses Waldhausen's theorem [19] on Heegaard splittings of connect sums of $S^1 \times S^2$, which is the primary 3-dimensional result that we will make use of.

Lemma 7. *Let P_1 and P_2 be vertices in $\mathcal{P}(\Sigma_g)$, with $\mathcal{M}^3(P_1, P_2) \cong \sharp^k(S_1 \times S_2)$ for some $k \leq g$. Then there exists a walk W in $\mathcal{P}(\Sigma)$ with $\mathcal{X}^4(W) \cong \natural^k(S^1 \times D^3)$.*

Proof. Note that P_1 and P_2 form a genus g Heegaard diagram for $\sharp^m(S^1 \times S^2)$. As an immediate consequence of Waldhausen's theorem [19], there exist vertices P_1' and P_2' in the same handlebody sets as P_1 and P_2, respectively, which are standard, in that there are k parallel sets of non-separating curves, and $g-k$ dual sets of non-separating curves in these pants decompositons. The path of length $g-k$ between these pants decompositions which turns each curve in P_1 to its corresponding dual in P_2 gives rise to a handle decomposition consisting of g 1-handles and $g-k$ 2-handles. These 2-handles are dual to 1-handles, so the resulting manifold has a handle decomposition consisting of just k 1-handles, and so is diffeomorphic to $\natural^k(S^1 \times D^3)$. $\square$

So far, the construction discussed produces a manifold whose boundary is the 3-manifold given by the first and last endpoints of the path. If the path is a loop, L, we can also construct a closed 4-manifold. We proceed just as in the above construction, but when the loop returns to the vertex that we start on, we glue the identical handlebodies together using the identity map (or equivalently, uniquely glue in $\natural^g(S^1 \times D^3)$). Further, if the loop is oriented, the resulting 4-manifold obtains an orientation. We denote the resulting closed orientable manifold by $\mathcal{X}^4_C(L)$.

We next seek to prove that every 4-manifold arises in this fashion. To do this, we will convert a handle decomposition into a loop in the pants complex - this is very similar to the handlebody proof that every 4-manifold admits a trisection [5]. Throughout the proof, the reader may find it helpful to consult Figures 14 and 15. These figures, read left to right, show how to obtain a handlebody diagram from a loop in the pants complex. The following proof illustrates the reverse of this, so that these figures, read right to left, provide examples of the procedure.

Theorem 2. *For every closed, smooth, orientable 4-manifold X^4, there exists a closed loop L in $\mathcal{P}(\Sigma)$ so that X is diffeomorphic to $\mathcal{X}^4_C(L)$.*

Proof. Let X^4 be an arbitrary closed 4-manifold. Fix a handlebody decomposition diagram for X. Let Σ_0 be a Heegaard splitting surface of the boundary of the 4-dimensional handlebody that we see after just attaching the 0- and 1-handles in the construction of X. Let $l = l_1 \cup \cdots \cup l_n$ be the framed link that describes how the 2-handles are attached. Project l onto Σ_0 and then stabilize the Heegaard surface Σ_0 to obtain a new Heegaard surface Σ in the following way. First, stabilize Σ_0 in order to make the l_i embedded as in Figure 11. If needed, stabilize further so that for each curve, l_i, there is a curve, α_i, embedded in the surface so that α_i intersects l in exactly one point. Call this resulting surface Σ. By twisting the l_i around the α_i as in Figure 12, we can ensure that the framing on each l_i is the same as the framing coming from the surface embedding, and we will assume that l is sitting in Σ in this way.

We now construct our loop L in $\mathcal{P}(\Sigma)$ with $\mathcal{X}^4_C(L) \cong X$. We will construct L so that the handlebody decomposition of $\mathcal{X}^4_C(L)$ that we see from Lemma 6 is identifiable with the given handlebody decomposition of X.

Suppose that g is the genus of Σ and k is the genus of Σ_0 (i.e. the number of 1-handles in the given handlebody decomposition of X). Our construction of W will take place in a few stages. We start by constructing the 1-handles of X^4. Take a pants decomposition of Σ that contains the cut system in Figure 9. By performing $g-k$ S-moves, and perhaps some A-moves, we arrive at a pants decomposition that contains the cut system in Figure 10, which we call Q. We call this walk W_1. At this point, via Lemma 6, we see that we have constructed a genus k 4-dimensional 1-handlebody, and all of

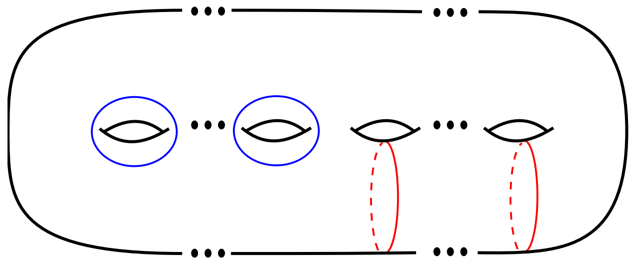

Figure 10: The handlebody Q used in Thereom 2

the 1-handles that have been cancelled are exactly the handles that do not appear in the given handlebody decomposition of X.

Let n_i denote the boundary of the punctured torus that is a regular neighborhood of $\alpha_i \cup l_i$. Note that the n_i together with all of the α_i form a collection of disjoint simple closed curves that bound disjoint properly embedded disks in $\mathcal{H}(Q)$. Let R be a pants decomposition obtained from extending the union of the n_i and α_j to a pants decomposition via Lemma 3, so that $\mathcal{H}(R) = \mathcal{H}(Q)$. By Lemma 1, we can get from Q to R, using a walk, W_2, with just A-moves. Note that $\mathcal{X}^4(W_1W_2) \cong \mathcal{X}^4(W_1) \cong \natural^k(S^1 \times D^3)$, as no new handles have been added.

Now we are in position to attach the desired 2-handles by doing S-moves. Namely, for each α_i curve do the S-move that turns α_i into l_i. Let W_3 be the walk starting at the vertex R that consists of this sequence of S-moves. By Lemma 6, we see that $\mathcal{X}^4(W_1W_2W_3)$ is diffeomorphic to the 0-, 1- and 2-handles in the handlebody decomposition of X. Since X is a closed 4-manifold, we must have that the boundary of $\mathcal{X}^4(W_1W_2W_3)$ is $\natural^m(S^1 \times D^3)$ for some m. Moreover, the first and last handlebodies of $W_1W_2W_3$ form a Heegaard splitting for the boundary. By Lemma 7, there exists a walk W_4 from the end of W_3 to the beginning of W_1 with $\mathcal{X}^4(W_4) = \natural^m(S^1 \times D^3)$. Here again, since 3- and 4-handles glue in uniquely by [11], we see that $\mathcal{X}^4_C(W_1W_2W_3W_4) \cong X$. $\square$

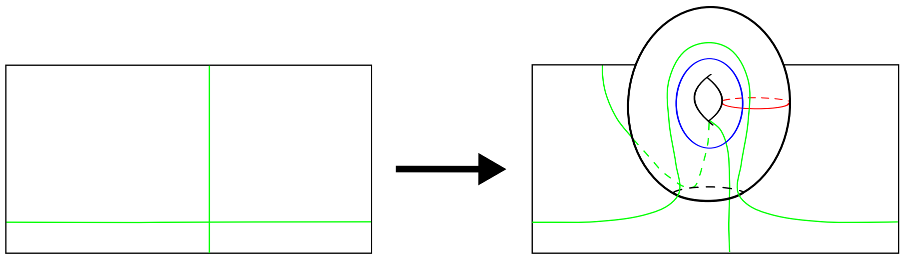

Figure 11: Stabilizing the Heegaard surface of the boundary of the 0- and 1-handles allows us to eliminate any intersections which arise between the 2-handles when they are projected onto the Heegaard surface.

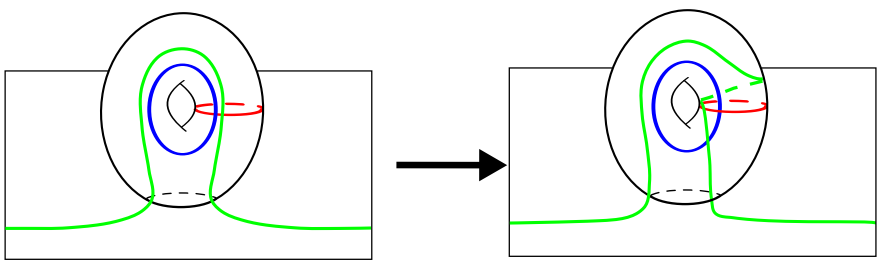

Figure 12: By twisting an attaching circle of a 2-handles around the corresponding dual α curve, we can make the handle framing match the surface framing.

4. The 2-skeleton of the pants complex and cobordisms of 4-manifolds

4.1. Elementary homotopies in the pants complex

Let X be a connected 2-dimensional CW-complex where all of the 2-cells are attached along a finite number of 1-cells. We say two cellular loops L_1 and L_2 in the 1-skeleton X^1 are *elementary homotopic* if they differ by a sequence of the following two moves or their inverses:

1) If somewhere in the loop, an edge e is traversed and then immediately traversed again in the opposite direction, we can remove these two occurrences of e from the loop.

2) If the loop transverses the boundary of a 2-cell, we can remove the traversal of the boundary of the 2-cell.

We will refer to these two moves (and their respective inverses) as Move (1) and Move (2). By completing a path which goes over part of a 2-cell to go completely around the 2-cell and then backwards using Move (1), we see that the equivalence relation of being elementary homotopic is the same as the relationship where instead of Move (2) we are allowed to replace part of the boundary of a 2-cell with the rest of the boundary as in Figure 21.

Lemma 8. *If X be a simply-connected 2-dimensional CW-complex where each 2-cell is attached along a finite number of 1-cells, then any two cellular loops in X_1 are elementary homotopic.*

Proof. Let $*$ be a base point for X. We show that any cellular loop is elementary homotopic to the constant loop at $*$, from which the result follows. Let T be a maximal spanning tree in X^1. We pick an orientation of each of the edges of X^1 that is not contained in T. Then by Seifert-van Kampen, we obtain a presentation for $\pi_1(X)$ with one generator for each edge, e, in $X^1 - T$. Here, the generator is given by first traversing the unique path in T from $*$ to the initial vertex of e (using the fixed orientation), then traversing e, and then returning to $*$ via the unique path in T from $*$ to the terminal vertex of e – we denote this element by g_e. By orienting the attaching region of each 2-cell, D_α, and choosing a base vertex on the boundary, we obtain a relation r_α. Namely, if as we traverse the boundary of D_α we see the sequence of distinguished edges $e_1^{\epsilon_1} \cdots e_k^{\epsilon_k}$ in $X^1 - T$, and $\epsilon_i = \pm 1$ indicates the orientation, then we have the relation $g_{e_1}^{\epsilon_1} \cdots g_{e_k}^{\epsilon_k}$. Note that there may be other edges in the boundary of the 2-cell which do not contribute to the relation.

Since $\pi_1(X) = 1$ we have $\langle g_e | r_\alpha \rangle = 1$. Let L be a loop in X_1. Whenever modifying L, we will again refer to the result as L. By using Move (1), modify L so that it contains $*$. Now we can consider L as an element of $\pi_1(X, *)$. Suppose that as we go around L, we see the distinguished edges in $X^1 - T$ in the order $e_1^{\epsilon_1} \cdots e_k^{\epsilon_l}$.

By repeatedly using Move (1), we make L equal to the loop $g_{e_1}^{\epsilon_1} \cdots g_{e_k}^{\epsilon_l}$ as follows: We modify L using Move (1) so that before each vertex v on an edge e_i in L, we have L first return to $*$ from v using the unique path in

T, and then return from $*$ to v again taking the same unique path in T. By again using Move (1) to remove any redundant edges, the resulting loop L is now exactly $g_{e_1}^{\epsilon_1} \cdots g_{e_k}^{\epsilon_l}$.

Now since $\pi_1(X) = \langle g_e | r_\alpha \rangle = 1$ we know that

$$g_{e_1}^{\epsilon_1} \cdots g_{e_k}^{\epsilon_l} = \prod_{i=1}^{m} h_i r_i h_i^{-1}$$

where each h_i is a word in the g_e and where each r_i is a relation coming from one of the 2-cells. Note that any occurrence of $g_e g_e^{-1}$ or $g_e^{-1} g_e$ in L can be eliminated by using Move (1).

It therefore only remains to show how we can remove a relation r_i from L. By using Move (1) to remove any redundant edges from r_i, we obtain the loop that starts at $*$, travels along the unique path in T to the base point of the boundary of the corresponding 2-cell D_α, traverses the boundary of D_α, and then returns in T to $*$. By using Move (2), this is elementary homotopic to just taking the path in T from $*$ to the base point of D_α and then returning in T to $*$. By applying Move (1), this can also be removed. $\square$

We seek to understand how our 4-manifolds change under elementary homotopy. To this end, we must understand the 4-manifolds that arise along all connected subsets of the boundaries of the 2-cells.

Note that the mapping class group of Σ acts on the set of handlebodies with boundary Σ, and that this action is transitive. If H is a handlebody with $\partial H = \Sigma$ and ϕ is a mapping class of Σ, we will denote the result of the action of ϕ on H by $\phi \cdot H$.

Lemma 9. *Let $W = P_1 \cdots P_n$, $W_2 = Q_1 \cdots Q_n$ be two walks in $\mathcal{P}(\Sigma)$, and let ϕ be a mapping class of ϕ such that $\phi \cdot \mathcal{H}(P_i) = \mathcal{H}(Q_i)$ for all $1 \leq i \leq n$. Then $\mathcal{X}^4(W_1) \cong \mathcal{X}^4(W_2)$. The analogous statement also holds for closed loops.*

Proof. This is an immediate application of [11] applied to each of the wedge pieces. $\square$

We now go through an extended example, analyzing the 4-manifolds obtained from the boundary of one of the two-cells in $\mathcal{P}(\Sigma)$. Many of the arguments will be repeated for the other 2-cells and this will form the core of the results that follow. Lemma 6 will be applied throughout this example and the examples that follow, in order to obtain handlebody decompositions so that we can recognise the relevant manifolds.

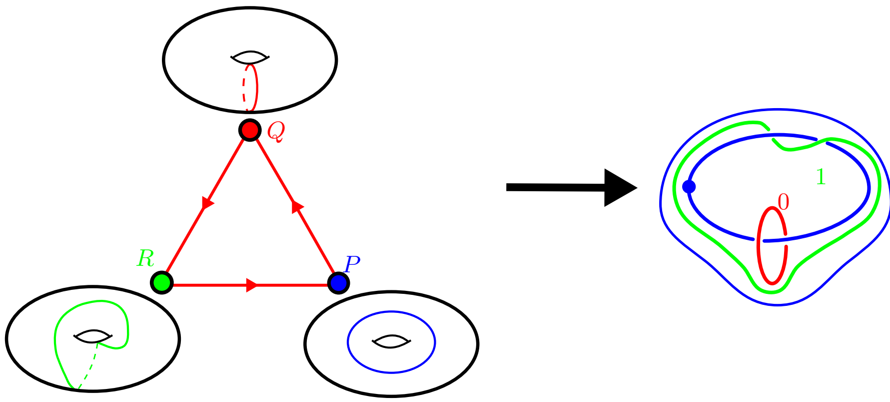

Figure 13: A 3S-triangle in $P(\Sigma_1)$ gives rise to the Kirby diagram on the right. After sliding the green curve over the red curve and cancelling the red-blue pair, we see that this manifold is $\mathbb{C}P^2$.

4.2. 3S-Triangles

Let Σ be a torus and recall that the pants decomposition $\mathcal{P}(\Sigma)$ is defined to have vertices as isotopy classes of essential curves, only S-edges, and 3S-triangles as 2-cells. Consider the oriented boundary of the 3S-triangle in Figure 13. We will be using Lemma 6 to convert loops in the pants complex to handlebody diagrams. After a handle slide, it is evident that $\mathcal{X}^4_C(PQRP) = \mathbb{C}P^2$. Note that if we instead traverse the triangle in the other direction, we obtain $\overline{\mathbb{C}P}^2$. By applying an appropriate mapping class, we can interchange any of the pants decompositions in the triangle. By Lemma 9, all of the resulting segments are diffeomorphic to the corresponding segments in $PQRP$. Thus, we have seen that any edge in the $PQRP$ triangles corresponds to a 4-ball, any pair of adjacent edges corresponds to $\mathbb{C}P^2 - \mathring{B}^4$ or $\overline{\mathbb{C}P}^2 - \mathring{B}^4$ depending on the orientation of the edges, and the whole triangle corresponds to $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$, again depending on the orientation. Note that for any set of curves on Σ that form a 3S-triangle Δ, there is a mapping class ϕ of Σ that sends the vertices of Δ to P, Q, R in some order, and therefore the above analysis carries over for any such Δ.

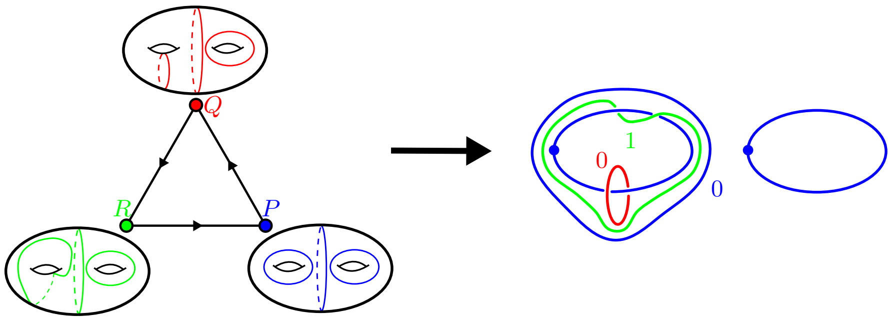

Figure 14: A 3S-triangle in $P(\Sigma_2)$ gives rise to the Kirby diagram on the right. After sliding the green curve over the red curve and cancelling the red-blue pair, we see that this manifold is $\mathbb{C}P^2\#(S^1 \times S^3)$.

An analogous analysis for 3S-triangles on higher genus surfaces is very similar. For example, in a genus 2 surface Σ in Figure 14 we have

$$\mathcal{X}^4_C(PQRP) = (S^1 \times S^3)\sharp\mathbb{C}P^2.$$

As in the torus example, we see by Lemma 9 that this analysis holds for any individual edge in the triangle, and any pair of adjacent edges, and any 3S-triangle in $\mathcal{P}(\Sigma)$. Similarly, this all goes through in the genus g case where an edge will give $\natural^{g-1}(S^1 \times D^3)$, a pair of adjacent edges will give $\natural^{g-1}(S^1 \times D^3)\sharp\mathbb{C}P^2$ or $\natural^{g-1}(S^1 \times D^3)\sharp\overline{\mathbb{C}P}^2$ depending on the orientation, and the whole triangle will give $\sharp^{g-1}(S^1 \times S^3)\sharp\mathbb{C}P^2$ or $\sharp^{g-1}(S^1 \times S^3)\sharp\overline{\mathbb{C}P}^2$ depending on the orientation.

4.3. 3A-Triangles, 4A-squares and 5A-Pentagons

The 3A-triangle and the 5A-pentagon both give rise to $\sharp^g(S^1 \times S^3)$ when Σ has genus g, since, by Lemma 6, the resulting manifold is built with a 0-handle, $g-$ 1-handles, g 3-handles, and a 4-handles. Further, again by Lemma 6 all of the edges and sequences of adjacent edges give rise to $\natural^g(S^1 \times D^3)$. The same also holds true for the 4A-square.

4.4. 4S-Squares

In this example, we look at the 4S-squares. We start with the genus 2 case shown in Figure 15. From the Kirby diagram, we see that $\mathcal{X}^4(PQ) = S^1 \times$

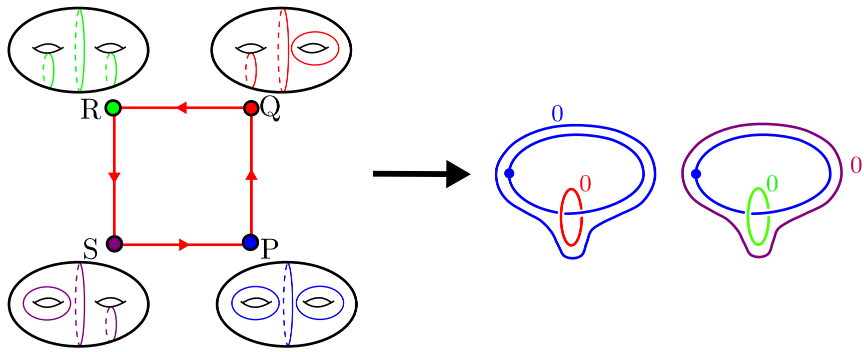

Figure 15: The 4S-square analyzed in Section 4.4. In $P(\Sigma_2)$, this square gives rise to the Kirby diagram on the right. After cancelling the 1-2 pairs, one recognizes this as a Kirby diagram for S^4 consisting of cancelling 2- and 3-handles.

D^3, $\mathcal{X}^4(PQR) = B^4$, $\mathcal{X}^4(PQRS) = S^2 \times D^2$, 0 and $\mathcal{X}^4_C(PQRSP) = S^4$. By the symmetry of the above diagrams, and Lemma 9, we find that the above analysis holds for all of the edges of the square, and all pairs and 3-tuples of adjacent edges. Again by Lemma 9, this holds for all such 4S-squares in $\mathcal{P}(\Sigma)$.

In the case of a genus g surface Σ, as in the previous examples, when $g > 2$ this simply adds more 1-handles to the above situation. So in this case the edges will result in $\natural^{g-1}(S^1 \times D^3)$, a pair of adjacent edges will give $\natural^{g-2}(S^1 \times D^3)$, three adjacent edges will give $\natural^{g-2}(S^1 \times D^3)\natural(S^2 \times D^2)$, and the whole square as a closed manifold will be $\sharp^{g-2}(S^1 \times S^3)$.

4.5. 4AS-Squares and 6AS-Hexagons

Next, we analyze the 4AS-square. Here our situation is pictured Figure 16, where vertices of the same color correspond to the same handlebody. Suppose that Σ has genus g. Then we have the following:

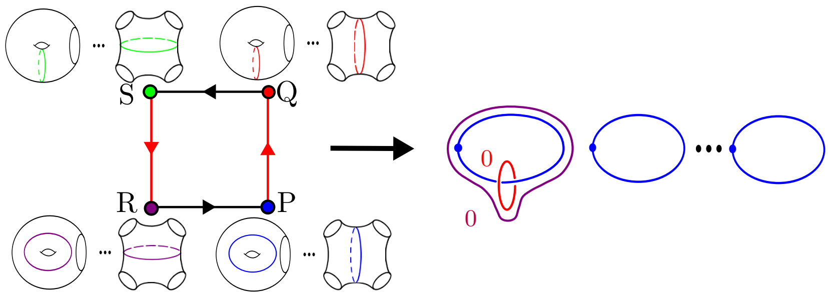

Figure 16: The 4CAS-Square analyzed in Section 4.5.

$$\mathcal{X}^4(PQ) = \mathcal{X}^4(RS) = \natural^{g-1}(S^1 \times D^3) \tag{1}$$
$$\mathcal{X}^4(QR) = \mathcal{X}^4(SP) = \natural^{g}(S^1 \times D^3) \tag{2}$$
$$\mathcal{X}^4(PQR) = \mathcal{X}^4(QRS) = \mathcal{X}^4(RSP) = \mathcal{X}^4(SPQ) = \natural^{g-1}(S^1 \times D^3) \tag{3}$$
$$\mathcal{X}^4(PQRS) = \mathcal{X}^4(RSPQ) = \natural^{g-1}(S^1 \times D^3) \tag{4}$$
$$\mathcal{X}^4_C(PQRSP) = \sharp^{g-1}(S^1 \times S^3) \tag{5}$$
$$\mathcal{X}^4(QRSP) = \mathcal{X}^4(SPQR) = \natural^{g-1}(S^1 \times D^3)\natural(S^2 \times D^2) \tag{6}$$

The 4-manifold that we obtain associated to a single edge is either $\natural^g(S^1 \times S^3)$ or $\natural^{g-1}(S^1 \times S^3)$ depending on whether the edge is an A-edge or an S-edge - thus 1 and 2 follow. The 4-manifold associated to any walk is the same as the 4-manifold associated to the walk obtained by ignoring any A-edges at the beginning or end of the walk, which yields 4. Then 5 follows from 4 by [11]. All of these could have instead have been obtained by applying Lemma 6. To see 6 we use Lemma 6 as in Figure 16.

The case of the 6AS-hexagon is completely analogous to the 4AS-square where we have the labeling in Figure 17. Namely, the two A-edges in between the S-edges do not contribute to changes in the topology and can be ignored as in the previous paragraph and thus everything is handled in an analogous fashion.

4.6. The 4-dimensional cobordism group

We now are in position to derive the following classical result from Hatcher's theorem that $\mathcal{P}(X)$ is simply-connected and our above analysis of the 2-cells:

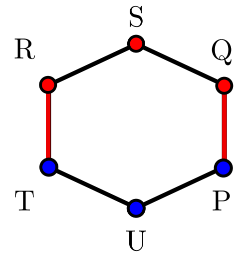

Figure 17: The 6AS-hexagon analyzed in Section 4.5.

Theorem 3. *Every closed 4-manifold is cobordant to the connect sum of some number of* $\mathbb{C}P^2$ *and* $\overline{\mathbb{C}P}^2$*.*

By a theorem of Thom [17], signature is a cobordism invariant. We remark that Thom's theorem has an elementary proof in terms of Lefschetz duality given on page 222 of [16]. It follows from Theorem 4 that the oriented cobordism group Ω_4 is isomorphic to $\mathbb{Z}$. Note, as an aside, that the fact that $\mathcal{P}(\Sigma)$ is connected together with our construction of $\mathcal{M}^3(P_1, P_2)$ immediately yields that $\Omega_3 = 0$.

We will need to understand how $\mathcal{X}^4_C(L)$ changes when we alter L by going over some 2-cell in $\mathcal{P}(\Sigma)$. We begin with the following preliminary observation whose proof follows immediately from the definition of our construction:

Lemma 10. *Let W_1, W_2 be walks in $\mathcal{P}(\Sigma)$ with endpoints P_1 and P_2, and let U and V be two walks in $\mathcal{P}(\Sigma)$ so that the end point of U is P_1 and the start point of V is P_2. Then*

$$\mathcal{X}^4(UW_2V) = (\mathcal{X}^4(UW_1V) - int(\mathcal{X}^4(W_1))) \cup_{\mathcal{M}^3(P_1,P_2)} \mathcal{X}^4(W_2)$$

This also holds in the case of $\mathcal{X}^4_C$ where the beginning of U is the end of V.

Note that there are no choices involved in the gluing as, by construction, $\partial\mathcal{X}^4(W_1) = \mathcal{M}^3(P_1, P_2) = \partial\mathcal{X}^4(W_2)$ as sets. We will be applying Lemma 10, in the case where $W_1 \cup W_2$ is the boundary of a 2-cell in $\mathcal{P}(\Sigma)$. This set up is pictured in Figure 21.

Recall that a 1-surgery on a 4-manifold X^4 is the result of taking an embedding $\phi : S^1 \times D^3 \hookrightarrow X$ and forming the new 4-manifold

$$X' = (X - \phi(S^1 \mathring{\times} D^3)) \cup_{\partial\phi} (D^2 \times S^2)$$

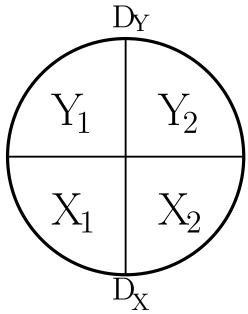

Figure 18: A schematic of the manifold Z used in Lemma 11. The portions labeled X_1 and Y_1 are diffeomorphic via a canonical diffeomorphism.

where we note that $\partial(D^2 \times S^2) = S^1 \times S^2 = \partial(S^1 \times D^3)$ and $\partial\phi$ denotes ϕ restricted to the boundary. In this case we say that X' is obtained from X by a 1-surgery. A 2-surgery on a 4-manifold is defined by switching the roles of $S^1 \times D^3$ and $D^2 \times S^2$ above. Note that if X' is obtained from X by a 1-surgery, then X' and X are cobordant via the *trace* of the surgery. Namely, given $\phi : S^1 \times D^3 \hookrightarrow X$, we can form

$$(X \times I) \cup_{S^1 \times D^3 \subset X \times \{1\}} D^2 \times D^3$$

which is a cobordism from X to X'. This can similarly be done if X' is obtained from X by a 2-surgery.

In our set up, we will not be seeing precisely the manifolds used in the definition of surgery, but nonetheless, the effect of cutting and pasting these pieces has the same effect as surgery. The following lemma will make this statement precise, but we will preface the lemma with an intuitive discussion following Figure 18. Here, we have a manifold broken into two pieces X and Y glued along their boundaries. These pieces are further broken into two pieces which are boundary summed. The portions labeled X_1 and Y_1 are diffeomorphic via a map extending their identifications on the boundary. If M is some other manifold and $X \subset M$ is an embedding, then the identification of the boundaries of X and Y allows us to replace X by Y to obtain a new manifold. The following lemma states that this operation actually only amounts to replacing X_2 by Y_2.

Lemma 11. *Let Z and M be oriented 4-manifolds with $Z = X \cup Y$ where $X = X_1 \natural_{D_X} X_2$ and $Y = Y_1 \natural_{D_Y} Y_2$ and where $\partial D_X = \partial D_Y$. Further, suppose that X_1 and Y_1 are diffeomorphic via some orientation-reversing diffeomorphism that extends the identification of ∂X_1 with ∂Y_1 (where we choose*

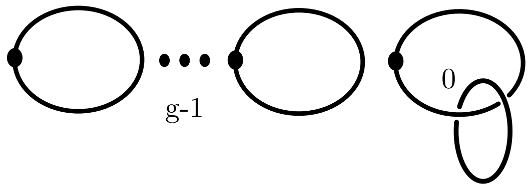

Figure 19: The handle decomposition of $\sharp^{g-1}(S^1 \times S^3)$ used for the surgery operation in Lemma 12.

some identification of D_X with D_Y). Assume also that we are given some orientation-preserving embedding $\phi : X \hookrightarrow M$ and let $M' = (M - \phi(X)) \cup_{\partial\phi} \overline{Y}$. Then $M' = (M - \phi(X_2)) \cup_{\partial\phi|_{X_2}} \overline{Y_2}$.

Proof. As is implicit in the statement, throughout we will use a fixed identification of D_X and D_Y extending the given identification $\partial D_X = \partial D_Y$. In particular, the map $\partial\phi_{X_2}$, this identification is used implicitly so that $\partial\phi_{X_2}$ is an identification of all of ∂X_1 with all of ∂X_2.

Instead of removing all of X and gluing back in Y to get M', we can first remove X_1 and glue back in Y_1, which gives us M back, by our assumption that X_1 and Y_1 are diffeomorphic via a diffeomorphism extending their boundary identification. Now we obtain M' by removing X_2 and gluing back in Y_2, which is exactly the statement of the lemma. □

The following two lemmas are the particular cases of of Lemma 11 which we will need. In particular, Lemma 12 will be applied to the case of homotoping a loop over a 4AS-square or a 6AS-hexagon, and Lemma 13 will be applied to the case of homotoping a loop over a 3S-Triangle.

Lemma 12. *Let X and M be oriented 4-manifolds with $X \cong \sharp^{g-1}(S^1 \times S^3)$ given by the Kirby diagram in Figure 19. Let $Z = X \cup Y$ where X is the union of the 0- and 1-handles and Y is the union of the 2-, 3-, and 4-handles. Let $\phi_1 : X \hookrightarrow M$ and $\phi_2 : Y \hookrightarrow M$ be orientation-preserving inclusions. Then*

$$(M - \phi_1(\mathring{X})) \cup_{\partial\phi_1} \overline{Y}$$

and

$$(M - \phi_2(\mathring{Y})) \cup_{\partial\phi_2} \overline{X}$$

are obtained from M by a single 1-surgery, and a single 2-surgery, respectively.

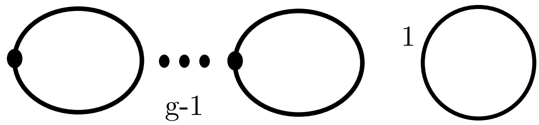

Figure 20: The handle decomposition of $\sharp^{g-1}(S^1 \times S^3)\sharp\mathbb{C}P^2$ used for the surgery operation in Lemma 13.

Proof. With notation as in Lemma 11, we take $X = X_1 \cup X_2$, where X_1 is a portion of the 0-handle together with the first $(g-1)$ 1-handles, and X_2 is the rest of the 0-handle together with the remaining 1-handle that is cancelled by the 2-handle in Z. Additionally, let $Y = Y_1 \cup Y_2$, where Y_1 is the 3-handles together with a portion of the 4-handle, and Y_2 is the rest of the 4-handle together with the 2-handle which cancels the 1-handle. Then the result follows from Lemma 11. □

If $X'\sharp\mathbb{C}P^2$ or $X'\sharp\overline{\mathbb{C}P}^2 = X$, then we call X' a (+)- or (-)-blowdown of X, respectively. Likewise, we call X a (+)- or (-)-blowup of X'. In the following construction, the pieces involved are not exactly those used in the definition of a blowup or a blowdown. However, as in the previous lemma the overall effect is to perform a blowup or a blowdown. The proof is the same word-for-word as the proof of Lemma 12 but applied to Figure 20.

Lemma 13. *Let Z and M be oriented 4-manifolds with $Z \cong \sharp^g(S^1 \times S^3)\sharp\mathbb{C}P^2$ given by the Kirby diagram in Figure 20. Let $Z = X \cup Y$ where X is the union of the 0- and 1-handles and the 2-handle which cancels the 1-handle. Y is the union of the rest of the 2-handles, as well as the 3-, and 4-handles. Let $\phi : X \hookrightarrow M$ be orientation-preserving inclusions. Then*

$$(X' - \phi_1(\mathring{X}_1)) \cup_{\partial\phi_1} \overline{X_2}$$

and

$$(X' - \phi_2(\mathring{X}_2)) \cup_{\partial\phi_2} \overline{X_1}$$

are a (-)-blowup, and a (-)-blowdown of X', respectively.

Following the previous analysis we now understand how manifolds change as we homotope loops over 2-cells in the pants complex. This leads us to our main application.

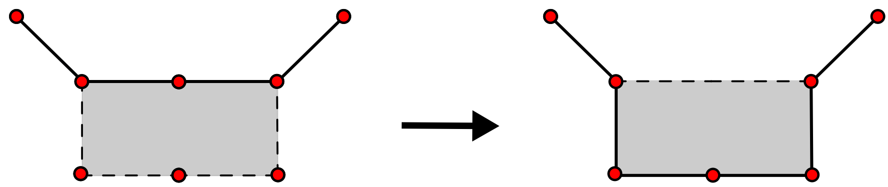

Figure 21: A homotopy of a path in a 2-complex consists of a sequence of operations of replacing a connected subset of the boundary of a 2-cell with the rest of the boundary.

Theorem 4. *Every smooth, oriented, closed manifold is cobordant to* $\coprod_m \mathbb{C}P^2 \coprod_n \overline{\mathbb{C}P}^2$.

Proof. Let Σ be a genus g surface, and let L be a loop in $\mathcal{P}(\Sigma)$. By Theorem 1, $\mathcal{P}(\Sigma)$ is simply-connected, so we know that there exists a cellular disk D in $\mathcal{P}(\Sigma)$ with boundary L. First, we alter L using D to a new loop called L', so that L' traverses a tree in $\mathcal{P}(\Sigma)$. We can do this by changing the sides of the polygons that L goes around, using D, as in Lemma 10. We will consider the different 2-cells following the order of the analysis in Section 4.

By Lemma 13, changing L using a 3S-triangle results in the manifold changing by a (+)- or a (-)-blowup or blowdown, depending on the orientation of the triangle and the partition of the edges. We may achieve a cobordism from $X^4_C(L)$ to the resulting manifold by introducing a disjoint copy of $\mathbb{C}P^2$ or $\overline{\mathbb{C}P}^2$ and forming the standard cobordism from a disjoint union to the connected sum, as illustrated in Figure 22.

Note that in Lemma 10, if W_1 and W_2 both consist only of A-moves, then $\mathcal{X}^4(W_1) \cong \mathcal{X}^4(W_2) \cong \natural^g S^1 \times D^3$ and thus by [11], the result of doing the replacement as in Lemma 10 does not change the manifold. Therefore, changing L using a 3A-triangle, a 5A-pentagon, or a 4A-square does not change the resulting 4-manifold at all. This modification of the loop therefore corresponds to the product cobordism on the manifolds.

Suppose we are changing L using a 4S-square. If we are in the situation where the edges are partitioned into two sets of two adjacent edges, then the effect of the move is to remove some set 4-dimensional 1-handlebody, and then to reinsert the same 4-dimensional 1-handlebody, so by [11], this does not change the resulting 4-manifold. If we are in the case where the edges are partitioned into two sets where one set has three adjacent edges, then this affects L by removing $\natural^{g-1}(S^1 \times D^3)$ and inserting $\natural^{g-2}(S^1 \times D^3)\natural(S^2 \times D^2)$

or vice versa, depending on which edges belong to L. By Lemma 12 this affects the resulting 4-manifold by performing either a 1- or a 2-surgery on $\mathcal{X}^4_C(L)$. We may achieve a cobordism between these manifolds using the trace of this surgery.

The 4AS-square is handled similarly. If we are in the situation where the edges are partitioned into two sets of two adjacent edges, then the effect of the move is to remove some set 4-dimensional 1-handlebody, and then to reinsert the same 4-dimensional 1-handlebody, so by [11], this does not change the resulting 4-manifold. Suppose we are in the case where the edges are partitioned into two sets where one set has three adjacent edges. If the set with a single edge is either PQ or RS, then this affects L by removing $\natural^g(S^1 \times D^3)$ and inserting $\natural^{g-1}(S^1 \times D^3)\natural(S^2 \times D^2)$ or vice versa, depending on which edges belong to L. By Lemma 12 this affects the resulting 4-manifold by performing either a 1- or a 2-surgery on $\mathcal{X}^4_C(L)$. If the set with a single edge is either QR or SP, then the effect of the move is to remove some set 4-dimensional 1-handlebody, and then to reinsert the same 4-dimensional 1-handlebody, so by [11], this does not change the resulting 4-manifold. In either case, we may achieve a cobordism between these manifolds using the trace of this surgery. The case of a 6AS-square is handled completely analogously.

After collapsing all of the 2-cells, we will have a loop L' that is traversing a tree in $\mathcal{P}(\Sigma)$ with $\mathcal{X}^4_C(L')$, $\mathcal{X}^4_C(L)$, and some number of $\mathbb{C}P^2$'s and $\overline{\mathbb{C}P}^2$'s cobounding a 5-manifold. We next collapse L' to just a point. To do this we proceed inductively on the size of the tree. Choose a leaf of the tree, so that L' must traverse this leaf in one direction, and then immediately turn back and go in the other direction. Let L'' be the loop obtained from L' by removing this redundant edge followed by its reverse. If the edge of the leaf is an A-edge, then $\mathcal{X}^4_C(L'')$ is equal to $\mathcal{X}^4_C(L')$, since removal of A-edges does not change the resulting 4-manifold. So we need only consider the case where this leaf edge is an S-edge.

Let W be a walk in $\mathcal{P}(\Sigma)$ that is an S-edge traversed twice in a row in opposite directions. The effect of removing the S-leaf from L' is the same as removing a copy of $\mathcal{X}^4(W)$ from $\mathcal{X}^4_C(L')$ and replacing the result with $\natural^g(S^1 \times D^3)$, which is the manifold with boundary associated to the constant path. But, in the same way that we analysed the boundaries of the 2-cells above, we find that $\mathcal{X}^4(W)$ is always $\natural^{g-1}(S^1 \times D^3)\natural(D^2 \times S^2)$. But then, by Lemma 12, $\mathcal{X}^4_C(L'')$ is obtained from $\mathcal{X}^4_C(L')$ by performing a 1-surgery. We again obtain a cobordism between these manifolds using the trace of the surgery. Then by induction on the number of leaves, we may repeat this process until we arrive at the constant path. In the end, we find that there is

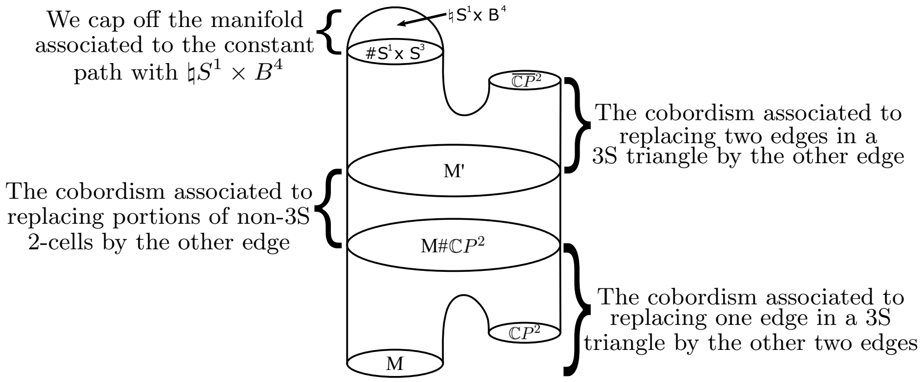

Figure 22: An illustration of the cobordism build in the proof of Theorem 4.

a 5-manifold whose boundary is a disjoint union of $\mathcal{X}^4_C(L)$, some number of $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$, and $\sharp^g S^1 \times S^3$. By capping off the $\sharp^g S^1 \times S^3$ with $\natural^g S^1 \times B^4$, the result follows. $\square$

5. An explicit cobordism

In the previous section, we have seen that a cellular homotopy of a loop $L \subset P(\Sigma)$ over a 2-cell which is not a 3S-triangle changes the manifold $X^4_C(L)$ by a (possibly trivial) surgery. Given a cellular homotopy between a loop L and a loop L' which does not pass over any 3S-triangles, we get a cobordism between $\mathcal{X}^4_C(L)$ and $\mathcal{X}^4_C(L')$ by composing the trace cobordisms obtained when the loop moves over each 2-cell.

The remainder of this section is dedicated to producing an explicit example of such a trace cobordism. In particular, we seek to produce a cobordism between S^4 and $S^2 \times S^2$ corresponding to the trace of a 1-surgery on S^4. This corresponds to a cobordism consisting of a single 5-dimensional 2-handle, so that the construction is determined by the isotopy type and the $\mathbb{Z}/2\mathbb{Z}$ valued framing of the 2-handle. Since S^4 is simply connected, and $S^2 \times S^2$ is spin, there is actually a unique cobordism from S^4 to $S^2 \times S^2$ consisting of a single 2-handle. Thus, once we have constructed this cobordism, we know it must correspond to the 5-manifold $(S^2 \times D^3) - \mathring{B}^5$.

We start the construction by describing loops in the pants complex corresponding to S^4 and $S^2 \times S^2$ coming from trisections of these manifolds. At the tops of Figures 23 and 24 we see trisection diagrams for S^4 and $S^2 \times S^2$, respectively. These give rise to the loops in the pants complex shown below each of these diagrams. These loops meet in a 6AS-hexagon as shown in Figure 25 with the rectangle on the left being the loop corresponding to S^4, and the long way around both rectangles representing $S^2 \times S^2$. Taking the homotopy of the loop representing S^4 over the 6AS-hexagon corresponds to doing a 1-surgery on S^4 to produce $S^2 \times S^2$. By the discussion in the previous paragraph this produces the cobordism $(S^2 \times D^3) - \mathring{B}^5$.

6. Signature

Thus far, we have primarily used information about the pants complex to derive information about 4-manifolds, but this process can also be reversed. For example, given a loop L in $\mathcal{P}(\Sigma)$ which bounds a disk D, we may derive information about the 2-cells that make up D from L.

By orienting L, the disk D inherits an orientation, and, in particular all of the 3S-triangles in D inherit an orientation as well. Each triangle in this disk is either positive or negative, namely those that give rise to $\mathbb{C}P^2$ are positive, and those that give rise to $\overline{\mathbb{C}P}^2$ are negative. Following Theorem 4, we see that summing the number of positive 3S-triangles in D and subtracting the negative 3S-triangles in D gives us the signature of $X^4_C(L)$. Defining $\sigma(L)$ to be $\sigma(X^4_C(L))$, we immediately obtain the following proposition.

Proposition 1. *Let L be a loop in the pants complex with $\sigma(L) = n$, then any disk bounded by L must contain at least n 3S-triangles.*

In the remainder of this section we show how $\sigma(L)$ can be calculated directly from L in the 1-skeleton of the pants complex. On the other hand, this invariant constrains the type of 2-cells which L can bound. One can view this as a particular instance of a theorem of Margalit [13], who showed that the 2-cells of the pants complex are determined by the combinatorics of its 1-skeleton.

Novikov additivity [1] states that, if X and X' are two 4-manifolds with diffeomorphic boundary, and Y is a manifold obtained by gluing X and X' along their boundaries by a orientation-reversing diffeomorphism, then $\sigma(Y) = \sigma(X) + \sigma(X')$. When the manifolds X_1 and X_2 are not glued along their whole boundary, but rather some submanifolds of their boundary, then we no longer have this additivity. However, there is a correction term that was identified by Wall [20]. This correction term was further identified with the Maslov index of a certain triple of Lagrangians in [4] (see also [14]).

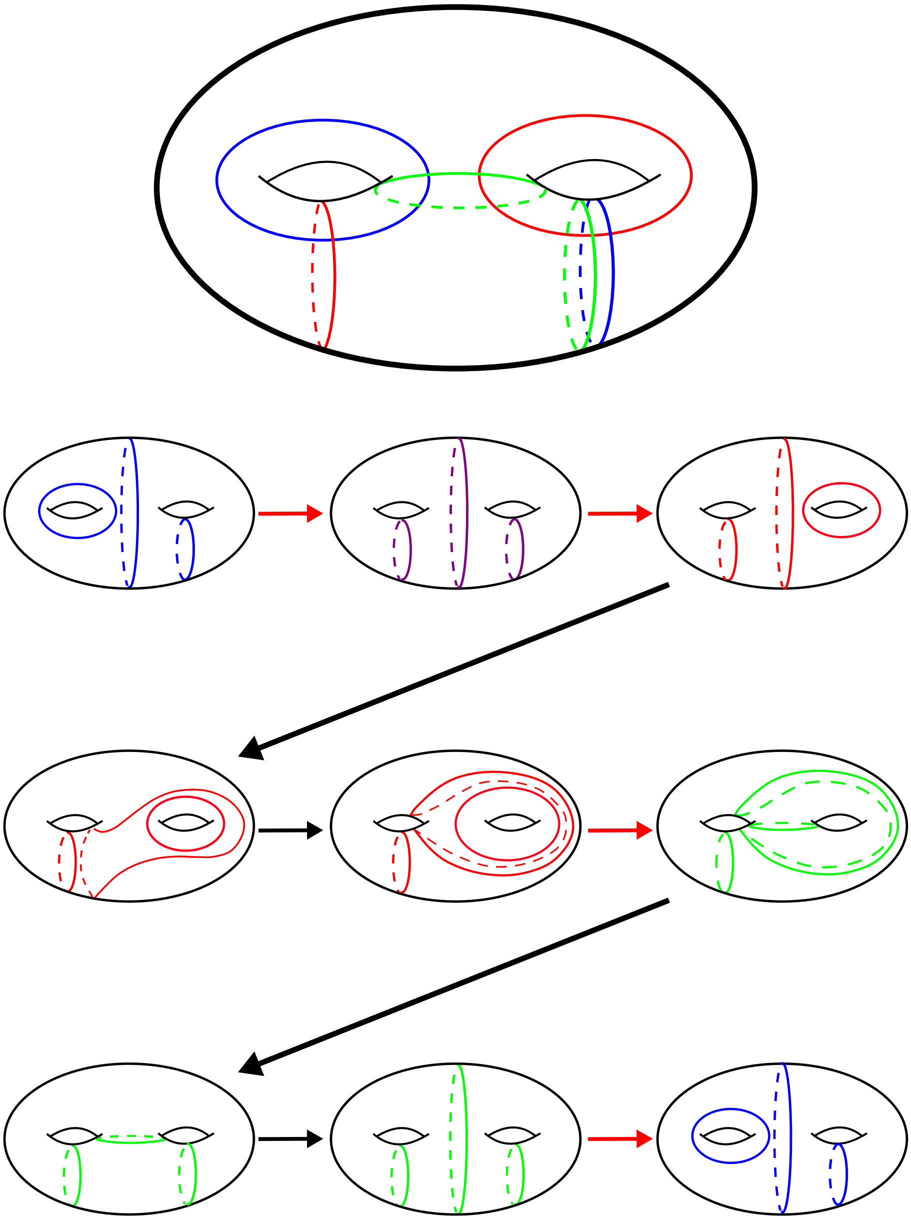

Figure 23: Top: An unbalanced trisection diagram for S^4. Bottom: A loop in the pants complex corresponding to this trisection.

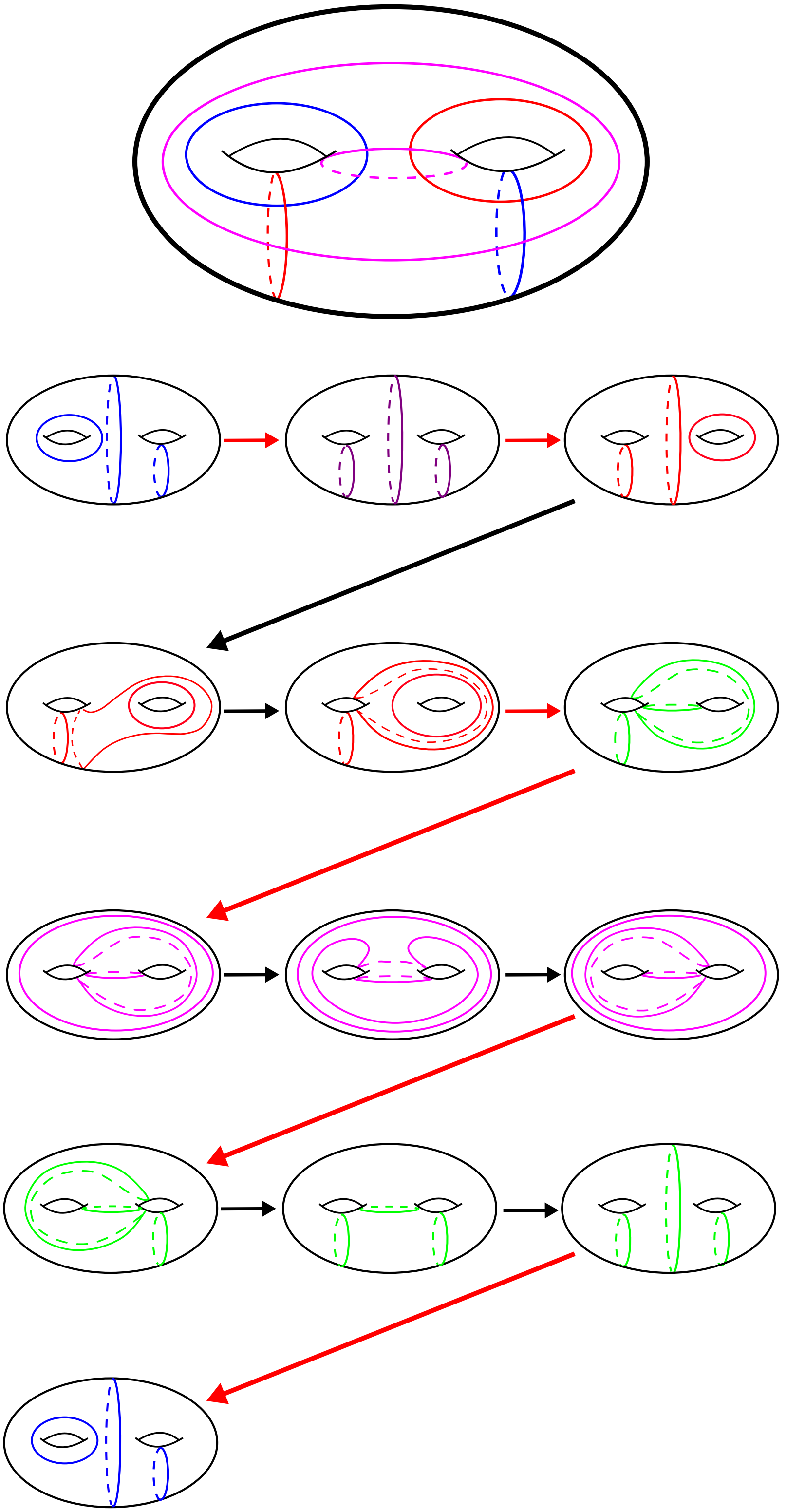

Figure 24: Top: A trisection diagram for $S^2 \times S^2$. Bottom: A loop in the

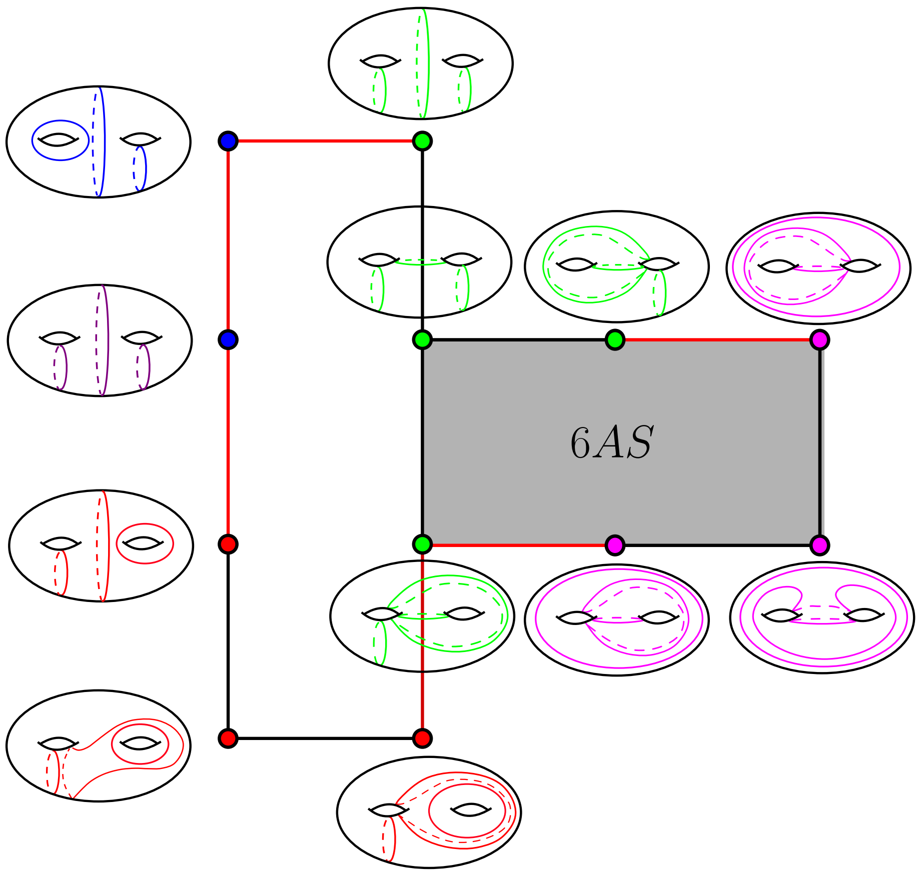

Figure 25: The loops for $S^2 \times S^2$ and S^4 meet in a $6AS$-hexagon. Taking the short way around the loop corresponds S^4 whereas the long way gives $S^2 \times S^2$.

Let (V, ψ) be a finite-dimensional vector space over $\mathbb{Q}$ together with a nonsingular symplectic form, and let $L_1, L_2, L_3 \subset V$ be three Lagrangians. The Maslov index $M(L_1, L_2, L_3) \in \mathbb{Z}$ is the signature of the singular symmetric form given by

$$\begin{aligned} \theta : L_1 \oplus L_2 \oplus L_3 \times L_1 \oplus L_2 \oplus L_3 &\to \mathbb{Q} \\ ((x_1, x_2, x_3), (y_1, y_2, y_3)) &\mapsto \sum_{i \neq j} (-1)^{i+j} \psi(x_i, y_j) \end{aligned}$$

Suppose that the boundaries of X and X' have both been Heegaard split with the same genus surfaces so that $\partial X = H_1 \cup_\Sigma H_2$ and $\partial X' = H_1' \cup_{\Sigma'} H_2'$. Let Y be the oriented 4-manifold that results from gluing H_2' to H_2 by an orientation-reversing diffeomorphism ϕ. Let L_1, L_2, and L_3 be the Lagrangians in $H_1(\Sigma;\mathbb{Q})$ that are the kernels of the inclusions of Σ as the boundary of H_1, H_2, and (using ϕ) H_1', respectively. Then we have $\sigma(Y) = \sigma(X) + \sigma(X') - M(L_1, L_2, L_3)$ which is the form of Wall additivity that we will use.

Proposition 2. *Let L be a loop in $\mathcal{P}(\Sigma)$ with vertices $P_1, P_2, \cdots P_n$ such that L_i is the Lagrangian subspace of $H_1(\Sigma;\mathbb{Q})$ given by the kernel of the map induced by inclusion $H_1(\Sigma;\mathbb{Q}) \to H_1(\mathcal{H}^3(P_i);\mathbb{Q})$ for $1 \le i \le n$. Then*

$$\sigma(\mathcal{X}_C^4(L)) = -\sum_{i=2}^{n-1} M(L_1, L_i, L_{i+1})$$

Proof. Since $\sigma(\natural^g S^1 \times D^3) = 0$, by Novikov additivity we have $\sigma(\mathcal{X}_C^4(L)) = \sigma(\mathcal{X}^4(P_1P_2\cdots P_n))$. The formula now follows from repeated use of Wall additivity. For $n = 3$ we have,

$$\begin{aligned}\sigma(\mathcal{X}_C^4(L)) &= \sigma(\mathcal{X}^4(P_1P_2P_3)) + \sigma(\natural^g S^1 \times D^3)\\ &= \sigma(\mathcal{X}^4(P_1P_2P_3))\\ &= \sigma(P_1P_2) + \sigma(P_2P_3) - M(L_1, L_2, L_3)\end{aligned}$$

For $n > 3$, we have by induction

$$\begin{aligned}\sigma(\mathcal{X}^4(P_1P_2\cdot P_n)) &= \sigma(\mathcal{X}^4(P_1P_2\cdots P_n)) + \sigma(\natural^g S^1 \times D^3)\\ &= \sigma(\mathcal{X}^4(P_1P_2\cdots P_n))\\ &= \sigma(\mathcal{X}^4(P_1P_2\cdots P_{n-1}))\\ &\quad + \sigma(\mathcal{X}^4(P_{n-1}P_n)) - M(L_1, L_{n-1}, L_n)\\ &= -\sum_{i=2}^{n-1} M(L_1, L_i, L_{i+1})\end{aligned}$$

□

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