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# Decomposition of Lagrangian classes on K3 surfaces

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We study the decomposability of a Lagrangian homology class on a K3 surface into a sum of classes represented by special Lagrangian submanifolds, and develop criteria for it in terms of lattice theory. As a result, we prove the decomposability on an arbitrary K3 surface with respect to the Kähler classes in dense subsets of the Kähler cone. Using the same technique, we show that the Kähler classes on a K3 surface which admit a special Lagrangian fibration form a dense subset also. This implies that there are infinitely many special Lagrangian 3-tori in any log Calabi-Yau 3-fold.

## 1. Introduction

Special Lagrangian submanifolds were introduced by Harvey-Lawson [HL82] as an important class of calibrated submanifolds. They are area minimizers within their homology classes, and a classical problem in geometric analysis is whether a given homology class can be represented by a special Lagrangian.

However, very few constructions of special Lagrangians are known. One method is to evolve a Lagrangian submanifold along the mean curvature flow, where the limit is a special Lagrangian if the flow converges. Such machinery was first introduced by Smoczyk [Smo00]. In general, the flow may not converge and can develop finite time singularities [Nev07]. Thomas and Yau [TY02] further studied the connection between stability and the mean curvature flow. In particular, there exists at most one special Lagrangian representative in each Hamiltonian isotopy class.

From a modern point of view, the mean curvature flow is used to study the Harder-Narasimhan property of the Fukaya category as suggested by Joyce [Joy15]. Many stability conditions have been constructed on the derived category of coherent sheaves on projective varieties. However, little is known about stability conditions on Fukaya categories beyond dimension one. It is a folklore conjecture that the stability condition on a Fukaya category is related to the complex structures on the underlying manifold. In particular, the stable objects would be given by special Lagrangians, and the Harder-Narasimhan property would imply the following statement at the homological level:

> *The homology class of a Lagrangian submanifold is a sum of classes represented by special Lagrangian submanifolds.*

In this paper, we prove the above statement for a dense subset of Kähler classes in the Kähler cone of an arbitrary K3 surface (not necessarily algebraic). Notice that the above statement is not always true: a counterexample can be found in [Wol05].

**Theorem 1.1 (⊆ Theorem 3.13).** *For every K3 surface $X$, its Kähler cone $\mathcal{K}_X$ contains a dense subset $\mathcal{S}_X$ such that every Lagrangian class decomposes as a sum of classes represented by special Lagrangian submanifolds with respect to a Ricci-flat Kähler form $\omega$ parametrized by $\mathcal{S}_X$. Moreover, the set $\mathcal{S}_X$ contains all rational Kähler classes, so this property holds in particular for polarized K3 surfaces.*

Another relevant aspect related to special Lagrangians comes from the Strominger-Yau-Zaslow conjecture [SYZ96]. It predicts that a Calabi-Yau manifold $X$ near large complex structure limit admits a special Lagrangian torus fibration. In the case of K3 surfaces this is already known, and the proof relies on a standard trick involving hyperkähler rotation [GW00] (see also [OO21, Section 4.4]). Our contribution is to study the distribution of the Kähler classes in the Kähler cone of a K3 surface $X$ with respect to which $X$ admits a special Lagrangian fibration.

**Theorem 1.2 (⊆ Theorem 3.15).** *For every K3 surface $X$, its Kähler cone $\mathcal{K}_X$ contains a dense subset $\mathcal{T}_X$ such that every Ricci-flat Kähler form $\omega$ parametrized by $\mathcal{T}_X$ admits a special Lagrangian fibration. Moreover, the set $\mathcal{T}_X$ contains all rational Kähler classes, so this property holds in particular for polarized K3 surfaces.*

In the case of log Calabi-Yau 3-folds, the above theorem in the algebraic case together with [CJL21] imply the following result.

**Corollary 1.3 (= Corollary 3.18).** *Let $Y$ be a Fano 3-fold and $D \in |-K_Y|$ be a smooth anti-canonical divisor. Then the log Calabi-Yau 3-fold $X = Y \backslash D$ contains infinitely many special Lagrangian 3-tori.*

This paper is organized as follows. In Section 2, we review some basic definitions and techniques which are necessary to our work. We also provide a sufficient condition for a Lagrangian class to be decomposed into special Lagrangian classes. In Section 3, we give criteria for the decomposability in terms of lattice theory and then prove the density property as a consequence. The SYZ conjecture for algebraic K3 surfaces is discussed in the end of the same section.

## 2. Lagrangian classes on K3 surfaces

In this section, we start with preliminaries about basic definitions and core machinery necessary to our work. The remaining part is devoted to developing a sufficient condition for a Lagrangian class to be a sum of classes of special Lagrangian submanifolds. Several parts of this section should be well-known to experts.

### 2.1. Lagrangians on Calabi-Yau manifolds

Let $X$ be a Calabi-Yau manifold of complex dimension $n$ and fix a Ricci-flat Kähler form $\omega$ on $X$. An *immersed Lagrangian submanifold* of $X$ with respect to $\omega$ is an immersion

$$\iota \colon L \longrightarrow X$$

of a connected and oriented $n$-dimensional real manifold $L$ such that $\iota^*\omega = 0$. Throughout the paper, we may call such an immersion briefly a *Lagrangian immersion*, or simply a *Lagrangian*. When there is no confusion, we may directly call $L$ a Lagrangian without specifying the immersion.

The Kähler form $\omega$ determines a Riemannian metric on $X$ and thus on $L$, which then induces a Riemannian volume form $dV_L$. On the other hand, the covariantly constant holomorphic top form $\Omega$ on $X$ restricts as a nowhere vanishing top form $\iota^*\Omega$ on $L$. So we can write

$$\iota^*\Omega = e^{i\phi} dV_L$$

where $\phi \colon L \to \mathbb{R}/\mathbb{Z}$ is a differentiable function on $L$ called the *phase*. In the case that $\iota$ is an embedding and the phase $\phi$ is a constant function, we call $L$ a *special Lagrangian submanifold*, or briefly a *special Lagrangian*. Note that this definition is independent of the choice of $\Omega$. It is well-known that special Lagrangians are volume minimizers among Lagrangians [HL82].

We call a homology class $\gamma \in H_n(X,\mathbb{Z})$ a *Lagrangian class* if it can be represented by a Lagrangian immersion, and we call it a *special Lagrangian class* if it is a sum of classes represented by special Lagrangians with positive coefficients. Note that, under this definition, a representative of a special Lagrangian class may be a union of special Lagrangian submanifolds. Using Poincaré duality, we consider a (special) Lagrangian class as a class in $H^n(X,\mathbb{Z})$ throughout the paper.

### 2.2. The hyperkähler rotation trick

A Kähler manifold $(X,\omega)$ of $\dim_{\mathbb{C}} X = 2n$ is *hyperkähler* if its holonomy group is in $\mathrm{Sp}(n)$. From the celebrated Calabi conjecture [Yau78], every holomorphic symplectic manifold is hyperkähler.

From the $\mathrm{Sp}(n)$-holonomy, there exists a covariantly costant holomorphic symplectic 2-form $\theta$ of norm $\sqrt{2}$ unique up to $S^1$-scaling. A *holomorphic Lagrangian* $L \subseteq X$ is a half-dimensional complex submanifold which satisfies $\theta|_L = 0$. Given any

$$\zeta \in \{z \in \mathbb{C} : |z| = 1\} \cong S^1$$

we denote by $X_\zeta$ the Kähler manifold with the same underlying space as $X$ and equipped with Kähler form and holomorphic symplectic 2-form given by

$$\begin{aligned} \omega_\zeta &:= \mathrm{Re}(\zeta\theta) \\ \theta_\zeta &:= \omega - i\mathrm{Im}(\zeta\theta). \end{aligned} \tag{2.1}$$

The following lemma is known as the hyperkähler rotation trick. It can be derived from (2.1) and [HL82, III Corollary 1.11].

**Lemma 2.1.** *Let $X$ be a hyperkähler manifold and let $L \subseteq X$ be a half-dimensional real submanifold.*

1. *If $L$ is a holomorphic Lagrangian in $X$, then the same underlying space of $L$ defines a special Lagrangian in $X_\zeta$ for any $\zeta \in S^1$.*
2. *If $L$ is a Lagrangian in $X$ such that $\mathrm{Im}(\zeta\theta)|_L = 0$ for some $\zeta \in S^1$, then $L$ is a holomorphic Lagrangian in $X_\zeta$.*

*In particular, if $\dim_{\mathbb{C}} X = 2$, then $L \subseteq X$ is a holomorphic Lagrangian if and only if it is a special Lagrangian up to a hyperkähler rotation.*

**Lemma 2.2.** *Let $X$ be a K3 surface equipped with a Ricci-flat Kähler form $\omega$ and a holomorphic 2-form $\theta$.*

(a) *If $C$ is a holomorphic curve in $X_\zeta$ for some $\zeta \in S^1$, then $C$ is a special Lagrangian in $X$.*

(b) *If $\gamma \in H^2(X, \mathbb{Z})$ is represented by an irreducible special Lagrangian submanifold, then $\gamma^2 \geq -2$.*

*Proof.* Part (a) is a special case of Lemma 2.1 (1). It can also be derived straightforwardly from (2.1). Part (b) follows from Lemma 2.1 and [BHPV04, VIII Proposition 3.7 (ii)]. □

**Remark 2.3.** In higher dimension, there exist special Lagrangians which never become holomorphic Lagrangians after any hyperkähler rotation. See [Hat19] for such examples.

### 2.3. Decomposing classes of type (1,1)

Here we prove a sufficient condition for classes of Hodge type $(1,1)$ to be decomposed into classes of smooth holomorphic curves on a K3 surface. Later on we will combine it with the hyperkähler rotation to develop a similar condition for Lagrangian classes.

**Lemma 2.4.** *Let $X$ be a K3 surface (notice that there is no projectivity assumption on $X$) equipped with a holomorphic 2-form $\theta$. Assume that $\gamma \in H^2(X, \mathbb{Z})$ satisfies $\int_\gamma \theta = 0$ and $\gamma^2 \geq -2$. Then $\gamma$ or $-\gamma$ can be written as*

$$n_1[C_1] + ... + n_k[C_k], \quad n_i \geq 0,$$

*where each $C_i \subseteq X$ is a smooth holomorphic curve.*

*Proof.* The condition $\int_\gamma \theta = 0$ means that

$$\gamma \in H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z}).$$

By [BHPV04, VIII Propositions 3.7 (i)], either $\gamma$ or $-\gamma$ is effective. Assume that $\gamma$ is effective. As a consequence of [BHPV04, VIII Propositions 3.8], we can decompose $\gamma$ into classes of irreducible curves as

$$\gamma = \sum_i n_i[C_i] + \sum_j m_j[E_j] + \sum_k \ell_k[D_k],$$

where $n_i$, $m_j$, $\ell$ are non-negative integers, $C_i^2 = -2$, $E_j^2 = 0$, and $D_k^2 > 0$. The curves $C_i$ are $(-2)$-curves and thus smooth. The linear system $|E_j|$ has no base-point, so we can apply Bertini's theorem [GH94, p.137] to choose a smooth irreducible representative of the class of $E_j$.

Suppose that $\ell_k \neq 0$ for some $k$. In this case, $X$ is algebraic by [BHPV04, IV Theorem 6.2]. Evidently, $D_k$ has no fix component, so we can apply [SD74, Corollary 3.2] to conclude that $|D_k|$ has no base-point. Again by Bertini's theorem we can choose a smooth irreducible representative of the class of $D_k$. This proves our claim for $\gamma$. In the case that $-\gamma$ is effective, an analogous argument yields our claim for $-\gamma$ instead. □

**Lemma 2.5.** *Let $X$ be a K3 surface equipped with a Ricci-flat Kähler form $\omega$ and a holomorphic 2-form $\theta$. Assume that $\gamma \in H^2(X, \mathbb{Z})$ is a Lagrangian class which satisfies $\int_\gamma \theta = 0$ and $\gamma^2 \geq -2$. Then $\gamma = 0$.*

*Proof.* By Lemma 2.4, we can write $\gamma = \sum_{i=1}^{k} n_i[C_i]$ where $n_i \geq 0$ for all $i$ and each $C_i \subseteq X$ is a holomorphic curve. As $[C_i] \cdot [\omega] > 0$ for all $i$, the condition $\gamma \cdot [\omega] = 0$ implies that $n_i = 0$ for all $i$ and thus $\gamma = 0$. □

### 2.4. Decomposing Lagrangian classes

Here we introduce the numerical condition for a Lagrangian class to be decomposed into classes of special Lagrangian submanifolds. Before showing the sufficiency of the condition, let us prove a lemma.

**Lemma 2.6.** *Let $X$ be a K3 surface equipped with a Ricci-flat Kähler form $\omega$ and a holomorphic 2-form $\theta$. Suppose that $\gamma \in H^2(X, \mathbb{Z})$ is a Lagrangian class such that $c := \int_\gamma \theta \neq 0$. Then $\int_\gamma \theta_\zeta = 0$ if and only if $\zeta^2 = \bar{c}/c$.*

*Proof.* Using the fact that $\int_\gamma \omega = 0$, we obtain

$$\begin{aligned} 0 = \int_\gamma \theta_\zeta &= \int_\gamma (\omega - i\mathrm{Im}(\zeta\theta)) = -i \int_\gamma \mathrm{Im}(\zeta\theta) \\ &= -\frac{1}{2}\int_\gamma (\zeta\theta - \overline{\zeta\theta}) = -\frac{1}{2}\left(\zeta \int_\gamma \theta - \frac{1}{\zeta}\int_\gamma \bar{\theta}\right) = -\frac{1}{2}\left(\zeta c - \frac{1}{\zeta}\bar{c}\right). \end{aligned}$$

This is equivalent to

$$\zeta c - \frac{1}{\zeta}\bar{c} = 0,$$

which holds if and only if $\zeta^2 = \bar{c}/c$. □

**Proposition 2.7.** *Let $X$ be a K3 surface equipped with a Ricci-flat Kähler form $\omega$. If $\gamma \in H^2(X,\mathbb{Z})$ is a Lagrangian class satisfying $\gamma^2 \geq -2$, then*

$$\gamma = \sum_{i=1}^{k} \alpha_i,$$

*where each $\alpha_i$ is represented by a special Lagrangian submanifold of the same phase. In other words, $\gamma$ is a special Lagrangian class.*

*Proof.* If $\int_\gamma \theta = 0$, then the conclusion follows from Lemma 2.5. Assume that $\int_\gamma \theta \neq 0$. Then there exists $\zeta$ such that $\int_\gamma \theta_\zeta = 0$ by Lemma 2.6. According to Lemma 2.4, either $\gamma$ or $-\gamma$ can be written as a sum

$$\sum_{i=1}^{k} \alpha_i, \tag{2.2}$$

where each $\alpha_i$ is represented by a smooth holomorphic curves on $X_\zeta$. Here we allow the terms in the sum to occur multiple times. Then by Lemma 2.2 (a), each $\alpha_i$ is represented by a special Lagrangian in $X$. Therefore, if $\gamma$ is equal to the sum in (2.2), then we are done. Otherwise, if $-\gamma$ is equal to (2.2), then write

$$\gamma = \sum_{i=1}^{k} (-\alpha_i)$$

and we have that the $-\alpha_i$ are holomorphic for $X_{-\zeta}$. Applying Lemma 2.2 (a) again, the classes $-\alpha_i$ are represented by special Lagrangians. □

## 3. Criteria for the decomposability

In this section, we give criteria which determine whether a Lagrangian class on a Kähler K3 surface is special, i.e. a sum of classes represented by special Lagrangian submanifolds, or not. The criteria are developed upon the lattice structure of the second cohomology group. They allow us to describe the locus in the moduli space about the Kähler K3 surfaces on which every Lagrangian class is special. In the end of the section, we discuss the existence of a special Lagrangian fibration on algebraic K3 surfaces.

### 3.1. Sublattice of Lagrangian classes

Let $X$ be a K3 surface. As a reminder, recall that the cohomology group $H^2(X,\mathbb{Z})$ equipped with the intersection product is isometric to the lattice

$$\Lambda_{\mathrm{K3}} := U^{\oplus 3} \oplus E_8^{\oplus 2},$$

where $U$ is the hyperbolic plane and $E_8$ is the unique negative definite unimodular even lattice of rank 8.

Now equip $X$ with a Ricci-flat Kähler form $\omega$ and covariantly constant holomorphic volume form $\theta$. We denote by $[\omega]^\perp$ the orthogonal complement of $[\omega]$ in

$$H^2(X,\mathbb{R}) = H^2(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

If a class $\gamma \in H^2(X,\mathbb{Z})$ is represented by a Lagrangian immersion $\iota\colon L \to X$, then the relation $\iota^*\omega = 0$ implies that

$$\gamma \cdot [\omega] = \int_L \iota^*\omega = 0.$$

The converse is true by [SW01, Corollary 2.4]. As a consequence, a class $\gamma \in H^2(X,\mathbb{Z})$ is Lagrangian if and only if $\gamma$ belongs to the sublattice

$$\mathrm{Lag}(X,\omega) := [\omega]^\perp \cap H^2(X,\mathbb{Z}), \tag{3.1}$$

which we call a *Lagrangian lattice*. Due to Yau's celebrated theorem [Yau78], there is a unique Ricci-flat metric in each Kähler class. We will no longer distinguish a Kähler class and a Ricci-flat Kähler form later on. We further define $\mathrm{SLag}(X,\omega)$ as the sublattice of $\mathrm{Lag}(X,\omega)$ generated by the classes of special Lagrangian submanifolds. In other words, $\mathrm{SLag}(X,\omega)$ consists of special Lagrangian classes.

Our main question is about whether $\mathrm{Lag}(X,\omega)$ and $\mathrm{SLag}(X,\omega)$ coincide or not. First of all, we have the following numerical characterization of $\mathrm{SLag}(X,\omega)$.

**Proposition 3.1.** $\mathrm{SLag}(X,\omega) = \langle \delta \in \mathrm{Lag}(X,\omega) : \delta^2 \geq -2\rangle$.

*Proof.* The containment ($\supseteq$) follows from Proposition 2.7. The other containment follows from Lemma 2.2 (b). $\square$

### 3.2. Characterization of Lagrangian lattices

Which sublattice $E \subseteq \Lambda_{\mathrm{K3}}$ appears as a Lagrangian lattice? Suppose that $E = \mathrm{Lag}(X, \omega)$ for some Kähler K3 surface $(X, \omega)$. Then clearly $E$ is proper and saturated (by proper we mean properly contained in $\Lambda_{\mathrm{K3}}$, and it could possibly be zero). Moreover, the subspace $(E^{\perp})_{\mathbb{R}} \subseteq (\Lambda_{\mathrm{K3}})_{\mathbb{R}}$ contains a vector with positive self-intersection, namely, the Kähler class $[\omega]$, which implies that $E^{\perp}$ itself contains a vector with positive self-intersection. In the following, we will show that these conditions are sufficient for $E$ to be a Lagrangian sublattice.

**Lemma 3.2.** *Let $\Lambda$ be a non-degenerate lattice and let $E \subsetneq \Lambda$ be a proper saturated sublattice. Suppose that the orthogonal complement $E^{\perp} \subseteq \Lambda$ contains a vector with positive self-intersection. Then there exists $v \in \Lambda_{\mathbb{R}} := \Lambda \otimes \mathbb{R}$ such that $v^2 > 0$ and $E = v^{\perp} \cap \Lambda$.*

*Proof.* By hypothesis, there exists $x \in E^{\perp}$ such that $x^2 > 0$. Suppose that $E^{\perp}$ is spanned by the basis $\{e_1, \ldots, e_m\} \subseteq \Lambda \setminus \{0\}$. Let $\{c_1, \ldots, c_m\} \subseteq \mathbb{R}$ be a set of real numbers ordered in such a way that the field extensions

$$\mathbb{Q} \subseteq \mathbb{Q}(c_1) \subseteq \ldots \subseteq \mathbb{Q}(c_1, \ldots, c_m)$$

are all transcendental extensions. (For example, one can choose $c_i = 2^{\pi^i}$.) Let

$$y = \sum_{i=1}^{m} c_i e_i,$$

and define

$$v := x + \epsilon y, \quad \epsilon \in \mathbb{Q} \setminus \{0\}.$$

Then $v^2 > 0$ if $|\epsilon| \ll 1$.

The inclusion $E \subseteq v^{\perp} \cap \Lambda$ is clear. We claim that $E = v^{\perp} \cap \Lambda$. Assume, to the contrary, that there exists $w \in v^{\perp} \cap \Lambda$ such that $w \notin E$. The condition $w \in v^{\perp}$ implies that

$$0 = v \cdot w = (x + \epsilon y) \cdot w = x \cdot w + \epsilon(y \cdot w),$$

which is equivalent to

$$-\frac{1}{\epsilon}(x \cdot w) = y \cdot w = \sum_{i=1}^{m} c_i(e_i \cdot w). \tag{3.2}$$

Note that $(E^\perp)^\perp = E$ since $\Lambda$ is non-degenerate and $E$ is saturated. If $(e_i \cdot w) = 0$ for all $i$, then $w \in (E^\perp)^\perp = E$ which contradicts to the assumption that $w \notin E$. Therefore, (3.2) gives a nontrivial algebraic relation among $c_1$, ..., $c_m$ over $\mathbb{Q}$, which is impossible. This completes the proof. □

Before proceeding to the next result, let us briefly review some background material. Our main reference is [Huy16, Chapter 6]. Let $\Lambda := \Lambda_{\mathrm{K3}}$. Recall that the period domain of K3 surfaces is defined as

$$\mathcal{D} := \{[x] \in \mathbb{P}(\Lambda_{\mathbb{C}}) : x^2 = 0,\ x \cdot \bar{x} > 0\}.$$

By the Torelli theorem, for every period point $[x] \in \mathcal{D}$, there exists a K3 surface $X$ together with an isometry

$$\varphi\colon H^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}},$$

which satisfies $\varphi(H^{2,0}(X, \mathbb{C})) = \mathbb{C}x$ after extending the scalars to $\mathbb{C}$. It is a general fact that, given any analytic submanifold $B \subseteq \mathcal{D}$, there exists a dense subset $NL(B) \subsetneq B$ consisting of countably many divisors, called the *Nother-Lefschetz locus*, such that the Picard numbers of the K3 surfaces parametrized by $B$ jump on $NL(B)$ and retain minimum on the complement.

**Proposition 3.3.** *Let $E \subsetneq \Lambda_{\mathrm{K3}}$ be a sublattice. Then there exists a Kähler K3 surface $(X, \omega)$ such that $\mathrm{Lag}(X, \omega) \cong E$ if and only if $E$ is proper, saturated, and such that $E^\perp \subseteq \Lambda_{\mathrm{K3}}$ contains a vector with positive self-intersection.*

*Proof.* The "only if" part is clear from the description before Lemma 3.2. To prove the "if" part, let us denote $\Lambda := \Lambda_{\mathrm{K3}}$. By Lemma 3.2, there exists $v \in \Lambda_{\mathbb{R}}$ such that $v^2 > 0$ and $E = v^\perp \cap \Lambda$. If we can find $x \in \mathcal{D}$ such that either $v$ or $-v$ corresponds to a Kähler class $[\omega]$ on a K3 surface $X$ parametrized by $x$, then

$$\mathrm{Lag}(X, \omega) \cong [\omega]^\perp \cap \Lambda \cong E$$

as desired. We divide the proof into two cases.

First assume that $rv \in \Lambda$ for some $r \in \mathbb{R}$. We can further assume that $rv$ is primitive in $\Lambda$ and let $d := (rv)^2$. Note that $d$ is a positive even integer since $v^2 > 0$ and $\Lambda$ is even. Then the locus

$$\mathcal{D}_d := \mathbb{P}(v^\perp) \cap \mathcal{D}$$

parametrizes polarized K3 surfaces of degree $d$, and a very general point $x \in \mathcal{D}_d$ corresponds to a K3 surface $X$ of Picard number one. On such $X$, either $rv$ or $-rv$ is the class of an ample line bundle and thus is a Kähler class, so we finished this case.

Now assume that $rv \notin \Lambda$ for any $r \in \mathbb{R} \setminus \{0\}$. A K3 surface $X$ parametrized by a very general point of the locus

$$\mathbb{P}(v^{\perp}) \cap \mathcal{D}$$

has Picard number zero. For such $X$, the Kähler cone and the positive cone coincide [Huy16, Chapter 8, Corollary 5.3]. In particular, $v^2 > 0$ implies that $v$ is a Kähler class. This completes the proof. □

**Corollary 3.4.** *There exists a Kähler K3 surface $(X, \omega)$ such that* $\mathrm{Lag}(X, \omega)$ *is isomorphic to the $E_8$ lattice.*

*Proof.* Consider $E_8$ as one of the $E_8$ copies in $\Lambda_{\mathrm{K3}} = U^{\oplus 3} \oplus E_8^{\oplus 2}$. Then we have $E_8^{\perp} \cong U^{\oplus 3} \oplus E_8$, where one can easily produce a vector with positive self-intersection from the copies of $U$'s. In particular, the existence is confirmed by Proposition 3.3. □

### 3.3. The lattice-theoretic criteria

Here we introduce our main result, namely, the criteria for determining whether $\mathrm{SLag}(X, \omega)$ is the whole $\mathrm{Lag}(X, \omega)$ or not. The criteria are established upon the following observation on lattices.

**Lemma 3.5.** *Let $L$ be an arbitrary lattice.*

(i) *If $L$ contains a vector with positive self-intersection, then every $\gamma \in L$ decomposes as $\gamma = \alpha + \beta$ with $\alpha^2 > 0$ and $\beta^2 > 0$.*

(ii) *If $L$ contains an isotropic vector $\delta$ such that $\delta \cdot \alpha \neq 0$ for some $\alpha \in L$, then $L$ contains a vector of positive self-intersection.*

(iii) *If $L$ contains no vector with positive self-intersection, then there is an orthogonal decomposition $L \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus N$ where each copy of $\mathbb{Z}$ is spanned by an isotropic vector and $N$ is negative definite.*

*Proof.* Assume that there exists $x \in L$ such that $x^2 > 0$. Let $\gamma \in L$ be an arbitrary vector and let $m$ be an integer. Define $\alpha = mx$ and $\beta = \gamma - mx$.

Then $\gamma = \alpha + \beta$ and $\alpha^2 = m^2(x^2) > 0$. Moreover, we have

$$\beta^2 = (\gamma - mx)^2 = \gamma^2 - 2m(\gamma \cdot x) + m^2(x^2),$$

which becomes positive by picking $m \gg 0$. Hence (i) is proved.

To prove (ii), let $m$ be any integer and consider the vector $m\delta + \alpha$. Since $\delta^2 = 0$, we have

$$(m\delta + \alpha)^2 = 2m(\alpha \cdot \delta) + \alpha^2.$$

If $\delta \cdot \alpha > 0$, then $m\delta + \alpha$ has positive self-intersection as $m \gg 0$. If $\delta \cdot \alpha < 0$, we can choose $m \ll 0$ instead.

If $L$ contains no vector with positive self-intersection, then (ii) implies that every isotropic vector in $\mathrm{Lag}(X, \omega)$ is orthogonal to every other vector. Therefore, we have the claimed decomposition, which proves (iii). □

**Theorem 3.6.** *Let $X$ be a K3 surface equipped with a Kähler form $\omega$.*

(1) *If $\mathrm{Lag}(X, \omega)$ contains a vector with positive self-intersection, then*

$$\mathrm{Lag}(X, \omega) = \mathrm{SLag}(X, \omega) = \langle \delta \in \mathrm{Lag}(X, \omega) : \delta^2 \geq -2 \rangle. \tag{3.3}$$

(2) *If $\mathrm{Lag}(X, \omega)$ contains no vector with positive self-intersection, then there is an orthogonal decomposition*

$$\mathrm{Lag}(X, \omega) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus N,$$

*where each copy of $\mathbb{Z}$ is spanned by an isotropic vector and $N$ is negative definite. In this case,* (3.3) *holds if and only if $N \subseteq \mathrm{SLag}(X, \omega)$.*

*Proof.* Assume that $\mathrm{Lag}(X, \omega)$ contains a vector with positive self-intersection. Then Lemma 3.5 (i) implies that every $\gamma \in \mathrm{Lag}(X, \omega)$ can be written as $\gamma = \alpha + \beta$ where $\alpha, \beta \in \mathrm{Lag}(X, \omega)$ satisfy $\alpha^2 > 0$ and $\beta^2 > 0$. Together with Proposition 3.1, we obtain

$$\mathrm{Lag}(X, \omega) \subseteq \langle \delta \in \mathrm{Lag}(X, \omega) : \delta^2 \geq -2 \rangle = \mathrm{SLag}(X, \omega) \subseteq \mathrm{Lag}(X, \omega),$$

which implies that the inclusions are actually equalities. This proves (1).

In condition (2), the orthogonal decomposition for $\mathrm{Lag}(X, \omega)$ follows immediately from Lemma 3.5 (iii). Note that the isotropic part is contained in $\mathrm{SLag}(X, \omega)$ due to Proposition 3.1. As a consequence, $\mathrm{Lag}(X, \omega) = \mathrm{SLag}(X, \omega)$ if and only if $\mathrm{Lag}(X, \omega) \subseteq \mathrm{SLag}(X, \omega)$, which holds if and only if $N \subseteq \mathrm{SLag}(X, \omega)$. □

**Remark 3.7.** In condition (2) of Theorem 3.6, the inclusion

$$N \subseteq \mathrm{SLag}(X,\omega) = \langle \delta \in \mathrm{Lag}(X,\omega) : \delta^2 \geq -2 \rangle \tag{3.4}$$

may hold or not. For instance, in [Wol05] Wolfson constructed an example such that $\mathrm{Lag}(X,\omega) \cong N \cong \langle -4 \rangle$ which cannot be contained in $\mathrm{SLag}(X,\omega)$. In particular, the equality $\mathrm{Lag}(X,\omega) = \mathrm{SLag}(X,\omega)$ does not hold for all possible complex structures with $\omega$ as a Kähler class. On the other hand, many sublattices of the K3 lattice $\Lambda_{\mathrm{K3}}$ appear as Lagrangian sublattices according to Proposition 3.3. For example, the situation $\mathrm{Lag}(X,\omega) \cong E_8$ can occur by Corollary 3.4. In this case, $N \cong E_8$ is generated by $(-2)$-vectors, so (3.4) holds, and thus $\mathrm{Lag}(X,\omega) = \mathrm{SLag}(X,\omega)$.

Recall that a K3 surface $X$ is algebraic if and only if it admits a Kähler form $\omega$ which is *rational*, i.e. the class $[\omega]$ belongs to $H^2(X,\mathbb{Q})$. In this case, we have:

**Corollary 3.8.** *Assume that $X$ is a K3 surface equipped with a rational Kähler form $\omega$. Then* $\mathrm{Lag}(X,\omega) = \mathrm{SLag}(X,\omega)$.

*Proof.* The lattice $\mathrm{Lag}(X,\omega)$ has signature $(2,19)$, so the conclusion follows immediately from Theorem 3.6 (1). □

**Corollary 3.9.** *Assume that $X$ is a K3 surface such that*

$$(H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X,\mathbb{Q}) \neq \{0\}. \tag{3.5}$$

*Let $\omega$ be any Kähler form $\omega$ on $X$. Then* $\mathrm{Lag}(X,\omega) = \mathrm{SLag}(X,\omega)$.

*Proof.* Let $v$ be a nonzero vector in $(H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X,\mathbb{Q})$. We have that $v \cdot \omega = 0$. Moreover, $v^2 > 0$ because $H^{2,0}(X) \oplus H^{0,2}(X)$ is positive definite. Then the conclusion follows from Theorem 3.6 (1). □

**Remark 3.10.** The K3 surfaces satisfying the assumption of Corollary 3.9 form a family of $\mathbb{R}$-dimension 20 in the moduli of K3 surfaces.

### 3.4. Distribution of Kähler classes giving the decomposition

Given a K3 surface $X$, for which class $[\omega]$ in the Kähler cone

$$\mathcal{K}_X \subseteq H^{1,1}(X,\mathbb{R})$$

do we have $\mathrm{Lag}(X,\omega) = \mathrm{SLag}(X,\omega)$? In the situation where $\mathrm{Lag}(X,\omega) = 0$, which does occur by Proposition 3.3, the sublattice $\mathrm{SLag}(X,\omega)$ must be zero as well, hence the equality holds trivially. Therefore, we will focus on the case that $\mathrm{Lag}(X,\omega) \neq 0$.

The cone $\mathcal{C} := \{x \in H^{1,1}(X,\mathbb{R}) : x^2 > 0\}$ consists of two connected components. Let $\mathcal{C}_X$ denote the positive cone, that is, the component which contains $\mathcal{K}_X$. We first show that, in fact, a similar density property can be seen in $\mathcal{C}_X$ already, in a sense made precise in the next proposition.

**Proposition 3.11.** *On a K3 surface X not satisfying the hypothesis of Corollary 3.9, the classes $w \in \mathcal{C}_X$ whose orthogonal complement $w^\perp \cap H^2(X,\mathbb{Z})$ contains a vector $v$ with $v^2 > 0$ form a dense subset $\mathcal{P}_X \subseteq \mathcal{C}_X$ in the analytic topology. This subset is a union of countably many hyperplane sections and it contains $\mathcal{C}_X \cap H^{1,1}(X,\mathbb{Q})$.*

The key idea behind the proof of the above proposition is the following lemma.

**Lemma 3.12.** *Let $\mathcal{C}^c$ be the complement of $\mathcal{C}$ in $H^{1,1}(X,\mathbb{R})$. Suppose that $\mathcal{A} \subseteq \mathcal{C}^c \setminus \{0\}$ is a dense subset. Then the union*

$$\bigcup_{a\in\mathcal{A}} a^\perp \cap \mathcal{C}_X$$

*is dense in $\mathcal{C}_X$.*

*Proof.* Let $x \in \mathcal{C}_X$. Then $x^\perp \cap \mathcal{C}_X = \emptyset$ by [BHPV04, IV Corollary 7.2]. In particular, we have $x^\perp \subseteq \mathcal{C}^c$. If, by contradiction, $x^\perp$ does not intersect $\overline{\mathcal{C}}^c$, then it would be contained in the boundary of $\mathcal{C}$, which cannot happen. So there exists $a \in \overline{\mathcal{C}}^c$ such that $a \cdot x = 0$. It follows that

$$\mathcal{C}_X \subseteq \bigcup_{a\in\overline{\mathcal{C}}^c} a^\perp \subseteq \bigcup_{a\in\mathcal{C}^c\setminus\{0\}} a^\perp.$$

Hence it is sufficient to show that

$$\bigcup_{a\in\mathcal{A}} a^\perp \subseteq \bigcup_{a\in\mathcal{C}^c\setminus\{0\}} a^\perp \tag{3.6}$$

is a dense subset. For simplicity of notation, let $V := H^{1,1}(X, \mathbb{R})$. Consider the diagram

$$\begin{array}{ccc} & \mathcal{U} := \{(a,z) \in \mathcal{C}^c \setminus \{0\} \times V : a^\perp \ni z\} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathcal{C}^c \setminus \{0\} & & V. \end{array}$$

Then $\pi_1$ is a vector bundle fibered in 19-dimensional real vector spaces, and the image of $\pi_2$ coincides with the right hand side of (3.6). In particular, $\pi_1$ is an open map. Because $\mathcal{A}$ is dense in $\mathcal{C}^c \setminus \{0\}$, $\pi_1^{-1}(\mathcal{A})$ is dense in $\mathcal{U}$ and thus

$$\bigcup_{a \in \mathcal{A}} a^\perp = \pi_2(\pi_1^{-1}(\mathcal{A}))$$

is dense in the image of $\pi_2$, which completes the proof. □

*Proof of Proposition 3.11.* The goal is to write $\mathcal{P}_X$ as in the statement of Lemma 3.12 for some dense $\mathcal{A} \subseteq \mathcal{C}^c \setminus \{0\}$. Define

$$\mathcal{Q}^+ := \{x \in H^2(X, \mathbb{Q}) : x^2 > 0\}.$$

First note that

$$\mathcal{P}_X = \bigcup_{x \in \mathcal{Q}^+} x^\perp \cap \mathcal{C}_X.$$

Indeed, if $w \in \mathcal{C}_X$ admits the existence of $v \in w^\perp \cap H^2(X, \mathbb{Z})$ with $v^2 > 0$, then $v \in \mathcal{Q}^+$ and $w \in v^\perp \cap \mathcal{C}_X$. Conversely, for every $w \in x^\perp \cap \mathcal{C}_X$, we can multiply $x$ by some integer $n$ to get $v := nx \in H^2(X, \mathbb{Z})$. Then $v^2 > 0$, and $w \in x^\perp$ implies that $v = nx \in w^\perp$.

Consider the orthogonal decomposition

$$H^2(X, \mathbb{R}) \cong H^{1,1}(X, \mathbb{R}) \oplus (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$$

together with the induced projection

$$\pi \colon H^2(X, \mathbb{R}) \longrightarrow H^{1,1}(X, \mathbb{R}).$$

It is easy to verify that $x^\perp \cap \mathcal{C}_X = \pi(x)^\perp \cap \mathcal{C}_X$ for all $x \in H^2(X, \mathbb{R})$, so

$$\mathcal{P}_X = \bigcup_{x \in \pi(\mathcal{Q}^+)} x^\perp \cap \mathcal{C}_X,$$

where the orthogonal complement $x^\perp$ is taken in $H^{1,1}(X,\mathbb{R})$. We define

$$\mathcal{A} := \pi(\mathcal{Q}^+) \cap \mathcal{C}^c.$$

Notice that automatically $0 \notin \mathcal{A}$ because $0 \notin \pi(\mathcal{Q}^+)$ since, by hypothesis,

$$(H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X,\mathbb{Q}) = 0.$$

By Lemma 3.12, we want to show that $\mathcal{A}$ is dense in $\mathcal{C}^c \setminus \{0\}$ and that

$$\mathcal{P}_X = \bigcup_{a\in\mathcal{A}} a^\perp \cap \mathcal{C}_X.$$

Let us start with showing that $\mathcal{A} \subseteq \mathcal{C}^c \setminus \{0\}$ is dense. Consider

$$\mathcal{Q}^+_{\mathbb{R}} := \{x \in H^2(X,\mathbb{R}) : x^2 > 0\}.$$

Note that $\mathcal{Q}^+ \subseteq \mathcal{Q}^+_{\mathbb{R}}$ is dense. We claim that

$$\pi(\mathcal{Q}^+_{\mathbb{R}}) = H^{1,1}(X,\mathbb{R}).$$

Indeed, the intersection pairing on $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ has signature $(2,0)$, so this space contains a vector $z$ such that $z^2 > 0$. For every $y \in H^{1,1}(X,\mathbb{R})$, define

$$x := y + cz, \quad c > 0.$$

Then $\pi(x) = y$ and $x$ lies in $\mathcal{Q}^+_{\mathbb{R}}$ for $c$ sufficiently large. As a result, $\pi$ restricts to a surjective map

$$\rho\colon \mathcal{Q}^+_{\mathbb{R}} \longrightarrow H^{1,1}(X,\mathbb{R}).$$

It follows that $\rho(\mathcal{Q}^+)$ is dense in $H^{1,1}(X,\mathbb{R})$, so $\mathcal{A} = \rho(\mathcal{Q}^+) \cap \mathcal{C}^c$ is dense in $\mathcal{C}^c \setminus \{0\}$.

Define $\mathcal{A}^c := \rho(\mathcal{Q}^+) \cap \mathcal{C}$, the complement of $\mathcal{A}$ in $\rho(\mathcal{Q}^+)$. Then

$$\mathcal{P}_X = \left(\bigcup_{a\in\mathcal{A}} a^\perp \cap \mathcal{C}_X\right) \cup \left(\bigcup_{b\in\mathcal{A}^c} b^\perp \cap \mathcal{C}_X\right).$$

The fact that $x \cdot y > 0$ for all $x, y \in \mathcal{C}_X$ [BHPV04, IV Corollary 7.2] implies that the latter subset is in fact empty. Hence

$$\mathcal{P}_X = \bigcup_{a\in\mathcal{A}} a^\perp \cap \mathcal{C}_X$$

and the density property follows by applying Lemma 3.12.

To prove the last sentence, first notice that $\mathcal{Q}^+$ is countable, so $\mathcal{A}$ is countable. Hence $\mathcal{P}_X$ is a union of countably many hyperplane sections. To see that it contains $\mathcal{C}_X \cap H^{1,1}(X, \mathbb{Q})$, assume that the latter set contains a nonzero element and rescale it as a primitive vector

$$w \in \mathcal{C}_X \cap H^{1,1}(X, \mathbb{Z}).$$

Let $m := w^2 > 0$. By Eichler, we can choose an isometry $H^2(X, \mathbb{Z}) \cong \Lambda_{\mathrm{K3}}$ such that

$$w = (e + mf, 0, 0, 0, 0) \in \Lambda_{\mathrm{K3}} = U^{\oplus 3} \oplus E_8^{\oplus 2}$$

where $\{e, f\}$ is a basis for the first copy of $U$ such that $e^2 = f^2 = 0$ and $e \cdot f = 1$. (See [GHS09, Proposition 3.3] for a more precise statement and for the reference to Eichler's original work.) It is easy to find $x \in \Lambda_{\mathrm{K3}}$ such that $x^2 > 0$ and $x \cdot w = 0$. For example, one can pick $x = e' + f'$ where $\{e', f'\}$ is the standard basis for the second copy of $U$. As a result, $x$ represents an element in $\mathcal{Q}^+$ such that $w \in x^\perp \cap \mathcal{C}_X$. Hence $w \in \mathcal{P}_X$. □

**Theorem 3.13.** *For every K3 surface $X$, there exists a subset $\mathcal{S}_X$ in the Kähler cone $\mathcal{K}_X$ which is dense in the analytic topology such that*

$$\mathrm{Lag}(X, \omega) = \mathrm{SLag}(X, \omega) \neq 0, \tag{3.7}$$

*for every $[\omega] \in \mathcal{S}_X$. Moreover, the following hold:*

- *If $X$ satisfies the hypothesis of Corollary 3.9, then $\mathcal{S}_X = \mathcal{K}_X$.*
- *Otherwise, $\mathcal{S}_X$ is a countable union of hyperplane sections of $\mathcal{K}_X$. Furthermore, it contains all rational Kähler classes, so (3.7) holds in particular for polarized K3 surfaces.*

*Proof.* First recall that $\mathcal{K}_X \subseteq \mathcal{C}_X$ is a sub-cone. Due to Proposition 3.11, the collection of $[\omega] \in \mathcal{K}_X$ such that $\mathrm{Lag}(X, \omega) = [\omega]^\perp \cap H^2(X, \mathbb{Z})$ contains a vector $v$ with $v^2 > 0$ form a dense subset

$$\mathcal{S}_X := \mathcal{P}_X \cap \mathcal{K}_X \subseteq \mathcal{K}_X,$$

which is a countable union of hyperplane sections and contains all rational Kähler classes. For every $[\omega] \in \mathcal{P}_X \cap \mathcal{K}_X$, we have $\mathrm{Lag}(X, \omega) = \mathrm{SLag}(X, \omega)$ by Theorem 3.6 (1), which is a nonzero lattice as it contains a positive vector. □

**Remark 3.14.** Let $\check{X}$ be an algebraic K3 surface. Consider the Mukai lattice $\widetilde{H}(\check{X},\mathbb{Z})$ and the Mukai vector

$$v\colon D^b\mathfrak{Coh}(\check{X}) \longrightarrow \widetilde{H}(\check{X},\mathbb{Z}). \tag{3.8}$$

Under the choice of a Bridgeland stability condition, every $E \in D^b\mathfrak{Coh}(\check{X})$ admits a Harder–Narasimhan filtration

$$\begin{array}{ccccccccccc} 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \rightarrow \cdots \rightarrow & E_{n-1} & \longrightarrow & E_n = E \\ \nwarrow & \swarrow & \nwarrow & \swarrow & & & \nwarrow & \swarrow & \\ & A_1 & & A_2 & & & & A_n & \end{array}$$

where each $A_i$ is a semistable object and thus satisfies $v(A_i)^2 \geq -2$ (see, for example, [BM14, Theorem 2.15]). Assume that $(X,\omega)$ is a Kähler K3 surface where the homological mirror symmetry $\mathfrak{Fuk}(X) \cong D^b\mathfrak{Coh}(\check{X})$ is proved, for instance [Sei15] (see also [She19, Remark 1.17]), and such that $\check{X}$ is constructed over $\mathbb{C}$. Notice that usually the mirrors are constructed over non-archimedean fields. Assume further that we have a commutative diagram as follows

$$\begin{array}{ccc} \mathfrak{Fuk}(X) & \xrightarrow{\sim} & D^b\mathfrak{Coh}(\check{X}) \\ \downarrow & & \downarrow v \\ [\omega]^\perp & \xrightarrow{\sim} & \widetilde{H}^{1,1}(\check{X},\mathbb{Z}), \end{array}$$

where $[\omega]^\perp$ is the orthogonal complement taken inside $H^2(X,\mathbb{Z})$. Since the Mukai vector (3.8) factors through the numerical Grothendieck group $K_{\mathrm{num}}(\check{X})$ and induces an isometry

$$K_{\mathrm{num}}(\check{X}) \xrightarrow{\sim} \widetilde{H}^{1,1}(\check{X},\mathbb{Z})$$

(see [AT14, §2.2]), one can prove Theorem 3.6 (1) for such an $X$.

### 3.5. Special Lagrangian fibrations on K3 surfaces

We now turn to the analysis of the Kähler classes in the Kähler cone of a K3 surface for which we have a special Lagrangian torus fibration.

**Theorem 3.15.** *The set $\mathcal{S}_X$ in Theorem 3.13 contains a subset $\mathcal{T}_X$ which is dense in $\mathcal{K}_X$ in the analytic topology such that every Kähler form $\omega$ from $\mathcal{T}_X$ guarantees the existence of a special Lagrangian fibration on $X$. Moreover, the following hold:*

- *If $X$ satisfies the hypothesis of Corollary 3.9, then $\mathcal{T}_X$ is a union of countably many hyperplane sections in $\mathcal{K}_X$.*
- *Otherwise, $\mathcal{T}_X$ is a union of countably many codimension two linear sections in $\mathcal{K}_X$.*

*In both cases, $\mathcal{T}_X$ contains all rational Kähler classes. In particular, every polarized K3 surface admits a special Lagrangian fibration.*

The proof of the theorem is based on the following proposition which is similar to Proposition 3.11.

**Proposition 3.16.** *On a K3 surface $X$, the classes $w \in \mathcal{C}_X$ whose orthogonal complement $w^\perp \cap H^2(X,\mathbb{Z})$ contains a nonzero vector $\ell$ with $\ell^2 = 0$ form a dense subset $\mathcal{R}_X \subseteq \mathcal{C}_X$ in the analytic topology. This subset is a union of countably many hyperplane sections and it contains $\mathcal{C}_X \cap H^{1,1}(X,\mathbb{Q})$.*

*Proof.* The proof is analogous to the proof of Proposition 3.11, but with the following changes. First of all, replace $\mathcal{Q}^+$ with

$$\mathcal{Q} := \{x \in H^2(X,\mathbb{Q}) : x^2 = 0\} \setminus \{0\},$$

so that

$$\mathcal{R}_X = \bigcup_{\ell\in\mathcal{Q}} \ell^\perp \cap \mathcal{C}_X = \bigcup_{\ell\in\pi(\mathcal{Q})} \ell^\perp \cap \mathcal{C}_X.$$

Since $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{Q}}$ is either empty or positive definite, we have $0 \notin \pi(\mathcal{Q})$, which implies that $\mathcal{B} := \pi(\mathcal{Q}) \cap \mathcal{C}^c$ does not contain $\{0\}$.

To show that $\mathcal{B} \subseteq \mathcal{C}^c \setminus \{0\}$ is dense, consider

$$\mathcal{Q}_{\mathbb{R}} := \{x \in H^2(X,\mathbb{R}) : x^2 = 0\} \setminus \{0\}.$$

Note that $\mathcal{Q} \subseteq \mathcal{Q}_{\mathbb{R}}$ is dense. We claim that

$$\pi(\mathcal{Q}_{\mathbb{R}}) = H^{1,1}(X,\mathbb{R}).$$

Indeed, the intersection pairing on $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ has signature $(2,0)$, so this space contains a vector $z$ such that $z^2 > 0$. For every $y \in$

$H^{1,1}(X,\mathbb{R})$, define

$$x := y - cz,$$

where $c := \sqrt{y^2/z^2} \in \mathbb{R}$. Then $\pi(x) = y$ and $x$ lies in $\mathcal{Q}_{\mathbb{R}}$. As a result, $\pi$ restricts to a surjective map

$$\eta\colon \mathcal{Q}_{\mathbb{R}} \longrightarrow H^{1,1}(X,\mathbb{R}).$$

It follows that $\eta(\mathcal{Q})$ is dense in $H^{1,1}(X,\mathbb{R})$, so $\mathcal{B} = \eta(\mathcal{Q}) \cap \mathcal{C}^c$ is dense in $\mathcal{C}^c \setminus \{0\}$.

Define $\mathcal{B}^c := \eta(\mathcal{Q}) \cap \mathcal{C}$, the complement of $\mathcal{B}$ in $\eta(\mathcal{Q})$. Then

$$\mathcal{R}_X = \left(\bigcup_{a\in\mathcal{B}} a^\perp \cap \mathcal{C}_X\right) \cup \left(\bigcup_{b\in\mathcal{B}^c} b^\perp \cap \mathcal{C}_X\right) = \bigcup_{a\in\mathcal{B}} a^\perp \cap \mathcal{C}_X.$$

The second equality follows from the fact that $x \cdot y > 0$ for all $x, y \in \mathcal{C}_X$ [BHPV04, IV Corollary 7.2]. By applying Lemma 3.12 again, we conclude that $\mathcal{R}_X$ is dense in $\mathcal{C}_X$.

Finally, $\mathcal{R}_X$ contains all the rational Kähler classes $[\omega]$ since we can always find a nonzero isotropic vector in $[\omega]^\perp$ (see the end of the proof of Proposition 3.11). □

**Lemma 3.17.** *Let $\mathcal{C}^c$ be the complement of $\mathcal{C}$ in $H^{1,1}(X,\mathbb{R})$. Suppose that $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}^c \setminus \{0\}$ are disjoint dense subsets. Then the union*

$$\bigcup_{(a,b)\in\mathcal{A}\times\mathcal{B}} a^\perp \cap b^\perp \cap \mathcal{C}_X$$

*is dense in $\mathcal{C}_X$.*

*Proof.* First of all, let $\Delta \subseteq (\mathcal{C}^c \setminus \{0\})^{\times 2}$ denote the diagonal and define

$$\mathcal{E} := (\mathcal{C}^c \setminus \{0\})^{\times 2} \setminus \Delta.$$

We claim that

$$\mathcal{C}_X \subseteq \bigcup_{(a,b)\in\mathcal{E}} a^\perp \cap b^\perp. \tag{3.9}$$

Indeed, for every $x \in \mathcal{C}_X$, we have $x^\perp \cap \mathcal{C}_X = \emptyset$ by [BHPV04, IV Corollary 7.2]. This implies that $x^\perp \subseteq \mathcal{C}^c$, as it was already explained in the proof of Lemma 3.12. So there exist $a, b \in \overline{\mathcal{C}}^c$, $a \neq b$ such that $a \cdot x = 0$ and $b \cdot x = 0$, implying the containment in (3.9).

By (3.9), it is sufficient to show that

$$\bigcup_{(a,b)\in\mathcal{A}\times\mathcal{B}} a^\perp \cap b^\perp \subseteq \bigcup_{(a,b)\in\mathcal{E}} a^\perp \cap b^\perp \tag{3.10}$$

is a dense subset. For simplicity of notation, let $V := H^{1,1}(X, \mathbb{R})$. Consider the diagram

$$\begin{array}{ccc} & \mathcal{U} := \{(a,b,z) \in \mathcal{E} \times V : a^\perp \cap b^\perp \ni z\} & \\ \swarrow^{\pi_{12}} & & \searrow^{\pi_3} \\ \mathcal{E} & & V. \end{array}$$

Then $\pi_{12}$ is a vector bundle fibered in 18-dimensional real vector spaces, and the image of $\pi_3$ coincides with the left hand side of (3.10). In particular, $\pi_{12}$ is an open map. Because $\mathcal{A} \times \mathcal{B}$ is dense in $\mathcal{E}$, $\pi_{12}^{-1}(\mathcal{A} \times \mathcal{B})$ is dense in $\mathcal{U}$ and thus

$$\bigcup_{(a,b)\in\mathcal{A}\times\mathcal{B}} a^\perp \cap b^\perp = \pi_3(\pi_{12}^{-1}(\mathcal{A} \times \mathcal{B}))$$

is dense in the image of $\pi_3$, which completes the proof. □

*Proof of Theorem 3.15.* Given $[\omega] \in \mathcal{K}_X$, assume there exist $\ell \in [\omega]^\perp$ such that $\ell$ is a nonzero isotropic vector. Then by Lemma 2.1 (2), we can perform a hyperkähler rotation to make $\ell$ a $(1,1)$-class, which still satisfies $\ell^2 = 0$. The linear system $|L|$ given by $\ell$ may contain smooth rational curves as base locus, which can be removed by applying suitable reflections with respect to $(-2)$-curves. (Notice that these are Hodge isometries.) Now $|L|$ induces an elliptic fibiration. By reversing the hyperkähler rotation, the fibration turns into a special Lagrangian torus fibration on $X$ with respect to $\omega$ (this strategy is well know, and was also used in [OO21]). So what we need to do is to study the distribution of Kähler classes for which such vector $\ell$ exist.

Suppose that we are in the situation of Corollary 3.9. Retain the notation from Proposition 3.16 and define

$$\mathcal{T}_X := \mathcal{R}_X \cap \mathcal{K}_X.$$

In this case, we have $\mathcal{T}_X \subseteq \mathcal{S}_X$ since $\mathcal{S}_X = \mathcal{K}_X$ by Theorem 3.13. By definition, every $[\omega] \in \mathcal{T}_X$ lies in $\mathcal{R}_X$, so there exists a nonzero isotropic vector $\ell \in [\omega]^\perp$ by Proposition 3.16.

Now assume that the hypotheses of Corollary 3.9 are not satisfied. Consider the two sets

$$\mathcal{Q}^+ := \{x \in H^2(X, \mathbb{Q}) : x^2 > 0\},\ \mathcal{Q} := \{x \in H^2(X, \mathbb{Q}) : x^2 = 0\} \setminus \{0\},$$

project them into $H^{1,1}(X, \mathbb{R})$ using the map

$$\pi\colon H^2(X, \mathbb{R}) \longrightarrow H^{1,1}(X, \mathbb{R}),$$

and let $\mathcal{A}$ and $\mathcal{B}$, respectively, be their intersections with $\mathcal{C}^c \setminus \{0\}$. From the proofs of Propositions 3.11 and 3.16, we know that both $\mathcal{A}$ and $\mathcal{B}$ are dense in $\mathcal{C}^c \setminus \{0\}$. We claim that $\mathcal{A}$ and $\mathcal{B}$ are disjoint from each other. Suppose not, that is, there exists

$$x \in \mathcal{A} \cap \mathcal{B} \subseteq H^{1,1}(X, \mathbb{R}).$$

So $\pi^{-1}(x)$ contains nonzero vectors $v, \ell \in H^2(X, \mathbb{Q})$ satisfying $v^2 > 0$ and $\ell^2 = 0$. In this case, we can write

$$v = x + s \quad \text{and} \quad \ell = x + t$$

for some $s, t \in (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$. Take the subtraction

$$\xi := v - \ell = (x + s) - (x + t) = s - t.$$

Note that $(v - \ell) \in H^2(X, \mathbb{Q}) \setminus \{0\}$ and that $(s - t) \in H^{2,0}(X) \oplus H^{0,2}(X)$. Hence $\xi$ is a nonzero vector in

$$(H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X, \mathbb{Q}).$$

But this contradicts to our hypothesis.

Now we can apply Lemma 3.17 to conclude that the set

$$\bigcup_{(a,b) \in \mathcal{A} \times \mathcal{B}} a^\perp \cap b^\perp \cap \mathcal{C}_X$$

is dense in $\mathcal{C}_X$. Define

$$\mathcal{T}_X := \bigcup_{(a,b) \in \mathcal{A} \times \mathcal{B}} a^\perp \cap b^\perp \cap \mathcal{K}_X$$

which is dense in $\mathcal{K}_X$. We note that this is a countable union of codimension two linear subspaces in $\mathcal{S}_X$, and that it contains all the rational Kähler classes. The latter is true because we can find two nonzero vectors $v, \ell$ in $[\omega]^\perp$ such that $v^2 > 0$ and $\ell^2 = 0$ (see the end of the proof of Proposition 3.11). $\square$

The following is an immediate application.

**Corollary 3.18.** *Let $Y$ be a Fano 3-fold and $D \in |-K_Y|$ be a smooth anti-canonical divisor. Then the log Calabi-Yau 3-fold $X = Y \backslash D$ contains infinitely many special Lagrangian 3-tori.*

*Proof.* Since $Y$ is Fano 3-fold, $D$ is an algebraic K3 surface [Bea04]. Then $D$ admits a special Lagrangian fibration by Theorem 3.15. Let $L \subseteq D$ be any of the special Lagrangian 2-torus. By [CJL21, Theorem 1.1], there exist infinitely many special Lagrangians in $X$ diffeomorphic to $S^1 \times L \cong T^3$. $\square$

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