The double Cayley Grassmannian

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We study the smooth projective symmetric variety of Picard number one that compactifies the exceptional complex Lie group G_2, by describing it in terms of vector bundles on the spinor variety of $Spin_{14}$. We call it the double Cayley Grassmannian because quite remarkably, it exhibits very similar properties to those of the Cayley Grassmannian (the other symmetric variety of type G_2), but doubled in a certain sense. We deduce among other things that all smooth projective symmetric varieties of Picard number one are infinitesimally rigid.

1. Introduction

Symmetric spaces have been of constant interest since their classification by Elie Cartan in 1926. In complex algebraic geometry, projective symmetric varieties of Picard number one have been classified by Alessandro Ruzzi in 2011 [18]. Some of them are in fact homogeneous under their full automorphism group. Some others are just hyperplane sections of homogeneous spaces.

The two remaining ones are more mysterious, among other things because of their connections with the exceptional group G_2. These connections prompted us to call the first of them the *Cayley Grassmannian*, and denote it CG; its geometry and its cohomology (including its small quantum cohomology) were studied in [5, 16]. The second one is the subject of the present paper; we will call it the *double Cayley Grassmannian*, and denote it DG.

This terminology is supported by the observation that many important properties of CG are also observed for DG, but *doubled* in a certain way. Let us give an overview of a few of them, first for the Cayley Grassmannian:

1) CG compactifies $G_2/SL_2 \times SL_2$, acted on by G_2,

2) CG parametrizes four dimensional subalgebras of the complex octonion algebra $\mathbb{O}$,

3) CG can be described as the zero locus of a general section of a rank 4 homogeneous vector bundle on the Grassmannian $G(4, V_7)$, where $V_7 \simeq Im\mathbb{O}$ is the natural representation of G_2,

4) its linear span in the Plücker embedding is $\mathbb{P}(\mathbb{C} \oplus S^2 V_7)$,

5) its G_2-equivariant Hilbert series is $(1 - t)^{-1}(1 - tV_{2\omega_1})^{-1}(1 - t^2 V_{2\omega_2})^{-1}$,

6) its topological Euler characteristic is $\chi_{top}(CG) = \binom{6}{2}$,

7) CG admits three orbits under the action of G_2, the complement of the open one being a hyperplane section, and the closed one being the quadric $\mathbb{Q}_5$,

8) if we blowup the closed orbit, we obtain the wonderful compactification of $G_2/SL_2 \times SL_2$, with the two exceptional divisors

$$E \simeq \mathbb{P}(Sym^2 C) \to \mathbb{Q}_5 \qquad \text{and} \qquad F \simeq \mathbb{P}(Sym^2 N) \to X_{ad}(G_2),$$

where $\mathbb{Q}_5 \simeq G_2/P_1$ and $X_{ad}(G_2) \simeq G_2/P_2$ are the two generalized Grassmannians of G_2, with their G_2-homogeneous rank two vector bundles: the *Cayley bundle* C over $\mathbb{Q}^5$ and the *null bundle* N over $X_{ad}(G_2)$.

We find it quite remarkable that the double Cayley Grassmannian DG exhibits the very same properties, in the following "double" form:

1) DG compactifies G_2, acted on by $G_2 \times G_2$,

2) DG parametrizes eight dimensional subalgebras of the complex bioctonion algebra $\mathbb{O} \otimes \mathbb{C}$,

3) DG can be described as the zero locus of a general section of a rank 7 homogeneous vector bundle on the spinor variety $\mathbb{S}_{14} = Spin_{14}/P_7$,

4) its linear span in the spinorial embedding is $\mathbb{P}(\mathbb{C} \oplus V_7 \otimes V_7')$, where V_7 and V_7' are the natural representations of the two copies of G_2,

5) its equivariant Hilbert series is $(1-t)^{-1}(1-tV_{\omega_1+\omega_1'})^{-1}(1 - t^2 V_{\omega_2+\omega_2'})^{-1}$,

6) its topological Euler characteristic is $\chi_{top}(DG) = 6^2$,

7) DG admits three orbits under the action of $G_2 \times G_2$, the complement of the open one being a hyperplane section, and the closed one being $\mathbb{Q}_5 \times \mathbb{Q}_5$,

8) if we blowup the closed orbit, we obtain the wonderful compactification of G_2, with the two exceptional divisors

$$E \simeq \mathbb{P}(C \boxtimes C') \to \mathbb{Q}_5 \times \mathbb{Q}_5 \quad \text{and} \quad F \simeq \mathbb{P}(N \boxtimes N') \to X_{ad}(G_2) \times X_{ad}(G_2).$$

The main body of the paper will be devoted to the proof of these properties. In a sense, the whole story is hidden in the observation, already found in [19, Proposition 40], that $Spin_{14}$ acts almost transitively on the projectivization of its half-spin representations, with generic stabilizer $G_2 \times G_2$. An important consequence is the *multiplicative double-point property* used in [1] in order to obtain a remarkable matrix factorization of the octic invariant of these representations. We will use this property in an essential way in order to understand the geometry of DG.

We have not been able to describe its cohomology, partly because the number of classes is too big. In principle one should be able to deduce it from the cohomology of its blowup along the closed orbit, which should be accessible using [7, 9, 20]. What we have been able to check is that DG is infinitesimally rigid, a question motivated by a longstanding interest for the rigidity properties of homogeneous and quasi-homogeneous spaces (see for example [4, 11, 12]). This concludes the proof of the following statement:

Proposition. *Every smooth projective symmetric variety of Picard number one is infinitesimally rigid.*

Along the way, when discussing the geometry of DG, we will meet two varieties, admitting an action of $G_2 \times G_2$, which are Fano manifolds of Picard number one, and as such would deserve special consideration (see Propositions 16 and 18). This illustrates, once again, the amazing wealth of beautiful geometric objects related to the exceptional Lie groups.

2. Geometric description

2.1. Fano symmetric varieties of Picard number one

Ruzzi proved in [18] that there exist exactly six smooth projective symmetric varieties of Picard number one which are not homogeneous. One of them is a completion of G_2, considered as the symmetric space $(G_2 \times G_2)/G_2$. From [18] we can extract the following information.

1) The symmetric space G_2 admits a unique smooth equivariant completion with Picard number one, that we denote DG.

2) The connected automorphism group of DG is $G_2 \times G_2$; it has index two inside the full automorphism group.

3) Under the action of $G_2 \times G_2$, the variety DG has exactly three orbits: the open one, a codimension one orbit $\mathcal{O}_1$, and a closed orbit $\mathcal{O}_4 \simeq \mathbb{Q}_5 \times \mathbb{Q}_5$. The closure D of $\mathcal{O}_1$ is singular along $\mathcal{O}_4$.

4) The blow up of DG along its closed orbit is the wonderful compactification of G_2.

5) Consider the spinor variety $\mathbb{S}_{14} \subset \mathbb{P}\Delta$, the closed $Spin_{14}$-orbit inside a projectivized half-spin representation; then DG can be realized as a linear section of $\mathbb{S}_{14}$ by a linear subspace of codimension 14.

The last statement provides a geometric realization of DG which is not so useful, since the linear subspace is highly non transverse (note that $\mathbb{S}_{14}$ has dimension 21). Our first observation is that a more satisfactory description can be given in terms of vector bundles.

2.2. Octonionic factorization

We will need some extra information on half-spin representations. Let V_{14} be a fourteen dimensional complex vector space endowed with a non degenerate quadratic form. Let Δ be one of the half-spin representations of $Spin_{14}$. Its dimension is 64, and the action of the 91-dimensional group $Spin_{14}$ on $\mathbb{P}\Delta$ is prehomogeneous.

Recall that if we fix a maximal isotropic subspace E of V_{14}, we can identify the half-spin representation Δ with the even part $\wedge^+ E$ of the exterior algebra $\wedge^\bullet E$. For $e_1, \ldots, e_7$ a basis of E, let us denote $e_{ij} = e_i \wedge e_j$, and so on. A general element of Δ is then

$$z = 1 + e_{1237} + e_{4567} + e_{123456}.$$

The stabilizer of z in $Spin_{14}$ is locally isomorphic to $G_2 \times G_2$ (see [19, Proposition 40] or [1, Proposition 2.1.1]). The following statement was proved in [1].

Proposition 1. *A general element z of Δ determines an orthogonal decomposition $V_{14} = V_7 \oplus V_7'$. This yields a factorization of Δ as $\Delta_8 \otimes \Delta_8'$, for Δ_8 and Δ_8' the spin representations of $Spin(V_7)$ and $Spin(V_7')$, such that $z = \delta \otimes \delta'$ for some general $\delta \in \Delta_8$ and $\delta' \in \Delta_8'$.*

Explicitely, for $z = 1 + e_{1237} + e_{4567} + e_{123456}$ we get an orthogonal decomposition of V_{14} as the direct sum of the two spaces

$$V_7 = \langle e_1, e_2, e_3, f_1, f_2, f_3, e_7 - f_7 \rangle, \qquad V_7' = \langle e_4, e_5, e_6, f_4, f_5, f_6, e_7 + f_7 \rangle,$$

such that each copy of G_2 acts naturally on one of them, and trivially on the other one. Moreover $\delta = 1 + e_{123}$ and $\delta' = 1 + e_{456}$. The stabilizer of δ (resp. δ') in $Spin(V_7)$ (resp. $Spin(V_7')$) is the corresponding G_2.

Let us analyze how Δ decomposes as a $G_2 \times G_2$-module. As a $Spin_7 \times Spin_7$-module, we have just mentionned that Δ is a tensor product $\Delta_8 \otimes \Delta_8'$ of eight-dimensional spin representations. Moreover we can identify Δ_8 with $\wedge^\bullet A$ and Δ_8' with $\wedge^\bullet A'$, where $A = \langle e_1, e_2, e_3 \rangle$ and $A' = \langle e_4, e_5, e_6 \rangle$. Now, the restriction of Δ_7 to G_2 decomposes as $\mathbb{C} \oplus V_7$, so that finally

$$\Delta \simeq V_7 \otimes V_7' \oplus V_7 \oplus V_7' \oplus \mathbb{C}.$$

The result of [18] is that DG is the (highly non transverse) intersection of $\mathbb{S}_{14}$ with $\mathbb{P}D_z$, where $D_z = V_7 \otimes V_7' \oplus \mathbb{C} \subset \Delta$.

The orthogonal to D_z can be described as follows. The Clifford multiplication yields a morphism $V_{14} \otimes \Delta \to \Delta^\vee$. The image of $V_{14} \otimes z$ is a subspace L_z of $\Delta^\vee$, of dimension 14, which must be stable under $G_2 \times G_2$. In particular it must coincide with the orthogonal of D_z. We can explicitly determine this subspace by computing a basis:

$$
\begin{array}{rclcrcl}
e_1.z & = & e_1 + e_{14567}, & \quad & f_1.z & = & e_{237} + e_{23456}, \\
e_2.z & = & e_2 + e_{24567}, & \quad & f_2.z & = & -e_{137} - e_{13456}, \\
e_3.z & = & e_3 + e_{34567}, & \quad & f_3.z & = & e_{127} + e_{12456}, \\
e_4.z & = & e_4 - e_{12347}, & \quad & f_4.z & = & e_{567} - e_{12356}, \\
e_5.z & = & e_5 - e_{12357}, & \quad & f_5.z & = & -e_{467} + e_{12346}, \\
e_6.z & = & e_6 - e_{12367}, & \quad & f_6.z & = & e_{457} - e_{12345}, \\
e_7.z & = & e_7 + e_{1234567}, & \quad & f_7.z & = & -e_{123} - e_{456}.
\end{array}
$$

2.3. Spinorial interpretation

Let us denote by $\mathcal{L}$ the very ample line bundle that defines the embedding of the spinor variety $\mathbb{S}_{14} \subset \mathbb{P}\Delta$. Recall that Δ is one of the half-spin representations of $Spin_{14}$, and its dimension is 64. The spinor variety $\mathbb{S}_{14}$ parametrizes one of the two families of maximal isotropic spaces in V_{14}, and the square

$\mathcal{L}^2$ defines the Plücker embedding

$$\mathbb{S}_{14} \hookrightarrow G(7, V_{14}) \subset \mathbb{P}(\wedge^7 V_{14}).$$

The tautological bundle on $G(7, V_{14})$ restricts to a rank seven vector bundle $\mathcal{U}$ on $\mathbb{S}_{14}$, such that $\det(\mathcal{U}) = \mathcal{L}^{-2}$. Moreover, $\mathcal{U} \otimes \mathcal{L}$ is an irreducible homogeneous vector bundle, and by the Borel-Weil theorem,

$$H^0(\mathbb{S}_{14}, \mathcal{L}) = \Delta^\vee \qquad \text{and} \qquad H^0(\mathbb{S}_{14}, \mathcal{U} \otimes \mathcal{L}) = \Delta.$$

Since $\mathcal{U} \otimes \mathcal{L}$ is irreducible and admits non zero sections, it is automatically globally generated. So a general section vanishes along a codimension seven subvariety of $\mathbb{S}_{14}$. Note that this family of zero loci is isotrivial, since $Spin_{14}$ acts on $\mathbb{P}\Delta$ with an open orbit (whose complement is a degree 7 hypersurface, see [1] for more details).

Proposition 2. *The zero locus of a general section of the vector bundle $\mathcal{U} \otimes \mathcal{L}$ on $\mathbb{S}_{14}$ is projectively isomorphic to DG.*

Proof. Let z be a general element of Δ, and s_z the associated section of $\mathcal{U} \otimes \mathcal{L}$. Let y be a pure spinor; in other words, $[y]$ is a point of $\mathbb{S}_{14}$. Then $s_z([y])$ is a linear homomorphism from $\mathcal{L}^\vee_{[y]} = \mathbb{C}y$ to $\mathcal{U}_{[y]}$. The latter is the subspace of V_{14} characterized as

$$\mathcal{U}_{[y]} = \{v \in V_{14}, \; v.y = 0\},$$

where $v.y \in \Delta^\vee$ denotes the Clifford product of the vector v by the spinor y (recall that the fact that $\mathcal{U}_{[y]}$ is maximal isotropic is equivalent to y being a pure spinor [10]). We claim that $s_z([y])$ is defined by the following formula:

$$s_z([y])(u) = \langle z, u.y \rangle, \qquad u \in V_{14}.$$

Note that the right hand side is a linear form in $u \in V_{14}$ that certainly vanishes on $\mathcal{U}_{[y]}$. Since it is maximal isotropic, $\mathcal{U}_{[y]} \simeq \mathcal{U}^\perp_{[y]}$. So the right hand side really defines an element of $\mathcal{U}_{[y]}$, depending linearly on $y \in [y]$, as required.

We have therefore defined a non trivial equivariant morphism from Δ to $H^0(\mathbb{S}_{14}, \mathcal{U} \otimes \mathcal{L})$. By the Schur Lemma, it must be an isomorphism, and the same one up to scalar as the one provided by the Borel-Weil theorem.

So the zero-locus of s_z is the set of points $[y] \in \mathbb{S}_{14}$ such that

$$\langle z, u.y \rangle = \langle u.z, y \rangle = 0 \quad \forall u \in V_{14}.$$

In other words, set theoretically it is the intersection of $\mathbb{S}_{14}$ with the orthogonal to the fourteen dimension subspace $V_{14}.z \subset \Delta^\vee$. This is exactly Ruzzi's description, and we are done. $\square$

Corollary 3. *DG is a prime Fano manifold of dimension* 14 *and index* 7.

Proof. That DG has Picard number one follows from [18]. The spinor variety $\mathbb{S}_{14}$ has index 12, while $\det(\mathcal{U} \otimes \mathcal{L}) = \det(\mathcal{U}) \otimes \mathcal{L}^7 = \mathcal{L}^5$. Of course the restriction of $\mathcal{L}$ cannot be divisible since by Kobayashi-Ochiai it cannot be bigger that 15, and DG would be a quadric if it was equal to 14. $\square$

Recall that the Chow ring of $\mathbb{S}_{14}$ has an integral basis of Schubert classes τ_μ indexed by strict partitions $\mu = (\mu_1 > \cdots > \mu_m > 0)$, with $\mu_1 \leq 6$. In particular τ_1 is the hyperplane class, and the Pieri formula states that

$$\tau_\mu \tau_1 = \sum_\nu \tau_\nu,$$

where the sum is over all strict partitions ν obtained by adding one to some part of μ (or adding a part equal to one). There is a more general version for the product of a Schubert class by a special class τ_k, with multiplicities given by certain powers of two [6]. A consequence is that the Chow ring of $\mathbb{S}_{14}$ is generated, over the rationals, by the three special classes τ_1, τ_3, τ_5.

Corollary 4. *The fundamental class of DG in the Chow ring of* $\mathbb{S}_{14}$ *is*

$$[DG] = c_7(\mathcal{U} \otimes \mathcal{L}) = \tau_{61} + \tau_{52} + \tau_{43} + \tau_{421} = 2\tau_1 \tau_3^2 + 2\tau_1^2 \tau_5 - 6\tau_1^4 \tau_3 + 3\tau_1^7.$$

Proof. By the Thom-Porteous formula $[DG] = c_7(\mathcal{U} \otimes \mathcal{L})$. Since

$$c_7(\mathcal{U} \otimes \mathcal{L}) = \sum_{i=0}^{7} c_i(\mathcal{U}) c_1(\mathcal{L})^{7-i},$$

a repeated application of the Pieri formula yields the result. $\square$

Another direct application is to rigidity questions, which attracted strong interests for homogeneous spaces and their subvarieties [4, 11].

Proposition 5. *DG is infinitesimally rigid.*

Proof. Since DG is Fano, its deformations are non obstructed and we just need to prove that $H^1(T_{DG}) = 0$. Then the usual computations with the Koszul complex and the Borel-Weil-Bott theorem yield the result. Indeed, the Koszul complex takes the form

$$0\longrightarrow\mathcal{L}^{-5}\longrightarrow\mathcal{U}\otimes\mathcal{L}^{-4}\longrightarrow\cdots\longrightarrow\mathcal{U}^\vee\otimes\mathcal{L}^{-1}\longrightarrow\mathcal{O}_{\mathbb{S}_{14}}\longrightarrow\mathcal{O}_{DG}\longrightarrow0.$$

First step. We first prove that $H^1(T\mathbb{S}_{14|DG}) = 0$ by tensoring the Koszul complex with $T\mathbb{S}_{14} = \wedge^2\mathcal{U}^\vee$, and then by checking that for any integer k, with $0 \le k \le 7$, the cohomology group

$$H^{k+1}(\mathbb{S}_{14}, \wedge^2\mathcal{U}^\vee \otimes \wedge^k\mathcal{U}^\vee \otimes \mathcal{L}^{-k}) = 0.$$

For $k = 0$ we just get the irreducible bundle $\wedge^2\mathcal{U}^\vee$, which is globally generated and has no higher cohomology by the Bott-Borel-Weil theorem. For $k > 0$, the tensor product $\wedge^2\mathcal{U}^\vee \otimes \wedge^k\mathcal{U}^\vee$ is the direct sum of at most three irreducible homogeneous bundles, of respective weights $\lambda_k = \epsilon_1 + \cdots + \epsilon_{k+2}$ (for $k \le 5$), $\mu_k = 2\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{k+1}$ (for $1 \le k \le 6$) and $\nu_k = 2\epsilon_1 + 2\epsilon_2 + \epsilon_3 + \cdots + \epsilon_k$ (for $k \ge 2$). Here we made the usual choice of positive roots $\epsilon_i \pm \epsilon_j$ for $1 \le i < j \le 7$, where $(\epsilon_1, \ldots, \epsilon_7)$ is an orthonormal basis. Following the Bott-Borel-Weil theorem, these bundles twisted by $\mathcal{L}^{-k}$ are acyclic if we can find roots $\varphi_k, \chi_k, \psi_k$ such that

$$\langle \lambda_k - k\omega_7 + \rho, \varphi_k \rangle = \langle \mu_k - k\omega_7 + \rho, \chi_k \rangle = \langle \nu_k - k\omega_7 + \rho, \psi_k \rangle = 0,$$

where ρ denotes the sum of the fundamental weights, and $\omega_7 = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_7)$. We will look for a root of the form $\varphi_k = \epsilon_i + \epsilon_j$, with $1 \le i < j \le 7$, so that we always have $\langle \omega_6, \varphi_k \rangle = 1$. Then the vanishing condition becomes $i + j + k = 14 + \delta$, with $\delta = 2$ for $j \le k + 2$, $\delta = 1$ for $i \le k + 2 < j$, and $\delta = 0$ for $k + 2 < i$. Solutions do exist for any $k = 1, \ldots, 5$: take respectively $(i, j) = (6, 7), (5, 7), (5, 7), (4, 7), (5, 6)$. Similarly we can choose the root χ_k, for $k = 1, \ldots, 6$, to be again of the form $\epsilon_i + \epsilon_j$ with $(i, j) = (6, 7), (5, 7), (5, 6), (4, 7), (3, 7), (3, 7)$. Finally for the root ψ_k we can choose $\epsilon_i + \epsilon_j$ with $(i, j) = (5, 7), (5, 6), (4, 7), (3, 7), (4, 6), (3, 6)$ for $k = 2, \ldots, 7$.

Second step. Then we need to compute $H^0(\mathcal{U} \otimes \mathcal{L}_{|DG})$. Using the same techniques as in the previous step, we check that the restriction morphism

$$H^0(\mathcal{U} \otimes \mathcal{L})\longrightarrow H^0(\mathcal{U} \otimes \mathcal{L}_{|DG})$$

is surjective, with kernel generated by the section that defines DG. In other words, $H^0(\mathcal{U} \otimes \mathcal{L}_{|DG}) \simeq \Delta/\mathbb{C}z$.

Third step. We conclude the proof by checking that the morphism

$$H^0(T\mathbb{S}_{14|DG}) \longrightarrow H^0(\mathcal{U} \otimes \mathcal{L}_{|DG})$$

is surjective. For this we simply observe that it factorizes the morphism from $H^0(T\mathbb{S}_{14}) \simeq \mathfrak{spin}_{14}$ to $\Delta/\mathbb{C}z$ given by $X \mapsto Xz \mod \mathbb{C}z$. Finally, the surjectivity of the latter morphism is equivalent to the fact that the orbit of $[z]$ is open in $\mathbb{P}(\Delta)$. $\qquad\square$

As we already mentionned in the introduction, this implies that all the smooth projective symmetric varieties of Picard number one are infinitesimally rigid (see [12]).

Question. Is DG globally rigid? There are very nice examples of linear sections (of codimension two and three) of the ten dimensional spinor variety $\mathbb{S}_{10}$, which are defined by the generic point of a representation with an open orbit, and turn out for this reason to be locally rigid. However, they are not globally rigid because the generic points of some smaller orbits still define smooth sections, but of a different type [4, 13]. In our case, what does happen if we replace the general point z of Δ by a general point of its invariant octic divisor? Since this divisor is the projective dual of the spinor variety in the dual representation, the zero-locus of a section defined by such a point should contain a special $\mathbb{P}^6$; is it its singular locus? An explicit representative is

$$z_1 = 1 + e_{1237} + e_{1587} + e_{2467} + e_{123456}.$$

In the case of the Cayley Grassmannian CG, when one passes to a section defined by a point of the exceptional divisor, the zero-locus immediately becomes singular (more precisely it becomes singular along a $\mathbb{P}^3$), so there is no immediate obstruction to global rigidity. Up to our knowledge the question of the global rigidity of CG remains open.

3. Octonionic interpretations

Consider the real algebra $\mathbb{C} \otimes_\mathbb{R} \mathbb{O}_\mathbb{R}$, with the obvious product. This is called the algebra of complex octonions, or bioctonions. Of course it is no longer a division algebra, but it is still what is called a *structurable algebra* [2]. We will consider this algebra with complex coefficients: in other words, we complexify once more.

Proposition 6. *The double Cayley Grassmannian DG parametrizes the eight-dimensional isotropic subalgebras of the complexified bioctonions.*

The main point is that complexifying the complex numbers, we just get the algebra $\mathbb{C} \oplus \mathbb{C}$. Indeed, if we denote by i and I the roots of -1 in our two copies of $\mathbb{C}$, then $E = (1 + iI)/2$ and $F = (1 - iI)/2$ are such that $E + F = 1$, $EF = FE = 0$, and $E^2 = E$ and $F^2 = F$. Hence an isomorphism

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}_{\mathbb{R}} \simeq \mathbb{O} \oplus \mathbb{O}.$$

An eight dimensional subspace of $\mathbb{O} \oplus \mathbb{O}$, which is transverse to this decomposition, can be written as the graph Γ_g of some $g \in GL(\mathbb{O}) \simeq GL_8$. Moreover, it contains the unit element if and only if $g(1) = 1$. And it is a subalgebra if and only if g verifies $g(xy) = g(x)g(y) \forall x, y \in \mathbb{O}$, which means that g belongs to G_2. It is then generated by the unit element, and its intersection L_g with $V_{14} = Im\mathbb{O} \oplus Im\mathbb{O}$. Note that Γ_g (respectively L_g) is isotropic with respect to the difference of the octonionic norms on the two copies of $\mathbb{O}$ (respectively $Im\mathbb{O}$). This yields an embedding of G_2 inside $Spin_{14}$, whose closure is exactly DG.

So DG parametrizes a certain family of subspaces of the bioctonions. These spaces must be isotropic subalgebras, since this condition is closed. So let us consider such a subalgebra A, and suppose it defines a point of DG, not on the open orbit. Let K, K' denote the kernels of the projections to the two copies of $\mathbb{O}$. They must be positive dimensional subspaces of $Im\mathbb{O}$, totally isotropic, and such that $KK \subset K$ and $K'K' \subset K'$. In particular $\mathbb{C}1 + K$ and $\mathbb{C}1 + K'$ are subalgebras of $\mathbb{O}$. Let $k = \dim K$ and $k' = \dim K'$. These are invariants of the $Spin_{14}$-action, and since this group has only three orbits on DG, there are at most two possibilities for the pair (k, k'), apart from the generic case $(k, k') = (0, 0)$.

First case: $(k, k') = (3, 3)$. Then $\mathbb{C}1 + K$ and $\mathbb{C}1 + K'$ are four dimensional subalgebras of $\mathbb{O}$. By [16, Proposition 2.7], the isotropic four dimensional subalgebras of $\mathbb{O}$ are parametrized by the quadric $\mathbb{Q}_5 = G_2/P_1$. Explicitly, if ℓ is an isotropic line in $Im\mathbb{O}$, then $K_\ell = \ell\mathbb{O} \cap Im\mathbb{O}$ is such a subalgebra, and they are all of this type.

When K and K' are given, then $K \oplus K'$ is isotropic of dimension six, so it is contained in exactly two maximal isotropic subspaces of V_{14}, one in each family. In particular there is exactly one in $\mathbb{S}_{14}$. This defines an embedding of $\mathbb{Q}_5 \times \mathbb{Q}_5$ inside $\mathbb{S}_{14}$. Since this is the unique $G_2 \times G_2$-equivariant embedding of $\mathbb{Q}_5 \times \mathbb{Q}_5$ in $\mathbb{P}\Delta$, it must factor through DG.

Second case: $(k, k') = (2, 2)$. We will show how to construct examples of this type. Since we know there is only one orbit which is neither closed nor open, this will necessarily provide us with representatives of this intermediate orbit $\mathcal{O}_1$. We start with two null-planes N and N'. Recall that $\mathbb{C}1 \oplus N^\perp$ is a six dimensional subalgebra of $\mathbb{O}$, a copy of the sextonion subalgebra [15]. Moreover it contains H, a copy of the quaternion algebra transverse to N. (Over the complex numbers, the quaternion algebra is just an algebra of rank two matrices, and N is isomorphic to its two-dimensional simple module.) Let us also choose H' in $\mathbb{C}1 \oplus N'^\perp$, transverse to N'. Consider

$$A = (N, 0) \oplus (0, N') \oplus \Delta_h,$$

where Δ_h is the graph of some morphism h from H to H'. Then A is an isotropic subalgebra of the bioctonions if and only if h is an algebra isomorphism.

We claim that A belongs to DG. Because of the $G_2 \times G_2$-equivariance, it is enough to exhibit just one such A that does belong to DG. To do this we shall start from an explicit null plane in $Im\mathbb{O}$. Let $u_1, \ldots, u_7$ be an orthonormal basis of $Im\mathbb{O}$, whose multiplication rule is encoded in a Fano plane, as in [16]. Then for example, $N = \langle u_1 + iu_2, u_4 - iu_5 \rangle$ is a null-plane. It is convenient to reindex this basis by letting $u_1 = v_{-1}, u_2 = v_2, u_3 = v_{-3}, u_4 = v_1, u_5 = v_{-2}, u_6 = v_3, u_7 = v_0$. Then we may suppose that the transformation rule between the basis $v_{-3}, v_{-2}, v_{-1}, v_0, v_1, v_2, v_3$ and $e_1, e_2, e_3, f_1, f_2, f_3, e_7 - f_7$ is given by

$$v_k = \frac{1}{\sqrt{2}}(e_k + f_k), \quad v_{-k} = \frac{i}{\sqrt{2}}(e_k - f_k), \quad v_0 = \frac{i}{\sqrt{2}}(e_7 - f_7).$$

After this change of basis, our null-plane of V_7 becomes $N = \langle e_1 + e_2, f_1 - f_2 \rangle$. Similarly, $N' = \langle e_4 + e_5, f_4 - f_5 \rangle$ is a null-plane in V_7'.

Remark. Note the connection with the *null triples* of [3].

Proposition 6 now follows. Indeed we have observed that DG parametrizes some isotropic subalgebras of the complexified bioctonions. By the previous discussion there are three orbits of those. Since DG contains precisely three orbits, the claim follows. □

Lemma 7. *The three dimensional projective space $\mathbb{P}(N \otimes N')$ is contained in DG. Moreover a spinor $x \in N \otimes N'$ is of type $(3, 3)$ if its tensor rank is one, and of type $(2, 2)$ if its tensor rank is two.*

Proof. We have the following correspondance between vectors in $N \otimes N'$ and in Δ:

$$
\begin{aligned}
(e_1 + e_2) \otimes (e_4 + e_5) &\mapsto & y_1 &= (e_1 + e_2)(e_4 + e_5), \\
(e_1 + e_2) \otimes (f_4 - f_5) &\mapsto & y_2 &= (e_1 + e_2)(e_4 + e_5)e_6 e_7, \\
(f_1 - f_2) \otimes (e_4 + e_5) &\mapsto & y_3 &= (e_1 + e_2)(e_4 + e_5)e_3 e_7, \\
(e_1 + e_2) \otimes (e_4 + e_5) &\mapsto & y_4 &= (e_1 + e_2)(e_4 + e_5)e_3 e_6.
\end{aligned}
$$

This allows to check that $N \otimes N'$ is orthogonal to L_z. So its projectivization will be contained in DG as soon as it only consists of pure spinors. Consider $y = t_1 y_1 + t_2 y_2 + t_3 y_3 + t_4 y_4$. A straightforward computation shows that y is annihilated by

$$
P_y = \langle e_1 + e_2, f_1 - f_2, e_4 + e_5, f_4 - f_5, p_3, p_6, p_7 \rangle,
$$

where $p_3 = t_4 e_6 + t_3 e_7 - t_1 f_3, p_6 = t_4 e_3 - t_2 e_7 + t_1 f_6, p_7 = t_3 e_3 + t_2 e_6 + t_1 f_7$. In particular y is the pure spinor associated (up to scalar) to the maximal isotropic space P_y. Note moreover that the intersection of P_y with $\langle f_1, \ldots, f_7 \rangle$ has dimension equal to two plus the corank of a size three skew-symmetric matrix; in particular this dimension is always odd, which means that y is a positive pure spinor. In other words, it is a point of DG. $\qquad \square$
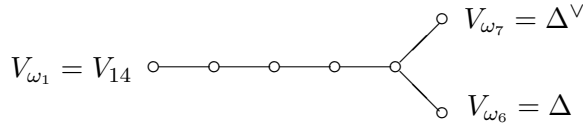
Recall that we denoted by D the closure of the codimension one orbit in DG. Necessarily, D must be the intersection of DG with the hyperplane $\mathbb{P}(V_7 \otimes V_7')$. Moreover, by the previous lemma D contains the union of the projective spaces $\mathbb{P}(N \otimes N')$, for N and N' null-planes in V_7 and V_7'. Since this union is obviously $G_2 \times G_2$-invariant, it has to coincide with D. (This describes D as the image of a projectivized Kempf collapsing). Moreover, for the very same reason the closed orbit $\mathcal{O}_4$ must be the union of the rank one elements $\mathbb{P}N \times \mathbb{P}N' \subset \mathbb{P}(N \otimes N')$. Since the intersection of two different tensor products $N_1 \otimes N_1'$ and $N_2 \otimes N_2'$ can only contain elements of rank one (or zero), we deduce the following statement.

Proposition 8. *Suppose that x belongs to $\mathcal{O}_1$. Then there exists a unique null-plane N_x in V_7, and a unique null-plane N_x' in V_7', such that x is contained in $\mathbb{P}(N_x \otimes N_x')$. Moreover, x has full rank in $\mathbb{P}(N_x \otimes N_x')$.*

Geometrically, this means that $\mathcal{O}_1$ fibers over a product of adjoint varieties $X_{ad}(G_2) \times X_{ad}(G_2)$, with fiber the complement of a smooth quadric in $\mathbb{P}^3$.

4. Postulation

Recall that the vertices of the Dynkin diagram D_7 are in bijective correspondence with the fundamental weights ω_i, or the fundamental representations V_{ω_i} of $Spin_{14}$, for $1 \le i \le 7$. We use the following indexation:

$$V_{\omega_1} = V_{14} \qquad\qquad\qquad \begin{matrix} V_{\omega_7} = \Delta^{\vee} \\ \\ V_{\omega_6} = \Delta \end{matrix}$$

One way to compute the cohomology groups on DG of $\mathcal{L}$ and its powers, is again to use the Koszul complex

$$(1) \qquad 0 \longrightarrow \wedge^7 \mathcal{E}^{\vee} \longrightarrow \cdots \longrightarrow \mathcal{E}^{\vee} \longrightarrow \mathcal{O}_{\mathbb{S}_{14}} \longrightarrow \mathcal{O}_{DG} \longrightarrow 0,$$

where $\mathcal{E} = \mathcal{U} \otimes \mathcal{L}$. For any $k \ge 0$ and $i \ge 0$, the bundle $\wedge^i \mathcal{E}^{\vee} \otimes \mathcal{L}^k = \wedge^i \mathcal{U}^{\vee} \otimes \mathcal{L}^{k-i}$ is irreducible, with highest weight θ_i given by $\theta_i = (k-i)\omega_7 + \omega_i$ for $0 \le i \le 5$ (and $\omega_0 = 0$ by convention), while $\theta_6 = (k-5)\omega_7 + \omega_6$ and $\theta_7 = (k-5)\omega_7$. One easily checks that these weights are either dominant or singular. By the Bott-Borel-Weil theorem this implies that $\mathcal{L}^k$ has no higher cohomology. Moreover we can compute the dimension of its space of global sections as the alternate sum of modules whose dimensions are given by the Weyl dimension formula, as follows:

$$\dim V_{k\omega_7} = \frac{(k+1)(k+2)(k+3)^2(k+4)^2(k+5)^3(k+6)^3(k+7)^3(k+8)^2(k+9)^2(k+10)(k+11)}{1 \times 2 \times 3^2 \times 4^2 \times 5^3 \times 6^3 \times 7^3 \times 8^2 \times 9^2 \times 10 \times 11},$$

$$\dim V_{(k-1)\omega_7+\omega_1} = \frac{k(k+1)(k+2)^2(k+3)^2(k+4)^3(k+5)^2(k+6)^3(k+7)^2(k+8)^2(k+9)(k+10)(k+11)}{3^2 \times 4^2 \times 5^3 \times 6^2 \times 7^2 \times 8^2 \times 9^2 \times 10 \times 11 \times 12},$$

$$\dim V_{(k-2)\omega_7+\omega_2} = \frac{(k-1)k(k+1)^2(k+2)^2(k+3)^2(k+4)^2(k+5)^3(k+6)^2(k+7)^2(k+8)^2(k+9)(k+11)}{2 \times 3^2 \times 4^2 \times 5^2 \times 6^2 \times 7^2 \times 8^2 \times 9^2 \times 10^2 \times 11},$$

$$\dim V_{(k-3)\omega_7+\omega_3} = \frac{(k-2)(k-1)k^2(k+1)(k+2)^2(k+3)^2(k+4)^3(k+5)^3(k+6)^2(k+7)(k+8)(k+9)(k+10)}{2 \times 3^2 \times 5^2 \times 6^2 \times 7^2 \times 8^3 \times 9^2 \times 10 \times 11 \times 12},$$

$$\dim V_{(k-4)\omega_7+\omega_4} = \frac{(k-3)(k-2)(k-1)k(k+1)^2(k+2)^3(k+3)^3(k+4)^2(k+5)^2(k+6)(k+7)^2(k+8)(k+9)}{2 \times 3 \times 4 \times 5^2 \times 6^3 \times 7^2 \times 8^2 \times 9^2 \times 10 \times 11 \times 12},$$

$$\dim V_{(k-5)\omega_7+\omega_5} = \frac{(k-4)(k-2)(k-1)^2k^2(k+1)^2(k+2)^3(k+3)^2(k+4)^2(k+5)^2(k+6)^2(k+7)(k+8)}{2 \times 3 \times 4 \times 5^2 \times 6^2 \times 7^2 \times 8^2 \times 9^2 \times 10^2 \times 11 \times 12},$$

$$\dim V_{(k-5)\omega_7+\omega_6} = \frac{(k-4)(k-3)(k-2)(k-1)^2k^2(k+1)^3(k+2)^2(k+3)^3(k+4)^2(k+5)^2(k+6)(k+7)}{2 \times 3 \times 4^2 \times 5^2 \times 6^3 \times 7^2 \times 8 \times 9^2 \times 10^2 \times 11 \times 12},$$

$$\dim V_{(k-5)\omega_7} = \frac{(k-4)(k-3)(k-2)^2(k-1)^2k^3(k+1)^3(k+2)^3(k+3)^2(k+4)^2(k+5)(k+6)}{1 \times 2 \times 3^2 \times 4^2 \times 5^3 \times 6^3 \times 7^3 \times 8^2 \times 9^2 \times 10 \times 11}.$$

Proposition 9. *For any $k \geq 0$ and $i > 0$, $H^i(DG, \mathcal{L}^k) = 0$. Moreover,*

$$h^0(DG, \mathcal{L}^k) = \frac{(k+1)(k+2)(k+3)^2(k+4)^2(k+5)(k+6)}{2^{10}3^5 5^2 7^2 11} P(k),$$

$$P(k) = 186k^6 + 3906k^5 + 34441k^4 + 163184k^3$$
$$+ 438545k^2 + 634858k + 388080.$$

Corollary 10. *The degree of $DG \subset \mathbb{P}^{49}$ is $4836 = 2^2 \times 3 \times 13 \times 31$.*

This could also have been deduced from the fundamental class of DG, by applying repeatedly the product formula by the hyperplane class.

Since DG is spherical, it is multiplicity free. As in [16], we can obtain the $G_2 \times G_2$-module structure of $H^0(DG, \mathcal{L}^k)$ by restricting to the hyperplane divisor D. Using the projecting bundle structure of its resolution, we get

$$H^0(D, \mathcal{L}_D^k) = H^0(X_{ad}(G_2) \times X'_{ad}(G_2), Sym^k(N \otimes N')^\vee).$$

By the Cauchy formula,

$$Sym^k(N \otimes N') = \bigoplus_{i+2j=k} Sym^i N \otimes (\det N)^j \otimes Sym^i N' \otimes (\det N')^j.$$

Since $Sym^i N^\vee$ is irreducible of highest weight $i\omega_1$, and $\det N^\vee$ of weight ω_2, the Borel-Weil theorem yields

$$H^0(D, \mathcal{L}_D^k) = \bigoplus_{i+2j=k} V_{i\omega_1+j\omega_2} \otimes V'_{i\omega_1+j\omega_2}.$$

We finally get (to be compared with Proposition 3.6 of [16]):

Proposition 11. *The equivariant Hilbert series of the double Cayley Grassmannian is*

$$H_{DG}^{G_2 \times G_2}(t) = (1-t)^{-1}(1 - tV_{\omega_1+\omega_1'})^{-1}(1 - t^2 V_{\omega_2+\omega_2'})^{-1}.$$

Here we use formally the Cartan multiplication of representations, according to the rule $V_\mu V_\nu = V_{\mu+\nu}$. Moreover we use it for $G_2 \times G_2$, so that $V_{\mu+\nu'}$ is the tensor product of the representation V_μ of the first copy of G_2, by the representation V_ν of the second copy.
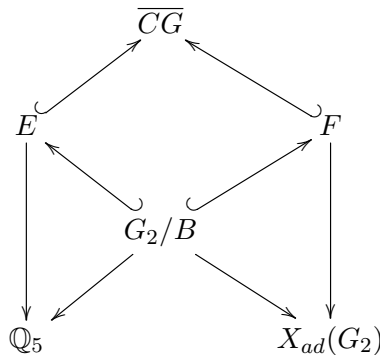
5. The wonderful compactification of G_2

Recall that the Cayley Grassmannian $CG \subset G(4, V_7)$ has a very similar G_2-orbits structure: a closed orbit $\mathcal{O}_3 \simeq \mathbb{Q}_5$, a codimension one orbit $\mathcal{O}_1$ whose closure is a hyperplane section H of CG, and an open orbit $\mathcal{O}_0 \simeq G_2/SL_2 \times SL_2$. Moreover, if we blow-up $\mathcal{O}_3 \subset CG$, we get the wonderful compactification of the symmetric space $\mathcal{O}_0$. Since we are in rank two, the proper orbit closures of this wonderful compactification $\overline{CG}$ are the two divisors F (the proper transform of H), E (the exceptional divisor), and their transverse intersection $E \cap F$. The two divisors support smooth projective fibrations:

$$E \simeq \mathbb{P}(Sym^2 C) \to \mathbb{Q}_5, \qquad F \simeq \mathbb{P}(Sym^2 N) \to X_{ad}(G_2),$$

where C denotes the so-called *Cayley bundle* over $\mathbb{Q}_5$, and N is the null-plane bundle over the adjoint variety $X_{ad}(G_2)$. Both are rank two irreducible homogeneous bundles. The latter is the restriction of the tautological bundle for the embedding of $X_{ad}(G_2)$ into $G(2, V_7)$. The former is defined by the conditions that $H^0(C) = 0$ and $H^0(C(1)) = \mathfrak{g}_2$; its first Chern class is the hyperplane class [17].

Observe that in particular, E and F both contain a conic fibration, preserved by G_2, which must therefore coincide with the closed orbit $E \cap F$. In fact, this closed orbit is nothing else than the full flag variety of G_2.
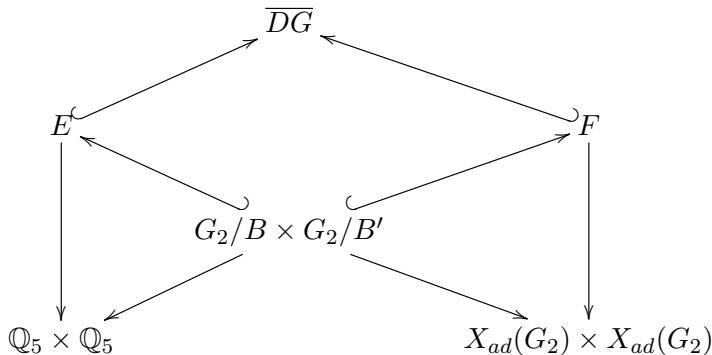
We have the following diagram:

The picture is strickingly similar for the double Cayley Grassmannian. Blowing-up the closed orbit $\mathcal{O}_4 \simeq \mathbb{Q}_5 \times \mathbb{Q}_5$, we get an exceptional divisor E, which is the projectivization of the normal bundle.

Lemma 12. *The normal bundle to the closed orbit in DG is $C \otimes C'$.*

Moreover the strict transform F of D is the total space of the projectivisation of $N \otimes N'$ over $X_{ad}(\mathfrak{g}_2) \times X_{ad}(G_2)$. Again each of these divisors contains a quadric surface bundle, which must coincide with the closed orbit $E \cap F$. In fact this closed orbit is nothing else than the product of two copies of the flag variety of G_2. We get the following diagram:

Proof. For a quick check of the Lemma we can argue as follows. The normal bundle N on $\mathbb{Q}_5 \times \mathbb{Q}_5$ we are looking for has rank four, and is by construction homogeneous under $G_2 \times G_2$, and symmetric with respect to the two quadrics. In particular it must be constructed from homogeneous bundles of rank at most two on the two quadrics. Since there are no non trivial extensions between line bundles on $\mathbb{Q}_5$, this quadric admits only two, up to twists, G_2-homogeneous bundles of rank at most two: the trivial line bundle and the Cayley bundle.

A possibility would be that $N = C(a,b) \oplus C'(b,a)$, where we denote by C and C' the two Cayley bundles induced from the two quadrics. But then we would get $\det(N) = (2a + 2b - 1, 2a + 2b - 1)$, while a computation with tangent bundles yields $\det(N) = (2, 2)$. So N must be a twist of $C \otimes C'$, and since this has the correct determinant, the twist must be trivial. $\square$

Remark. Exactly as in the case of CG, there also exists another contraction of $\overline{DG}$ to another variety, contracting the divisor D. But the result of this contraction is singular.

6. Betti numbers

In this section we compute the Betti numbers of DG. We would like to be able to compute its cohomology ring.

6.1. Torus action

Let T be a maximal torus of $G_2 \times G_2$.

Proposition 13. *The torus T acts on DG with exactly 36 fixed points, all contained in the closed orbit $\mathbb{Q}_5 \times \mathbb{Q}_5$.*

Proof. Recall that the linear span of DG is the projectivization of $V_7 \otimes V_7'$. Moreover, G_2 acts on V_7 with weights $0, \pm\alpha_1, \pm\alpha_2, \pm\alpha_3$ with $\alpha_1 + \alpha_2 + \alpha_3 = 0$. The weights of the action of $G_2 \times G_2$ on $V_7 \otimes V_7' \oplus \mathbb{C}$ are thus $\pm\alpha_i, \pm\alpha_j', \pm\alpha_i \pm \alpha_j'$, all with multiplicity one, and 0 with multiplicity two. Let W_0 be the two-dimensional zero weight space. To ensure that T acts on DG with finitely many fixed points, the only thing we need to check is that the projective line $\mathbb{P}W_0$ is not contained in DG. But this is clear, since this line contains $[z]$, which is not contained in $\mathbb{S}_{14}$ and a fortiori not in DG.

We claim, more precisely, that:

1) every T-fixed point with non zero weight is contained in DG,

2) $DG \cap \mathbb{P}W_0$ is empty.

The first statement is clear, since Δ being minuscule, each fixed point in $\mathbb{P}\Delta$ of a maximal torus of $Spin_{14}$ is contained in $\mathbb{S}_{14}$. Since T is a subtorus of a maximal torus T_+ of $Spin_{14}$, this remains true for all the T-fixed points with non zero weight, just because they are also T_+-fixed points.

To check the second statement, we may suppose that $e_1, e_2, e_3, f_1, f_2, f_3, e_7 - f_7$ are T-eigenvectors in V_7, with weights $\alpha_1, \alpha_2, \alpha_3, -\alpha_1, -\alpha_2, -\alpha_3, 0$; and similarly for V_7'. Then the T-invariants in L_z are $e_7.z$ and $f_7.z$. From that we deduce that

$$W_0 = \langle 1 + e_{123456}, e_{1237} + e_{4567} \rangle.$$

We need to check that W_0 contains no pure spinor. Observe that if an element of Δ of the form $1 + \omega_2 + \omega_4 + \omega_6$ is a pure spinor, then ω_4 must be proportional to $\omega_2 \wedge \omega_2$ and ω_6 must be proportional to $\omega_2 \wedge \omega_2 \wedge \omega_2$. This already rules out all the points of W_0 except the multiples $x_0 = e_{1237} + e_{4567}$. But recall that a spinor x is pure when the space of elements $v \in V_{14}$ such that $vx = 0$ is seven dimensional. A straightforward check shows that x_0 is only killed by (multiples of) e_7, hence is not pure. $\square$

An immediate consequence is:

Corollary 14. *The Chow ring of DG is free of rank* 36.

Explicitly, the T-fixed points correspond to the weight vectors in Δ of type e_{ij}, $e_{ii'j7}$, $e_{ijj'7}$, $e_{ii'jj'}$ where $1 \leq i, i' \leq 3$ and $4 \leq j, j' \leq 6$. Note that two fixed points e_{ij} (respectively e_{ijkl}) and e_{abcd} are connected by a T-stable line if and only if $\{i, j\} \subset \{a, b, c, d\}$ (respectively $\{a, b, c, d\}$ and $\{i, j, k, l\}$ have three elements in common).

6.2. Schubert varieties

Since the maximal torus T of $G_2 \times G_2$ acts on DG with finitely many fixed points, the Bialynicki-Birula decomposition yields, for any choice of a general rank one subtorus, a stratification of DG into affine spaces, which is uniquely defined up to conjugation. The closures of those affine spaces will be called Schubert varieties. Their classes in the (equivariant) Chow ring, called the (equivariant) Schubert classes, form a basis. A priori, we should be able to describe these equivariant Schubert classes by localization, and then their multiplication rule. A more modest goal would be to compute a Pieri formula in the classical Chow ring. This would allow to get the degrees of the Schubert varieties, which would give lots of informations on the restriction map from the spinor variety. In the case of CG, the restriction map from the ambient Grassmannian is surjective, so the multiplicative structure of the Chow ring of CG can be deduced.

In the case of a wonderful compactification $\bar{G}$ of an adjoint semisimple group G, the Schubert classes are indexed by $W \times W$ and the Betti numbers are given by the following formula:

$$b_{2i}(\bar{G}) = \#\{(u, v) \in W \times W, \ \ell(u) + \ell(v) + m(v) = i\},$$

where ℓ is the classical length function, and m is the simple length function, defined as the number of simple roots that are sent to negative roots [7]. Recall that the Weyl group of G_2 is isomorphic to the dihedral group D_6, and in particular has 12 elements: two elements in each length from 1 to 5, and one element of length 0 and 6. All have simple length 1, except the maximal one (whose simple length are 0 and 2). This yields the even Betti numbers of $\bar{G}_2$:

$$b_{2\bullet}(\bar{G}_2) = 1, 2, 4, 8, 12, 16, 19, 20, 19, 16, 12, 8, 4, 2, 1.$$

In order to deduce the Betti numbers of DG, we just need to recall that $\bar{G}_2$ can be obtained by blowing-up $\mathbb{Q}_5 \times \mathbb{Q}_5$ in DG. This modifies the Betti numbers by the Betti numbers of a $(\mathbb{P}^2 - \mathbb{P}^0)$-bundle over $\mathbb{Q}_5 \times \mathbb{Q}_5$. We readily deduce:

Proposition 15. *The Poincaré polynomial of the variety DG is*

$$P_{DG}(t) = \frac{1 - t^{12}}{1 - t^2}(1 + t^6 + t^8 + t^{10} + t^{12} + t^{18}).$$

In other words the odd Betti nubers of DG are zero, and the even ones are

$$b_{2\bullet}(DG) = 1, 1, 1, 2, 3, 4, 4, 4, 4, 4, 3, 2, 1, 1, 1.$$

Note that, as a consequence, the restriction map on cohomology from $\mathbb{S}_{14}$ cannot be surjective in degree four. In fact there is an obvious special cohomology class of degree four, that of the closed orbit $\mathbb{Q}_5 \times \mathbb{Q}_5$. Its degree is $4\binom{10}{5} = 1008$, while the degrees of the restrictions to DG of the degree four Schubert classes can be computed to be

$$\int_{DG} \tau_4 h^{10} = 1260, \qquad \int_{DG} \tau_{31} h^{10} = 1780.$$

So the class of $\mathbb{Q}_5 \times \mathbb{Q}_5$ is certainly not an integral combination of the restrictions of τ_4 and τ_{31}, and probably not a combination at all.

Question. By pull-back, the Chow ring of DG embeds inside the Chow ring of $\overline{DG}$. Moreover, $\overline{DG}$ being the wonderful compactification of G_2, its equivariant cohomology ring can be extracted from [20] or [9]. Can we deduce that of DG? The Bialynicki-Birula decomposition of the wonderful compactification has been studied in [7]. Can one extract a Pieri formula, and push it down to DG?

7. Some incidences

7.1. Incidences for the Cayley Grassmannian

Let us briefly consider the Cayley Grassmannian $CG \subset G(4, V_7)$, defined by the general three-form ω. The latter also defines a global section of $Q^{\vee}(1)$ on $G(2, V_7)$, whose zero locus is the adjoint variety of G_2. Consider the incidence

diagram

$$
\begin{array}{ccc}
 & CI_{10} & \\
{\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\
CG & & G(5, V_7)
\end{array}
$$

where CI_{10} parametrizes the pairs $(U_4 \subset U_5)$ such that U_4 belongs to CG. In particular CI_{10} is a $\mathbb{P}^2$-bundle over CG. For $U_5 \subset V_7$, the restriction of ω to U_5 is dual to a skew-symmetric degree two tensor which can be of rank two or four. In the latter case, the support of this tensor is a hyperplane $U_4 \subset U_5$ on which ω vanishes, and it is the only such hyperplane; this implies that q is birational. The former case occurs over a locus X_7 of codimension three, and the corresponding fibers of q are projective planes. We conclude that q is just the blowup of $X_7 \simeq OG(2, V_7)$.
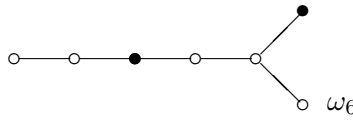
There is a slightly different incidence diagram

$$
\begin{array}{ccc}
 & CI_{11} & \\
{\scriptstyle r}\swarrow & & \searrow{\scriptstyle s} \\
CG & & X_{ad}(G_2) \subset G(2, V_7)
\end{array}
$$

where the fibers of s are del Pezzo fourfolds of degree five, and the fibers of r are conics in $X_{ad}(G_2)$. As observed by Kuznetsov, this allows to interprete the Cayley Grassmannian CG as the Hilbert scheme of conics on the adjoint variety of G_2.

7.2. Incidences with DG

What are the analogs of those incidences when we switch to DG? Recall that DG is defined by a general element of Δ, which defines a global section of the irreducible homogeneous vector bundle $\mathcal{E}_{\omega_6} = U \otimes \mathcal{L}$ over $\mathbb{S}_{14}$. Over each flag variety F of $Spin_{14}$, there is an irreducible homogeneous vector bundle $\mathcal{E}_{\omega_6}^F$ whose space of sections is Δ.

Consider for example the flag variety $OF = OF(k, 7, V_{14})$ for $k \leq 5$, parametrizing flags of isotropic spaces of dimensions k and 7, with its two projections to $\mathbb{S}_{14}$ and $OG = OG(k, V_{14})$. The ranks of $\mathcal{E}_{\omega_6}^{OF}$ and $\mathcal{E}_{\omega_6}^{OG}$ can be read on the following weighted Dynkin diagram (where $k = 3$):

The flag variety OF is defined by the two marked vertices. When we suppress those two vertices, the connected component of the remaining diagram containing the vertex associated to ω_6 has type A_{6-k}. So $\mathcal{E}^{OF}_{\omega_6}$, which corresponds to the natural representation, has rank $7-k$. In more concrete terms, note that OF has two tautological bundles U_k and U, of rank k and 7, respectively; then $\mathcal{E}^{OF}_{\omega_6} = (U/U_k) \otimes L$, where L denotes the pull-back of the spinor line bundle from $\mathbb{S}_{14}$.
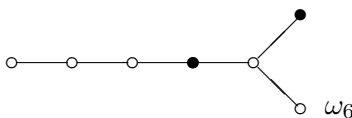
Similarly, the orthogonal Grassmannian OG is defined by the rightmost of the two marked vertices. When we suppress this vertex, the connected component of the remaining diagram containing the vertex associated to ω_6 has type D_{7-k}. So $\mathcal{E}^{OG}_{\omega_6}$, which corresponds to a half-spin representation, has rank 2^{6-k}.

Our general element $z \in \Delta$ defines a general section s_z of the bundle $\mathcal{E}^{OF}_{\omega_6}$, whose zero locus we denote by OF_z. The fibers of the projection to $\mathbb{S}_{14}$ are Grassmannians $G(k, 7)$, and the restriction of $\mathcal{E}^{OF}_{\omega_6}$ to each fiber is isomorphic to the quotient tautological bundle. In particular, if the restriction of s_z to such a fiber is non identically zero, it vanishes on a copy of $G(k-1, 6)$. So the general fiber of the projection from OF_z to $\mathbb{S}_{14}$ is $G(k-1, 6)$, and the special fiber is $G(k, 7)$ over DG.
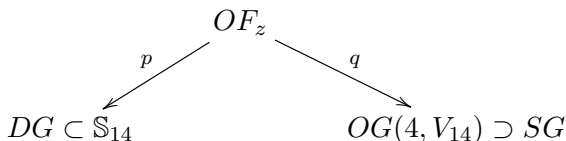
Similarly the projection of OF to OG is a spin manifold $\mathbb{S}_{14-2k}$, and the restriction of $\mathcal{E}^{OF}_{\omega_6}$ to each fiber is isomorphic to a spinor bundle. The zero-locus of s_z to such a fiber depends on its type as an element of the half-spin representation of $Spin_{14-2k}$. In fact this representation has finitely many orbits, so there is an induced stratification of OG by orbital degeneracy loci of s_z, and the type of the fiber of the projection from OF_z to OG depends on the strata. Let us discuss two cases a little further.

7.3. Incidence with 4-planes

The case where $k = 4$ is special because $Spin_6 = SL_4$, and in this case the bundle $\mathcal{E}^{OG}_{\omega_6}$ is just a rank four bundle defined by a natural representation of SL_4, as can be read from the weighted diagram

$$\omega_6$$

Similarly $\mathcal{E}_{\omega_6}^{OF}$ is defined by a natural representation of SL_3, so on each fiber of the projection from OF_z to OG, the section s_z vanishes either at one point, or everywhere. We thus get a diagram

$$OF_z$$
$$p \swarrow \qquad \searrow q$$
$$DG \subset \mathbb{S}_{14} \qquad\qquad OG(4, V_{14}) \supset SG$$

where q is the blowup of a codimension four subvariety $SG \subset OG(4, V_{14})$, while p is a $G(3,6)$-fibration over the complement of DG in $\mathbb{S}_{14}$, with special fibers $G(4,7)$ over DG. The weights of the rank four bundle $\mathcal{E}_{\omega_6}^{OG}$ are ω_6, $s_6(\omega_6) = \omega_5 - \omega_6$, $s_5 s_6(\omega_6) = \omega_4 - \omega_5 + \omega_7$ and $s_7 s_5 s_6(\omega_6) = \omega_4 - \omega_7$, hence $\det(\mathcal{E}_{\omega_6}^{OG}) = \mathcal{O}(2)$.

We can then compute the Poincaré polynomial of OF_z in two ways: first in terms of those of DG, $G(3,6)$ and $G(3,7)$, that we already know; second, in terms of those of $\mathbb{S}_{14}$ and SG. The latter can thus be computed and we get:

Proposition 16. *The variety SG is a Fano manifold of dimension 26, Picard number 1, and index 7, admitting an action of $G_2 \times G_2$. Its Poincaré polynomial is*

$$P_{SG}(t) = \frac{1 - t^{10}}{1 - t^2}(1 + t^6)^2 \Big(\frac{1 - t^{16}}{1 - t^4}(1 + t^8 + t^{10} + t^{12} + t^{20}) + t^{16} \Big).$$

This means the odd Betti numbers of SG are zero, and the even ones are

$$b_{2\bullet}(SG) = 1,1,2,4,6,8,12,16,20,25,29,33,35,36,35,33,29,25,20,16,12,8,6,4,2,1,1.$$

The topological Euler characteristic is 420. It would be interesting to know if the action of $G_2 \times G_2$ is quasi-homogeneous.

7.4. Incidence with 2-planes

Over the orthogonal Grassmannian $OG(2, 14)$, the bundle $\mathcal{F} = \mathcal{E}_{\omega_6}^{OG}$ has rank 16 and is induced from a half-spin representation of $Spin_{10}$. Since $OG(2, 14)$

has dimension 21, the general section of $\mathcal{E}_{\omega_6}^{OG}$ defined by z must vanish in dimension 5 (or possibly, nowhere), and its zero locus Z_z must be stable under the action of $G_2 \times G_2$.

Proposition 17. *Z_z is the disjoint union of two copies of $X_{ad}(G_2)$.*

Proof. Recall that our general element z of Δ determined an orthogonal decomposition $V_{14} = V_7 \oplus V_7'$ and a tensor decomposition $\Delta = \Delta_7 \otimes \Delta_7'$ such that $z = \delta \otimes \delta'$ for some general elements δ and δ' of Δ_7 and Δ_7'.

Given an isotropic plane P, consider the Plücker line $\wedge^2 P$. The image of the Clifford multiplication map

$$\wedge^2 P \otimes \Delta \subset \wedge^2 V_{14} \otimes \Delta \longrightarrow \Delta$$

is a sixteen dimensional space $\mathcal{G}_P \subset \Delta$, and we can identify $\mathcal{G}$ with $\mathcal{F}^\vee$ (recall that Δ is self-dual). This implies that P belongs to Z_z if and only if $\mathcal{G}_P \subset z^\perp$.

Now suppose that $P \subset V_7$. The Clifford action of P on $\Delta = \Delta_7 \otimes \Delta_7'$ is just given by its action of Δ_7, so we deduce that P belongs to Z_z if and only if $\wedge^2 P.\Delta_7 \subset \delta^\perp$. This is a codimension two condition on $OG(2, V_7)$, that defines the adjoint variety $X_{ad}(G_2)$.

We conclude that Z_z contains the disjoint union of $X_{ad}(G_2)$ and $X_{ad}'(G_2)$, the adjoint varieties of our two copies of G_2. In order to prove equality, we just need to check that Z_z has at most two connected components. For this we can use the Koszul resolution of the structure sheaf of Z_z. A direct computation shows that the only non zero cohomology groups of the wedge powers of the dual of $\mathcal{E}_{\omega_6}^{OG}$ are $H^0(\wedge^0(\mathcal{E}_{\omega_6}^{OG})^\vee) = H^4(\wedge^4(\mathcal{E}_{\omega_6}^{OG})^\vee) = \mathbb{C}$. We readily deduce that $h^0(\mathcal{O}_{Z_z}) = 2$, and this concludes the proof. $\qquad \square$

Taking the incidence between DG and Z_z we get the following diagram:

$$
\begin{array}{ccc}
& DI_{13} & \\
{\scriptstyle 2:1} \swarrow & & \searrow {\scriptstyle t} \\
DG \longleftarrow D & & X_{ad}(G_2) \amalg X_{ad}'(G_2)
\end{array}
$$

where the fibers of t are codimension two linear sections of $\mathbb{S}_{10}$.
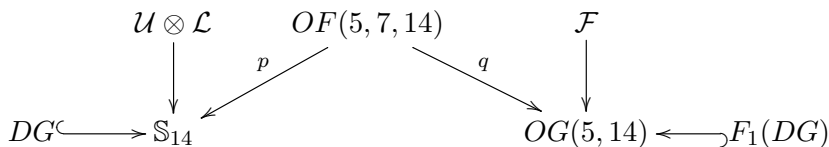
8. Linear subspaces

Since DG has dimension 14 and index 7, the expected dimension of the space of lines on DG is $14 + 7 - 3 = 18$. The expected dimension of the space of lines through a general point is 5.

Proposition 18. *The variety $F_1(DG)$ of lines on DG is a smooth Fano manifold of dimension* 18, *Picard number one, and index* 4.

Proof. By [14, Theorem 4.3], the variety of lines on $\mathbb{S}_{14}$ is the orthogonal Grassmannian $OG(5,14)$, whose dimension is 30. The weights $\omega_1, \omega_6, \omega_7$ define irreducible homogeneous vector bundles of ranks 5, 2, 2 on $OG(5,14)$: the first one is $\mathcal{V}^\vee$, the dual of the tautological bundle, and we denote the other ones by $\mathcal{E}_6$ and $\mathcal{E}_7$. Their determinant line bundles are all equal to $\mathcal{O}(1)$, the restriction of the Plücker line bundle. Note moreover that

$$\mathcal{E}_6 \otimes \mathcal{E}_7 \simeq (\mathcal{V}^\perp/\mathcal{V})(1).$$

Consider the incidence diagram, where $OF(5,7,14) = D_7/P_{5,7}$,

$$
\begin{array}{ccccc}
\mathcal{U} \otimes \mathcal{L} & OF(5,7,14) & & \mathcal{F} \\
\downarrow & {}^{p}\swarrow \quad \searrow^{q} & & \downarrow \\
DG \hookrightarrow \mathbb{S}_{14} & & OG(5,14) \longleftarrow F_1(DG)
\end{array}
$$

Since DG is defined by a general section s of the bundle $\mathcal{E} = \mathcal{U} \otimes \mathcal{L}$ on $\mathbb{S}_{14}$, its variety of lines $F_1(DG)$ will be defined by a section of the bundle $\mathcal{F} = q_* p^* \mathcal{E}$ on $OG(5,14)$. Obviously there is an exact sequence
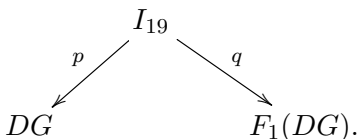
$$0 \longrightarrow q^*\mathcal{V} \otimes p^*\mathcal{L} \longrightarrow p^*(\mathcal{U} \otimes \mathcal{L}) \longrightarrow (p^*\mathcal{U}/q^*\mathcal{V}) \otimes p^*\mathcal{L} \longrightarrow 0$$

on $OF(5,7,14)$. This pushes forward on $OG(5,14)$ to

$$0 \longrightarrow \mathcal{V} \otimes \mathcal{E}_6 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}_7 \longrightarrow 0.$$

We deduce that $\mathcal{F}$ has rank 12, and that the space of its global sections is again Δ. By construction $\mathcal{F}$ is globally generated, so $F_1(DG)$ is smooth of dimension $30 - 12 = 18$, being the zero-locus of a general section. Since moreover $\det \mathcal{F} = \mathcal{O}(4)$, we deduce that $F_1(DG)$ is Fano of index 4.

In order to check that $F_1(DG)$ has Picard number one, consider the point-line incidence correspondence

$$
\begin{array}{ccc}
& I_{19} & \\
{}^{p}\swarrow & & \searrow^{q} \\
DG & & F_1(DG).
\end{array}
$$

Of course q is just a $\mathbb{P}^1$-bundle. The fibers of p are of three different types, over the three orbits in DG. A computation shows that the fiber over the closed orbit is the union of two copies of $\mathbb{P}^2 \times \mathbb{P}^3$ blown-up at one point, while the other orbits are irreducible. We could in principle compute the Hodge polynomials of the three fibers and deduce that of $F_1(DG)$, but the simple fact that the fiber over the codimension one orbit $\mathcal{O}_1 \subset DG$ is irreducible already implies that the Picard number of $F_1(DG)$ is one, as claimed. $\square$

The generic fiber of p is the variety of lines in DG through a general point. It is isomorphic to its image in the tangent space, the *variety of minimal rational tangents* (VMRT).

Proposition 19. *The VMRT at a general point of DG is a copy of the adjoint variety $X_{ad}(G_2) \subset \mathbb{P}\mathfrak{g}_2$.*

Proof. The general point x of DG has stabilizer G_2, so the VMRT at x is a five dimensional subvariety, stable under G_2, and equivariantly embedded inside $\mathbb{P}T_x DG = \mathbb{P}^{13}$. This VMRT must contain a closed G_2-orbit, and it contains no fixed point because the restriction of $D_z = V_7 \oplus V_7' \oplus \mathbb{C}$ to the diagonal G_2 contains a unique stable plane, but the corresponding line, since it contains $[z]$, is not contained in DG. Since the minimal non trivial closed G_2-orbits are $G_2/P_1 = \mathbb{Q}_5$ and $G_2/P_2 = X_{ad}(G_2)$, both of dimension five, the VMRT must be one of these. Since it is equivariantly embedded inside $\mathbb{P}^{13}$, it must be the second one. $\square$

It was already observed in [8] that the VMRT at a general point of the wonderful compactification of an adjoint simple algebraic group is a copy of its adjoint variety (except in type A). The only special feature in our situation is that the minimal rational curves are lines in the spinor variety.

Corollary 20. *DG contains planes, but no higher dimensional linear spaces, passing through the general point.*

Proof. This follows from the fact that $X_{ad}(G_2)$ contains no plane, since its variety of lines through a given point is a cubic rational normal curve by [14, Theorem 4.3]. $\square$

In fact we have seen in Lemma 7 that DG also contains a ten-dimensional family of $\mathbb{P}^3$'s, parametrized by $X_{ad}(G_2) \times X_{ad}(G_2)$. But they only cover the codimension one orbit closure (and there is exactly one of them through the general point).

9. Some numerology

Let us conclude this paper by a couple of slightly esoteric observations. The Cayley Grassmannian and its double appear in two series of compactifications of symmetric spaces, as follows:

$$X = \quad SL_3/SO_3 \subset \mathbb{P}^5, \quad SO_5/GL_2 \subset \mathbb{Q}^3 \times \mathbb{Q}^3, \quad G_2/SO_4 \subset CG,$$
$$Y = \quad\quad PSL_3 \subset \mathbb{P}^8, \quad\quad\quad SO_5 \subset \mathbb{S}_{10}, \quad\quad\quad\quad G_2 \subset DG.$$

Each of these compactifications contains a unique closed orbit, and blowing it yields the wonderful compactification. Let $a = 1, 2, 4$ for the three members of each series. The closed orbit Z in the first series has dimension $a + 1$ and codimension 3. The closed orbit Z' in the second series is isomorphic to $Z \times Z$, so its dimension is $2a + 2$, while its codimension is 4. In fact each Z in the series admits a homogeneous rank two vector bundle C such that its normal bundle is isomorphic to Sym^2C, while the normal bundle to Z' is isomorphic to $C \boxtimes C$.

The Weyl group W has cardinality $2a + 4$. Recall that the Chow ring of the wonderful compactification has a basis indexed by $W \times W$, so that the Euler topological characteristic $\chi_{top}(\bar{G}) = (\#W)^2 = 4(a + 2)^2$. A computation shows that the minimal compactification Y has Euler characteristic

$$\chi_{top}(Y) = \frac{1}{4}\chi_{top}(\bar{G}) = (a + 2)^2.$$

Does it admit a natural basis indexed by $\bar{W} \times \bar{W}$, where $\bar{W} = W/\mathbb{Z}_2$?

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