

The universal sheaf as an operator

ANDREI NEGUȚ

We compute the universal sheaf of moduli spaces \mathcal{M} of sheaves on a surface S , as an operator $\Lambda = \{\text{symmetric polynomials}\} \rightarrow K(\mathcal{M})$, thus generalizing the viewpoint of [4] to arbitrary rank and general smooth surfaces.

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1. Introduction

Fix a natural number r . The moduli space \mathcal{M}^f of rank r framed sheaves on the plane is an algebro-geometric incarnation of the instanton moduli space that gives rise to supersymmetric $\mathcal{N} = 2$ $U(r)$ -gauge theory on \mathbb{R}^4 in the Ω -background. In [17], the partition function of this theory was expressed in terms of equivariant integrals over \mathcal{M}^f . The present note is concerned with the deformation from cohomology to K -theory (over \mathbb{Q}), which corresponds to supersymmetric $\mathcal{N} = 1$ $U(r)$ -gauge theory on $\mathbb{R}^4 \times S^1$. In this setting, [4] considered the universal sheaf:

$$\begin{array}{c} \mathcal{U} \\ \vdots \\ \mathcal{M}^f \times \mathbb{A}^2 \end{array}$$

and its exterior powers $\mathcal{U}_1 \otimes \dots \otimes \mathcal{U}_k$ on $\mathcal{M}^f \times \mathbb{A}^{2k}$, where \mathcal{U}_i denotes the pull-back of \mathcal{U} from the i -th factor of $\mathbb{A}^{2k} = (\mathbb{A}^2)^k$. These exterior powers

yield operators:

$$(1.1) \quad K_T(\mathcal{M}^f) \xrightarrow{V} \Lambda_{\mathbb{A}^2} = \bigoplus_{k=0}^{\infty} K_{\mathbb{C}^* \times \mathbb{C}^* \times \mathfrak{S}(k)}(\mathbb{A}^{2k})$$

(the action $T \curvearrowright \mathcal{M}^f$ is explained in Subsection 2.14, the action $\mathbb{C}^* \times \mathbb{C}^* \curvearrowright \mathbb{A}^2$ is the usual scaling, and the symmetric group $\mathfrak{S}(k)$ acts on $\mathbb{A}^{2k} = (\mathbb{A}^2)^k$ by permutations) given by:

$$(1.2) \quad V = \bigoplus_{k=0}^{\infty} \rho_{k*} \left(\mathcal{U}_1 \otimes \dots \otimes \mathcal{U}_k \otimes \pi_k^* \right)$$

The maps in (1.2) are the natural projection maps:

$$(1.3) \quad \begin{array}{ccc} & \mathcal{M}^f \times \mathbb{A}^{2k} & \\ \pi_k \swarrow & & \searrow \rho_k \\ \mathcal{M}^f & & \mathbb{A}^{2k} \end{array}$$

As shown in *loc. cit.*, the operator $W(m) : K_T(\mathcal{M}^f) \rightarrow K_T(\mathcal{M}^f)$ that encodes the contribution of bifundamental matter to the gauge theory at hand factors as:

$$(1.4) \quad W(m) = V^* \cdot m^{\text{deg}} \cdot V$$

(up to a renormalization that will not concern us in the present paper) where $\text{deg} : \Lambda_{\mathbb{A}^2} \rightarrow \Lambda_{\mathbb{A}^2}$ is the operator which scales the k -th direct summand in (1.1) by k .

In [4], from whom we borrowed both the main construction and the title of the present paper, the authors compute the $r = 1$ case of V as follows: the moduli space $\mathcal{M}^f|_{r=1}$ is isomorphic to the Hilbert scheme of points on \mathbb{A}^2 , and its K -theory is naturally identified with $\Lambda_{\mathbb{A}^2}$ (see [2, 13]). Then [4] computes V as an explicit exponential in the usual bosonic realization of $\Lambda_{\mathbb{A}^2}$, times the famous ∇ operator (see [1]). The resulting formula for V yields a geometric incarnation of a combinatorial identity from [9], and implies the formula for Φ_m computed in [5].

In the present paper, we take a somewhat different route toward computing the operator V , for general r . We recall the actions of the elliptic Hall algebra

\mathcal{A} on the domain and target of the map (1.1), which were studied in [7] and [8, 27]:

$$(1.5) \quad \Lambda_{\mathbb{A}^2} \overset{\Psi}{\curvearrowright} \mathcal{A} \overset{\Phi}{\curvearrowright} K_T(\mathcal{M}^f)$$

(we refer to [19] for an overview of our viewpoint on these actions). We will recall the definition of the elliptic Hall algebra \mathcal{A} in Subsection 2.6, and in Subsection 2.8 we will introduce the subalgebra:

$$(1.6) \quad \mathcal{A}^{(r)} \subset \mathcal{A}$$

Intuitively, $\mathcal{A}^{(r)}$ is half of \mathcal{A} with respect to a certain triangular decomposition. Consider the following modification of the diagram (1.3):

$$\begin{array}{ccc} \mathcal{M}^f \times \{\text{origin}\} & \xrightarrow{\iota} & \mathcal{M}^f \times \mathbb{A}^{2k} \\ \parallel & \swarrow \pi_k & \searrow \rho_k \\ \mathcal{M}^f & & \mathbb{A}^{2k} \end{array}$$

from which it is easy to see that:

$$(1.7) \quad \Gamma_{\mathbb{A}^2} = \bigoplus_{k=0}^{\infty} \iota^* (\mathcal{U}_1 \otimes \dots \otimes \mathcal{U}_k \otimes \rho_k^*)^{\mathfrak{S}(k)} : \Lambda_{\mathbb{A}^2} \longrightarrow K_T(\mathcal{M}^f)$$

is given by $\Gamma_{\mathbb{A}^2} = V^* \circ [(1 - q_1)(1 - q_2)]^{\text{deg}}$. Thus, we will compute $\Gamma_{\mathbb{A}^2}$ instead of V .

Theorem 1.1. *For any r , the operator $\Gamma_{\mathbb{A}^2}$ commutes with the $\mathcal{A}^{(r)}$ -action:*

$$(1.8) \quad \begin{array}{ccc} \Lambda_{\mathbb{A}^2} & \xrightarrow{\Gamma_{\mathbb{A}^2}} & K_T(\mathcal{M}^f) \\ \Psi(a) \downarrow & & \downarrow \Phi(a) \\ \Lambda_{\mathbb{A}^2} & \xrightarrow{\Gamma_{\mathbb{A}^2}} & K_T(\mathcal{M}^f) \end{array}$$

for all $a \in \mathcal{A}^{(r)}$. Moreover, after localization to $\text{Frac}(\text{Rep}_T)$, the operator $\Gamma_{\mathbb{A}^2}$ is uniquely determined by the commutativity of diagram (1.8).

The point of view in Theorem 1.1, namely of determining an operator through its commutation with an algebra action rather than through explicit formulas, was used in [21, 24] to compute the operator (1.4). However,

the strength of this approach is that it allows us to generalize from the local situation of the moduli spaces \mathcal{M}^f of framed sheaves on the affine plane to the global case of moduli spaces of stable sheaves \mathcal{M}^s on a general smooth projective surface S . We will review the necessary setup in Section 3 (including Assumptions A and S, subject to which we make all the following claims), but the idea is to consider the operator:

$$(1.9) \quad \Lambda_S = \bigoplus_{k=0}^{\infty} K_{\mathfrak{S}(k)}(S^k) \xrightarrow{\Gamma_S} K(\mathcal{M}^s)$$

explicitly given by:

$$(1.10) \quad \Gamma_S = \bigoplus_{k=0}^{\infty} \pi_{k*} \left(\mathcal{U}_1 \otimes \dots \otimes \mathcal{U}_k \otimes \rho_k^* \right)^{\mathfrak{S}(k)}$$

where the notions in the right-hand side of (1.10) are defined just like their counterparts in (1.2) (Assumption A ensures that there exists a universal sheaf \mathcal{U} on $\mathcal{M}^s \times S$, and we fix such a sheaf throughout the present paper). As for the analogues of the action (1.5) for a general surface, the former was worked out in [25]:

$$(1.11) \quad \mathcal{A} \xrightarrow{\Phi} \text{Hom}(K(\mathcal{M}^s), K(\mathcal{M}^s \times S))$$

and provided a generalization of the classical Heisenberg algebra action on the cohomology groups of Hilbert schemes ([12, 16]). We will also provide an analogue:

$$(1.12) \quad \mathcal{A} \xrightarrow{\Psi} \text{Hom}(\Lambda_S, \Lambda_S \times K(S))$$

of the second action from (1.5), where by a slight abuse of notation, we write:

$$\Lambda_S \times K(S) = \bigoplus_{k=0}^{\infty} K_{\mathfrak{S}(k)}(S^k \times S)$$

(the symmetric group $\mathfrak{S}(k)$ only acts on S^k). Then our main result for a smooth surface S , subject to the assumptions in Subsection 3.1, is the following:

Theorem 1.2. *For any r , the operator (1.10) commutes with the $\mathcal{A}^{(r)}$ -action:*

$$(1.13) \quad \begin{array}{ccc} \Lambda_S & \xrightarrow{\Gamma_S} & K(\mathcal{M}^s) \\ \Psi(a) \downarrow & & \downarrow \Phi(a) \\ \Lambda_S \times K(S) & \xrightarrow{\Gamma_S} & K(\mathcal{M}^s \times S) \end{array}$$

for all $a \in \mathcal{A}^{(r)}$.

As an $\mathcal{A}^{(r)}$ -module, $K_T(\mathcal{M}^f)$ is generated by a single element, namely the fundamental class of the component parametrizing framed sheaves with $c_2 = 0$ (this plays an important role in the uniqueness statement of Theorem 1.1). Meanwhile, we will show in Proposition 3.30 that $K(\mathcal{M}^s)$ has countably many generators, namely the fundamental classes $\mathbf{1}_d$ of the components $\mathcal{M}_d^s \subset \mathcal{M}^s$ parametrizing stable sheaves with $c_2 = d$. In our language, we have $\Gamma_S(1) = \prod_{d \in \mathbb{Z}} \mathbf{1}_d$, where $1 \in K_{\mathfrak{S}(0)}(S^0) \cong \mathbb{Q}$.

2. The case of the affine plane

2.1.

Even before dealing with \mathbb{A}^2 , let us discuss the situation of $\mathfrak{S}(k)$ -equivariant coherent sheaves on a point \circ , which is just another word for finite-dimensional $\mathfrak{S}(k)$ -modules. We have the well-known Frobenius character isomorphism:

$$(2.1) \quad K_{\mathfrak{S}(k)}(\circ) \cong \left\{ \text{degree } k \text{ symmetric polynomials in } x_1, x_2, \dots \right\}$$

given as a sum over partitions $\lambda = (\lambda_1 \geq \dots \geq \lambda_t)$ of size k by the formula:

$$(2.2) \quad M \mapsto \sum_{|\lambda|=k} \frac{p_\lambda}{z_\lambda} \cdot \text{Tr}_M(\omega_\lambda)$$

where we let $\omega_\lambda \in \mathfrak{S}(k)$ be any permutation of cycle type λ , and define:

$$(2.3) \quad p_\lambda = p_{\lambda_1} \cdots p_{\lambda_t}$$

with $p_n = x_1^n + x_2^n + \dots$ being the power sum functions. In (2.2), we let $z_\lambda = \lambda! \prod_i \lambda_i$, where $\lambda!$ is the product of factorials of the number of times each integer appears in λ . It is useful to take the direct sum of (2.1) over all $k \in \mathbb{N} \sqcup 0$, and obtain:

$$(2.4) \quad \Lambda := \bigoplus_{k=0}^{\infty} K_{\mathfrak{S}(k)}(\circ) \cong \left\{ \text{symmetric polynomials in } x_1, x_2, \dots \right\}$$

This is beneficial because Λ is manifestly a commutative ring, in fact the polynomial ring generated by p_1, p_2, \dots . In terms of representations of $\mathfrak{S}(k)$, the operation of multiplication by power sum functions corresponds to parabolic induction:

$$(2.5) \quad K_{\mathfrak{S}(l)}(\circ) \xrightarrow{p_k} K_{\mathfrak{S}(k+l)}(\circ), \quad M \mapsto \text{Ind}_{\mathfrak{S}(k) \times \mathfrak{S}(l)}^{\mathfrak{S}(k+l)}(p_k \boxtimes M)$$

However, the ring Λ is also endowed with a symmetric pairing, determined by $\langle p_\lambda, p_\mu \rangle = \delta_\lambda^\mu z_\lambda$, or, in the language of finite-dimensional $\mathfrak{S}(k)$ -modules:

$$(2.6) \quad \langle M, M' \rangle = \dim \text{Hom}_{\mathfrak{S}(k)}(M, M')$$

With this in mind, Frobenius reciprocity states that the adjoints of the operators (2.5) are the parabolic restriction operators:

$$(2.7) \quad K_{\mathfrak{S}(k+l)}(\circ) \xrightarrow{p_k^\dagger} K_{\mathfrak{S}(l)}(\circ), \quad M \mapsto \text{Hom}_{\mathfrak{S}(k)} \left(p_k, \text{Res}_{\mathfrak{S}(k) \times \mathfrak{S}(l)}^{\mathfrak{S}(k+l)}(M) \right)$$

A reformulation of the main result of [10] is the following:

Theorem 2.2. *The operators $p_k, p_k^\dagger : \Lambda \rightarrow \Lambda$ satisfy the relations:*

$$[p_k^\dagger, p_l] = k \delta_k^l \cdot \text{Id}$$

for all $k, l \in \mathbb{N}$, as well as the obvious relations $[p_k, p_l] = [p_k^\dagger, p_l^\dagger] = 0$.

2.3.

We will follow the presentation of [4] in the present Subsection, and we will recycle the notation used in the previous Subsection. We will consider \mathbb{A}^2 with the standard action of $\mathbb{C}^* \times \mathbb{C}^*$ that dilates the coordinate axes,

and then the induced action of $\mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{A}^{2k} = (\mathbb{A}^2)^k$ commutes with the action of $\mathfrak{S}(k)$ that permutes the factors. Then we will consider the analogue of (2.4):

$$\Lambda_{\mathbb{A}^2} = \bigoplus_{k=0}^{\infty} K_{\mathbb{C}^* \times \mathbb{C}^* \times \mathfrak{S}(k)}(\mathbb{A}^{2k})$$

If we let q_1 and q_2 denote the elementary characters of $\mathbb{C}^* \times \mathbb{C}^*$, then the inclusion of the origin $\circ \hookrightarrow \mathbb{A}^2$ induces a map:

$$\Lambda \otimes_{\mathbb{Q}} \mathbb{Q}[q_1^{\pm 1}, q_2^{\pm 1}] \rightarrow \Lambda_{\mathbb{A}^2}$$

which sends a finite-dimensional $\mathfrak{S}(k)$ -module to the same module supported at the origin $\circ^k \hookrightarrow \mathbb{A}^{2k}$. With this in mind, we may consider the following elements:

$$[p_k \otimes \mathcal{O}_{\circ^k}] \in K_{\mathbb{C}^* \times \mathbb{C}^* \times \mathfrak{S}(k)}(\mathbb{A}^{2k})$$

(the skyscraper sheaf at the origin tensored with the $\mathfrak{S}(k)$ -character p_k) which induce the following analogues of the operators (2.5) and (2.7):

$$(2.8) \quad K_{\mathbb{C}^* \times \mathbb{C}^* \times \mathfrak{S}(l)}(\mathbb{A}^{2l}) \xrightleftharpoons[p_k^\dagger]{p_k} K_{\mathbb{C}^* \times \mathbb{C}^* \times \mathfrak{S}(k+l)}(\mathbb{A}^{2(k+l)})$$

given by:

$$(2.9) \quad p_k(M) = \text{Ind}_{\mathfrak{S}(k) \times \mathfrak{S}(l)}^{\mathfrak{S}(k+l)} \left(\underbrace{[p_k \otimes \mathcal{O}_{\circ^k}]}_{\text{sheaf on } \mathbb{A}^{2k}} \boxtimes \underbrace{M}_{\text{sheaf on } \mathbb{A}^{2l}} \right)$$

$$(2.10) \quad p_k^\dagger(M) = \text{Hom}_{\mathfrak{S}(k)} \left(p_k, \text{Res}_{\mathfrak{S}(k) \times \mathfrak{S}(l-k)}^{\mathfrak{S}(l)}(M) \Big|_{\circ^k \times \mathbb{A}^l} \right)$$

It is easy to see that the operators (2.9) and (2.10) are adjoint with respect to the pairing on $\Lambda_{\mathbb{A}^2}$ given by formula (2.6), with the caveat that the symbol “dim Hom” must be understood to mean the $\mathbb{C}^* \times \mathbb{C}^*$ character of the space of $\mathfrak{S}(k)$ -equivariant global homomorphisms over \mathbb{A}^{2k} . With respect to this pairing, we have:

$$(2.11) \quad \langle p_\lambda, p_\mu \rangle_{\mathbb{A}^2} = \delta_\lambda^\mu z_\lambda \prod_{i=1}^t [(1 - q_1^{\lambda_i})(1 - q_2^{\lambda_i})]$$

for any $\lambda = (\lambda_1 \geq \dots \geq \lambda_t)$. The natural analogue of Theorem 2.2 reads:

Proposition 2.4. *The operators $p_k, p_k^\dagger : \Lambda_{\mathbb{A}^2} \rightarrow \Lambda_{\mathbb{A}^2}$ satisfy the relations:*

$$(2.12) \quad [p_k^\dagger, p_l] = k\delta_k^l(1 - q_1^k)(1 - q_2^k) \cdot \text{Id}$$

for all $k, l \in \mathbb{N}$, as well as the obvious relations $[p_k, p_l] = [p_k^\dagger, p_l^\dagger] = 0$.

Proof. See [4], although the proof is analogous to that of Proposition 3.14. □

2.5.

Two very important classes of symmetric polynomials are the elementary and complete ones, whose generating series are given by:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{h_k}{z^k} &= \exp \left[\sum_{k=1}^{\infty} \frac{p_k}{kz^k} \right] = \prod_{i=1}^{\infty} \left(1 - \frac{x_i}{z} \right)^{-1} \\ \sum_{k=0}^{\infty} \frac{e_k}{(-z)^k} &= \exp \left[- \sum_{k=1}^{\infty} \frac{p_k}{kz^k} \right] = \prod_{i=1}^{\infty} \left(1 - \frac{x_i}{z} \right) \end{aligned}$$

As elements of Λ and $\Lambda_{\mathbb{A}^2}$ (i.e. as $\mathfrak{S}(k)$ -modules or $\mathfrak{S}(k)$ -modules supported at the origin of \mathbb{A}^{2k} , respectively), the symmetric polynomials h_k and e_k correspond to the trivial and sign one-dimensional representation, respectively. Let:

$$h_k^\dagger, e_k^\dagger : \Lambda_{\mathbb{A}^2} \rightarrow \Lambda_{\mathbb{A}^2}$$

denote the adjoints, with respect to the pairing (2.11), of the operators of multiplication by h_k and e_k (respectively). Clearly, we have:

$$\begin{aligned} \sum_{k=0}^{\infty} h_k^\dagger z^k &= \exp \left[\sum_{k=1}^{\infty} \frac{p_k^\dagger z^k}{k} \right] \\ \sum_{k=0}^{\infty} e_k^\dagger (-z)^k &= \exp \left[- \sum_{k=1}^{\infty} \frac{p_k^\dagger z^k}{k} \right] \end{aligned}$$

The following computations are simple consequences of (2.12):

$$\begin{aligned}
 (2.13) \quad & \exp \left[- \sum_{k=1}^{\infty} \frac{p_k^\dagger z^k}{k} \right] \exp \left[\sum_{k=1}^{\infty} \frac{p_k w^{-k}}{k} \right] \\
 & = \exp \left[\sum_{k=1}^{\infty} \frac{p_k w^{-k}}{k} \right] \exp \left[- \sum_{k=1}^{\infty} \frac{p_k^\dagger z^k}{k} \right] \zeta \left(\frac{z}{w} \right)^{-1}
 \end{aligned}$$

$$\begin{aligned}
 (2.14) \quad & \exp \left[\sum_{k=1}^{\infty} \frac{p_k^\dagger z^k}{k} \right] \exp \left[- \sum_{k=1}^{\infty} \frac{p_k w^{-k}}{k q^k} \right] \\
 & = \exp \left[- \sum_{k=1}^{\infty} \frac{p_k w^{-k}}{k q^k} \right] \exp \left[\sum_{k=1}^{\infty} \frac{p_k^\dagger z^k}{k} \right] \zeta \left(\frac{w}{z} \right)^{-1}
 \end{aligned}$$

where we let $q = q_1 q_2$ and write:

$$(2.15) \quad \zeta(x) = \frac{(1 - x q_1)(1 - x q_2)}{(1 - x)(1 - x q)} = \exp \left[\sum_{k=1}^{\infty} \frac{x^k}{k} \cdot (1 - q_1^k)(1 - q_2^k) \right]$$

Note that $\zeta(x) = \zeta\left(\frac{1}{xq}\right)$.

2.6.

Consider the following half planes in \mathbb{Z}^2 :

$$\mathbb{Z}_+^2 = \{(n, m) \in \mathbb{Z}^2 \text{ s.t. } n > 0 \text{ or } n = 0, m > 0\}$$

$$\mathbb{Z}_-^2 = \{(n, m) \in \mathbb{Z}^2 \text{ s.t. } n < 0 \text{ or } n = 0, m < 0\}$$

The following is a model for the Hall algebra of the category of coherent sheaves over an elliptic curve, as defined in [3] (although we follow the normalization of [21]).

Definition 2.7. *Consider the algebra:*

$$\mathcal{A}_{\text{loc}} = \mathbb{Q}(q_1, q_2) \left\langle P_{n,m}, c_1^{\pm 1}, c_2^{\pm 1} \right\rangle_{(n,m) \in \mathbb{Z}^2 \setminus (0,0)} \Big/_{\substack{c_1, c_2 \text{ central, and} \\ \text{relations (2.16), (2.17)}}}$$

where we impose the following relations. The first of these is:

$$(2.16) \quad [P_{n,m}, P_{n',m'}] = \delta_{n+n'}^0 \frac{d(1 - q_1^d)(1 - q_2^d)}{q^{-d} - 1} (1 - c_1^{-n} c_2^{-m})$$

if $nm' = n'm$ and $(n, m) \in \mathbb{Z}_+^2$, with $d = \gcd(m, n)$. The second relation states that whenever $nm' > n'm$ and the triangle with vertices $(0, 0)$, (n, m) , $(n + n', m + m')$ contains no lattice points inside nor on one of the edges, then we have the relation:

$$(2.17) \quad [P_{n,m}, P_{n',m'}] = \frac{(1 - q_1^d)(1 - q_2^d)}{q^{-1} - 1} Q_{n+n', m+m'}$$

$$\cdot \begin{cases} c_1^n c_2^m & \text{if } (n, m) \in \mathbb{Z}_+^2, (n', m') \in \mathbb{Z}_+^2, (n + n', m + m') \in \mathbb{Z}_+^2 \\ c_1^{-n'} c_2^{-m'} & \text{if } (n, m) \in \mathbb{Z}_-^2, (n', m') \in \mathbb{Z}_+^2, (n + n', m + m') \in \mathbb{Z}_-^2 \\ 1 & \text{otherwise} \end{cases}$$

where $d = \gcd(n, m) \gcd(n', m')$ (by the assumption on the triangle, we note that at most one of the pairs (n, m) , (n', m') , $(n + n', m + m')$ can fail to be coprime), and:

$$(2.18) \quad \sum_{k=0}^{\infty} Q_{ka, kb} \cdot x^k = \exp \left[\sum_{k=1}^{\infty} \frac{P_{ka, kb}}{k} \cdot x^k (1 - q^{-k}) \right]$$

for all coprime integers a, b . Note that $Q_{0,0} = 1$.

2.8.

Let us consider $H_{n,m} \in \mathcal{A}_{\text{loc}}$ defined for all coprime integers a, b by:

$$(2.19) \quad \sum_{k=0}^{\infty} H_{ka, kb} \cdot x^k = \exp \left[\sum_{k=1}^{\infty} \frac{P_{ka, kb}}{k} \cdot x^k \right]$$

In other words, for every fixed pair of coprime integers a, b , the elements $H_{ka, kb}$ will be to complete symmetric functions as the elements $P_{ka, kb}$ are to power-sum functions. In the present paper, we will work with the subalgebra:

$$(2.20) \quad \mathcal{A}_{\text{loc}} \supset \mathcal{A} = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}] \left\langle H_{n,m}, c_1^{\pm 1}, c_2^{\pm 1} \right\rangle_{(n,m) \in \mathbb{Z}^2 \setminus (0,0)}$$

We note a slight abuse in (2.20): the notation implies that the structure constants of products of $H_{n,m}$'s lie in $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$, but this is not quite true.

The reason is the presence of denominators in (2.16) and (2.17). However, in (2.17), the denominator is canceled by $Q_{n,m}$, which is by definition a multiple of $1 - q^{-1}$. In (2.16), the denominator will be canceled by the numerator in all representations in which:

$$(c_1, c_2) \mapsto (q^r, 1) \text{ or } (1, q^{-1})$$

which will be the case throughout the present paper. However, for all $r \in \mathbb{N}$, the following subalgebra of \mathcal{A} is unambiguously well-defined over $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$:

$$\mathcal{A}^{(r)} = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}] \left\langle H_{n,m} \right\rangle_{m > -nr}$$

The subalgebra $\mathcal{A}^{(r)}$ is half of \mathcal{A} with respect to a triangular decomposition.

2.9.

The following is obtained by combining the action of [7] with the explicit formulas obtained in [18] (see Theorem 2.15 of [20] for the explicit formulas, although the normalization of *loc. cit.* is somewhat different from that of the present paper).

Theorem 2.10. *There is an action $\mathcal{A} \curvearrowright^{\Psi} \Lambda_{\mathbb{A}^2}$ given by:*

$$(2.21) \quad c_1 \mapsto 1, \quad c_2 \mapsto q^{-1},$$

$$(2.22) \quad P_{0,m} \mapsto p_m, \quad P_{0,-m} \mapsto -q^m \cdot p_m^\dagger$$

while for all $n > 0$ and $m \in \mathbb{Z}$, we have:

$$(2.23) \quad H_{n,m} \mapsto \int_{|z_1| \gg \dots \gg |z_n|} \frac{\prod_{i=1}^n z_i^{\lfloor \frac{mi}{n} \rfloor - \lfloor \frac{m(i-1)}{n} \rfloor}}{\prod_{i=1}^{n-1} \left(1 - \frac{z_{i+1}q}{z_i}\right) \prod_{i < j} \zeta\left(\frac{z_j}{z_i}\right)}$$

$$\exp \left[\sum_{k=1}^{\infty} \frac{z_1^{-k} + \dots + z_n^{-k}}{k} \cdot p_k \right] \exp \left[- \sum_{k=1}^{\infty} \frac{z_1^k + \dots + z_n^k}{k} \cdot p_k^\dagger \right] \prod_{a=1}^n \frac{dz_a}{2\pi i z_a}$$

and:

$$(2.24) \quad H_{-n,m} \mapsto \int_{|z_1| \ll \dots \ll |z_n|} \frac{(-q)^n \prod_{i=1}^n z_i^{\lfloor \frac{mi}{n} \rfloor - \lfloor \frac{m(i-1)}{n} \rfloor}}{\prod_{i=1}^{n-1} \left(1 - \frac{z_{i+1}q}{z_i}\right) \prod_{i < j} \zeta\left(\frac{z_j}{z_i}\right)}$$

$$\exp \left[- \sum_{k=1}^{\infty} \frac{z_1^{-k} + \dots + z_n^{-k}}{k \cdot q^k} \cdot p_k \right] \exp \left[\sum_{k=1}^{\infty} \frac{z_1^k + \dots + z_n^k}{k} \cdot p_k^\dagger \right] \prod_{a=1}^n \frac{dz_a}{2\pi i z_a}$$

The integrals go over concentric circles, contained inside each other in the order depicted in the subscript of each integral, and far away from each other relative to the size of the parameters q_1 and q_2 (which are assumed to be complex numbers).

Proof. We will sketch the proof, in order to prepare for the analogous argument in Theorem 3.20. There is a well-known triangular decomposition:

$$\mathcal{A} = \mathcal{A}^+ \otimes \mathcal{A}^0 \otimes \mathcal{A}^-$$

where \mathcal{A}^\pm are the subalgebras of \mathcal{A} generated by $H_{\pm n,m}$ for $(n,m) \in \mathbb{N} \times \mathbb{Z}$, and \mathcal{A}^0 is generated by $P_{0,\pm m}$ and the central elements c_1, c_2 . The main result of [26] implies that, in order to show that formulas (2.21)–(2.24) yield an action $\mathcal{A} \curvearrowright \Lambda_{\mathbb{A}^2}$, one needs to prove the following two things:

- Formulas (2.23) and (2.24) induce actions of the subalgebras \mathcal{A}^+ and \mathcal{A}^- on $\Lambda_{\mathbb{A}^2}$.
- The particular cases of (2.16) and (2.17) when $n, n' \in \{-1, 0, 1\}$ hold, i.e.:

$$(2.25) \quad \left[\Psi(P_{0,\pm m}), \Psi(P_{0,\pm m'}) \right] = 0$$

$$(2.26) \quad \left[\Psi(P_{0,m}), \Psi(P_{0,-m'}) \right] = \delta_m^m m (1 - q_1^m) (1 - q_2^m) q^m$$

$$(2.27) \quad \left[\Psi(H_{\pm 1,k}), \Psi(P_{0,\pm m}) \right] = -(1 - q_1^m) (1 - q_2^m) \cdot \Psi(H_{\pm 1,k \pm m})$$

$$(2.28) \quad \left[\Psi(H_{\pm 1,k}), \Psi(P_{0,\mp m}) \right] = (1 - q_1^m) (1 - q_2^m) q^{m\delta_\pm^\pm} \cdot \Psi(H_{\pm 1,k \mp m})$$

(2.29)

$$\left[\Psi(H_{1,k}), \Psi(H_{-1,k'}) \right] = \frac{(1 - q_1)(1 - q_2)}{q^{-1} - 1} \begin{cases} \Psi(A_{k+k'}) & \text{if } k + k' > 0 \\ 1 - q^k & \text{if } k + k' = 0 \\ -q^k \Psi(B_{-k-k'}) & \text{if } k + k' < 0 \end{cases}$$

for all $m, m' \in \mathbb{N}$ and $k, k' \in \mathbb{Z}$, where in the last expression, we write:

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{A_m}{x^m} &= \exp \left[\sum_{m=1}^{\infty} \frac{p_m}{m x^m} (1 - q^{-m}) \right] \\ \sum_{m=0}^{\infty} \frac{B_m}{x^m} &= \exp \left[\sum_{m=1}^{\infty} \frac{p_m^\dagger}{m x^m} (1 - q^m) \right] \end{aligned}$$

The second bullet is a consequence of straightforward computations using Proposition 2.4, which we leave as exercises to the interested reader. As for the first bullet, we note that formula (2.24) reads:

$$(2.30) \quad \Psi(H_{-n,m}) = \int_{|z_1| \ll \dots \ll |z_n|} r_{n,m}(z_1, \dots, z_n) X(z_1, \dots, z_n)$$

where $r_{n,m}(z_1, \dots, z_n)$ (resp. $X(z_1, \dots, z_n)$) is the rational function (resp. the expression) in z_1, \dots, z_n on the first (resp. second) line of (2.24). If we assume q_1 and q_2 to be complex numbers with absolute value greater than 1, then one can move the contours in the integral of (2.30) to $|z_1| = \dots = |z_n|$, without picking up any new poles. Once one does this, because $X(z_1, \dots, z_n)$ is symmetric in z_1, \dots, z_n , then replacing $r_{n,m}$ with its symmetrization only changes the value of the integral by an overall factor of $n!$. Explicitly, this means that (2.30) is equivalent to:

$$(2.31) \quad \Psi(H_{-n,m}) = \frac{1}{n!} \int_{|z_1| = \dots = |z_n|} R_{n,m}(z_1, \dots, z_n) X(z_1, \dots, z_n)$$

where $R_{n,m} = \text{Sym } r_{n,m}$. An elementary application of (2.14) shows that:

$$X(z_1, \dots, z_n) X(z_{n+1}, \dots, z_{n+n'}) = X(z_1, \dots, z_{n+n'}) \prod_{\substack{1 \leq i \leq n \\ n+1 \leq j \leq n+n'}} \zeta \left(\frac{z_j}{z_i} \right)^{-1}$$

Therefore, by applying (2.31) twice, we obtain:

$$(2.32) \quad \Psi(H_{-n,m})\Psi(H_{-n',m'}) = \frac{1}{(n+n)!} \int_{|z_1|=\dots=|z_{n+n'}|} (R_{n,m} * R_{n',m'})(z_1, \dots, z_{n+n'}) X(z_1, \dots, z_{n+n'})$$

where $R_{n,m} * R_{n',m'}$ denotes the rational function in $z_1, \dots, z_{n+n'}$ given by:

$$\frac{1}{n!n'} \cdot \text{Sym} \left[R_{n,m}(z_1, \dots, z_n) R_{n',m'}(z_{n+1}, \dots, z_{n+n'}) \prod_{\substack{1 \leq i \leq n \\ n+1 \leq j \leq n+n'}} \zeta \left(\frac{z_i}{z_j} \right) \right]$$

The operation $*$ gives rise to an associative product on the vector space \mathcal{S} of symmetric rational functions with certain poles ([7]), called the shuffle product. It was shown in [18] that the operation:

$$(\mathcal{A}^-, \cdot) \rightarrow (\mathcal{S}, *) \quad H_{-n,m} \mapsto R_{n,m}$$

induces an algebra homomorphism. As we have seen by comparing formulas (2.31) and (2.32), the operation:

$$(\mathcal{S}, *) \rightarrow (\text{End}(\Lambda_{\mathbb{A}^2}), \cdot) \quad R_{n,m} \mapsto \text{RHS of (2.31)}$$

is also an algebra homomorphism. Composing the aforementioned homomorphisms implies that formulas (2.31) give a well-defined action of \mathcal{A}^- on $\Lambda_{\mathbb{A}^2}$. The fact that formulas (2.23) give rise to a well-defined action of \mathcal{A}^+ on $\Lambda_{\mathbb{A}^2}$ is proved analogously. \square

2.11.

We will use the symbol X to refer to the totality of the variables x_1, x_2, \dots , and thus we will denote the complete and elementary symmetric functions by:

$$(2.33) \quad \sum_{k=0}^{\infty} \frac{h_k}{z^k} = \wedge^\bullet \left(-\frac{X}{z} \right)$$

$$(2.34) \quad \sum_{k=0}^{\infty} \frac{e_k}{(-z)^k} = \wedge^\bullet \left(\frac{X}{z} \right)$$

where \wedge^\bullet is a multiplicative symbol determined by the property that if a vector space V has torus character χ , then $\wedge^\bullet(\chi)$ denotes the torus character

of the total exterior power $\wedge^\bullet(V)$. Elements of $\Lambda_{\mathbb{A}^2}$ will generally be denoted by $f[X]$. We will adopt plethystic notation, according to which one defines:

$$(2.35) \quad f[X \pm (1 - q_1)(1 - q_2)z] \in \Lambda_{\mathbb{A}^2}[z]$$

to be the image of $f[X]$ under the ring homomorphism $\Lambda_{\mathbb{A}^2} \rightarrow \Lambda_{\mathbb{A}^2}[z]$ that sends:

$$(2.36) \quad p_n \mapsto p_n \pm (1 - q_1^n)(1 - q_2^n)z^n$$

In other words, one computes the plethysm (2.35) by expanding $f[X]$ in the basis (2.3), and then replacing each p_n therein according to (2.36). The reader may find a description of plethysm in the language of equivariant K -theory in Proposition 3.27. The following is a well-known and straightforward exercise:

Proposition 2.12. *For any $f[X] \in \Lambda_{\mathbb{A}^2}$ and any variable z , we have:*

$$(2.37) \quad f[X \pm (1 - q_1)(1 - q_2)z] = \exp \left[\pm \sum_{k=1}^{\infty} \frac{p_k^\dagger z^k}{k} \right] \cdot f[X]$$

where p_k^\dagger is the adjoint operator defined in Subsection 2.3.

2.13.

Using (2.33), (2.34) and (2.37), formulas (2.23)–(2.24) take the form:

$$(2.38) \quad \Psi(H_{n,m})(f[X]) = \int_{0, X \prec |z_n| \prec \dots \prec |z_1| \prec \infty} \frac{\prod_{i=1}^n z_i^{\lfloor \frac{mi}{n} \rfloor - \lfloor \frac{m(i-1)}{n} \rfloor}}{\prod_{i=1}^{n-1} \left(1 - \frac{z_{i+1}q}{z_i}\right) \prod_{i < j} \zeta \left(\frac{z_j}{z_i}\right)} \wedge^\bullet \left(-\frac{X}{z_1}\right) \dots \wedge^\bullet \left(-\frac{X}{z_n}\right) \cdot f \left[X - (1 - q_1)(1 - q_2) \sum_{i=1}^n z_i \right] \prod_{a=1}^n \frac{dz_a}{2\pi i z_a}$$

and:

$$\Psi(H_{-n,m})(f[X]) = \int_{0, X \prec |z_1| \prec \dots \prec |z_n| \prec \infty} \frac{(-q)^n \prod_{i=1}^n z_i^{\lfloor \frac{mi}{n} \rfloor - \lfloor \frac{m(i-1)}{n} \rfloor}}{\prod_{i=1}^{n-1} \left(1 - \frac{z_{i+1}q}{z_i}\right) \prod_{i < j} \zeta \left(\frac{z_j}{z_i}\right)}$$

$$(2.39) \quad \wedge^\bullet \left(\frac{X}{z_1 q} \right) \dots \wedge^\bullet \left(\frac{X}{z_n q} \right) \cdot f \left[X + (1 - q_1)(1 - q_2) \sum_{i=1}^n z_i \right] \prod_{a=1}^n \frac{dz_a}{2\pi i z_a}$$

Above, the notation $0, X \prec |z_n| \prec \dots \prec |z_1| \prec \infty$ means that we integrate the variables z_1, \dots, z_n over concentric circles that go in the prescribed order, and are contained between the poles at $0, x_1, x_2, \dots$ and the pole at ∞ . Indeed, the variables z_i must have absolute value larger than the variables x_1, x_2, \dots , in order for us to be able to replace the symbols p_k in (2.23)–(2.24) by $x_1^k + x_2^k + \dots$.

2.14.

We will work over an algebraically closed field of characteristic 0, henceforth denoted by \mathbb{C} . Fix a line $\infty \subset \mathbb{P}^2$, and let us write $\mathbb{A}^2 = \mathbb{P}^2 \setminus \infty$ for the complement.

Definition 2.15. Fix $r \in \mathbb{N}$. For any $d \geq 0$, consider the moduli space:

$$(2.40) \quad \mathcal{M}_d^f = \left\{ (\mathcal{F}, \phi), \mathcal{F} \text{ a torsion free sheaf on } \mathbb{P}^2, \mathcal{F}|_\infty \cong \mathcal{O}_\infty^{\oplus r}, c_2(\mathcal{F}) = d \right\}$$

It is a smooth quasiprojective algebraic variety of dimension $2rd$.

An isomorphism ϕ as in (2.40) is called a framing of the torsion-free sheaf \mathcal{F} , and the pair (\mathcal{F}, ϕ) is called a framed sheaf. We will write:

$$\mathcal{M}^f = \bigsqcup_{d=0}^\infty \mathcal{M}_d^f$$

(the rank r of our sheaves will be fixed throughout the present paper). The torus:

$$T = \mathbb{C}^* \times \mathbb{C}^* \times (\mathbb{C}^*)^r$$

acts on \mathcal{M}^f as follows: the first two factors $\mathbb{C}^* \times \mathbb{C}^*$ act on sheaves by their underlying action on the standard coordinate directions of \mathbb{A}^2 , while $(\mathbb{C}^*)^r$ acts by multiplication on the isomorphism ϕ in (2.40). Therefore, we may

consider:

$$(2.41) \quad K_T(\mathcal{M}^f) = \prod_{d=0}^{\infty} K_T(\mathcal{M}_d^f)$$

Let $\circ \in \mathbb{A}^2$ denote the origin, and let us consider the derived restriction:

$$\begin{array}{c} \mathcal{U}_\circ \\ \vdots \\ \downarrow \\ \mathcal{M}^f \end{array}$$

of the universal sheaf \mathcal{U} on $\mathcal{M}^f \times \mathbb{A}^2$ to $\mathcal{M}^f \times \{\circ\} \cong \mathcal{M}^f$.

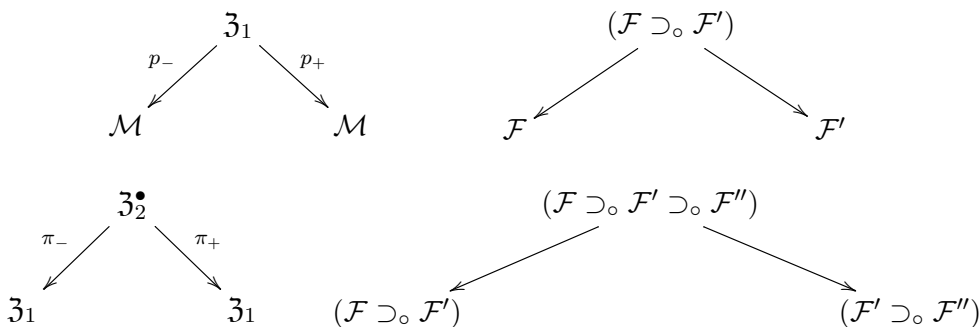
2.16.

We will now define certain operators on $K_T(\mathcal{M})$, which were shown in [19] to give rise to the elliptic Hall algebra action that was discovered earlier in [8, 27].

Definition 2.17. *The following moduli spaces are smooth quasiprojective varieties:*

$$\begin{aligned} \mathfrak{Z}_1 &= \{(\mathcal{F} \supset_\circ \mathcal{F}')\} \\ \mathfrak{Z}_2^\bullet &= \{(\mathcal{F} \supset_\circ \mathcal{F}' \supset_\circ \mathcal{F}'')\} \end{aligned}$$

where $\mathcal{F} \supset_\circ \mathcal{F}'$ means that $\mathcal{F} \supset \mathcal{F}'$ (as framed sheaves) and the quotient \mathcal{F}/\mathcal{F}' is isomorphic to the length 1 coherent sheaf supported at $\circ \in \mathbb{A}^2$. Consider the maps:



and the line bundles:

$$\begin{array}{ccc}
 \mathcal{L} & & \mathcal{F}_\circ/\mathcal{F}'_\circ \\
 \vdots & & \vdots \\
 \mathfrak{Z}_1 & & (\mathcal{F} \supset_\circ \mathcal{F}') \\
 \\
 \mathcal{L}_1, \mathcal{L}_2 & & \mathcal{F}'_\circ/\mathcal{F}''_\circ, \mathcal{F}_\circ/\mathcal{F}'_\circ \\
 \vdots & & \vdots \\
 \mathfrak{Z}_2^\bullet & & (\mathcal{F} \supset_\circ \mathcal{F}' \supset_\circ \mathcal{F}'')
 \end{array}$$

The smoothness of these moduli spaces is proved by analogy with the corresponding statements in Definition 3.10. However, all we need at the moment is the structure of \mathfrak{Z}_1 and \mathfrak{Z}_2^\bullet as dg schemes, which was developed in [19]. The following is the main result of *loc. cit.* (see also [25] for notation closer to ours):

Theorem 2.18. *There exists an action $\mathcal{A} \overset{\Phi}{\curvearrowright} K_T(\mathcal{M}^f)$ given by:*

$$(2.42) \quad c_1 \mapsto q^r, \quad c_2 \mapsto 1,$$

$$(2.43) \quad P_{0,m} \mapsto \text{tensoring with } p_m(\mathcal{U}_\circ)$$

$$(2.44) \quad P_{0,-m} \mapsto \text{tensoring with } -q^m \cdot p_m(\mathcal{U}_\circ^\vee)$$

¹ while for all $n > 0$ and $m \in \mathbb{Z}$, we have:

$$(2.45) \quad H_{n,m} \mapsto p_{-*} \left[\mathcal{L}^{d_n} \otimes \pi_{-*} \pi_+^* \left[\mathcal{L}^{d_{n-1}} \otimes \dots \otimes \pi_{-*} \pi_+^* \left[\mathcal{L}^{d_1} \otimes p_+^* \right] \dots \right] \right]$$

and:

$$(2.46) \quad H_{-n,m} \mapsto \left[\frac{\det \mathcal{U}_\circ}{(-q)^{r-1}} \right]^n \otimes p_{+*} \left[\mathcal{L}^{d_{1-r}} \otimes \dots \otimes \pi_{+*} \pi_-^* \left[\mathcal{L}^{d_{n-r}} \otimes p_-^* \right] \dots \right]$$

where $d_i = \lfloor \frac{mi}{n} \rfloor - \lfloor \frac{m(i-1)}{n} \rfloor$.

¹Above, $p_m(\mathcal{U})$ means the m -th power sum functor: if $\mathcal{U}_\circ = \sum_i \pm y_i \in K_T(\mathcal{M}^f)$, then:

$$p_m(\mathcal{U}_\circ) = \sum_i \pm y_i^m \in K_T(\mathcal{M}^f)$$

2.19.

Recall the decomposition (2.41), and consider the class of the structure sheaf:

$$(2.47) \quad \mathbf{1}_d \in K_T(\mathcal{M}_d^f)$$

For any symmetric polynomial $f[X] \in \Lambda_{\mathbb{A}^2}$, we consider the so-called universal class:

$$(2.48) \quad f[\mathcal{U}_o] \in K_T(\mathcal{M}_d^f)$$

by applying the symmetric polynomial f to the Chern roots of the universal sheaf \mathcal{U}_o on \mathcal{M}_d^f . It is well-known that $K_T(\mathcal{M}_d^f)$ is spanned by universal classes for every $d \geq 0$, i.e. by (2.48) as $f[X]$ ranges over $\Lambda_{\mathbb{A}^2}$ (this fact holds for all Nakajima quiver varieties, of which \mathcal{M}_d^f is an example). Then formulas (2.43) imply that $K_T(\mathcal{M}_d^f)$ is generated by the operators $P_{0,1}, P_{0,2}, \dots$ acting on the class (2.47), for every $d \geq 0$. This also happens in the case of general surfaces, as we will see in Section 3.

Proposition 2.20. ([23]) *In terms of universal classes, (2.45)–(2.46) read:*

$$(2.49) \quad \begin{aligned} \Phi(H_{n,m})(f[\mathcal{U}_o]) &= \int_{\mathcal{U}_o \prec |z_n| \prec \dots \prec |z_1| \prec 0, \infty} \frac{\prod_{i=1}^n z_i^{\lfloor \frac{mi}{n} \rfloor - \lfloor \frac{m(i-1)}{n} \rfloor}}{\prod_{i=1}^{n-1} \left(1 - \frac{z_{i+1}q}{z_i}\right) \prod_{i < j} \zeta\left(\frac{z_j}{z_i}\right)} \\ &\wedge^\bullet \left(-\frac{\mathcal{U}_o}{z_1}\right) \dots \wedge^\bullet \left(-\frac{\mathcal{U}_o}{z_n}\right) \otimes f \left[\mathcal{U}_o - (1 - q_1)(1 - q_2) \sum_{i=1}^n z_i \right] \prod_{a=1}^n \frac{dz_a}{2\pi i z_a} \end{aligned}$$

and:

$$(2.50) \quad \begin{aligned} \Phi(H_{-n,m})(f[\mathcal{U}_o]) &= \int_{\mathcal{U}_o \prec |z_1| \prec \dots \prec |z_n| \prec 0, \infty} \frac{(-q)^n \prod_{i=1}^n z_i^{\lfloor \frac{mi}{n} \rfloor - \lfloor \frac{m(i-1)}{n} \rfloor}}{\prod_{i=1}^{n-1} \left(1 - \frac{z_{i+1}q}{z_i}\right) \prod_{i < j} \zeta\left(\frac{z_j}{z_i}\right)} \\ &\wedge^\bullet \left(\frac{\mathcal{U}_o}{z_1 q}\right) \dots \wedge^\bullet \left(\frac{\mathcal{U}_o}{z_n q}\right) \otimes f \left[\mathcal{U}_o + (1 - q_1)(1 - q_2) \sum_{i=1}^n z_i \right] \prod_{a=1}^n \frac{dz_a}{2\pi i z_a} \end{aligned}$$

where $\wedge^\bullet \left(\frac{\mathcal{U}_o}{z}\right) = \sum_{i=0}^\infty (-z)^{-i} [\wedge^i(\mathcal{U}_o)]$.

Recall from the last paragraph of Subsection 2.13 that the notation $\mathcal{U} \prec |z_n| \prec \dots \prec |z_1| \prec 0, \infty$ means that we integrate the variables z_1, \dots, z_n over concentric circles that go in the prescribed order, and are contained between the Chern roots of the universal sheaf \mathcal{U}_o and the poles at 0 and ∞ .

Proof. of Theorem 1.1: It is easy to see that the operator $\Gamma_{\mathbb{A}^2}$ of (1.7) is given by:

$$\Gamma_{\mathbb{A}^2}(f[X]) = f[\mathcal{U}_o]$$

in the notations of Subsections 2.11 and 2.19, respectively. The fact that $\Gamma_{\mathbb{A}^2}$ commutes with $P_{0,m}$ for any $m > 0$ is an immediate consequence of comparing (2.22) and (2.43). As for the fact that Γ intertwines $\Psi(H_{\pm n,m})$ with $\Phi(H_{\pm n,m})$ for all $n > 0$ and $m > \mp nr$, this follows by comparing formulas (2.38)–(2.39) with (2.49)–(2.50): either of these formulas involve one and the same integrand, the distinction between them being the location of the contours. Specifically, the contours in (2.38)–(2.39) differ from the ones in (2.49)–(2.50) only in which side of the contour the pole at 0 lies. The integrals are equal because the integrands are regular at 0 in each variable among z_1, \dots, z_n , which is easily seen to be the case for (2.49)–(2.50) when $m > \mp nr$.

Concerning the uniqueness statement, let us show that there exists at most a unique:

$$(2.51) \quad \Gamma = \prod_{d=0}^{\infty} \Gamma_d \quad \text{with} \quad \Gamma_d : \Lambda_{\mathbb{A}^2, \text{loc}} \rightarrow K_T(\mathcal{M}_d^f)_{\text{loc}}$$

where:

$$\begin{aligned} \Lambda_{\mathbb{A}^2, \text{loc}} &= \Lambda_{\mathbb{A}^2} \otimes_{\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]} \mathbb{Q}(q_1, q_2) \\ K_T(\mathcal{M}_d^f)_{\text{loc}} &= K_T(\mathcal{M}_d^f) \otimes_{K_T(o)} \text{Frac}(K_T(o)) \end{aligned}$$

such that Γ is determined by the facts that $\Gamma_0(1) = \mathbf{1}_0$ and that Γ commutes with the action of $\mathcal{A}^{(r)}$, in the sense of diagram (1.8). The commutativity with the operators $P_{0,m}$ for $m > 0$ uniquely determine Γ_0 . Meanwhile, we have the following.

Claim 2.21. *Any class $\gamma \in K_T(\mathcal{M}_d^f)_{\text{loc}}$ is uniquely determined by the collection:*

$$\left\{ \Phi(H_{1,m_1} \dots H_{1,m_d})(\gamma) \in K_T(\mathcal{M}_0^f)_{\text{loc}} \right\}$$

as m_1, \dots, m_d range over the integers $> -r$.

The commutativity of diagram (1.8) implies that:

$$\Phi(H_{1,m_1} \dots H_{1,m_d})(\Gamma_d(f)) = \Gamma_0(\Psi(H_{1,m_1} \dots H_{1,m_d})(f))$$

for all $f \in \Lambda_{\mathbb{A}^2, \text{loc}}$. Since we have already seen that Γ_0 is uniquely determined, then Claim 2.21 implies that $\Gamma_d(f)$ is uniquely determined, for all $d \geq 0$ and all f .

Proof. of Claim 2.21: Let $\mathbb{F} = \text{Frac}(K_T(\circ)) = K_T(\mathcal{M}_0^f)_{\text{loc}}$, where the last equality is due to the fact that \mathcal{M}_0^f is a point. The Thomason equivariant localization theorem gives us the following isomorphism of \mathbb{F} -vector spaces:

$$(2.52) \quad K_T(\mathcal{M}_d^f)_{\text{loc}} \cong \bigoplus_{\lambda \text{ of size } d} \mathbb{F} \cdot |\lambda\rangle$$

where $|\lambda\rangle$ denotes the (renormalized) skyscraper sheaf at the T -fixed point of \mathcal{M}_d^f indexed by an r -partition λ of size d (i.e. an r -tuple of partitions of total size d , we refer to [19, 21] for a discussion of the connection between fixed points and r -partitions). The classes $|\lambda\rangle$ form an orthogonal basis of (2.52), with respect to the equivariant Euler characteristic pairing. Since the adjoint of $H_{1,m}$ with respect to this pairing is a multiple of $H_{-1,m+r}$, the claim is equivalent to proving that:

$$\left\{ \Phi(H_{-1,m_1} \dots H_{-1,m_d})(K_T(\mathcal{M}_0^f)_{\text{loc}}) \right\}_{m_i > 0} \text{ span } K_T(\mathcal{M}_d^f)_{\text{loc}}$$

We may prove this claim by induction on d , and it suffices to establish that:

$$(2.53) \quad \left\{ \Phi(H_{-1,m})(K_T(\mathcal{M}_{d-1}^f)_{\text{loc}}) \right\}_{m > 0} \text{ span } K_T(\mathcal{M}_d^f)_{\text{loc}}$$

To prove the claim above, let's consider an r -partition μ of size $d - 1$. We have:

$$(2.54) \quad \Phi(H_{-1,m})|\mu\rangle = \sum_{\lambda = \mu + \blacksquare} \chi_{\blacksquare}^m \cdot \tau_{\mu}^{\lambda} |\lambda\rangle$$

where the right-hand side goes over all r -partitions obtained by adding a single box \blacksquare to μ , and if this box is located at coordinates (x, y) in the i -th constituent partition of μ , its weight is defined by $\chi_{\blacksquare} = u_i q_1^x q_2^y$ (see [19, 21] for the aforementioned notions and formulas for the coefficients τ_{μ}^{λ} that appear in (2.54), but we remark that they do not depend on m). Since there are only finitely many ways to add a single box to the r -partition μ , and all of these boxes have different weights, it is clear that there exists a \mathbb{F} -linear combination H of the operators $H_{-1,1}, H_{-1,2}, \dots$ such that $\Phi(H)|\mu\rangle = |\lambda\rangle$ for any fixed λ . This completes the proof of (2.53). □

□

3. The case of general surfaces

3.1.

Consider a smooth projective surface S with an ample divisor H , and also fix $(r, c_1) \in \mathbb{N} \times H^2(S, \mathbb{Z})$. Consider the moduli space \mathcal{M}^s of H -stable sheaves on the surface S with the numerical invariants r, c_1 and any c_2 . We make the following:

(3.1) **Assumption A:** $\gcd(r, c_1 \cdot H) = 1$, and

(3.2) **Assumption S:** either $\begin{cases} \mathcal{K}_S \cong \mathcal{O}_S & \text{or} \\ c_1(\mathcal{K}_S) \cdot H < 0 \end{cases}$

Assumption A implies that \mathcal{M}^s is representable, i.e. there exists a universal sheaf:

$$\begin{array}{c} \mathcal{U} \\ \vdots \\ \downarrow \\ \mathcal{M}^s \times S \end{array}$$

We fix a choice of \mathcal{U} throughout this paper. If we let $\mathcal{M}_d^s \subset \mathcal{M}^s$ denote the subspace of sheaves with $c_2 = d$, then we have a disconnected union:

$$\mathcal{M}^s = \bigsqcup_{d=\lceil \frac{r-1}{2r} c_1^2 \rceil}^{\infty} \mathcal{M}_d^s$$

where the fact that d is bounded below is a consequence of Bogomolov’s inequality. Each \mathcal{M}_d^s is projective (by Assumption A) and smooth (by Assumption S). In the present paper, we will work with the K –theory groups:

$$K(\mathcal{M}^s) = \prod_{d=\lceil \frac{r-1}{2r} c_1^2 \rceil}^{\infty} K(\mathcal{M}_d^s)$$

We refer the reader to [22, 25] for an introduction to basic facts on the moduli space of stable sheaves, as pertains to the present paper.

3.2.

Since $K(\mathcal{M}^s \times S) \not\cong K(\mathcal{M}^s)$, as opposed from the case $S = \mathbb{A}^2$ studied previously, we must take care what we mean by “algebras acting on K –theory groups”. In Definitions 3.3 and 3.5, we let X be any smooth quasiprojective algebraic variety.

Definition 3.3. A *weak action* $\mathcal{A} \curvearrowright_S^{\Phi} K(X)$ is an abelian group homomorphism:

$$(3.3) \quad \mathcal{A} \xrightarrow{\Phi} \text{Hom}(K(X), K(X \times S))$$

such that:

- $\Phi(1)$ is the standard pull-back map;
- for all $a \in \mathcal{A}$ and $f \in \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]^{\text{Sym}}$, we require $\Phi(f \cdot a)$ to equal the composition:

$$(3.4) \quad K(X) \xrightarrow{\Phi(a)} K(X \times S) \xrightarrow{\text{Id}_X \boxtimes f(q_1, q_2)} K(X \times S)$$

where q_1, q_2 are identified with the Chern roots of $[\Omega_S^1] \in K(S)$;

- for all $a, b \in \mathcal{A}$, we require $\Phi(ab)$ to equal the composition:

$$(3.5) \quad K(X) \xrightarrow{\Phi(b)} K(X \times S) \xrightarrow{\Phi(a) \boxtimes \text{Id}_S} K(X \times S \times S) \xrightarrow{\text{Id}_X \boxtimes \Delta^*} K(X \times S)$$

(where $\Delta : S \hookrightarrow S \times S$ is the diagonal).

We will apply the definition above when $X = \mathcal{M}^s$ or $X = \bigsqcup_{k=0}^\infty S^k$.

3.4.

Note that in the definition of a weak action, the composition of operators $\Phi(a)$ and $\Phi(b)$ only records what happens on the diagonal of $S \times S$. To understand the behavior off the diagonal, we introduce the following stronger notion.

Definition 3.5. *A weak action as in Definition 3.3 is called strong if, for all $a, b \in \mathcal{A}$, we have the following equality of operators $K(\overline{X}) \rightarrow K(X \times S \times S)$:*

$$(3.6) \quad [\Phi(a), \Phi(b)] = (\text{Id}_X \boxtimes \Delta)_* \left[\Phi \left(\frac{[a, b]}{(1 - q_1)(1 - q_2)} \right) \right]$$

where the left-hand side of (3.6) denotes the difference of the compositions:

$$(3.7) \quad K(X) \xrightarrow{\Phi^{(2)}(b)} K(X \times S_2) \xrightarrow{\Phi^{(1)}(a) \boxtimes \text{Id}_{S_2}} K(X \times S_1 \times S_2)$$

$$(3.8) \quad K(X) \xrightarrow{\Phi^{(1)}(a)} K(X \times S_1) \xrightarrow{\Phi^{(2)}(b) \boxtimes \text{Id}_{S_1}} K(X \times S_1 \times S_2)$$

(we write S_i instead of S and $K(X) \xrightarrow{\Phi^{(i)}(a)} K(X \times S_i)$ instead of $\Phi(a)$, $\forall i \in \{1, 2\}$, in order to better illustrate the two factors of S involved in (3.7)–(3.8)).

The right-hand side of (3.6) is well-defined, because (see [23]) the commutator of any two elements of \mathcal{A} is a multiple of $(1 - q_1)(1 - q_2)$. The following operators:

$$(3.9) \quad [\Phi(a), \Phi(b)]_{\text{red}} = \Phi \left(\frac{[a, b]}{(1 - q_1)(1 - q_2)} \right)$$

which act between $K(X) \rightarrow K(X \times S)$, will be called the reduced commutators.

Remark 3.6. Consider any $\alpha, \beta, \gamma : K(X) \rightarrow K(X \times S)$, and any $f \in \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]^{\text{Sym}}$. Let us define the operators:

$$f\alpha, \alpha\beta : K(X) \rightarrow K(X \times S)$$

and:

$$[\alpha, \beta] : K(X) \rightarrow K(X \times S \times S)$$

by replacing $\Phi(a)$ and $\Phi(b)$ in (3.4), (3.5), (3.7), (3.8) with α and β . Then we have the following associativity properties:

$$(3.10) \quad (f\alpha)\beta = f(\alpha\beta), \quad (\alpha\beta)\gamma = \alpha(\beta\gamma)$$

Moreover, assume that the commutator of any two of α, β, γ is supported on the diagonal $\Delta \subset S \times S$, i.e. we have the following equality $K(X) \rightarrow K(X \times S \times S)$:

$$(3.11) \quad [\alpha, \beta] = (\text{Id}_X \boxtimes \Delta)_*([\alpha, \beta]_{\text{red}})$$

for some operator $[\alpha, \beta]_{\text{red}} : K(X) \rightarrow K(X \times S)$, and the analogous formulas for the pairs (β, γ) and (α, γ) .² Then the following Leibniz rule holds:

$$(3.12) \quad [\alpha\beta, \gamma]_{\text{red}} = \alpha[\beta, \gamma]_{\text{red}} + [\alpha, \gamma]_{\text{red}}\beta$$

and the following Jacobi identity holds:

$$(3.13) \quad \sum_{\text{cyclic}} [\alpha, [\beta, \gamma]_{\text{red}}]_{\text{red}} = 0$$

The claims (3.10), (3.12) and (3.13) are straightforward exercises.

3.7.

Let us apply Definitions 3.3 and 3.5 to the case $X = \mathcal{M}^f, S = \mathbb{A}^2$ and that of T -equivariant K -theory. In this case, composing the action map:

$$\mathcal{A} \xrightarrow{\Phi} \text{Hom}(K_T(\mathcal{M}^f), K_T(\mathcal{M}^f \times \mathbb{A}^2))$$

²If (3.11) holds for some operator $[\alpha, \beta]_{\text{red}}$, then this operator is unique, due to the fact that the map $(\text{Id}_X \boxtimes \Delta)_*$ has a left inverse given by $(\text{Id}_X \boxtimes \text{proj}_1)_*$

with the restriction to the origin $\circ \in \mathbb{A}^2$ (which is an isomorphism), we obtain:

$$\mathcal{A} \xrightarrow{\Phi'} \text{End}(K_T(\mathcal{M}^f))$$

Property (3.4) states that q_1 and q_2 are the equivariant Chern roots of $[\Omega_{\mathbb{A}^2}^1]$, property (3.5) states that $\Phi'(ab) = \Phi'(a)\Phi'(b)$, while property (3.6) states that:

$$[\Phi'(a), \Phi'(b)] = \Phi'([a, b])$$

the reason being that $\Delta^*\Delta_* = (1 - q_1)(1 - q_2)$ if $\Delta : \mathbb{A}^2 \hookrightarrow \mathbb{A}^2 \times \mathbb{A}^2$ is the diagonal. The conclusion is that Φ' yields an honest action of \mathcal{A} on $K_T(\mathcal{M}^f)$.

Remark 3.8. *Definitions 3.3 and 3.5 are inspired by the Heisenberg algebra action on the cohomology groups of Hilbert schemes that was developed by Grojnowski ([12]) and Nakajima ([16]). This construction can be interpreted as “operators on the cohomology groups of Hilbert schemes of points on a surface S , indexed by a cohomology class on S ”. Indeed, if:*

$$(3.14) \quad \Phi(a)^{(\gamma)} : K(\mathcal{M}^S) \rightarrow K(\mathcal{M}^S)$$

denotes the composition:

$$K(\mathcal{M}^S) \xrightarrow{\Phi(a)} K(\mathcal{M}^S \times S) \xrightarrow{\text{Id}_{\mathcal{M}^S} \boxtimes \gamma} K(\mathcal{M}^S \times S) \xrightarrow{\pi_*} K(\mathcal{M}^S)$$

for any $\gamma \in K(S)$ (where $\pi : \mathcal{M}^S \times S \rightarrow \mathcal{M}^S$ is the projection), then (3.6) reads:

$$\left[\Phi(a)^{(\gamma)}, \Phi(b)^{(\delta)} \right] = \Phi \left(\frac{[a, b]}{(1 - q_1)(1 - q_2)} \right)^{(\gamma\delta)}$$

for any $\gamma, \delta \in K(S)$. The particular case of the formula above when $a = P_{n,0}$ and $b = P_{n',0}$ yields precisely a Heisenberg algebra action in the sense of loc. cit. However, since in K -theory one does not have a Künneth decomposition, the datum of the homomorphism $\Phi(a)$ is stronger than the totality of the endomorphisms (3.14).

3.9.

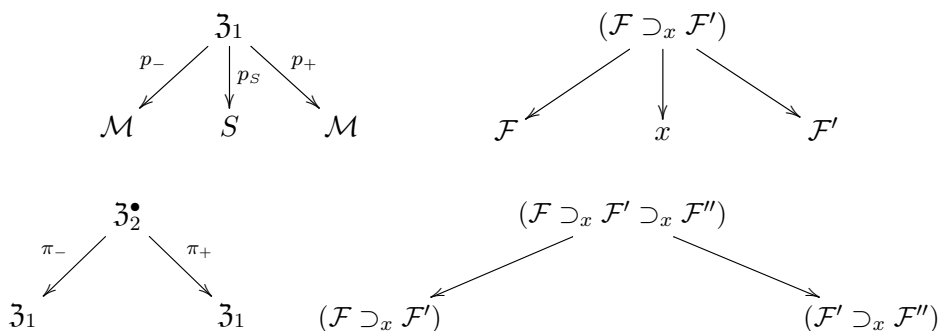
Let us present the analogues of the correspondences of Subsection 2.16 with $(\mathcal{M}^f, \mathbb{A}^2)$ replaced by (\mathcal{M}^S, S) , and use them to construct an action $\mathcal{A} \curvearrowright_S K(\mathcal{M}^S)$.

Definition 3.10. *The following moduli spaces are smooth projective varieties:*

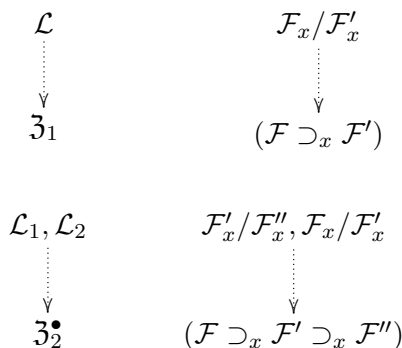
$$\mathfrak{Z}_1 = \{(\mathcal{F} \supset_x \mathcal{F}') \text{ for some } x \in S\}$$

$$\mathfrak{Z}_2^\bullet = \{(\mathcal{F} \supset_x \mathcal{F}' \supset_x \mathcal{F}'') \text{ for some } x \in S\}$$

Consider the maps:



and the line bundles:



We refer the reader to [22] for the statements pertaining to \mathfrak{Z}_1 (although they were known for a long time, see [6]) and to [23] for the statements pertaining to \mathfrak{Z}_2^\bullet .

3.11.

The following analogue of Theorem 2.18 was proved in [25].

Theorem 3.12. *There exists a strong action $\mathcal{A} \overset{\Phi}{\curvearrowright}_S K(\mathcal{M}^s)$ given by:*

$$(3.15) \quad c_1 \mapsto q^r, \quad c_2 \mapsto 1,$$

(recall from Definition 3.3 that q_1, q_2 are identified with the Chern roots of $[\Omega_S^1]$, hence $q = q_1 q_2$ is identified with the canonical line bundle $[\mathcal{K}_S]$) and:

$$(3.16) \quad P_{0,m} \mapsto \left[K(\mathcal{M}^s) \xrightarrow{\text{pull-back}} K(\mathcal{M}^s \times S) \xrightarrow{\otimes p_m(\mathcal{U})} K(\mathcal{M}^s \times S) \right]$$

$$(3.17) \quad P_{0,-m} \mapsto \left[K(\mathcal{M}^s) \xrightarrow{\text{pull-back}} K(\mathcal{M}^s \times S) \xrightarrow{\otimes (-q^m p_m(\mathcal{U}^\vee))} K(\mathcal{M}^s \times S) \right]$$

while for all $n > 0$ and $m \in \mathbb{Z}$, we have:

$$(3.18) \quad H_{n,m} \mapsto (p_- \times p_S)_* \left[\mathcal{L}_n^{d_n} \otimes \pi_{-*} \pi_+^* \left[\mathcal{L}_{n-1}^{d_{n-1}} \otimes \dots \pi_{-*} \pi_+^* \left[\mathcal{L}_1^{d_1} \otimes p_+^* \right] \dots \right] \right]$$

and:

$$(3.19) \quad H_{-n,m} \mapsto \left[\frac{\det \mathcal{U}}{(-q)^{r-1}} \right]^n \otimes (p_+ \times p_S)_* \left[\mathcal{L}_1^{d_1-r} \otimes \dots \otimes \pi_{+*} \pi_-^* \left[\mathcal{L}_n^{d_n-r} \otimes p_-^* \right] \dots \right]$$

where $d_i = \lfloor \frac{mi}{n} \rfloor - \lfloor \frac{m(i-1)}{n} \rfloor$.

3.13.

The analogue of the ring of symmetric functions for an arbitrary surface is:

$$\Lambda_S = \bigoplus_{k=0}^{\infty} K_{\mathfrak{S}(k)}(S^k)$$

where $\mathfrak{S}(k)$ permutes the factors of S^k . We will (slightly abusively) write:

$$\begin{aligned} \Lambda_S \times K(S) &= \bigoplus_{k=0}^{\infty} K_{\mathfrak{S}(k)}(S^k \times S) \\ \Lambda_S \times K(S \times S) &= \bigoplus_{k=0}^{\infty} K_{\mathfrak{S}(k)}(S^k \times S \times S) \end{aligned}$$

where $\mathfrak{S}(k)$ does not act on the last factor of S or $S \times S$. There exist analogues of the operators (2.8) of parabolic induction and restriction, respectively:

$$(3.20) \quad p_k, p_k^\dagger : \Lambda_S \longrightarrow \Lambda_S \times K(S)$$

which take the form:

$$(3.21) \quad K_{\mathfrak{S}(l)}(S^l) \xrightarrow{p_k} K_{\mathfrak{S}(k+l)}(S^{k+l} \times S)$$

$$(3.22) \quad K_{\mathfrak{S}(k+l)}(S^{k+l}) \xrightarrow{p_k^\dagger} K_{\mathfrak{S}(l)}(S^l \times S)$$

and are explicitly given by:

$$(3.23) \quad p_k(M) = \text{Ind}_{\mathfrak{S}(k) \times \mathfrak{S}(l)}^{\mathfrak{S}(k+l)} \left(\underbrace{[p_k \otimes \mathcal{O}_{\Delta_{1\dots k\bullet}}]}_{\text{sheaf on } S^k \times S} \boxtimes \underbrace{M}_{\text{sheaf on } S^l} \right)$$

$$(3.24) \quad p_k^\dagger(M) = \text{Hom}_{\mathfrak{S}(k)} \left(p_k, \text{Res}_{\mathfrak{S}(k) \times \mathfrak{S}(l)}^{\mathfrak{S}(k+l)}(M) \Big|_{\Delta_{1\dots k\bullet \times S^l}} \right)$$

where $\Delta_{1\dots k\bullet} \subset S^k \times S$ is the small diagonal. In the right-hand side of (3.24), we implicitly pull-back M from S^{k+l} to $S^{k+l} \times S$, and then restrict it to the diagonal obtained by identifying the first k and the last factor, thus obtaining a sheaf on $S^l \times S$.

Proposition 3.14. *The operators p_k, p_k^\dagger give rise to a strong action (in the sense of Definitions 3.3 and 3.5) of the infinite-dimensional Heisenberg algebra, i.e.:*

$$(3.25) \quad [p_k^\dagger, p_l] = (\text{Id}_{\Lambda_S} \boxtimes \Delta)_* \left(\rho^* \left[\delta_k^l k \frac{(1 - q_1^k)(1 - q_2^k)}{(1 - q_1)(1 - q_2)} \right] \otimes \pi^* \right)$$

as well as $[p_k, p_l] = [p_k^\dagger, p_l^\dagger] = 0$, as homomorphisms $\Lambda_S \rightarrow \Lambda_S \times K(S \times S)$, where:

$$\begin{array}{ccc} & \bigsqcup_{n=0}^\infty S^n \times S & \\ \pi \swarrow & & \searrow \rho \\ \bigsqcup_{n=0}^\infty S^n & & S \end{array}$$

are the standard projections. In the left-hand side of (3.25), the operator p_k^\dagger (respectively p_l) acts only on the first (respectively second) factor of $S \times S$.

Remark 3.15. *Proposition 3.14 actually holds for an arbitrary smooth variety S , although we will only need the surface case. If S has dimension d , then the constant in the square brackets in the right-hand side of (3.25) must be replaced by:*

$$\delta_k^l \frac{(1 - q_1^k)(1 - q_2^k) \dots (1 - q_d^k)}{(1 - q_1)(1 - q_2) \dots (1 - q_d)}$$

where $q_1 + q_2 + \dots + q_d = [\Omega_S^1] \in K(S)$. The proof below applies to arbitrary d .

Proof. Recall that elements of $K(S^n)$ represent vector bundles M on S^n . Given any permutation $\sigma = \{1, \dots, n\} \rightarrow \{a_1, \dots, a_n\}$, we will use the notation $M_{a_1 \dots a_n}$ for the vector bundle $\sigma^*(M)$ on S^n and the associated K -theory class. Similarly, elements of $K_{\mathfrak{S}(n)}(S^n)$ represent $\mathfrak{S}(n)$ -equivariant vector bundles, i.e. vector bundles M on S^n endowed with isomorphisms $M \cong \sigma^*(M)$ for all $\sigma \in \mathfrak{S}(n)$ that respect the product of permutations. In this language, formula (3.23) reads:

$$(3.26) \quad p_l(M) = \bigoplus_{\text{shuffles}}^{(l,n)} [p_l \otimes \mathcal{O}_{\Delta_{a_1 \dots a_l \bullet}}] \boxtimes M_{b_1 \dots b_n}$$

where the right-hand side is a vector bundle on $S^{n+l} \times S$ (the index \bullet represents the last factor of S) with the action of $\mathfrak{S}(n+l)$ given by permutation of the indices. The term “ (l, n) -shuffles” above and henceforth refers to the set of all partitions:

$$(3.27) \quad \{a_1 < \dots < a_l\} \sqcup \{b_1 < \dots < b_n\} = \{1, \dots, n+l\}$$

Iterating (3.26) twice implies that:

$$(3.28) \quad p_k p_l(M) = \bigoplus_{\text{shuffles}}^{(k,l,n)} [p_k \otimes \mathcal{O}_{\Delta_{a_1 \dots a_k \circ}}] \otimes [p_l \otimes \mathcal{O}_{\Delta_{b_1 \dots b_l \bullet}}] \boxtimes M_{c_1 \dots c_n}$$

where (k, l, n) -shuffles are partitions of $\{1, \dots, n+k+l\}$ into three sets, of sizes k, l and n , respectively. The right-hand side of (3.28) is a vector bundle on $S^{n+k+l} \times S \times S$, where the latter two factors of S are indexed by the symbols \circ and \bullet , respectively. It is clear that the right-hand side of (3.28) is symmetric under permuting $k \leftrightarrow l$, if we also permute $\circ \leftrightarrow \bullet$. This implies $[p_k, p_l] = 0$, and the statement that $[p_k^\dagger, p_l^\dagger] = 0$ is analogous. As for the

commutator (3.25), we note that (3.24) implies:

$$(3.29) \quad p_k^\dagger p_l(M) = \text{Hom}_{\mathfrak{S}(k)} \left(p_k, \bigoplus_{\text{shuffles}}^{(l,n)} [p_l \otimes \mathcal{O}_{\Delta_{a_1 \dots a_l \bullet}}] \boxtimes M_{b_1 \dots b_n} \Big|_{\Delta_{1 \dots k \circ} \times S^{n+l-k}} \right)$$

as a vector bundle on $S^{n+l-k} \times S \times S$ (the indices \bullet and \circ represent the two latter factors of S) with the action of $\mathfrak{S}(n+l-k)$ given by permutation of the indices $> k$. As a virtual representation of $\mathfrak{S}(l)$, the power sum p_l has the property that:

$$(3.30) \quad p_l \Big|_{\mathfrak{S}(i) \times \mathfrak{S}(l-i)} = 0$$

for all $i \in \{1, \dots, l-1\}$. We will call any shuffle as in (3.27) of “type i ” if:

$$\{a_1, \dots, a_i\} \sqcup \{b_1, \dots, b_{k-i}\} = \{1, \dots, k\}$$

Because of (3.30), the only shuffles which contribute non-trivially to (3.29) are those of type 0 and type l . The shuffles of type 0 correspond to the case when $\{1, \dots, k\} \subset \{b_1, \dots, b_n\}$, and their contribution to (3.29) may be identified with:

$$(3.31) \quad p_l p_k^\dagger(M) = \bigoplus_{(l,n)\text{-shuffles}}^{\text{type 0}} [p_l \otimes \mathcal{O}_{\Delta_{a_1 \dots a_l \bullet}}] \boxtimes \text{Hom}_{\mathfrak{S}(k)} \left(p_k, M_{b_1 \dots b_n} \Big|_{\Delta_{1 \dots k \circ} \times S^{n-k}} \right)$$

Therefore, the difference between (3.29) and (3.31) consists precisely of the sum over type l shuffles, i.e. those such that $\{a_1, \dots, a_l\} \subset \{1, \dots, k\}$. However, if $k > l$, then the $\text{Hom}_{\mathfrak{S}(k)}(p_k, \dots)$ space in (3.29) vanishes because of (3.30) for $k \leftrightarrow l$. Therefore, the only shuffle which has a non-zero contribution to the difference of (3.29) and (3.31) is the one corresponding to $\{a_1, \dots, a_l\} = \{1, \dots, k\}$. We thus conclude that:

$$(3.32) \quad [p_k^\dagger, p_l](M) = M \boxtimes \delta_k^l \text{Hom}_{\mathfrak{S}(k)} \left(p_k, p_k \otimes \mathcal{O}_{\Delta_{1 \dots k \bullet}} \Big|_{\Delta_{1 \dots k \circ}} \right)$$

In K -theory, the restriction of a regular subvariety (in the situation above, the small diagonal $\Delta_{1 \dots k} : S \hookrightarrow S^k$) to itself is equal to the exterior algebra

of the conormal bundle to this subvariety. In the situation of (3.32), this leads to:

$$(3.33) \quad [p_k^\dagger, p_l](M) = M \boxtimes \Delta_* \left[\delta_k^l \text{Hom}_{\mathfrak{S}(k)} \left(p_k, p_k \otimes \wedge^\bullet(N_{S|S^k}^*) \right) \right]$$

where $\Delta \hookrightarrow S \times S$ is the diagonal that identifies the points \bullet and \circ . Recall that:

$$\text{Hom}_{\mathfrak{S}(k)}(p_k, V) = \text{Tr}_V(\omega_k)$$

where $\omega_k \in \mathfrak{S}(k)$ is a k -cycle. Therefore, (3.33) implies (3.25) because of the well-known fact that $\text{Tr}_{p_k}(\omega_k) = k$ and the claim below:

Claim 3.16. *If $N_{S|S^k}$ denotes the normal bundle of the small diagonal in S^k , then:*

$$\text{Tr}_{\wedge^\bullet(N_{S|S^k}^*)}(\omega_k) = \frac{(1 - q_1^k)(1 - q_2^k)}{(1 - q_1)(1 - q_2)}$$

where $q_1 + q_2 = [\Omega_S^1]$.

The normal bundle arises from the short exact sequence:

$$0 \longrightarrow TS \xrightarrow{\text{diag}} (TS)^{\oplus k} \longrightarrow N_{S|S^k} \longrightarrow 0$$

where the $\mathfrak{S}(k)$ -action permutes the factors of TS . Therefore, we have:

$$\wedge^\bullet(N_{S|S^k}^*) = \frac{\wedge^\bullet((\Omega_S^1)^{\oplus k})}{\wedge^\bullet(\Omega_S^1)}$$

Since the denominator of the expression above is a trivial $\mathfrak{S}(k)$ -module with K -theory class $(1 - q_1)(1 - q_2)$, it remains to show that:

$$(3.34) \quad \text{Tr}_{\wedge^\bullet((T^*S)^{\oplus k})}(\omega_k) = (1 - q_1^k)(1 - q_2^k)$$

By the splitting principle, we can assume $\Omega_S^1 \cong \mathcal{L}_1 \oplus \mathcal{L}_2$, where \mathcal{L}_1 and \mathcal{L}_2 are line bundles with K -theory classes q_1 and q_2 . If we let l_a denote a local section of the line bundle \mathcal{L}_a , then a basis for the local sections of $\wedge^\bullet(\mathcal{L}_1^{\oplus k} \oplus \mathcal{L}_2^{\oplus k})$ consists of:

$$(3.35) \quad l_1^{(i_1)} \wedge l_1^{(i_2)} \wedge \dots \wedge l_1^{(i_a)} \wedge l_2^{(j_1)} \wedge l_2^{(j_2)} \wedge \dots \wedge l_2^{(j_b)}$$

where $l_a^{(i)}$ denotes the section l_a on the i -th copy of \mathcal{L}_a inside $\mathcal{L}_a^{\oplus k}$. The cycle ω_k acts on the basis (3.35) by increasing the indices $i_a, j_b \in \mathbb{Z}/n\mathbb{Z}$ by

1. Therefore, there are only 4 basis elements which are unchanged by ω_k , namely the cases $(a, b) = (0, 0), (k, 0), (0, k), (k, k)$ of (3.35). These 4 basis elements contribute precisely $1, -q_1^k, -q_2^k, q^k$ (respectively) to the trace, thus implying (3.34). \square

3.17.

By analogy with Subsection 2.5, we have:

$$(3.36) \quad \sum_{k=0}^{\infty} \frac{h_k}{z^k} = \exp \left[\sum_{k=1}^{\infty} \frac{p_k}{kz^k} \right] \Big|_{\Delta}$$

$$(3.37) \quad \sum_{k=0}^{\infty} \frac{e_k}{(-z)^k} = \exp \left[- \sum_{k=1}^{\infty} \frac{p_k}{kz^k} \right] \Big|_{\Delta}$$

where h_k, e_k are operators defined by analogy with (3.23), specifically:

$$h_k(M) = \text{Ind}_{\mathfrak{S}(k) \times \mathfrak{S}(l)}^{\mathfrak{S}(k+l)} \left([\text{triv}_{\mathfrak{S}(k)} \otimes \mathcal{O}_{\Delta_{1\dots k \bullet}}] \boxtimes M \right)$$

$$e_k(M) = \text{Ind}_{\mathfrak{S}(k) \times \mathfrak{S}(l)}^{\mathfrak{S}(k+l)} \left([\text{sign}_{\mathfrak{S}(k)} \otimes \mathcal{O}_{\Delta_{1\dots k \bullet}}] \boxtimes M \right)$$

for all $M \in K_{\mathfrak{S}(l)}(S^l)$. Moreover, we consider:

$$(3.38) \quad \sum_{k=0}^{\infty} h_k^{\dagger} z^k = \exp \left[\sum_{k=1}^{\infty} \frac{p_k^{\dagger} z^k}{k} \right] \Big|_{\Delta}$$

$$(3.39) \quad \sum_{k=0}^{\infty} e_k^{\dagger} (-z)^k = \exp \left[- \sum_{k=1}^{\infty} \frac{p_k^{\dagger} z^k}{k} \right] \Big|_{\Delta}$$

which by analogy with (3.24) satisfy:

$$(3.40) \quad h_k^{\dagger}(M) = \text{Hom}_{\mathfrak{S}(k)} \left(\text{triv}_{\mathfrak{S}(k)}, \text{Res}_{\mathfrak{S}(k) \times \mathfrak{S}(l)}^{\mathfrak{S}(k+l)}(M) \Big|_{\Delta_{1\dots k \bullet \times S^l}} \right)$$

$$(3.41) \quad e_k^{\dagger}(M) = \text{Hom}_{\mathfrak{S}(k)} \left(\text{sign}_{\mathfrak{S}(k)}, \text{Res}_{\mathfrak{S}(k) \times \mathfrak{S}(l)}^{\mathfrak{S}(k+l)}(M) \Big|_{\Delta_{1\dots k \bullet \times S^l}} \right)$$

for all $M \in K_{\mathfrak{S}(k+l)}(S^{k+l})$.

Remark 3.18. In [14], a similar result to (a categorification of) Proposition 3.14 was proved by using certain operators $\Lambda_S \rightarrow \Lambda_S$ indexed by classes

$\gamma \in K(S)$. While similar in overall shape with our:

$$e_k^{(\gamma)} : \Lambda_S \xrightarrow{e_k} \Lambda_S \times K(S) \xrightarrow{\text{Id}_{\Lambda_S} \boxtimes \gamma} \Lambda_S \times K(S) \xrightarrow{\pi_*} \Lambda_S$$

and their adjoints, the operators of loc. cit. are not linear in γ . Linearity is necessary in order for operators indexed by $\gamma \in K(S)$ to “glue” to an operator:

$$\Lambda_S \rightarrow \Lambda_S \times K(S)$$

which is required for the framework of Definitions 3.3 and 3.5.

3.19.

We have the following global analogue of the rational function (2.15):

$$(3.42) \quad \zeta^S(x) = \wedge^\bullet(-x \cdot \mathcal{O}_\Delta) \in K(S \times S)(x)$$

whose restriction to the diagonal is precisely:

$$\wedge^\bullet\left(-x \cdot ([\mathcal{O}_S] - [\Omega_S^1] + [\mathcal{K}_S])\right) = \frac{(1 - xq_1)(1 - xq_2)}{(1 - x)(1 - xq)} = \zeta(x)$$

where $[\Omega_S^1] = q_1 + q_2 \in K(S)$. We have the following analogue of Theorem 2.10:

Theorem 3.20. *There is a strong action $\mathcal{A} \curvearrowright_S \Lambda_S$ given by:*

$$(3.43) \quad c_1 \mapsto 1, \quad c_2 \mapsto q^{-1},$$

$$(3.44) \quad P_{0,m} \mapsto p_m, \quad P_{0,-m} \mapsto -mq^m \cdot p_m^\dagger,$$

while for all $n > 0$ and $m \in \mathbb{Z}$, we have:

$$(3.45) \quad H_{n,m} \mapsto \int_{|z_1| \gg \dots \gg |z_n|} \frac{\prod_{i=1}^n z_i^{\lfloor \frac{mi}{n} \rfloor - \lfloor \frac{m(i-1)}{n} \rfloor}}{\prod_{i=1}^{n-1} \left(1 - \frac{z_{i+1}q}{z_i}\right) \prod_{i < j} \zeta\left(\frac{z_i}{z_j}\right)} \exp\left[\sum_{k=1}^\infty \frac{z_1^{-k} + \dots + z_n^{-k}}{k} \cdot p_k\right] \exp\left[-\sum_{k=1}^\infty \frac{z_1^k + \dots + z_n^k}{k} \cdot p_k^\dagger\right] \Big|_\Delta \prod_{a=1}^n \frac{dz_a}{2\pi i z_a}$$

and:

$$\begin{aligned}
 H_{-n,m} &\mapsto \int_{|z_1| \ll \dots \ll |z_n|} \frac{(-q)^n \prod_{i=1}^n z_i^{\lfloor \frac{mi}{n} \rfloor - \lfloor \frac{m(i-1)}{n} \rfloor}}{\prod_{i=1}^{n-1} \left(1 - \frac{z_{i+1}q}{z_i}\right) \prod_{i < j} \zeta\left(\frac{z_j}{z_i}\right)} \\
 (3.46) \quad &\exp \left[- \sum_{k=1}^{\infty} \frac{z_1^{-k} + \dots + z_n^{-k}}{k \cdot q^k} \cdot p_k \right] \exp \left[\sum_{k=1}^{\infty} \frac{z_1^k + \dots + z_n^k}{k} \cdot p_k^\dagger \right] \Big|_{\Delta} \prod_{a=1}^n \frac{dz_a}{2\pi i z_a}
 \end{aligned}$$

(see the last sentence of Theorem 2.10 for the meaning of the contours).

Proof. Let us first show that the assignments (3.43)–(3.46) give rise to a weak action. As shown in [25], this can be achieved by establishing that the two bullets in the proof of Theorem 2.10 hold. The first bullet is proved almost word-for-word as in the aforementioned Theorem, with the minor modification that the parameters q_1 and q_2 are now identified with the Chern roots of Ω_S^1 . As for the second bullet, it is an immediate consequence of the following analogues of (2.25)–(2.29):

$$(3.47)$$

$$[\Psi(P_{0,\pm m}), \Psi(P_{0,\pm m'})] = 0$$

$$(3.48)$$

$$[\Psi(P_{0,m}), \Psi(P_{0,-m'})] = \Delta_* \left(\rho^* \left[\delta_{m'}^m m \frac{(1 - q_1^m)(1 - q_2^m)}{(1 - q_1)(1 - q_2)} \right] \pi^* \right)$$

$$(3.49)$$

$$[\Psi(H_{\pm 1,k}), \Psi(P_{0,\pm m})] = \Delta_* \left(-\rho^* \left[\frac{(1 - q_1^m)(1 - q_2^m)}{(1 - q_1)(1 - q_2)} \right] \Psi(H_{\pm 1,k \pm m}) \right)$$

$$(3.50)$$

$$[\Psi(H_{\pm 1,k}), \Psi(P_{0,\mp m})] = \Delta_* \left(\rho^* \left[\frac{(1 - q_1^m)(1 - q_2^m)q^{m\delta_{\pm}^+}}{(1 - q_1)(1 - q_2)} \right] \Psi(H_{\pm 1,k \mp m}) \right)$$

$$(3.51)$$

$$[\Psi(H_{1,k}), \Psi(H_{-1,k'})] = \Delta_* \left(\frac{1}{q^{-1} - 1} \begin{cases} \Psi(A_{k+k'}) & \text{if } k + k' > 0 \\ \rho^*(1 - q^k)\pi^* & \text{if } k + k' = 0 \\ -\Psi(q^k B_{-k-k'}) & \text{if } k + k' < 0 \end{cases} \right)$$

(in the context of a weak action, we only need the restriction of formulas (3.47)–(3.51) to the diagonal $\Delta \subset S \times S$ where $\pi : \mathcal{M}^s \times S \rightarrow \mathcal{M}^s$ and $\rho :$

$\mathcal{M}^s \times S \rightarrow S$ are the standard projections, and $A_m, B_m : \Lambda_S \rightarrow \Lambda_S \times K(S)$ are defined by:

$$\sum_{m=0}^{\infty} \frac{A_m}{x^m} = \exp \left[\sum_{m=1}^{\infty} \frac{p_m}{mx^m} (1 - q^{-m}) \right] \Big|_{\Delta}$$

$$\sum_{m=0}^{\infty} \frac{B_m}{x^m} = \exp \left[\sum_{m=1}^{\infty} \frac{p_m^\dagger}{mx^m} (1 - q^m) \right] \Big|_{\Delta}$$

Formulas (3.47)–(3.51) are equalities of homomorphisms $K(\mathcal{M}^s) \rightarrow K(\mathcal{M}^s \times S \times S)$, which are straightforward consequences of Proposition 3.14 (in fact, the first two of these formulas are trivial). Therefore, let us prove (3.51) as an illustration, and leave the remaining formulas as exercises to the interested reader. We have:

$$\Psi(H_{1,k}) = \sum_{|\lambda| - |\mu| = k}^{\lambda, \mu \text{ partitions}} \frac{(-1)^{|\mu|}}{z_\lambda z_\mu} p_\lambda p_\mu^\dagger \Big|_{\Delta}$$

$$\Psi(H_{-1,k'}) = \sum_{|\lambda'| - |\mu'| = k'}^{\lambda', \mu' \text{ partitions}} \frac{(-q)^{1 - |\lambda'|}}{z_{\lambda'} z_{\mu'}} p_{\lambda'} p_{\mu'}^\dagger \Big|_{\Delta}$$

Therefore, we have:

$$[\Psi(H_{1,k}), \Psi(H_{-1,k'})]_{\text{red}}$$

$$= \sum_{|\lambda| - |\mu| = k, |\lambda'| - |\mu'| = k'}^{\lambda, \mu, \lambda', \mu' \text{ partitions}} \frac{(-1)^{|\mu|} (-q)^{1 - |\lambda'|}}{z_\lambda z_\mu z_{\lambda'} z_{\mu'}} \left[p_\lambda p_\mu^\dagger \Big|_{\Delta}, p_{\lambda'} p_{\mu'}^\dagger \Big|_{\Delta} \right]_{\text{red}}$$

Formula (3.12) allows us to compute the reduced commutator in the right-hand side. Specifically, this reduced commutator picks up a contribution from the pairing of any $k \in \lambda$ (respectively $l \in \mu$) and any $k \in \mu'$ (respectively $l \in \lambda'$), and this contribution is a scalar due to (3.25). Therefore, one can write the right-hand side of the expression above as a linear combination of expressions of the form:

$$p_{\tilde{\lambda}} p_{\tilde{\mu}}^\dagger \Big|_{\Delta}$$

whose coefficients are symmetric Laurent polynomials in q_1 and q_2 . However, the right-hand side of (3.51) is also a linear combination of expressions of the same form. The fact that the two sides of (3.51) are equal in the case of $S = \mathbb{A}^2$ (when q_1, q_2 are formal parameters) implies that they are equal

in the case of an arbitrary surface (when q_1, q_2 are specialized to the Chern roots of the cotangent bundle).

Now that we have showed that formulas (3.43)–(3.46) give rise to a weak action $\mathcal{A} \curvearrowright K(\mathcal{M}^s)$, let us show that they in fact produce a strong action. This entails proving (3.6) for all $x, y \in \mathcal{A}$. Because of the Leibniz rule (3.12), it suffices to consider the case when x, y range among the generators of \mathcal{A} , namely the $H_{n,m}$'s. For illustration, let us start with the case $x = H_{-1,m}$ with $y = H_{-1,m'}$:

$$(3.52) \quad \Phi(H_{-1,m}) = \int (-q) \exp \left[-\frac{p_k}{kz^k q^k} \right] \exp \left[\frac{p_k^\dagger z^k}{k} \right] \Big|_{\Delta} \frac{dz}{2\pi iz}$$

Below, we will consider two copies $S_1 = S_2 = S$ of the surface, and write:

$$p_k^{(i)}, p_k^{\dagger, (i)}, \Phi^{(i)}(H_{-1,m}) : \Lambda_S \rightarrow \Lambda_S \times K(S_i) \quad \text{and} \quad q^{(i)} = [\mathcal{K}_{S_i}] \in K(S_i)$$

for each $i \in \{1, 2\}$. By applying relation (3.52) twice, we have:

$$(3.53) \quad \left(\Phi^{(1)}(H_{-1,m}) \boxtimes \text{Id}_{S_2} \right) \circ \Phi^{(2)}(H_{-1,m'}) = \int_{|z_1| \ll |z_2|} q^{(1)} q^{(2)} z_1^m z_2^{m'} \\ \exp \left[-\frac{p_k^{(1)}}{kz_1^k q^{(1)k}} \right] \exp \left[\frac{p_k^{\dagger, (1)} z_1^k}{k} \right] \Big|_{\Delta} \\ \exp \left[-\frac{p_k^{(2)}}{kz_2^k q^{(2)k}} \right] \exp \left[\frac{p_k^{\dagger, (2)} z_2^k}{k} \right] \Big|_{\Delta} \frac{dz_1}{2\pi iz_1} \frac{dz_2}{2\pi iz_2}$$

As a consequence of (3.25), we have the following analogue of (2.14):

$$\exp \left[\sum_{k=1}^{\infty} \frac{p_k^{\dagger, (1)} z_1^k}{k} \right] \exp \left[-\sum_{k=1}^{\infty} \frac{p_k^{(2)}}{kz_2^k q^{(2)k}} \right] = \\ = \exp \left[-\sum_{k=1}^{\infty} \frac{p_k^{(2)}}{kz_2^k q^{(2)k}} \right] \exp \left[\sum_{k=1}^{\infty} \frac{p_k^{\dagger, (1)} z_1^k}{k} \right] \zeta^S \left(\frac{z_2}{z_1} \right)^{-1}$$

as an equality of operators $\text{Hom}(\Lambda_S, \Lambda_S \times K(S_1 \times S_2))$, where ζ^S is the rational function (3.42). Therefore, formula (3.53) may be rewritten as:

$$(3.54) \quad \begin{aligned} & \left(\Phi^{(1)}(H_{-1,m}) \boxtimes \text{Id}_{S_2} \right) \circ \Phi^{(2)}(H_{-1,m'}) = \int_{|z_1| \ll |z_2|} q^{(1)} q^{(2)} z_1^m z_2^{m'} \zeta^S \left(\frac{z_2}{z_1} \right)^{-1} \\ & \exp \left[- \sum_{k=1}^{\infty} \left(\frac{p_k^{(1)}}{k z_1^k q^{(1)k}} + \frac{p_k^{(2)}}{k z_2^k q^{(2)k}} \right) \right] \\ & \exp \left[\sum_{k=1}^{\infty} \left(\frac{p_k^{\dagger,(1)} z_1^k}{k} + \frac{p_k^{\dagger,(2)} z_2^k}{k} \right) \right] \frac{dz_1}{2\pi i z_1} \frac{dz_2}{2\pi i z_2} \end{aligned}$$

Similarly, we have:

$$(3.55) \quad \begin{aligned} & \left(\Phi^{(2)}(H_{-1,m'}) \boxtimes \text{Id}_{S_1} \right) \circ \Phi^{(1)}(H_{-1,m}) = \int_{|z_1| \gg |z_2|} q^{(1)} q^{(2)} z_1^m z_2^{m'} \zeta^S \left(\frac{z_1}{z_2} \right)^{-1} \\ & \exp \left[- \sum_{k=1}^{\infty} \left(\frac{p_k^{(1)}}{k z_1^k q^{(1)k}} + \frac{p_k^{(2)}}{k z_2^k q^{(2)k}} \right) \right] \\ & \exp \left[\sum_{k=1}^{\infty} \left(\frac{p_k^{\dagger,(1)} z_1^k}{k} + \frac{p_k^{\dagger,(2)} z_2^k}{k} \right) \right] \frac{dz_1}{2\pi i z_1} \frac{dz_2}{2\pi i z_2} \end{aligned}$$

However, Remark 3.17 of [22] gives us the following formula:

$$(3.56) \quad \zeta^S(x)^{-1} = 1 - [\mathcal{O}_\Delta] \otimes \frac{x}{(1 - xq_1)(1 - xq_2)}$$

where q_1 and q_2 are the Chern roots of Ω_S^1 on the diagonal inside $S \times S$. Expanding formula (3.56) in either positive or negative powers of x shows that it is always equal to 1 times a multiple of $[\mathcal{O}_\Delta]$. Therefore, the difference of (3.54) and (3.55) is a multiple of $[\mathcal{O}_\Delta]$, and we conclude that:

$$(3.57) \quad \left[\Phi^{(1)}(H_{-1,m}), \Phi^{(2)}(H_{-1,m'}) \right] = (\text{Id}_{\Lambda_S} \boxtimes \Delta)_*(A)$$

where A is a certain difference of integrals of rational functions in q_1 and q_2 , times the symmetric expression:

$$\exp \left[- \sum_{k=1}^{\infty} \left(\frac{p_k}{kz_1^k q^k} + \frac{p_k}{kz_2^k q^k} \right) \right] \exp \left[\sum_{k=1}^{\infty} \left(\frac{p_k^\dagger z_1^k}{k} + \frac{p_k^\dagger z_2^k}{k} \right) \right] \Big|_{\Delta} : \Lambda_S \rightarrow \Lambda_S \times K(S)$$

However, because of Theorem 2.10, we have:

$$(3.58) \quad A = \Phi \left(\frac{[H_{-1,m}, H_{-1,m'}]}{(1 - q_1)(1 - q_2)} \right)$$

in the $S = \mathbb{A}^2$ case (when q_1, q_2 are formal parameters). Therefore, formula (3.58) also holds in the situation at hand (when q_1, q_2 are the Chern roots of Ω_S^1). The generalization of the argument above to any $x = H_{n,m}$ and $y = H_{n',m'}$ where n and n' have the same sign is straightforward, and we refer the interested reader to the proof of “Conjecture 5.7 subject to Assumption B” from [21].

Let us now show how to prove (3.6) for $x = H_{n,m}$ and $y = H_{n',m'}$ where n and n' have opposite signs, and we will do so by induction on $|n| + |n'|$. The base cases of the induction are precisely (3.47)–(3.51), so let us assume without loss of generality that $n \geq 2$. There exist u, u', v, v' with $u + u' = n$, $v + v' = m$, $u, u' > 0$ such that:

$$[H_{u,v}, H_{u',v'}] = (1 - q_1)(1 - q_2) (c \cdot H_{n,m} + \dots)$$

(see [3]) where the ellipsis denotes a sum of products of $H_{u'',v''}$ with $0 < u'' < n$, and c is a product of expressions of the form $1 + q + \dots + q^{d-1}$. Since u and u' have the same sign, the argument in the previous paragraph implies that:

$$\Psi(H_{n,m}) = c^{-1} \left[\Psi(H_{u,v}), \Psi(H_{u',v'}) \right]_{\text{red}} - c^{-1} \left(\Psi(\dots) \Big|_{\Delta} \right)$$

(see (3.9) for the definition of the reduced commutator). We note that c is invertible in $K(S)$ because $q - 1$ is nilpotent. Therefore, the Leibniz rule

(3.12) and the Jacobi identity (3.13) imply that:

$$\begin{aligned} & \left[\Psi(H_{n,m}), \Psi(H_{n',m'}) \right]_{\text{red}} = c^{-1} \left[\Psi(H_{u,v}), \left[\Psi(H_{u',v'}), \Psi(H_{n',m'}) \right]_{\text{red}} \right]_{\text{red}} - \\ & - c^{-1} \left[\Psi(H_{u',v'}), \left[\Psi(H_{u,v}), \Psi(H_{n',m'}) \right]_{\text{red}} \right]_{\text{red}} - c^{-1} \left[\Psi(\dots) \Big|_{\Delta}, \Psi(H_{n',m'}) \right]_{\text{red}} \end{aligned}$$

By the induction hypothesis, the right-hand side of the expression is equal to:

$$c^{-1} \Psi \left(\frac{[H_{u,v}, [H_{u',v'}, H_{n',m'}]] - [H_{u',v'}, [H_{u,v}, H_{n',m'}]]}{(1 - q_1)^2(1 - q_2)^2} - \frac{[\dots, H_{n',m'}]}{(1 - q_1)(1 - q_2)} \right)$$

The usual Leibniz rule and Jacobi identity in \mathcal{A} show that the expression above is

$$\Psi \left(\frac{[H_{n,m}, H_{n',m'}]}{(1 - q_1)(1 - q_2)} \right)$$

as required. □

3.21.

Akin with Subsection 2.11, we will denote elements of Λ_S by $f[X]$, and write:

$$(3.59) \quad \sum_{k=0}^{\infty} \frac{h_k}{z^k} = \wedge \bullet \left(-\frac{X}{z} \right)$$

$$(3.60) \quad \sum_{k=0}^{\infty} \frac{e_k}{(-z)^k} = \wedge \bullet \left(\frac{X}{z} \right)$$

The following notion is analogous to Proposition 2.12:

Definition 3.22. *For any $f[X] \in \Lambda_S$ and any variable z , define:*

$$(3.61) \quad f[X \pm \mathcal{O}_{\Delta}z] = \exp \left[\pm \sum_{k=1}^{\infty} \frac{p_k^{\dagger} z^k}{k} \right] \Big|_{\Delta} \cdot f[X]$$

as an element of $\Lambda_S \times K(S)[z]$.

Using (3.59), (3.60), (3.61), we may rewrite formulas (3.45) and (3.46) as:

(3.62)

$$\begin{aligned} \Psi(H_{n,m})(f[X]) &= \int_{0, X \prec |z_n| \prec \dots \prec |z_1| \prec \infty} \frac{\prod_{i=1}^n z_i^{\lfloor \frac{mi}{n} \rfloor - \lfloor \frac{m(i-1)}{n} \rfloor}}{\prod_{i=1}^{n-1} \left(1 - \frac{z_{i+1}q}{z_i}\right) \prod_{i < j} \zeta\left(\frac{z_j}{z_i}\right)} \\ &\wedge \bullet \left(-\frac{X}{z_1}\right) \dots \wedge \bullet \left(-\frac{X}{z_n}\right) \cdot f \left[X - \mathcal{O}_\Delta \sum_{i=1}^n z_i \right] \prod_{a=1}^n \frac{dz_a}{2\pi i z_a} \end{aligned}$$

and:

(3.63)

$$\begin{aligned} \Psi(H_{-n,m})(f[X]) &= \int_{0, X \prec |z_1| \prec \dots \prec |z_n| \prec \infty} \frac{(-q)^n \prod_{i=1}^n z_i^{\lfloor \frac{mi}{n} \rfloor - \lfloor \frac{m(i-1)}{n} \rfloor}}{\prod_{i=1}^{n-1} \left(1 - \frac{z_{i+1}q}{z_i}\right) \prod_{i < j} \zeta\left(\frac{z_j}{z_i}\right)} \\ &\wedge \bullet \left(\frac{X}{z_1 q}\right) \dots \wedge \bullet \left(\frac{X}{z_n q}\right) \cdot f \left[X + \mathcal{O}_\Delta \sum_{i=1}^n z_i \right] \prod_{a=1}^n \frac{dz_a}{2\pi i z_a} \end{aligned}$$

These formulas are proved by analogy with (2.38) and (2.39). The contours of integration are explained in the last paragraph of Subsection 2.13.

3.23.

We will now bridge Theorems 3.12 and 3.20. To any element $f[X] \in K_{\mathfrak{S}(k)}(S^k) \subset \Lambda_S$, we may associate universal classes on \mathcal{M}^s via the construction (1.9):

$$(3.64) \quad f[X] \xrightarrow{\Gamma_S} f[\mathcal{U}] := \pi_* \left(\mathcal{U}_1 \otimes \dots \otimes \mathcal{U}_k \otimes \rho^*(f[X]) \right)^{\mathfrak{S}(k)}$$

where \mathcal{U} is the universal sheaf on $\mathcal{M}^s \times S$, \mathcal{U}_i denotes its pull-back to $\mathcal{M}^s \times S^k$ via the i -th projection map $S^k \rightarrow S$, and $\pi : \mathcal{M}^s \times S^k \rightarrow \mathcal{M}^s$, $\rho : \mathcal{M}^s \times S^k \rightarrow S^k$ are the projection maps. Just like in the case of \mathbb{A}^2 , the universal classes generate the K -theory groups of moduli spaces of stable sheaves, as the following result shows.

Lemma 3.24. *Any element of $K(\mathcal{M}_d^s)$ is of the form (3.64), for some $k \gg d$.*

Proof. Theorem 1 of [15] shows that the class of the diagonal:

$$\Delta_{\mathcal{M}_d^s} \hookrightarrow \mathcal{M}_d^s \times \mathcal{M}_d^s$$

can be expressed as a Chern class of the virtual bundle:

$$\begin{array}{ccc} \mathcal{E} & \longleftarrow & \sum_{i=0}^2 (-1)^i \text{Ext}^i(\mathcal{F}, \mathcal{F}') \\ \downarrow & & \downarrow \\ \mathcal{M}_d^s \times \mathcal{M}_d^s & \longleftarrow & (\mathcal{F}, \mathcal{F}') \end{array}$$

Although the result of *loc. cit.* is stated in cohomology, it holds at the level of Chow groups $A^*(\mathcal{M}_d^s)$ with rational coefficients. Also, while this result is stated for a surface with trivial canonical class (the top option in Assumption S), an analogous argument works for a surface with negative canonical class (the bottom option in Assumption S), as shown in [11] for Hilbert schemes. Based on these facts, it is standard to show that any element of $A^*(\mathcal{M}_d^s)$ can be written in the form:

$$\pi_* \left(\text{ch}(\mathcal{U}_1) \cdot \dots \cdot \text{ch}(\mathcal{U}_k) \cdot \rho^*(g) \right) \in A^*(\mathcal{M}_d^s)$$

for some $k \gg d$ and some $g \in A^*(S^k)$. Since the Chern character $K(\mathcal{M}_d^s) \rightarrow A^*(\mathcal{M}_d^s)$ is an isomorphism (over \mathbb{Q} , as a consequence of Assumption S), then a simple application of the Grothendieck-Hirzebruch-Riemann-Roch theorem shows that any element of $K(\mathcal{M}_d^s)$ can be written as:

$$\pi_* \left(\mathcal{U}_1 \otimes \dots \otimes \mathcal{U}_k \otimes \rho^*(g) \right)$$

for some $g \in K(S^k)$. The formula above is equal to (3.64) for $f[X] = \sum_{\sigma \in \mathfrak{S}(k)} \sigma^*(g)$. □

Remark 3.25. *It would be very interesting to prove Lemma 3.24 without Assumption S (although in this case, one should rather work with a dg scheme model of the moduli space of stable sheaves, instead of the singular scheme \mathcal{M}^s). However, the argument provided above requires Assumption S in several crucial places.*

3.26.

We will need a geometric incarnation on the plethysm operation (3.61):

Proposition 3.27. *For any $f[X] \in K_{\mathfrak{S}(n)}(S^n)$, we have the following identity:*

$$(3.65) \quad f[\mathcal{U} \pm \mathcal{O}_\Delta z] = \pi_* \left[(\mathcal{U}_1 \pm \mathcal{O}_{\Delta_{1\bullet}} z) \dots (\mathcal{U}_k \pm \mathcal{O}_{\Delta_{n\bullet}} z) \otimes f[X] \right]^{\mathfrak{S}(n)}$$

as elements of $K(\mathcal{M}^s \times S)[z]$, where $\pi : \mathcal{M}^s \times S^n \times S \rightarrow \mathcal{M}^s \times S$ is the projection map that forgets the middle n factors of S , and \bullet indicates the surviving factor of S .

Proof. Let us prove (3.65) in the case $\pm = +$, as the case $\pm = -$ is analogous, and so we leave it to the interested reader. The right-hand side of (3.65) is:

$$\begin{aligned} & \sum_{k=0}^n z^k \cdot \left(\sum_{\{a_1, \dots, a_k\} \subset \{1, \dots, n\}} \pi_* \left[\bigotimes_{s \in \{1, \dots, n\} \setminus \{a_1, \dots, a_n\}} \mathcal{U}_s \otimes \mathcal{O}_{\Delta_{a_1 \dots a_k \bullet}} \otimes f[X] \right] \right)^{\mathfrak{S}(n)} \\ &= \sum_{k=0}^n z^k \cdot \left(\pi_*^{(k)} \left[\mathcal{U}_1 \otimes \dots \otimes \mathcal{U}_{n-k} \otimes f[X] \Big|_{S^{n-k} \times \Delta_{n-k+1 \dots n \bullet}} \right] \right)^{\mathfrak{S}(n-k) \times \mathfrak{S}(k)} \end{aligned}$$

where $\pi^{(k)} : \mathcal{M}^s \times S^{n-k} \times S \rightarrow \mathcal{M}^s \times S$ is the projection map that forgets the middle $n - k$ factors of S (the two rows above are equal because of Frobenius reciprocity). By (3.40), the bottom row of the expression above is equal to:

$$\begin{aligned} & \sum_{k=0}^n z^k \cdot \pi_*^{(k)} \left[\mathcal{U}_1 \otimes \dots \otimes \mathcal{U}_{n-k} \otimes h_k^\dagger(f[X]) \right]^{\mathfrak{S}(n-k)} \stackrel{(3.64)}{=} \\ &= \sum_{k=0}^n z^k \cdot \Gamma_S \left(h_k^\dagger(f[X]) \right) \stackrel{(3.36)}{=} \Gamma_S \left(\exp \left[\sum_{k=1}^\infty \frac{p_k^\dagger z^k}{k} \right] \Big|_\Delta \cdot f[X] \right) \\ &= \Gamma_S (f[X + \mathcal{O}_\Delta z]) \end{aligned}$$

which is equal to the left-hand side of (3.65). □

The following is Proposition 5.12 of [23] (see also Theorem 3.16 of [21] for the $S = \mathbb{A}^2$ case). In *loc. cit.*, the objects denoted by $f[\dots]$ in formulas (3.66) and (3.67) were actually understood to be the right-hand side of (3.65). The fact they match our current notion of plethysm (3.61) is the content of Proposition 3.27.

Proposition 3.28. *In terms of universal classes, (3.18)–(3.19) read:*

$$(3.66) \quad \Phi(H_{n,m})(f[\mathcal{U}]) = \int_{\mathcal{U} \prec |z_n| \prec \dots \prec |z_1| \prec 0, \infty} \frac{\prod_{i=1}^n z_i^{\lfloor \frac{mi}{n} \rfloor - \lfloor \frac{m(i-1)}{n} \rfloor}}{\prod_{i=1}^{n-1} \left(1 - \frac{z_{i+1}q}{z_i}\right) \prod_{i < j} \zeta\left(\frac{z_j}{z_i}\right)} \\ \wedge^\bullet \left(-\frac{\mathcal{U}}{z_1}\right) \dots \wedge^\bullet \left(-\frac{\mathcal{U}}{z_n}\right) \otimes f \left[\mathcal{U} - \mathcal{O}_\Delta \sum_{i=1}^n z_i \right] \prod_{a=1}^n \frac{dz_a}{2\pi i z_a}$$

and:

$$(3.67) \quad \Phi(H_{-n,m})(f[\mathcal{U}]) = \int_{\mathcal{U} \prec |z_1| \prec \dots \prec |z_n| \prec 0, \infty} \frac{(-q)^n \prod_{i=1}^n z_i^{\lfloor \frac{mi}{n} \rfloor - \lfloor \frac{m(i-1)}{n} \rfloor}}{\prod_{i=1}^{n-1} \left(1 - \frac{z_{i+1}q}{z_i}\right) \prod_{i < j} \zeta\left(\frac{z_j}{z_i}\right)} \\ \wedge^\bullet \left(\frac{\mathcal{U}}{z_1q}\right) \dots \wedge^\bullet \left(\frac{\mathcal{U}}{z_nq}\right) \otimes f \left[\mathcal{U} + \mathcal{O}_\Delta \sum_{i=1}^n z_i \right] \prod_{a=1}^n \frac{dz_a}{2\pi i z_a}$$

as elements of $K(\mathcal{M}^s \times S)$, where $q_1 + q_2 = [\Omega_S^1]$ and $q = q_1q_2 = [K_S]$.

Proof. of Theorem 1.2: Comparing formulas (3.62)–(3.63) with (3.66)–(3.67), we observe that they are one and the same integral but over slightly different contours. The difference is in which side of the z_1, \dots, z_n contours the pole at 0 lies. As we explained in the proof of Theorem 1.1, when $m > \pm nr$, the integrand is actually regular at 0, so formulas (3.62)–(3.63) produce the same result as (3.66)–(3.67). □

3.29.

Consider a weak action $\mathcal{A} \overset{\Phi}{\curvearrowright}_S K(X)$. Given any element $v \in K(X)$, the submodule generated by v will refer to the subset $\mathcal{A} \cdot v \subset K(X)$ consisting of linear combinations of the following operators applied to v :

$$K(X) \xrightarrow{\Phi(a_k)} K(X \times S) \xrightarrow{\Phi(a_{k-1}) \boxtimes \text{Id}_S} \dots \\ \dots \xrightarrow{\Phi(a_1) \boxtimes \text{Id}_{S^{k-1}}} K(X \times S^k) \xrightarrow{\text{Id}_X \boxtimes \gamma} K(X \times S^k) \xrightarrow{\pi_*} K(X)$$

where $k \in \mathbb{N}$, $a_1, \dots, a_{k-1}, a_k \in \mathcal{A}$, $\gamma \in K(S^k)$ are arbitrary, and $\pi : X \times S^k \rightarrow X$ denotes the projection. For the action of Theorem 3.12, we consider the

submodules:

$$(3.68) \quad \mathcal{A} \cdot \mathbf{1}_d \subset K(\mathcal{M}^s)$$

where $\mathbf{1}_d \in K(\mathcal{M}_d^s)$ denotes the structure sheaf of the subvariety $\mathcal{M}_d^s \subset \mathcal{M}^s$. Since the algebra \mathcal{A} contains the operators (3.16) of tensor product with the universal sheaf, then Lemma 3.24 implies that:

$$K(\mathcal{M}_d^s) \subset \mathcal{A} \cdot \mathbf{1}_d$$

for all $d \geq \lceil \frac{r-1}{2r} c_1^2 \rceil$. Therefore, we conclude the following:

Proposition 3.30. *The $\mathcal{A}^{(r)}$ -module $K(\mathcal{M}^s)$ is generated by $\{\mathbf{1}_d\}_{d \geq \lceil \frac{r-1}{2r} c_1^2 \rceil}$, i.e.*

$$(3.69) \quad K(\mathcal{M}^s) = \bigcup_{d=\lceil \frac{r-1}{2r} c_1^2 \rceil}^{\infty} \mathcal{A} \cdot \mathbf{1}_d$$

Moreover, these submodules are contained inside each other, i.e.:

$$(3.70) \quad \mathcal{A} \cdot \mathbf{1}_d \supset \mathcal{A} \cdot \mathbf{1}_{d-1}$$

for all d .

The final claim in Proposition 3.30 follows from:

$$(3.71) \quad \Phi(H_{n,m})^{(\circ)} \cdot \mathbf{1}_d = \begin{cases} \mathbf{1}_{d-n} & \text{if } m = 0 \\ 0 & \text{if } m \in \{-1, \dots, -nr + 1\} \end{cases}$$

for the skyscraper sheaf $\circ \in K(S)$ at any point on S . Formula (3.71) is an immediate consequence of relation (3.66) for $f = 1$ (see also Proposition 4.15 of [19]).

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DEPARTMENT OF MATHEMATICS, MIT
CAMBRIDGE, MA 02139, USA

E-mail address: `andrei.negut@gmail.com`

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