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Local rigidity, contact homeomorphisms, and conformal factors

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We show that if the image of a Legendrian submanifold under a contact homeomorphism (*i.e.* a homeomorphism that is a C^0-limit of contactomorphisms) is smooth then it is Legendrian, assuming only positive local lower bounds on the conformal factors of the approximating contactomorphisms. More generally the analogous result holds for coisotropic submanifolds in the sense of [H15]. This is a contact version of the Humilière-Leclercq-Seyfaddini coisotropic rigidity theorem in C^0 symplectic geometry, and the proof adapts the author's recent re-proof of that result in [U22] based on a notion of local rigidity of points on locally closed subsets. We also provide two different flavors of examples showing that a contact homeomorphism can map a submanifold that is transverse to the contact structure to one that is smooth and tangent to the contact structure at a point.

1. Introduction

The Eliashberg-Gromov symplectic rigidity theorem, stating that the group of symplectic diffeomorphisms of a symplectic manifold is C^0-closed in the group of all diffeomorphisms, led to the notion of a symplectic homeomorphism as a homeomorphism which is a C^0-limit of symplectic diffeomorphisms, and to the field of "C^0 symplectic topology," studying the properties of symplectic manifolds that are invariant under symplectic homeomorphisms. Analogous ideas in the setting of contact manifolds have only fairly recently begun to be developed. In particular [MüSp14], following older ideas of Eliashberg, gave the first full proof in the literature of the contact version of the Eliashberg-Gromov theorem, and [Mü19] gave an alternative proof based on a characterization of contact diffeomorphisms in terms of their effect on a version of Eliashberg's shape invariant.

Throughout this paper, a **contact homeomorphism** of a contact manifold (Y, ξ) is by definition a homeomorphism of Y that arises as limit of some sequence of contact diffeomorphisms with respect to the C^0 (compact-open) topology (this differs from the usage in [MüSp15], in which the contact homeomorphism group is a certain subgroup of the group of topological automorphisms mentioned at the end of this paragraph, and thus is significantly smaller than what we define as the contact homeomorphism group). If ξ is cooriented, say with $\xi = \ker \alpha$ where $\alpha \in \Omega^1(Y)$, it is well-known that questions about contact diffeomorphisms of (Y, ξ) can be converted to questions about symplectic diffeomorphisms of the symplectization $(\mathbb{R} \times Y, d(e^r\alpha))$ (where r is the coordinate on $\mathbb{R}$). Specifically, a diffeomorphism $\psi\colon Y \to Y$ obeys $\psi^*\alpha = f\alpha$ for a smooth function $f\colon Y \to (0, \infty)$ if and only if the diffeomorphism $\Psi_f\colon \mathbb{R} \times Y \to \mathbb{R} \times Y$ defined by $\Psi_f(r, y) = (r - \log f(y), \psi(y))$ is a symplectomorphism. An analogous device is *not* in general available for contact homeomorphisms, essentially because of the dependence of Ψ_f above on the conformal factor f, which in turn depends on the derivative of ψ. It is quite possible for a sequence $\{\psi_m\}$ of contactomorphisms to C^0-converge to a homeomorphism ψ while the logarithms of the conformal factors f_m given by $\psi_m^*\alpha = f_m\alpha$ are unbounded (see Section 5.2 for one family of examples), in which case the Ψ_{f_m} do not converge. On the other hand, in [MüSp15] the authors consider the more restricted class of "topological automorphisms" of a contact manifold, defined to be limits C^0-limits of sequences of contactomorphisms $\{\psi_m\}_{m=1}^{\infty}$ with the property that the corresponding conformal factors f_m converge uniformly.

The main question motivating this paper is the following:

Question 1.1. Let (Y, ξ) be a contact manifold and $\psi\colon Y \to Y$ a contact homeomorphism. Suppose that $\Lambda \subset Y$ is a Legendrian submanifold such that $\psi(\Lambda)$ is a smooth submanifold. Must $\psi(\Lambda)$ be Legendrian?

More generally we will consider the situation where Λ is coisotropic in the sense of Definition 4.1 (this is the same definition used in [H15]); under this definition, Legendrian submanifolds are precisely the coisotropic submanifolds of dimension $\frac{1}{2}(\dim Y - 1)$. In the C^0 symplectic world, the main result of [HLS15] asserts that the image under a symplectic homeomorphism of a coisotropic submanifold of a symplectic manifold is coisotropic provided that is is smooth. In [RZ18, Theorem 1.3] this is used to deduce an affirmative answer to Question 1.1 (and also its analogue for coisotropic submanifolds) in the special case that ψ is a topological automorphism in the sense of [MüSp15].

However the authors of [RZ18] express doubt (in their Remark 4.4) that the same conclusion continues to hold when one considers fully general contact homeomorphisms. To indicate why one indeed should not blithely assume that obvious analogues of C^0 symplectic results hold in the C^0 contact context, note that [LS94, Theorem 2] shows that a smooth embedding of a compact n-dimensional manifold into $\mathbb{R}^{2n}$ that is a C^0-limit of Lagrangian embeddings is itself Lagrangian, whereas *any* smooth embedding of an n-dimensional manifold into a $(2n+1)$-dimensional contact manifold that satisfies a mild homotopy-theoretic hypothesis can be C^0-approximated by Legendrian embeddings (see [Et, Theorem 2.5] if $n = 1$ and [CE12, Theorem 7.25] if $n > 1$). It is not clear from the proofs of the latter results whether the approximating Legendrian embeddings can be arranged to be the restrictions of a uniformly convergent sequence of contactomorphisms.

While we do not resolve Question 1.1 here, we do give an affirmative answer under a significantly weaker hypothesis on the conformal factors of the approximating sequence $\{\psi_m\}$ for the contact homeomorphism ψ than in [RZ18, Theorem 1.3] (which required these conformal factors to converge uniformly). Specifically we will require ψ to be **bounded below near** all points of C in the sense of Definition 2.5; for example if α is a contact form for ξ and if the approximating sequence ψ_m has $\psi_m^*\alpha = f_m\alpha$ this will hold if the functions $|f_m|$ satisfy m-independent positive lower bounds on some neighborhood of C. Our main result is then:

Theorem 1.2. Suppose that (Y, ξ) is a contact manifold and $\psi\colon\ Y \to Y$ is a contact homeomorphism. If C is a coisotropic submanifold of Y such that $\psi(C)$ is smooth and such that ψ is bounded below near every point of C, then $\psi(C)$ is coisotropic.

Remark 1.3. A contact form α on a $(2n+1)$-dimensional manifold Y induces a Borel measure μ_α on Y by setting $\mu_\alpha(E) = \int_E \alpha \wedge (d\alpha)^{\wedge n}$. If

$\psi\colon Y \to Y$ is a contactomorphism, say with $\psi^*\alpha = f\alpha$, then evidently one has

$$\mu_\alpha(\psi(E)) = \int_E |f|^{n+1}\alpha \wedge (d\alpha)^{\wedge n}$$

for all Borel sets E. Thus imposing local lower bounds on the absolute values of the functions f_m given by $\psi_m^*\alpha = f_m\alpha$ for an approximating sequence ψ_m for a contact homeomorphism ψ amounts to imposing lower bounds on the ratios $\frac{\mu_\alpha(\psi_m(U))}{\mu_\alpha(U)}$ for appropriate open sets U, or equivalently local lower bounds on the Jacobian determinants of the ψ_m when these are expressed in local coordinates.

If $\{\psi_m\}_{m=1}^\infty$ is any sequence of contactomorphisms such that both ψ_m and ψ_m^{-1} obey uniform local bounds on their Lipschitz constants, then the Arzelà-Ascoli theorem implies that some subsequence of $\{\psi_m\}$ converges in the compact-open topology to a contact homeomorphism ψ, and then both ψ and ψ^{-1} will be bounded below near every point.

Neither the hypothesis nor the conclusion of Theorem 1.2 is manifestly preserved under replacing ψ by ψ^{-1}; rather, applying Theorem 1.2 to ψ^{-1} leads to the statement that if ψ is a contact homeomorphism that is bounded *above* near all points of N and if N and $\psi(N)$ are both smooth submanifolds with N *not* coisotropic then $\psi(N)$ is also not coisotropic. The following theorem, proven in Section 5, shows however that the situation is different if instead of asking whether the whole image $\psi(N)$ is coisotropic one just asks whether it is coisotropic at an isolated point.

Theorem 1.4. For any contact manifold (Y, ξ) of dimension $2n+1 \geq 3$, there exist contact homeomorphisms $\psi\colon Y \to Y$ and smooth n-dimensional submanifolds $\Lambda \subset Y$ such that $\psi(\Lambda)$ is a smooth submanifold and, for some point $p \in \Lambda$, we have $T_p\Lambda \not\subset \xi_p$ but $T_{\psi(p)}\psi(\Lambda) \subset \xi_p$. In fact, ψ can be chosen to have any of the following properties:

(i) ψ is bounded both above and below near p; or

(ii) ψ is bounded below but not above near p; or

(iii) ψ restricts to Λ as a smooth map, and is bounded above but not below near p.

Remark 1.5. In [Mas16, Section 5.3.3], Massot suggests an alternative definition of a contact homeomorphism of a compact contact manifold (Y, ξ) as a homeomorphism of Y that is bi-Lipschitz with respect to one and hence any Carnot-Carathéodory distance induced by ξ (declaring the distance between two points to be the infimal length of a Legendrian arc connecting them, as measured by an auxiliary Riemannian metric). It is noted on [Mas16, p. 89] that this is probably a different notion than the one based on C^0-limits that we use in this paper, and the examples in Section 5.2 (which are the ones that we use to prove variation (iii) of Theorem 1.4) confirm this expectation. Indeed, restricting to the three-dimensional case for ease of notation, these contact homeomorphisms ψ are given in the hypersurface $\{y = 0\}$ within a Darboux cube $((-1,1)^3, \ker(dz - ydx))$ by $\psi(x, 0, z) = (x, 0, g(z))$ where the function $g\colon \mathbb{R} \to \mathbb{R}$, a general formula for appears in Proposition 5.11, typically has $g'(0) = 0$ and can even be arranged to vanish to infinite order at 0 as in Example 5.14. In particular ψ preserves the z axis and restricts to it as a non-bi-Lipschitz function (with respect either to the standard metric or the restriction of the Carnot-Carathéodory distance).

1.1. Outline of the paper

The proof of Theorem 1.2 is a contact version of the author's recent reproof in [U22] of the Humilière-Leclercq-Seyfaddini theorem [HLS15] on C^0-rigidity of coisotropic submanifolds of symplectic manifolds. As in [U22], the plan is to characterize coisotropic submanifolds in terms of a notion that we call "local rigidity" and prove that this notion is invariant under the appropriate class of homeomorphisms. The difference in length between Sections 2 through 4 of this paper and [U22, Sections 1 and 2] is explained by a combination of the contact geometric case being objectively more complicated and the theory of coisotropic submanifolds in contact geometry being less well-developed than that of their counterparts in symplectic geometry.[1]

Section 2 defines our notion of a point p on a locally closed subset N of a contact manifold being locally rigid with respect to N. The symplectic version of this from [U22] was in terms of the Hofer energy needed to locally disjoin arbitrarily small neighborhoods of p from N, and the contact version introduced here is essentially the same but with the Shelukhin norm [Sh16] on the identity component of the contactomorphism group (defined using absolute values of contact Hamiltonians) used in place of the Hofer norm. A crucial fact about local rigidity is then Proposition 2.6, asserting that if p is locally rigid with respect to N and if ψ is a contact homeomorphism of

[1]One indication of this underdevelopment is that the very recent sources [RZ18],[LdL19], [Mü19] all have conflicting definitions of a coisotropic submanifold of a contact manifold; as mentioned earlier our definition is that used in [H15] and [RZ18].

Y that is bounded below near p in the sense of Definition 2.5, then $\psi(p)$ is locally rigid with respect to $\psi(N)$. The need for the boundedness hypothesis can be understood in terms of the fact that the Shelukhin norm $\|\cdot\|$ (unlike the Hofer norm in symplectic topology) is not conjugation-invariant; rather $\|\psi \circ \phi \circ \psi^{-1}\|$ can be bounded in terms of $\|\phi\|$ and the conformal factor of ψ.

Section 3 proves Corollary 3.4, asserting that points on Legendrian submanifolds Λ are always locally rigid; this is the only point in the paper that depends on pseudoholomorphic curve techniques. To prove it we show in Theorem 3.3 that, under suitable assumptions, there is a positive lower bound on the Shelukhin norm of a contactomorphism that disjoins a given pre-Lagrangian submanifold from Λ; this follows from Lemma 3.6 which establishes a lower bound for the Hofer norm of a symplectomorphism of the symplectization that disjoins a compact Lagrangian submanifold from $\mathbb{R} \times \Lambda$, using a number of technical ingredients from [DS16] and references therein. Lemma 3.6 requires a rather restrictive hypothesis—hypertightness in the sense of Definition 3.1—on Λ, but because our definition of local rigidity is indeed local we can use tubular neighborhood theorems to deduce relevant information from Lemma 3.6 about any Legendrian submanifold, even one that is not closed as a subset. We also observe in Corollary 3.5 that Theorem 3.3 implies, in the special case of hypertight Legendrians, [RZ18, Conjecture 1.10] on the contact analogue of Chekanov-Hofer pseudometrics on orbits of submanifolds.

The proof of Theorem 1.2 is completed at the end of Section 4, in which we characterize coisotropic submanifolds in terms of local rigidity. Proposition 4.11 shows that a point p on a submanifold C is locally rigid only if C is coisotropic at p; this follows by a variation on arguments from [U14],[RZ18]. Unlike in the symplectic case (see [U22, Theorem 2.1]) it is not known to the author whether the converse to this holds, except in the case that C is Legendrian in which case the converse is already given by Corollary 3.4. The reason is that, on non-Legendrian coisotropic submanifolds C of contact manifolds (Y, ξ), there are two fundamentally different types of points p: those for which $T_pC \subset \xi_p$, and those at which C is transverse to ξ. However, as we show in Corollary 4.10, points of the latter type form an open dense subset of C, and moreover any such point is contained in a Legendrian submanifold that is in turn contained in C. (The behavior of C near those points p where $T_pC \subset \xi_p$ can, on the other hand, be quite complicated, cf. [H15].) Given Corollary 3.4 and Proposition 4.11, it then follows that the coisotropic submanifolds are precisely those submanifolds of a contact manifold for which an open and dense subset of the points are locally rigid;

Proposition 2.6 proves that this property is preserved under contact homeomorphisms that are bounded below near every point of the submanifold, thus proving Theorem 1.2.

The final Section 5 explains the examples referenced in Theorem 1.4, whose proof is completed at the very end of the paper. One of these constructions (see Section 5.1) is obtained by a straightforward modification of a construction from [BO16, Section 4]; this yields a contact homeomorphism ψ that is bounded both above and below, which maps a codimension-two contact submanifold Z (the locus where $x_1 = y_1 = 0$ in the notation of Proposition 5.1) to an explicit non-contact submanifold, though the behavior of the contact homeomorphism away from this contact submanifold seems difficult to understand. By restricting to submanifolds of Z one obtains in Corollary 5.2 the examples indicated in item (i) of Theorem 1.4, as well as similar examples which, instead of being Legendrian, are coisotropic of some codimension smaller than $n+1$. The other construction (in Section 5.2) is perhaps more distinctively contact-geometric, and uses the flow of an explicit time-dependent contact Hamiltonian vector field on the complement of a Legendrian torus T that extends continuously over the torus and whose flow contracts small neighborhoods of T by increasingly large factors as one approaches T. This construction, unlike the other one, leads to $\psi|_\Lambda$ being a smooth map (not just to $\psi(\Lambda)$ being a smooth submanifold). In fact the approximating sequence ψ_m to ψ has the property that, where Λ is as in Proposition 1.4, $\psi_m|_\Lambda$ converges to $\psi|_\Lambda$ in C^1 (conceivably this could be improved to C^∞ for a different choice of approximating sequence). We obtain a rather clearer global understanding of the examples in Section 5.2 than we do of those in Section 5.1; in fact for a variation on the construction that results in $\psi(\Lambda)$ only being a C^1-submanifold rather than a smooth one we are even able to write down an explicit formula for ψ in Example 5.12.

Note that for ψ as in either Section 5.1 or Section 5.2 (corresponding to variations (i) and (iii) of Theorem 1.4), Theorem 1.2 is applicable to ψ^{-1}, and shows that ψ^{-1} cannot map a Legendrian submanifold to a non-Legendrian submanifold, whereas by Theorem 1.4 ψ^{-1} does map the submanifold $\psi(\Lambda)$ that is Legendrian *at a point*[2] to a non-Legendrian submanifold.

[2]In fact, inspection of the examples shows that $\psi(\Lambda)$ has a codimension-one submanifold consisting of points at which it is Legendrian.

2. Local rigidity and boundedness

Our proof of Theorem 1.2 is based on characterizing coisotropic submanifolds of contact manifolds in terms of a notion of local rigidity. In the symplectic context similar ideas were developed in [U22]; the arguments in the contact context require somewhat more care due to issues relating to conformal factors. These issues lead to an addditional hypothesis in our invariance statement, namely Proposition 2.6, compared to the symplectic case ([U22, Proposition 1.4]), and this is the reason for the boundedness hypothesis in Theorem 1.2.

Local rigidity is, true to its name, a local property; consequently there is no need to make any compactness or coorientability hypotheses on our contact manifold (Y,ξ), because we can always localize to subsets U having compact closure with $\xi|_{\bar{U}}$ coorientable.

If W is an open subset of a contact manifold (Y,ξ) let $\mathcal{C}_W(Y,\xi)$ denote the space of smooth time-dependent contact vector fields $\mathbb{V}=(V_t)_{t\in[0,1]}$ having compact support contained in $[0,1]\times W$. For $t\in[0,1]$ we write $\psi^{\mathbb{V},t}$ for the time-t flow of such a vector field. A choice of contact form α for $\xi|_W$ (assuming that one exists, *i.e.* that $\xi|_W$ is coorientable, as will be true for small enough W) sets up a one-to-one correspondence between $\mathcal{C}_W(Y,\xi)$ and the space of smooth functions $H\colon [0,1]\times Y\to\mathbb{R}$ having compact support contained in $[0,1]\times W$, by setting $H(t,\cdot)=\alpha(V_t)$.

Definition 2.1. Given a contact manifold (Y,ξ), a subset $N\subset Y$, open subsets $U,W\subset Y$ with $\bar{U}\subset W$ and $N\cap W$ closed as a subset of W, and a one-form α on W with $\ker\alpha=\xi|_W$, we define the **α-disjunction energy of U and N rel W** as

$$e_\alpha^W(U,N)=\inf\left\{\int_0^1\max_W|\alpha(V_t)|dt\ \middle|\ \mathbb{V}=(V_t)_{t\in[0,1]}\in\mathcal{C}_W(Y,\xi),\ \psi^{\mathbb{V},1}(\bar{U})\cap N=\varnothing\right\}.$$

We record some straightforward properties of this quantity, leaving proofs to the reader:

Proposition 2.2. For $(Y,\xi),N,U,W,\alpha$ as in Definition 2.1:

(i) If $\beta = f\alpha$ is another contact form inducing the contact structure $\xi|_W$ on W, then

$$\left(\inf_W |f|\right) e^W_\alpha(U,N) \leq e^W_\beta(U,N) \leq \left(\sup_W |f|\right) e^W_\alpha(U,N).$$

(ii) If $\phi\colon Y \to Y'$ is an isocontact embedding between contact manifolds of the same dimension and if α' is a contact form on $\phi(W)$, then

$$e^W_{\phi^*\alpha'}(U,N) = e^{\phi(W)}_{\alpha'}(\phi(U),\phi(N)).$$

(iii) If $N \cap W = N' \cap W$ then $e^W_\alpha(U,N) = e^W_\alpha(U,N')$.

(iv) If $W \subset W'$, if $\alpha'|_W = \alpha$, and if $N \cap W'$ is closed in W' then $e^{W'}_{\alpha'}(U,N) \leq e^W_\alpha(U,N)$.

(v) If $N \subset N'$ with $N' \cap W$ closed in W then $e^W_\alpha(U,N') \geq e^W_\alpha(U,N)$.

Definition 2.3. Let (Y,ξ) be a contact manifold, $N \subset Y$ a locally closed subset, and $p \in N$. We say p is **locally rigid with respect to** N if there is a neighborhood W of p in Y having compact closure such that $N \cap W$ is closed in W and, for every neighborhood U of p with $\bar{U} \subset W$, we have $e^W_\alpha(U,N) > 0$ for one and hence any contact form α for $\xi|_W$ that extends continuously to $\bar{W}$.

(That our definition of local rigidity is independent of the choice of contact form on $\bar{W}$ representing ξ is immediate from Proposition 2.2 (i) and the requirement in the definition that $\bar{W}$ be compact.)

Here are some quick consequences of Proposition 2.2 and Definition 2.3:

Proposition 2.4. For (Y,ξ) a contact manifold, $N, N' \subset Y$ locally closed, and $p \in N$:

(i) If $N \subset N'$ and p is locally rigid with respect to N then p is locally rigid with respect to N'.

(ii) If p is locally rigid with respect to N and if $(\hat{Y},\hat{\xi})$ is another contact manifold containing a locally closed subset $\hat{N}$ and a point $\hat{p} \in \hat{N}$ such that there is a contactomorphism ϕ between neighborhoods V of p in Y and $\hat{V}$ of $\hat{p}$ in $\hat{Y}$ satisfying $\phi(p) = \hat{p}$ and $\phi(N \cap V) = \hat{N} \cap \hat{V}$, then $\hat{p}$ is locally rigid with respect to $\hat{N}$.

Proof. (i) follows immediately from Proposition 2.2 (v).

For (ii), first note that Proposition 2.2 (iv) implies that if p is locally rigid with respect to N then $e^W_\alpha(U,N) > 0$ for *all* sufficiently small precompact neighborhoods W of p and all open $U \subset W$ with $p \in U$ and $\bar{U} \subset W$, and for any contact form α for the restriction of ξ to the closure of such a neighborhood. If necessary, shrink the open subset V in the assumption of (ii) so that ξ has coorientable restriction to a neighborhood of $\bar{V}$. Choosing a sufficiently small W that in particular is contained in V, and letting $\hat{\alpha}$ be an arbitrary contact form for the restriction of $\hat{\xi}$ to a neighborhood of $\phi(\bar{V})$, Proposition 2.2 (ii) and (iii) then imply that $e^{\phi(W)}_{\hat{\alpha}}(\phi(U),\hat{N}) = e^W_{\phi^*\hat{\alpha}}(U,N)$ for every neighborhood U of p having $\bar{U} \subset W$, which suffices to prove the local rigidity of $\hat{p} = \phi(p)$ with respect to $\hat{N}$. □

As mentioned in the introduction, a **contact homeomorphism** of a contact manifold (Y,ξ) is by definition a homeomorphism $\psi\colon Y \to Y$ that is a limit of a sequence of contact diffeomorphisms with respect to the compact-open topology. (Throughout the paper we refer to convergence with respect to the compact-open topology as "C^0-convergence.") Note that since the homeomorphism group of Y is a topological group with respect to the compact-open topology by [Ar46, Theorem 4], the contact homeomorphisms of (Y,ξ) form a subgroup of the homeomorphism group.

Definition 2.5. Let (Y,ξ) be a contact manifold, let $p \in Y$, and let $\psi\colon Y \to Y$ be a contact homeomorphism.

(A) We say that ψ is **bounded below near** p if there are:
  (i) a sequence $\{\psi_m\}$ of contactomorphisms C^0-converging to ψ;
  (ii) a neighborhood $\mathcal{O}$ of p such that the closure $\bar{\mathcal{O}}$ is compact and $\xi|_{\bar{\mathcal{O}}}$ is coorientable; and
  (iii) contact forms α and α' for the restrictions of ξ to neighborhoods of $\bar{\mathcal{O}}$ and $\psi(\bar{\mathcal{O}})$, respectively, such that $(\psi_m^*\alpha')|_{\bar{\mathcal{O}}} = f_m\alpha|_{\bar{\mathcal{O}}}$ where

$$\inf_{m\in\mathbb{Z}_+}\inf_{p\in\bar{\mathcal{O}}}|f_m(p)| > 0. \tag{2.1}$$

(B) We say that ψ is **bounded above near** p if there are $\{\psi_m\}$, $\mathcal{O}$, α, α' as in (A)(i-iii) such that, instead of (2.1), we have $\sup_{m\in\mathbb{Z}_+}\sup_{p\in\bar{\mathcal{O}}}|f_m(p)| < \infty$.

Evidently ψ is bounded below near p if and only if ψ^{-1} is bounded above near $\psi(p)$.

Proposition 2.6. Let (Y,ξ) be a contact manifold, let $N \subset Y$ be locally closed, and suppose that $p \in N$ is locally rigid with respect to N. If $\psi\colon\, Y \to Y$ is a contact homeomorphism that is bounded below near p then $\psi(p)$ is locally rigid with respect to $\psi(N)$.

Proof. Let $\{\psi_m\}_{m=1}^{\infty}$ be a sequence of contactomorphims of Y, $\mathcal{O}$ a neighborhood of p, and α, α' contact forms on $\bar{\mathcal{O}}$ and $\psi(\bar{\mathcal{O}})$ as in Definition 2.5(A). Let $W \subset \mathcal{O}$ be a precompact neighborhood of p such that $N \cap W$ is closed in W and, for every neighborhood U of p with $\bar{U} \subset W$, we have $e_{\alpha}^{W}(U,N) > 0$. (As in the proof of Proposition 2.4(ii) we are free to assume that W is small enough to be contained in $\mathcal{O}$ by Proposition 2.2(iv).) Let $W' = \psi(W)$, and suppose that U' is an arbitrary neighborhood of $\psi(p)$ such that $\overline{U'} \subset W'$. It suffices to show that $e_{\alpha'}^{W'}(U', \psi(N)) > 0$.

So suppose that $\mathbb{V} = (V_t)_{t\in[0,1]}$ is a time-dependent contact vector field supported in W' such that $\psi^{\mathbb{V},1}(\overline{U'}) \cap \psi(N) = \varnothing$. Thus

$$(\psi^{-1} \circ \psi^{\mathbb{V},1} \circ \psi)(\overline{\psi^{-1}(U')}) \cap N = \varnothing. \tag{2.2}$$

Using [Ar46, Theorem 4], the fact that $\psi_m \to \psi$ in the compact-open topology implies that likewise $\psi_m^{-1} \circ \psi^{\mathbb{V},1} \circ \psi_m \to \psi^{-1} \circ \psi^{\mathbb{V},1} \circ \psi$ in the compact-open topology. So for all sufficiently large m, (2.2) implies that $(\psi_m^{-1} \circ \psi^{\mathbb{V},1} \circ \psi_m)(\overline{\psi^{-1}(U')}) \cap N = \varnothing$, *i.e.*,

$$\psi^{\psi_{m*}^{-1}\mathbb{V},1}(\overline{\psi^{-1}(U')}) \cap N = \varnothing.$$

Moreover since $\mathbb{V}$ has compact support within $[0,1] \times \psi(W)$, if m is sufficiently large (so that $\psi_m \circ \psi^{-1}$ is close enough to the identity) then the support of $\psi_{m*}^{-1}\mathbb{V}$ will be contained in $[0,1] \times W$. Hence for sufficiently large m

$$\int_0^1 \max_W |\alpha(\psi_{m*}^{-1} V_t)| dt \geq e_{\alpha}^{W}(\psi^{-1}(U'), N). \tag{2.3}$$

Now for $x \in Y$,

$$|\alpha'_{\psi_m(x)}(V_t)| = |(\psi_m^{*}\alpha')_x(\psi_{m*}^{-1} V_t)| = |f_m(x)\alpha_x(\psi_{m*}^{-1} V_t)|$$

where $f_m\colon\, \bar{\mathcal{O}} \to \mathbb{R}$ are as in Definition 2.5. So if we write $c = \inf_m \inf_{\bar{\mathcal{O}}} |f_m|$ (which is strictly positive by (2.1)) we find from (2.3) that

$$\int_0^1 \max_{W'} |\alpha'(V_t)| dt \geq c \int_0^1 \max_W |\alpha(\psi_{m*}^{-1} V_t)| dt \geq c e_{\alpha}^{W}(\psi^{-1} U', N) > 0.$$

Since $(V_t)_{t\in[0,1]}$ was arbitrary subject to its support being compactly contained in $[0,1]\times W'$ and its time-one map disjoining $\overline{U'}$ from $\psi(N)$, this suffices to show that $e^{W'}_{\alpha'}(U',\psi(N))>0$. $\square$

To give a little more context for Definition 2.5, we provide a criterion that allows one to see that some contact homeomorphisms ψ are *not* bounded below near a point without checking every sequence of contactomorphisms that C^0-converges to ψ. We apply this to some specific examples in Corollary 5.10.

Proposition 2.7. Let (Y,ξ) be a $(2n+1)$-dimensional contact manifold and suppose that a contact homeomorphism $\psi\colon Y\to Y$ is bounded below near $p\in Y$. Then for a sufficiently small neighborhood U of p with $\bar{U}$ compact and for one and hence every choice of contact forms α for $\xi|_{\bar{U}}$ and α' for $\xi|_{\psi(\bar{U})}$ there is $\delta>0$ (depending on α,α') such that, for every nonempty open subset $V\subset U$, we have

$$\frac{\int_{\psi(V)}\alpha'\wedge(d\alpha')^{\wedge n}}{\int_V\alpha\wedge(d\alpha)^{\wedge n}}\geq\delta. \tag{2.4}$$

Proof. We may choose U such that both U and $\psi(U)$ have closures contained in Darboux charts around p and $\psi(p)$ respectively, and take α and α' to be the respective pullbacks of the standard contact form $dz-\sum_j y_j dx_j$ on $\mathbb{R}^{2n+1}$ via these charts. We also assume that U is contained in a set $\mathcal{O}$ as in Definition 2.5. Since $B\mapsto\int_B\alpha\wedge(d\alpha)^{\wedge n}$ and $B\mapsto\int_{\psi(B)}\alpha'\wedge(d\alpha')^{\wedge n}$ both define Borel measures on a neighborhood of $\bar{U}$, they are each uniquely determined by their values in the special case where B is the preimage under the Darboux chart of a (sufficiently small) product of intervals. If C is any such product of intervals, denote by $\frac{1}{2}C$ the product of the intervals with the same centers but half the lengths, and identify C and $\frac{1}{2}C$ with their preimages under our Darboux chart around p. Since α is identified with the standard contact form on $\mathbb{R}^{2n+1}$ we have

$$\int_C\alpha\wedge(d\alpha)^{\wedge n}=2^{2n+1}\int_{\frac{1}{2}C}\alpha\wedge(d\alpha)^{\wedge n}. \tag{2.5}$$

Let ψ_m be as in Definition 2.5, with $\psi_m^*\alpha'=f_m\alpha$ on $\bar{\mathcal{O}}$ where $f_m\colon\bar{\mathcal{O}}\to\mathbb{R}$ has $|f_m|\geq c$ for some $c>0$ which is independent of m. Since $\psi_m\to\psi$ in the compact-open topology, for any product of intervals C we will have

$\psi^{-1}(\psi_m(\frac{1}{2}C)) \subset C$ for all m sufficiently large, and hence for large m

$$\int_{\psi(C)} \alpha' \wedge (d\alpha')^{\wedge n} \geq \int_{\psi_m(\frac{1}{2}C)} \alpha' \wedge (d\alpha')^{\wedge n} = \left| \int_{\frac{1}{2}C} (f_m\alpha) \wedge (d(f_m\alpha))^{\wedge n} \right|$$
$$\geq c^{n+1} \int_{\frac{1}{2}C} \alpha \wedge (d\alpha)^{\wedge n} = \frac{c^{n+1}}{2^{2n+1}} \int_C \alpha \wedge (d\alpha)^{\wedge n}.$$

So (2.4) holds with $\delta = \frac{c^{n+1}}{2^{2n+1}}$ whenever V is any (preimage under our Darboux chart of a) product of open intervals, and hence it also holds with this same value of δ for arbitrary open $V \subset U$ by standard approximation arguments. □

It would be interesting to know if the converse to Proposition 2.7 also holds.

3. Hypertightness and local rigidity for Legendrians

The key result of this section that is used in the rest of the paper is Corollary 3.4, asserting that points on arbitrary Legendrian submanifolds are locally rigid. This is directly analogous to [U22, Corollary 2.5] for Lagrangian submanifolds of symplectic manifolds, and the proof strategy is the same: we will prove the result for a restricted class of Legendrians using pseudoholomorphic curve methods (Theorem 3.3, analogous to [U22, Lemma 2.4]), and then exploit the fact that local rigidity is a local property to deduce the result in general via a tubular neighborhood theorem. To identify the restricted class we introduce the following terminology, borrowed from [CCD19]:

Definition 3.1. A Legendrian submanifold Λ of a contact manifold (Y, ξ) is **hypertight** if there is a contact form α for ξ whose Reeb vector field R_α obeys the following properties:

- Every closed orbit of R_α is noncontractible.
- Every Reeb chord for Λ (*i.e.*, every $\gamma\colon [0, T] \to Y$ such that $\gamma'(t) = R_\alpha(\gamma(t))$ and $\gamma(0), \gamma(1) \in \Lambda$) represents a nontrivial element of $\pi_1(Y, \Lambda)$.

This is a rather restrictive definition, but for our purposes it is sufficient that at least one example with Λ and Y both compact exists in every dimension:

Example 3.2. If $Y = ST^*T^{n+1}$ is the unit contangent bundle of $(n+1)$-torus, and if Λ is either connected component of the unit conormal bundle of the codimension-one torus $\{1\} \times T^n$, then by using the standard contact form whose Reeb flow is the geodesic flow of the flat metric on T^{n+1} we see that Λ is a hypertight Legendrian submanifold of Y (and $\dim \Lambda = n$). (See [EHS95, Section 3.2] for a somewhat more general family of examples.)

Here is one of our key technical results.

Theorem 3.3. If Λ is a closed, hypertight Legendrian submanifold of a closed contact manifold (Y, ξ), with contact form α as in Definition 3.1, then for every open subset U of Y with $U \cap \Lambda \neq \varnothing$ we have $e^Y_\alpha(U, \Lambda) > 0$

We will prove Theorem 3.3 in Section 3.1, with the main ingredient being the pseudoholomorphic-curves-based Lemma 3.6 which is proven in Section 3.2. Before proceeding to the proof let us extract two consequences from Theorem 3.3. Most significantly for the proof of Theorem 1.2, we have:

Corollary 3.4. If (Y, ξ) is any contact manifold and $\Lambda \subset Y$ is any Legendrian submanifold then every point on Λ is locally rigid with respect to Λ.

Proof. Let $\Lambda' \subset Y'$ be a closed hypertight Legendrian submanifold of some closed contact manifold (Y', ξ') with $\dim Y' = \dim Y$ (as exists by Example 3.2).

If $p \in \Lambda$ then the Legendrian neighborhood theorem (see [KM97, Proposition 43.18] for a version which does not require compactness of Λ) gives a contactomorphism Ψ from a neighborhood of W of p in Y to an open set $W' \subset Y'$, such that $\Psi(\Lambda \cap W) = \Lambda' \cap W'$. It follows immediately from Theorem 3.3 that $\Psi(p)$ is locally rigid with respect to Λ', and then Proposition 2.4 (ii) applied with $\phi = \Psi^{-1}$ shows that p is locally rigid with respect to Λ. □

Our other consequence of Theorem 3.3 concerns the contact version of the Chekanov-Hofer metric on the orbit of a submanifold under the identity component of the contactomorphism group, as considered in [RZ18]. Given a smooth manifold Y with a global contact form α and $\xi = \ker \alpha$, following [Sh16] one defines a norm $\|\cdot\|_\alpha$ on the identity component $\mathrm{Cont}_0(Y, \xi)$ of the contactomorphism group by

$$\|\psi\|_\alpha = \inf\left\{ \int_0^1 \max_Y |\alpha(V_t)|dt \,\middle|\, \mathbb{V} = (V_t)_{t\in[0,1]} \in \mathcal{C}_Y(Y,\xi),\ \psi^{\mathbb{V},1} = \psi \right\}$$

(with notation as in Section 2). If $N \subset Y$ is a closed subset, one can then let $\mathcal{L}(N) = \{\psi(N)|\psi \in \mathrm{Cont}_0(Y,\xi)\}$ and, analogously to [C00], define a pseudometric δ_α on $\mathcal{L}(N)$ by $\delta_\alpha(N_1,N_2) = \inf\{\|\psi\|_\alpha|\psi(N_1) = N_2\}$. [RZ18, Conjecture 1.10] states that δ_α is non-degenerate when N is a closed connected Legendrian submanifold. Theorem 3.3 quickly implies a special case:

Corollary 3.5. Let Λ be a closed hypertight Legendrian submanifold of a closed contact manifold (Y,ξ), and let $\xi = \ker\alpha$. Then the Shelukhin-Chekanov-Hofer pseudometric δ_α is non-degenerate on $\mathcal{L}(\Lambda)$.

Proof. By [RZ18, Proposition 5.1(4)] it suffices to check that $\delta_\alpha(\Lambda,\Lambda') > 0$ whenever $\Lambda' \in \mathcal{L}(\Lambda)$ with $\Lambda' \neq \Lambda$. Fix such an element Λ' and choose $\phi \in \mathrm{Cont}_0(Y,\xi)$ with $\phi(\Lambda) = \Lambda'$; since Λ is a closed manifold the fact that $\Lambda' \neq \Lambda$ implies that $\Lambda' \not\subset \Lambda$, so there is an open subset U of Y such that $U \cap \Lambda \neq \varnothing$ and $\phi(\bar{U}) \cap \Lambda = \varnothing$. Let $f\colon\, Y \to (0,\infty)$ be the smooth function such that $\phi^*\alpha = f\alpha$. We will show that

$$\delta_\alpha(\Lambda,\Lambda') \geq (\min_Y f)e^Y_\alpha(U,\Lambda), \tag{3.1}$$

which will suffice to prove the result since Theorem 3.3 shows that $e^Y_\alpha(U,\Lambda) > 0$.

So let $\psi \in \mathrm{Cont}_0(Y,\xi)$ be arbitrary subject to the condition that $\psi(\Lambda) = \Lambda'$. Since also $\phi(\Lambda) = \Lambda'$ we have $\phi^{-1}\psi(\Lambda) = \Lambda$, and since $\phi(\bar{U}) \cap \Lambda = \varnothing$ we obtain

$$\phi^{-1}\psi\phi(\bar{U}) \cap \Lambda = \phi^{-1}\psi(\phi(\bar{U}) \cap \Lambda) = \varnothing.$$

Thus $\|\phi^{-1}\psi\phi\|_\alpha \geq e^Y_\alpha(U,\Lambda)$. So we obtain

$$\begin{aligned}(\min_Y f)e^Y_\alpha(U,\Lambda) &\leq (\min_Y f)\|\phi^{-1}\psi\phi\|_\alpha = (\min_Y f)\|\psi\|_{\phi^{-1*}\alpha} \\ &\leq \|\psi\|_\alpha\end{aligned}$$

where the equality uses [Sh16, Theorem A(iv)] and the last inequality uses [Sh16, Lemma 10]. Since this holds for all ψ with $\psi(\Lambda) = \Lambda'$ we have proven (3.1). ∎

3.1. Proof of Theorem 3.3

What we will in fact show is that there is a positive lower bound on $\int_0^1 |\alpha(V_t)|dt$ for all time-dependent contact vector fields $(V_t)_{t\in[0,1]}$ whose time-one maps disjoin a given compact *pre-Lagrangian* submanifold L from our hypertight Legendrian Λ if L and Λ have nonempty transverse intersection; this will imply that $e_\alpha^Y(U,\Lambda) > 0$ by choosing L to be contained in U. Recall here that, continuing to write $\dim Y = 2n+1$, a pre-Lagrangian submanifold L of (Y,ξ) is an $(n+1)$-dimensional submanifold that is transverse to ξ such that some contact form β for ξ obeys $d\beta|_L = 0$. Perhaps after reversing the sign of β we can write $\beta = e^g\alpha$ for some $g\colon Y \to \mathbb{R}$, and then in the symplectization $(\mathbb{R}\times Y, d(e^r\alpha))$ the pre-Lagrangian $L \subset Y$ will lift to a Lagrangian submanifold $\hat{L} = \{(g(q),q)|q \in L\}$.

The key lemma is a lower bound on the Hofer norm of a Hamiltonian diffeomorphism of $\mathbb{R}\times Y$ that is required to disjoin a general compact Lagrangian submanifold $P \subset \mathbb{R}\times Y$ from the Lagrangian submanifold $\mathbb{R}\times\Lambda$. (Eventually we will take P to be a lift $\hat{L}$ of the pre-Lagrangian L mentioned in the previous paragraph.) If $K\colon [0,1]\times(\mathbb{R}\times Y) \to \mathbb{R}$ is a smooth function we write its Hamiltonian vector field (with respect to the symplectic form $d(e^r\alpha)$) at time t as Z_{K_t}, so $d(e^r\alpha)(\cdot, Z_{K_t}) = d(K(t,\cdot))$, and we write σ_K^t for the time-t flow of this time-dependent vector field (assuming that this flow exists).

Lemma 3.6. Let Λ be a closed hypertight Legendrian submanifold of a closed contact manifold (Y,ξ), and let $\alpha \in \Omega^1(Y)$ be as in Definition 3.1. Also let P be a **compact** Lagrangian submanifold of the symplectization $(\mathbb{R}\times Y, d(e^r\alpha))$ whose intersection with $\mathbb{R}\times\Lambda$ is nonempty and transverse. Then there is $\hbar > 0$, depending only on Λ, α, P, such that, for any compactly supported Hamiltonian $K\colon [0,1]\times(\mathbb{R}\times Y)\to\mathbb{R}$ with $\sigma_K^1(P)\cap(\mathbb{R}\times\Lambda) = \varnothing$, we have $\int_0^1 \max_{\mathbb{R}\times Y}|K(t,\cdot)|dt \geq \hbar$.

The proof of Lemma 3.6 will occupy Section 3.2.

Remark 3.7. The assumption that Λ is hypertight cannot be completely dispensed with in Lemma 3.6. If $\dim Y \geq 5$ and $(Y, \ker\alpha)$ is overtwisted, then according to [Mur13][3] there exist closed exact Lagrangian submanifolds $P \subset \mathbb{R}\times Y$. Letting $F_t\colon \mathbb{R}\times Y \to \mathbb{R}\times Y$ denote the map $(r,y)\mapsto(r-t,y)$,

[3]The equivalence of the condition in [Mur13] to overtwistedness is proven in [CMP19]. If one allows Y to be noncompact there is a much earlier example in [Mul90].

the exactness of P implies that the Lagrangian submanifolds $F_t(P)$ are all Hamiltonian isotopic. Arguing as in [C00, Proof of Proposition 11] this implies that the Chekanov-Hofer pseudometric on the orbit of P is degenerate, and hence identically zero by [C00, Theorem 2]. So if Λ is a Legendrian submanifold (say contained in a small Darboux chart of Y) such that $\mathbb{R}\times\Lambda$ intersects P transversely and if $\sigma\colon \mathbb{R}\times Y\to\mathbb{R}\times Y$ is a Hamiltonian diffeomorphism such that $\sigma(P)\cap(\mathbb{R}\times\Lambda)=\varnothing$, then there are Hamiltonians $K\colon \mathbb{R}\times(\mathbb{R}\times Y)\to\mathbb{R}$ having $\int_0^1\max|K(t,\cdot)|dt$ as small as one likes such that $\sigma_K^1(P)=\sigma(P)$ and hence $\sigma_K^1(P)\cap(\mathbb{R}\times\Lambda)=\varnothing$.

Note that by the main result of [AH09] a contact form on (what is now called) a compact overtwisted contact manifold always admits contractible periodic Reeb orbits, and thus cannot contain a hypertight Legendrian.

Conclusion of the proof of Theorem 3.3. Let U be an open subset of Y with nonempty intersection with Λ and let $p\in U\cap\Lambda$. By [Mü19, Lemma 4.7], there is a compact pre-Lagrangian submanifold L of Y that is contained in U and intersects Λ transversely at p; by an easy general position argument we can arrange for all other intersections of L and Λ to be transverse. Choose $g\colon Y\to\mathbb{R}$ so that $d(e^g\alpha)|_L=0$ and $g(p)=0$, and let $\hat{L}=\{(g(q),q)|q\in L\}$, so that $\hat{L}$ is a compact Lagrangian submanifold of $(\mathbb{R}\times Y, d(e^r\alpha))$ whose intersection with $\mathbb{R}\times\Lambda$ is transverse and contains the point $(0,p)$.

Time-dependent contact vector fields $(V_t)_{t\in[0,1]}$ are in one-to-one correspondence with smooth functions $H\colon [0,1]\times Y\to\mathbb{R}$ (by setting $H(t,y)=\alpha_y(V_t)$); given $H\colon [0,1]\times Y\to\mathbb{R}$ and $t\in[0,1]$ let ϕ_H^t denote the time-t map of the corresponding time-dependent vector field.

Our goal is then to provide a positive lower bound for $\int_0^1\max_Y|H(t,\cdot)|dt$ for all $H\colon [0,1]\times Y\to\mathbb{R}$ with the property that $\phi_H^1(\bar{U})\cap\Lambda=\varnothing$. This property obviously implies that $\phi_H^1(L)\cap\Lambda=\varnothing$ where L is the pre-Lagrangian contained in U from the first paragraph of the proof. A standard calculation ([MüSp15, Section 4]) shows that, if $h_t\colon Y\to\mathbb{R}$ are the smooth functions obeying $\phi_H^{t*}\alpha=e^{h_t}\alpha$, then the (symplectic) Hamiltonian $\hat{H}\colon [0,1]\times\mathbb{R}\times Y\to\mathbb{R}$ defined by $\hat{H}(t,r,y)=e^rH(t,y)$ obeys $\sigma_{\hat{H}}^t(r,y)=(r-h_t(y),\phi_H^t(y))$. Hence in particular $\sigma_{\hat{H}}^1(\hat{L})\cap(\mathbb{R}\times\Lambda)=\varnothing$. This Hamiltonian $\hat{H}$ is not compactly supported; to obtain a compactly supported Hamiltonian one can multiply $\hat{H}$ by a cutoff function χ that is equal to 1 on $[0,1]\times[-M,M]\times Y$ for a value M large enough that $|g(y)-h_t(y)|<M$ for all $(t,y)\in[0,1]\times L$, as then $\chi\hat{H}$ and $\hat{H}$ will coincide on a neighborhood of $\cup_{t\in[0,1]}\sigma_{\hat{H}}^t(\Lambda)$ and so $\sigma_{\chi\hat{H}}^1(\Lambda)=\sigma_{\hat{H}}^1(\Lambda)$.

Now as shown in [U15, Proof of Theorem 1.3], the function

$$K(t,(r,y)) = \chi\hat{H}(1-t, \sigma^{1-t}_{\chi\hat{H}}(\sigma^1_{\chi\hat{H}})^{-1}(r,y)),$$

which generates the Hamiltonian flow $\sigma^t_K = \sigma^1_{\chi\hat{H}} \circ (\sigma^{1-t}_{\chi\hat{H}})^{-1}$, has the useful properties that $\sigma^1_K = \sigma^1_{\chi\hat{H}}$ and $K(t,\sigma^t_K(r,y)) = \chi\hat{H}(1-t,(r,y))$ for all t,r,y. If we now let $K'\colon [0,1]\times(\mathbb{R}\times Y)\to\mathbb{R}$ be a smooth function that is supported on a small neighborhood of $\cup_t\{t\}\times\sigma^t_K(\hat{L})$ and coincides with K on a smaller neighborhood of $\cup_t\{t\}\times\sigma^t_K(\hat{L})$ then we will have $\sigma^1_{K'}(\hat{L}) = \sigma^1_K(\hat{L}) = \sigma^1_{\chi\hat{H}}(\hat{L})$, and (by taking the first neighborhood small enough) we can arrange that, for all t,

$$\max_{\mathbb{R}\times Y}|K'(t,\cdot)| \le \max_{(r,y)\in\hat{L}}|K(t,\sigma^t_K(r,y))| + \frac{\hbar}{2} = \max_{\hat{L}}|\chi\hat{H}(1-t,\cdot)| + \frac{\hbar}{2}$$

where $\hbar$ is the value from Lemma 3.6 (applied with $P=\hat{L}$). But by construction, for all t,

$$\max_{\hat{L}}|\chi\hat{H}(t,\cdot)| \le e^{\max_L g}\max_Y|H(t,\cdot)|.$$

So since $\sigma^1_{K'}(\hat{L})\cap(\mathbb{R}\times\Lambda) = \sigma^1_{\chi\hat{H}}(\hat{L})\cap(\mathbb{R}\times\Lambda) = \varnothing$, Lemma 3.6 gives

$$\hbar \le \int_0^1 \max_{\mathbb{R}\times Y}|K'(t,\cdot)|dt \le \frac{\hbar}{2} + e^{\max_L g}\int_0^1\max_Y|H(t,\cdot)|dt.$$

Since H was arbitrary subject to the assumption that $\phi^1_H(\bar{U})\cap\Lambda=\varnothing$ this shows that

$$e^Y_\alpha(U,\Lambda) \ge \frac{\hbar}{2}e^{-\max_L g} > 0.$$

□

3.2. Proof of Lemma 3.6

3.2.1. Pseudoholomorphic curve preliminaries. We begin by establishing some general properties of pseudoholomorphic curves in the symplectic manifold $(\mathbb{R}\times Y, d(e^r\alpha))$; as above r is the coordinate on $\mathbb{R}$ and we abbreviate $\omega = d(e^r\alpha)$.

Recall that if α is a contact form on a compact smooth manifold Y, an almost complex structure J on $\mathbb{R}\times Y$ is said to be **cylindrical** provided that it is invariant under translations $(r, y) \mapsto (r+s, y)$, that $J\partial_r$ is equal to the Reeb vector field of α, and that J preserves $\ker\alpha \subset T(\{0\}\times Y)$ and restricts to it $d\alpha$-compatibly. Some references such as [AFM15] use the term "SFT-like" instead of "cylindrical." A cylindrical almost complex structure is automatically compatible with the symplectic form ω, and is also compatible with the (non-closed) 2-form $\Omega := dr\wedge\alpha + d\alpha$, in the sense that the formula

$$\mu(v, w) = \Omega(v, Jw)$$

defines a Riemannian metric on $\mathbb{R}\times Y$. Because J and μ are both $\mathbb{R}$-invariant and Y is compact, it is easy to see that the triple (V, J, μ) is tame in the sense of [Si94, Definition 4.1.1].

The value $\hbar$ whose existence is asserted by Lemma 3.6 will be $\frac{1}{2}c$ where c is as in the following proposition, for a suitable choice of J.

Proposition 3.8. Let J be a cylindrical almost complex structure on $\mathbb{R}\times Y$. Then there is $c > 0$ such that for every nonconstant J-holomorphic map $v\colon (D^2, S^1) \to (\mathbb{R}\times Y, P)$ and every nonconstant finite-energy J-holomorphic map $w\colon (\mathbb{R}\times[0,1], \mathbb{R}\times\{0\}, \mathbb{R}\times\{1\}) \to (\mathbb{R}\times Y, P, \mathbb{R}\times\Lambda)$ we have $\int_{D^2} v^*\omega > c$ and $\int_{\mathbb{R}\times[0,1]} w^*\omega > c$.

Proof. Write $\lambda = e^r\alpha$ (r being the $\mathbb{R}$-coordinate on $\mathbb{R}\times Y$), so that $\omega = d\lambda$ with $\lambda|_{\mathbb{R}\times\Lambda} = 0$ and $d(\lambda|_P) = 0$. With v and w as in the proposition, Stokes' theorem implies that $\int_{D^2} v^*\omega = \int_{S^1} v^*\lambda$, and that $\int_{\mathbb{R}\times[0,1]} w^*\omega = \int_{\mathbb{R}\times\{0\}} w^*\lambda$. Since v, w are nonconstant and J-holomorphic these integrals are nonzero. Now $v|_{\partial D^2}$ is a loop in P, and since w has finite energy it follows from [Oh15b, Lemma 14.1.3] that $w|_{\mathbb{R}\times\{0\}}$ compactifies to an arc $\bar{w}\colon \mathbb{R}\cup\{\pm\infty\} \to P$ with $\bar{w}(\pm\infty) \in P\cap(\mathbb{R}\times\Lambda)$. Since $\lambda|_P$ is closed, the fact that $\int_{\partial D^2} v^*\lambda$ and $\int_{\mathbb{R}\times\{0\}} w^*\lambda$ are nonzero implies that neither $v|_{\partial D^2}$ nor $\bar{w}$ can be a contractible loop in P.

Let us use the metric μ described above the proposition to measure distances in $\mathbb{R}\times Y$. Let $r_0 > 0$ be such that the (extrinsic) diameter of every noncontractible loop in P is greater than r_0, and such that every path in P which is either a noncontractible loop in P or a path between distinct points of $P\cap(\mathbb{R}\times\Lambda)$ contains a point that is a distance greater than r_0 away from $\mathbb{R}\times\Lambda$. (The existence of such an r_0 follows readily from the compactness of P and the transversality of P and $\mathbb{R}\times\Lambda$.) Now the quadruple (V, J, P, μ) is tame in the sense of [Si94, Definition 4.7.1] since the compactness of P ensures that we can use a suitable constant multiple of ω in the role of ω_x from [Si94, Definition 4.7.1,(T'2)] simultaneously for all $x \in P$. Let r_1 be the minimum of r_0 and the distance r_P from [Si94, Definition 4.7.1] (applied with P in place of the manifold denoted there by W).

By construction, the image of $v|_{\partial D^2}$ or of $w|_{\mathbb{R}\times\{0\}}$ contains a point $x_0 \in P$ that has distance greater than r_1 from $\mathbb{R}\times\Lambda$, and this image is not contained in the ball $B(x_0, r_1)$ of radius r_1 around x_0. For generic r_2 that are slightly smaller than r_1 (specifically, r_2^2 should be a regular value of the composition of v or w with the function given by squared distance from x_0), $v|_{v^{-1}(B(x_0,r_2))}$ or $w|_{w^{-1}(B(x_0,r_2))}$ satisfies the hypotheses[4] of [Si94, Proposition 4.7.2(ii)] and hence has μ-area at least $C_4 r_2^2$ for a certain constant C_4. Since J is ω-compatible and P is compact, there is a constant $a > 0$ such that any J-holomorphic curve $u\colon \Sigma \to \mathbb{R}\times Y$ whose image is contained in an r_1-neighborhood of P will have $\int_\Sigma u^*\omega \geq a\mathrm{Area}_\mu(u)$. Thus our proposition holds with c equal to any constant that is less than $aC_4 r_1^2$. □

The following consequence of hypertightness will be a crucial ingredient in the proof of Lemma 3.6.

Lemma 3.9. ([DS16]) Let Λ be a closed, hypertight Legendrian submanifold of a closed contact manifold (Y,ξ), and let α be a contact form as in Definition 3.1. Fix $\kappa, E \in \mathbb{R}$, and fix a cylindrical almost complex structure J on the symplectization $(\mathbb{R}\times Y, d(e^r\alpha))$. Suppose that $\tilde{u}_m\colon \mathbb{R}\times[0,1] \to \mathbb{R}\times Y$ is a sequence of smooth maps such that, for each m:

(i) the map $\tilde{u}_m$ is transverse to the hypersurface $\{\kappa\}\times Y$;

(ii) The preimage $Z_m := \tilde{u}_m^{-1}((-\infty,\kappa]\times Y)$ is a compact subset of $\mathbb{R}\times(0,1]$, and the restriction $u_m := \tilde{u}_m|_{Z_m}$ is J-holomorphic;

(iii) $u_m(Z_m \cap \mathbb{R}\times\{1\}) \subset (-\infty,\kappa]\times\Lambda$

(iv) The **Hofer energy**

$$E_{Hof}(u_m) := \sup\left\{ \int_{Z_m} u_m^*(d(\phi(r)\alpha)) \,\middle|\, \phi \in C^\infty(\mathbb{R},[0,1]), \phi' \geq 0 \right\}$$

is bounded above by E.

Then there is a compact subset $K \subset \mathbb{R}\times Y$, independent of m, such that $u_m(Z_m) \subset K$ for all m.

[4] In the case of w we are using here that $w^{-1}(B(x_0,r_2))$ is disjoint from $\mathbb{R}\times\{1\}$ since $B(x_0,r_2)$ is disjoint from $\mathbb{R}\times\Lambda$.

Proof. This follows directly from the analysis leading to [DS16, Proposition 4.8] which is an analogue for curves with Lagrangian boundary conditions of [AFM15, Theorem 5.3].[5] Since [DS16, Propositions 4.7 and 4.8] are only formally stated for particular sequences $\tilde{u}_m$ introduced at the start of [DS16, Section 4], we summarize the argument so as to make clear that it applies under the general hypotheses in Lemma 3.9. Broadly the point is that the analysis in [DS16] is applicable to arbitrary sequences of J-holomorphic curves with compact domains that are subsets of $\mathbb{R} \times [0,1]$ and with Hofer energy bounded above—indeed it very explicitly follows the analysis in [AFM15, Section 6] that is used to prove the quite general [AFM15, Theorem 5.3]—and in particular the behavior of the $\tilde{u}_m$ outside of the region on which they are J-holomorphic is not used to obtain the conclusions that we require.

To begin the summary, let us write $u_m = (a_m, f_m)$ where $a_m\colon Z_m \to \mathbb{R}$ and $f_m\colon Z_m \to Y$. Since the u_m by assumption have image contained in $(-\infty, \kappa] \times Y$, it suffices to prove that $\inf_m \inf_{Z_m} a_m > -\infty$; more specifically we shall suppose for contradiction that $\inf_m \inf_{Z_m} a_m = -\infty$ and deduce from this the existence of a homotopically trivial Reeb chord or closed Reeb orbit, contrary to the hypertightness assumption.

So suppose that $\inf_m \inf_{Z_m} a_m = -\infty$. Then as in the first paragraph of the proof of [DS16, Proposition 4.7], the general [DS16, Proposition 4.6] applies to the u_m to show that, for a subsequence $\{u_{m_k}\}$, there are $\rho_k < s_k - k < s_k < \kappa$ such that $a_{m_k}^{-1}([\rho_k, s_k])$ contains a disjoint union of cylinders and strips[6] each running between $a_{m_k}^{-1}(\{\rho_k\})$ and $a_{m_k}^{-1}(\{s_k\})$, with u_{m_k} mapping the boundaries of the strips to $(\{\rho_k, s_k\} \times Y) \cup ([\rho_k, s_k] \times \Lambda)$, and moreover with $\int_{a_{m_k}^{-1}([\rho_k,s_k])} f_{m_k}^* d\alpha < \frac{E}{k}$ where E is our bound on the Hofer energy. Perhaps after passing to a further subsequence, by restricting attention to

[5]In the corresponding discussion in [DS16], one has two Legendrian submanifolds Λ', Λ (denoted there by Λ_0, Λ_1, respectively) rather than just Λ, and Z_m is allowed to intersect $\mathbb{R} \times \{0\}$; assumption (iii) is then supplemented by the condition that $u_m(\partial Z_m \cap \mathbb{R} \times \{0\}) \subset (-\infty, \kappa] \times \Lambda'$. Thus the J-holomorphic curves in Lemma 3.9 are essentially special cases of those in [DS16, Propositions 4.6,4.7,4.8], namely the ones with $Z_m \cap \mathbb{R} \times \{0\} = \varnothing$, so that Λ' plays no role.

[6]This disjoint union would be written in the notation of [DS16] as $Z_{\rho_k}^{s_k}(u_{m_k})$. One should take $[\rho_k, s_k]$ to be a suitable long interval that is disjoint from the "jumps" of u_{m_k} and has $Z_{\rho_k}^{s_k}(u_{m_k})$ disjoint from the "δ-essential local minima" of u_{m_k}, as is possible by [DS16, Proposition 4.6(1)]; see the discussion immediately preceding [DS16, Proposition 4.6] for the definitions of "jumps" and "δ-essential local minima."

one of these strips for each k we obtain (as in the conclusion of [DS16, Proposition 4.7]):

- subdomains $C_k \subset Z_{m_k}$ which are images of biholomorphisms $g_k : [-\ell_k, \ell_k] \times \mathcal{I} \to C_k$ where $\ell_k \to \infty$ and where $\mathcal{I}$ is either $[0,1]$ or S^1, independently of k. In case $\mathcal{I} = [0,1]$, so that $C_k \subset \mathbb{R} \times (0,1]$, the biholomorphism g_k will have the property that $g_k([-\ell_k, \ell_k] \times \{0,1\}) \subset \mathbb{R} \times \{1\}$.
- $\mathbb{R}$-shifts $w_k \colon C_k \to \mathbb{R} \times Y$ of the $u_{m_k}|_{C_k}$ (*i.e.*, compositions of $u_{m_k}|_{C_k}$ with the maps $(r,y) \mapsto (r + c_k, y)$ for a suitable sequence c_k) such that $\int_{C_k} w_k^* d\alpha \to 0$, and $\pm r \circ w_k(g_k(\pm\ell_k, t)) \to \infty$ uniformly in t.

The proof of [DS16, Proposition 4.8] then uses relative versions of several facts about pseudoholomorphic curves in symplectizations from [Ab14] to show that the proof of [AFM15, Theorem 5.3] extends to the present setting to yield a subsequence of the w_k so that, if g_k is the biholomorphism identifying $[-\ell_k, \ell_k] \times \mathcal{I}$ with C_k, then $w_k \circ g_k$ converges in C^∞_{loc} to a trivial cylinder over a closed Reeb orbit (if $\mathcal{I} = S^1$) or Reeb chord for Λ (if $\mathcal{I} = [0,1]$). In either case let us denote this Reeb orbit or Reeb chord by $\gamma \colon \mathcal{I} \to Y$.

Recall that the hypothesis of the lemma involves smooth maps $\tilde{u}_m \colon \mathbb{R} \times [0,1] \to \mathbb{R} \times Y$ with $u_m = \tilde{u}_m|_{Z_m}$, and that $C_k \subset Z_{m_k} \subset \mathbb{R} \times [0,1]$. So trivially $u_{m_k}|_{C_k} = \tilde{u}_{m_k}|_{C_k}$. Writing $\tilde{u}_{m_k}$ in coordinates as $(\tilde{a}_{m_k}, \tilde{f}_{m_k})$, since w_k is an $\mathbb{R}$-shift of $u_{m_k}|_{C_k}$ with $w_k \circ g_k$ converging in C^∞_{loc} to a trivial cylinder over γ it follows that $\tilde{f}_{m_k} \circ g_k|_{\{0\}\times\mathcal{I}}$ converges in C^∞ to γ. In particular γ represents the same class in $\pi_1(Y)$ or $\pi_1(Y,\Lambda)$ as does $\tilde{f}_{m_k} \circ g_k|_{\{0\}\times\mathcal{I}}$ for large k. But since $\pi_1(\mathbb{R}\times[0,1]) = \pi_1(\mathbb{R}\times[0,1], \mathbb{R}\times\{1\}) = \{0\}$, so that in particular $g_k|_{\{0\}\times\mathcal{I}}$ represents the trivial class therein, the class represented by $\tilde{f}_{m_k} \circ g_k|_{\{0\}\times\mathcal{I}}$ in $\pi_1(Y)$ or $\pi_1(Y,\Lambda)$ is likewise trivial. So γ is either a homotopically nontrivial closed Reeb orbit or a homotopically nontrivial Reeb chord, contradicting hypertightness and thus proving the lemma. □

3.2.2. Setting up the moduli spaces. We now explain the types of maps $\tilde{u}_m \colon \mathbb{R}\times[0,1] \to \mathbb{R}\times Y$ to which we will apply Lemma 3.9. These will be solutions to the Floer-type equation $(*_{\hat{J},R})$, and the properties of certain spaces $\mathcal{M}(R;\hat{J})$ of such solutions, as summarized in Proposition 3.11 at the end of this subsection, provide the foundation for our proof of Lemma 3.6.

Let Y, α, P, Λ, K be as in the statement of Lemma 3.6. Fix throughout the discussion:

- a smooth cylindrical almost complex structure J on $\mathbb{R}\times Y$;

- A number $d > 0$ such that the support of K (which was assumed compact) is contained in $[0,1] \times (-d,d) \times Y$, and such that $P \subset (-d,d) \times Y$; and
- a smooth function $\beta \colon \mathbb{R} \to [0,1]$ such that $\beta(s) = 1$ for $s \le 1$, $\beta(s) = 0$ for $s \ge 2$, and $\beta'(s) < 0$ for all $s \in (1,2)$.

For transversality purposes we will need to consider a class of domain-dependent perturbations of J. This can be done in various ways; we will adopt an approach similar to [F88, p. 807], though with an additional restriction on the support of the perturbations. Given a sequence of positive numbers $\epsilon = \{\epsilon_k\}_{k=0}^{\infty}$, let $\mathcal{S}_J^{\epsilon}$ denote the space of smoothly $\mathbb{R} \times [0,1]$-parametrized families of smooth sections[7] $S = \{S_{s,t}\}_{(s,t)\in\mathbb{R}\times[0,1]}$ of the bundle $End(T(\mathbb{R} \times Y)) \to \mathbb{R} \times Y$ having the following properties:

(i) $S_{s,t}J + JS_{s,t} = 0$ for each $(s,t) \in \mathbb{R} \times [0,1]$;

(ii) For $(s,t) \in \mathbb{R} \times [0,1]$ and $(r,y) \in \mathbb{R} \times Y$, and $v, w \in T_{(r,y)}\mathbb{R} \times Y$,

$$\omega(S_{s,t}v, Jw) + \omega(Jv, S_{s,t}w) = 0.$$

(iii) $S_{s,t} = 0$ whenever $(s,t) \notin (1,2) \times (0,1)$;

(iv) For all $(s,t) \in \mathbb{R} \times [0,1]$, $(S_{s,t})_{(r,y)} = 0$ when $(r,y) \notin (-d,d) \times Y$.

(v) $\sum_{k=0}^{\infty} \epsilon_k \|S\|_{C^k} < \infty$, where $\|S\|_{C^k}$ is the C^k norm on sections of $\pi^*End(T(\mathbb{R} \times Y))$ that is naturally induced by the Riemannian metric associated to ω and J.

Then $\mathcal{S}_J^{\epsilon}$ is a Banach space with norm $\sum_{k=0}^{\infty} \epsilon_k \|\cdot\|_{C^k}$. Conditions (i) and (ii) imply that, for each (s,t), the endomorphism $J\exp(S_{s,t})$ of $T(\mathbb{R} \times Y)$ is an almost complex structure which leaves ω invariant, so provided that S is small enough as to ensure that $\omega(v, J\exp(S_{s,t})v) > 0$ for all nonzero v, it will hold that each $J\exp(S_{s,t})$ is an ω-compatible almost complex structure. So fix a neighborhood $\mathcal{U}^{\epsilon}$ of 0 in $\mathcal{S}_J^{\epsilon}$ such that each $J\exp(S_{s,t})$ is an ω-compatible almost complex structure whenever $S \in \mathcal{U}^{\epsilon}$.

The proof of [F88, Lemma 5.1] extends without change to show that if the sequence $\epsilon = \{\epsilon_k\}$ decays sufficiently rapidly then $\mathcal{S}_J^{\epsilon}$ is L^2-dense in the L^2-closure of the space of smooth sections of $\pi^*End(T(\mathbb{R} \times Y))$ that satisfy (i) and (ii) and have support contained in $(1,2) \times (0,1) \times (-d,d) \times Y$. Fix

[7] Equivalently, S can be regarded as a section of $\pi^*End(T(\mathbb{R} \times Y))$ where $\pi \colon (\mathbb{R} \times [0,1]) \times (\mathbb{R} \times Y) \to \mathbb{R} \times Y$ is the projection.

throughout the rest of the proof such a rapidly-decaying sequence ϵ and let

$$\hat{\mathcal{J}} = \left\{ \{J\exp(S_{s,t})\}_{(s,t)\in\mathbb{R}\times[0,1]} \,|S \in \mathcal{U}^{\epsilon} \subset \mathcal{S}^{\epsilon}_J \right\}. \tag{3.2}$$

Thus each $\hat{J} = \{\hat{J}_{s,t}\}_{(s,t)\in\mathbb{R}\times[0,1]} \in \hat{\mathcal{J}}$ is a family of smooth ω-compatible almost complex structures $\hat{J}_{s,t}$ that coincide with J outside $(-d,d)\times Y$ for all $(s,t)\in\mathbb{R}\times[0,1]$, and which coincide with J everywhere in $\mathbb{R}\times Y$ for all $(s,t)\notin(1,2)\times(0,1)$.

Given $R\geq 1$, define $\beta_R\in C^\infty(\mathbb{R},[0,1])$ by

$$\beta_R(s) = \beta(|s|-(R-1));$$

thus $\beta_R(s)=1$ for $|s|\leq R$, $\beta_R(s)=0$ for $|s|\geq R+1$, and β'_R is nowhere-zero on $(-R-1,-R)\cup(R,R+1)$. Similarly if $\hat{J}\in\hat{\mathcal{J}}$ and $R\geq 1$, define an $(\mathbb{R}\times[0,1])$-parametrized family of almost complex structures $\hat{J}^R = \{\hat{J}^R_{s,t}\}$ by:

$$\hat{J}^R_{s,t} = \begin{cases} J & s\leq R \\ \hat{J}_{s-R+1,t} & s\geq R \end{cases}$$

(Item (iii) in the definition of $\mathcal{S}^\epsilon_J$ ensures that this formula is consistent for $s=R$. Note also that $\hat{J}_1 = \hat{J}$.)

Having defined β_R and $\hat{J}^R$ for $R\geq 1$, we extend this definition to $0\leq R\leq 1$ by setting:

$$\beta_R = R\beta_1, \qquad \hat{J}_R = \hat{J} \quad \text{for } 0\leq R\leq 1.$$

We will consider solutions $\tilde{u}\colon \mathbb{R}\times[0,1]\to\mathbb{R}\times Y$ of the following partial differential equation, analogous to the one used in [Al08, Definition 4.29]:

$$\frac{\partial\tilde{u}}{\partial s}(s,t) + \hat{J}^R_{s,t}(\tilde{u}(s,t))\left(\frac{\partial\tilde{u}}{\partial t}(s,t) - \beta_R(s)Z_{K_t}(\tilde{u}(s,t))\right) = 0. \tag{$*_{\hat{J},R}$}$$

(Recall our convention that $\iota_{Z_{K_t}}\omega = -dK_t$.) A solution $\tilde{u}$ to this equation is automatically smooth by standard elliptic regularity results.[8] We also point out that any solution $\tilde{u}$ of $(*_{\hat{J},R})$ has the following properties:

(i) If $R=0$ then $\tilde{u}$ satisfies the standard Cauchy-Riemann equation $\frac{\partial\tilde{u}}{\partial s} + \hat{J}_{s,t}\frac{\partial\tilde{u}}{\partial t} = 0$.

[8]For instance one could apply [McSa17, Lemma B.4.1] to the map $(s,t)\mapsto(s,t,\tilde{u}(s,t))$, which is pseudoholomorphic with respect to an almost complex structure that sends ∂_s to $\partial_t + \beta_R(s)Z_{K_t}$ and restricts as $\hat{J}^R_{s,t}$ to each $\{(s,t)\}\times T(\mathbb{R}\times Y)$.

(ii) The restriction of $\tilde{u}$ to $\{(s,t)||s|\geq R+1\}\cup\tilde{u}^{-1}\big(((-\infty,-d]\cup[d,\infty))\times\mathbb{R}\big)$ is a J-holomorphic curve (where J is our original, unperturbed cylindrical almost complex structure).

Another property of solutions to $(*_{\hat{J},R})$ is the following, which will be used when we consider transversality issues in Proposition 3.17. (This plays a role analogous to [Al08, Claim 4.33].)

Proposition 3.10. Let $\mathcal{I}_R$ denote the interval $(R,R+1)$ if $R\geq 1$, and $(1,2)$ if $0\leq R\leq 1$. If $\tilde{u}$ is a solution to $(*_{\hat{J},R})$ with $R\neq 0$, and if $\tilde{u}(\mathbb{R}\times\{0\})\subset P$ and $\tilde{u}(\mathbb{R}\times\{1\})\subset\mathbb{R}\times\Lambda$, then there is a point $(s_0,t_0)\in\mathcal{I}_R\times[0,1]$ such that $\frac{\partial\tilde{u}}{\partial s}(s_0,t_0)\neq 0$ and $\tilde{u}(s_0,t_0)\in(-d,d)\times Y$.

Proof. Suppose the proposition were false, and fix $s_0\in\mathcal{I}_R$. Let $T=\{t\in[0,1)|u(s_0,t)=u(s_0,0)\}$. We claim that $T=[0,1)$. Clearly T is closed (relative to $[0,1)$), and of course $0\in T$, so to show that $T=[0,1)$ it suffices to show that T is relatively open. If $t_0\in T$, then in particular $u(s_0,t_0)$ lies in the subset P of $(-d,d)\times Y$, so our contradiction assumption (and the continuity of u) implies that there is some neighborhood U of (s_0,t_0) in $\mathcal{I}_R\times[0,1]$ such that $\frac{\partial u}{\partial s}\big|_U\equiv 0$. By shrinking U if necessary we may assume that U is a product of intervals $I\times J$, in which case it follows that there is a C^∞ function $v\colon J\to\mathbb{R}\times Y$ so that $\tilde{u}(s,t)=v(t)$ for all $(s,t)\in I\times J$. Then $(*_{\hat{J},R})$ yields $\frac{dv}{dt}-\beta_R(s)Z_{K_t}(v(t))=0$ for all such (s,t). But since $R>0$, our construction ensures that β_R restricts injectively to $\mathcal{I}_R$, while $\frac{dv}{dt}$ and $Z_{K_t}(v(t))$ depend only on t. So choosing $s_1,s_2\in I$ so that $\beta_R(s_1)\neq\beta_R(s_2)$ we obtain $\beta_R(s_1)Z_{K_t}(v(t))=\beta_R(s_2)Z_{K_t}(v(t))=\frac{dv}{dt}$, which forces $Z_{K_t}(v(t))$ to be zero for each $t\in J$, and then forces $\frac{dv}{dt}$ to be zero for each $t\in J$. Thus J is a (relatively) open interval around t_0 throughout which $\frac{dv}{dt}=\frac{\partial\tilde{u}}{\partial t}(s_0,\cdot)=0$, in view of which t_0 has a neighborhood throughout which $t\mapsto\tilde{u}(s_0,t)$ is constant. This proves that T is relatively open, and hence equal to $[0,1)$ since the latter is connected.

So we have shown that if the proposition were false then $\tilde{u}$ would restrict as a constant v_0 to $\{s_0\}\times[0,1)$ (and hence by continuity also to $\{s_0\}\times[0,1]$), and moreover that $Z_{K_t}(v_0)=0$ for all t. But this conclusion is incompatible with the hypothesis of Lemma 3.6: we would have $v_0=\tilde{u}(s_0,0)\in P$ and $v_0=\tilde{u}(s_0,1)\in\mathbb{R}\times\Lambda$, and because $Z_{K_t}(v_0)=0$ we would also have $\sigma_K^1(v_0)=v_0$. So v_0 would be an element of $\sigma_K^1(P)\cap(\mathbb{R}\times\Lambda)$, which is empty by hypothesis. This contradiction proves the proposition. □

We now turn to describing the particular spaces of solutions to $(*_{\hat{J},R})$ that we will consider. Fix an intersection point $p\in(\mathbb{R}\times\Lambda)\cap P$. If $\tilde{u}\colon\mathbb{R}\times$

$[0,1] \to \mathbb{R}\times Y$ is a continuous map such that $\tilde{u}(\mathbb{R}\times\{0\}) \subset P$, $\tilde{u}(\mathbb{R}\times\{1\}) \subset \mathbb{R}\times\Lambda$, and $\tilde{u}(s,t) \to p$ uniformly in t as $s\to\pm\infty$, then $\tilde{u}$ extends continuously to a map $\bar{u}\colon [-\infty,\infty]\times[0,1]\to\mathbb{R}\times Y$ with $\bar{u}(\pm\infty,\cdot)\equiv p$, and thus determines an element in the set denoted $\pi_2(p,p)$ in the notation of [Oh97, Definition 13.9.3]. This set can also be regarded as consisting of the based homotopy classes of loops in the space of paths from P to $\mathbb{R}\times\Lambda$, with basepoint the constant path at p. Let 0_p denote the class of the constant map to p in $\pi_2(p,p)$.

We define, for $R\geq 0$ and $\hat{J}\in\hat{\mathcal{J}}$,

$$\mathcal{M}(R;\hat{J}) = \left\{ \tilde{u}\in C^\infty(\mathbb{R}\times[0,1],\mathbb{R}\times Y) \;\middle|\; \begin{array}{c} \tilde{u}(\mathbb{R}\times\{0\})\subset P,\ \tilde{u}(\mathbb{R}\times\{1\})\subset\mathbb{R}\times\Lambda, \\ \tilde{u}(s,t)\to p \text{ uniformly in } t \text{ as } s\to\pm\infty, \\ \tilde{u} \text{ satisfies } (*_{\hat{J},R}),\ [\tilde{u}]=0_p\in\pi_2(p,p), \\ \int_{\mathbb{R}\times[0,1]}\omega\left(\frac{\partial\tilde{u}}{\partial s},\hat{J}^R_{s,t}\frac{\partial\tilde{u}}{\partial s}\right)dsdt<\infty \end{array} \right\}.$$

The key step in proving Lemma 3.6 is the following:

Proposition 3.11. Under the hypotheses of Lemma 3.6:

(A) There is a compact subset $\mathcal{K}\subset\mathbb{R}\times Y$ such that for every $R\geq 0$, every $\hat{J}\in\hat{\mathcal{J}}$, and every $\tilde{u}\in\mathcal{M}(R;\hat{J})$ we have $\tilde{u}(\mathbb{R}\times[0,1])\subset\mathcal{K}$.

(B) Assume that $\int_0^1\max|K(t,\cdot)|<\hbar$ where $\hbar=\frac{c}{2}$ for c as in Proposition 3.8. There is then $\hat{J}\in\hat{\mathcal{J}}$ such that, for all $R\geq 0$, $\mathcal{M}(R;\hat{J})\neq\varnothing$.

The proof of Proposition 3.11(A) will be completed near the end of the following subsection (just before Proposition 3.14), and that of Proposition 3.11(B) will be completed at the start of Section 3.2.5, immediately after which we will complete the proof of Lemma 3.6.

3.2.3. Energy and compactness. The *energy* of an element $\tilde{u}\in\mathcal{M}(R;\hat{J})$ is defined to be $E(\tilde{u})=\int_{\mathbb{R}\times[0,1]}\omega\left(\frac{\partial\tilde{u}}{\partial s},\hat{J}^R_{s,t}\frac{\partial\tilde{u}}{\partial s}\right)dsdt$. That this is finite is part of the definition of $\mathcal{M}(R;\hat{J})$, and standard results (*e.g.*, [Oh15b, Proposition 14.1.5]) imply that any $\tilde{u}\in\mathcal{M}(R;\hat{J})$ has $\frac{\partial\tilde{u}}{\partial s}(s,t)\to 0$ uniformly exponentially quickly as $s\to\pm\infty$. In particular it follows that $\int_{\mathbb{R}\times[0,1]}\tilde{u}^*\omega$ converges absolutely. Since $\tilde{u}$ represents the class $0_p\in\pi_2(p,p)$ (so the restrictions $\tilde{u}|_{\mathbb{R}\times\{0\}}$ and $\tilde{u}|_{\mathbb{R}\times\{1\}}$ can be compactified to give contractible loops based at p in the Lagrangian submanifolds P and $\mathbb{R}\times\Lambda$, respectively) we then see from Stokes' theorem that

$$\int_{\mathbb{R}\times[0,1]}\tilde{u}^*\omega=0\quad\text{for all }\tilde{u}\in\mathcal{M}(R;\hat{J}). \tag{3.3}$$

This leads to the following general estimate on $E(\tilde{u})$:

Proposition 3.12. If $\tilde{u} \in \mathcal{M}(R; \hat{J})$ then

$$E(\tilde{u}) \leq 2\min\{R, 1\} \int_0^1 \max|K(t,\cdot)|dt.$$

Proof. Since $\tilde{u}$ solves $(*_{\hat{J},R})$ we have $\hat{J}^R_{s,t}\frac{\partial \tilde{u}}{\partial s} = \frac{\partial \tilde{u}}{\partial t} - \beta_R(s)Z_{K_t}$. Thus:

$$\begin{aligned}
E(\tilde{u}) &= \int_0^1 \int_{-\infty}^{\infty} \omega\left(\frac{\partial \tilde{u}}{\partial s}, \frac{\partial \tilde{u}}{\partial t} - \beta_R(s)Z_{K_t}\right) dsdt \\
&= \int_0^1 \int_{-\infty}^{\infty} \omega\left(\frac{\partial \tilde{u}}{\partial s}, \frac{\partial \tilde{u}}{\partial t}\right) dsdt - \int_0^1 \int_{-\infty}^{\infty} \beta_R(s)(dK_t)_{\tilde{u}(s,t)}\left(\frac{\partial \tilde{u}}{\partial s}\right) dsdt \\
&= \int_{\mathbb{R}\times[0,1]} \tilde{u}^*\omega - \int_0^1 \int_{-\infty}^{\infty} \frac{d}{ds}\left(\beta_R(s)K_t(\tilde{u}(s,t))\right) dsdt \\
&\quad + \int_0^1 \int_{-\infty}^{\infty} \beta_R'(s)K_t(u(s,t))dsdt \\
&= 0 + 0 + \int_0^1 \int_{-\infty}^{0} \beta_R'(s)K_t(\tilde{u}(s,t))dsdt \\
&\quad + \int_0^1 \int_0^{\infty} \beta_R'(s)K_t(\tilde{u}(s,t))dsdt \\
&\leq \min\{R,1\} \int_0^1 \max K(t,\cdot)dt + \min\{R,1\} \int_0^1 (-\min K(t,\cdot))dt \\
&= 2\min\{R,1\} \int_0^1 \max|K(t,\cdot)|dt.
\end{aligned}$$

Here the first 0 on the fourth line follows from (3.3), and the second 0 on the fourth line follows from the fact that $\beta_R(s) = 0$ for $|s| > R+1$. Also the penultimate line follows from the facts that β_R increases monotonically from 0 to $\min\{R,1\}$ on $(-\infty, 0]$, and decreases monotonically from $\min\{R,1\}$ to 0 on $[0,\infty)$. □

Proposition 3.13. Let $R \geq 0$, $\hat{J} \in \mathcal{J}$, and $\tilde{u} \in \mathcal{M}(R; \hat{J})$. Suppose that $\kappa < -d$ is a regular value both of $\pi_{\mathbb{R}} \circ \tilde{u} \colon \mathbb{R} \times [0,1] \to \mathbb{R}$ and of $\pi_{\mathbb{R}} \circ (\tilde{u}|_{\mathbb{R}\times\{1\}}) : \mathbb{R} \times \{1\} \to \mathbb{R}$ where $\pi_{\mathbb{R}} \colon \mathbb{R} \times Y \to \mathbb{R}$ is the projection. Let $Z = \tilde{u}^{-1}((-\infty,\kappa] \times Y)$ and $u = \tilde{u}|_Z$. Then for every smooth $\phi \colon \mathbb{R} \to [0,\infty)$ we have

$$\int_Z u^* d(\phi(r)\alpha) \leq 2e^{-\kappa}\phi(\kappa) \int_0^1 \max|K(t,\cdot)|dt.$$

Proof. The hypothesis implies that Z is a compact smooth manifold with corners, with codimension-one boundary strata given by $Z \cap (\mathbb{R} \times \{1\})$ (which maps by u to $\mathbb{R} \times \Lambda$) and $\tilde{u}^{-1}(\{\kappa\} \times Y)$. (Note that $Z \cap (\mathbb{R} \times \{0\}) = \varnothing$ since $P \subset (-d, d) \times Y$ and $\kappa < -d$.) Since $\phi(r)\alpha$ vanishes on $T(\mathbb{R} \times \Lambda)$ it then follows from Stokes' theorem that

$$\begin{aligned}\int_Z u^* d(\phi(r)\alpha) &= \int_{u^{-1}(\{\kappa\}\times Y)} u^*(\phi(r)\alpha) = \phi(\kappa) \int_{u^{-1}(\{\kappa\}\times Y)} u^*\alpha \\ &= e^{-\kappa}\phi(\kappa) \int_{u^{-1}(\{\kappa\}\times Y)} u^*(e^r\alpha) = e^{-\kappa}\phi(\kappa) \int_Z u^* d(e^r\alpha). \end{aligned} \tag{3.4}$$

Now since $\tilde{u}(Z)$ is disjoint from the support of $K(t, \cdot)$ for each t, the facts that $\tilde{u}$ obeys $(*_{\hat{J},R})$ and $\tilde{u}|_Z = u$ imply that $\frac{\partial u}{\partial s} + \hat{J}^R_{s,t}\frac{\partial u}{\partial t} = 0$ for all $(s, t) \in Z$. So (recalling $\omega = d(e^r\alpha)$) we obtain

$$\begin{aligned}\int_Z u^* d(e^r\alpha) &= \int_Z \omega\left(\frac{\partial \tilde{u}}{\partial s}, \hat{J}^R_{s,t}\frac{\partial \tilde{u}}{\partial s}\right) dsdt \\ &\le \int_{\mathbb{R}\times[0,1]} \omega\left(\frac{\partial \tilde{u}}{\partial s}, \hat{J}^R_{s,t}\frac{\partial \tilde{u}}{\partial s}\right) dsdt = E(\tilde{u})\end{aligned} \tag{3.5}$$

where the inequality holds because the integrand is nonnegative. The proposition then follows immediately upon combining (3.4),(3.5), and Proposition 3.12. □

Proof of Proposition 3.11(A). We first observe that, for any such $\tilde{u}$, we have $\tilde{u}(\mathbb{R} \times [0, 1]) \subset (-\infty, d) \times Y$. Indeed $\tilde{u}^{-1}([d, \infty) \times Y)$ is compact due to the asymptotic conditions on $\tilde{u}$ (as $p \in (-d, d) \times Y$), and $\tilde{u}^{-1}([d, \infty) \times Y)$ is disjoint from $\mathbb{R} \times \{0\}$ since $P \subset (-d, d) \times Y$. Moreover the Hamiltonian term in $(*_{\hat{J},R})$ vanishes throughout $\tilde{u}^{-1}([d, \infty) \times Y)$, so a maximum principle such as [KS02, Lemma 5.5] implies that $\pi_{\mathbb{R}} \circ \tilde{u}|_{\tilde{u}^{-1}([d,\infty)\times Y)}$ can have no local maximum. This forces $\tilde{u}^{-1}([d, \infty) \times Y)$ to be empty by the maximum value theorem, confirming the claim in the first sentence of the proof.

Now supposing Proposition 3.11(A) to be false, there would be sequences $\{R_m\}$ in $[0, \infty)$, $\{\hat{J}_m\}$ in $\hat{\mathcal{J}}$, and $\{\tilde{u}_m\}$ in $\mathcal{M}(R_m; \hat{J}_m)$ such that $\inf_m \inf_{\mathbb{R}\times[0,1]} \pi_{\mathbb{R}} \circ \tilde{u}_m = -\infty$. Choose $\kappa \in (-d - 1, -d)$ to be a common regular value of each of the $\pi_{\mathbb{R}} \circ \tilde{u}_m$ and $\pi_{\mathbb{R}} \circ \tilde{u}_m|_{\mathbb{R}\times\{1\}}$. (Such a κ exists by Sard's theorem.) Since, throughout $(-\infty, -d] \times Y$, each $\hat{J}^R_{s,t}$ coincides with J and $K(t, \cdot)$ is identically zero, the maps $u_m|_{\tilde{u}_m^{-1}((-\infty,\kappa]\times Y)}$ are each J-holomorphic, and by Proposition 3.13 they have Hofer energy bounded above by $2e^{-\kappa} \int_0^1 \max |K(t, \cdot)|dt$. So Lemma 3.9 applies to the sequence $\{\tilde{u}_m\}$, and gives a contradiction which completes the proof. □

The following compactness result will be key to Proposition 3.11(B).

Proposition 3.14. Let $\hbar = \frac{c}{2}$ where c is the constant of Proposition 3.8, applied with J equal to the almost complex structure that was fixed after the proof of Lemma 3.9, and choose any $\hat{J} \in \hat{\mathcal{J}}$ and $R_0 \geq 0$. If $\int_0^1 \max|K(t,\cdot)| < \hbar$ then

$$\tilde{\mathcal{M}}([0, R_0]; \hat{J}) := \{(R, \tilde{u}) | 0 \leq r \leq R_0,\ \tilde{u} \in \mathcal{M}(R; \hat{J})\}$$

is a compact subset of $[0, R_0] \times C^\infty(\mathbb{R} \times [0,1], \mathbb{R} \times Y)$.

Proof. Let $\{(R_m, \tilde{u}_m)\}_{m=1}^\infty$ be a sequence in $\tilde{\mathcal{M}}([0, R_0]; \hat{J})$; passing to a subsequence we may assume that $R_m \to \bar{R} \in [0, R_0]$. Each $\tilde{u}_m$ has energy at most c by Proposition 3.12 and has image contained in the compact set $\mathcal{K}$ by Proposition 3.11(A); moreover the Hamiltonian perturbations $\beta_R(s)Z_{K_t}$ appearing in $(*_{\hat{J},R})$ as well as the differences $\hat{J}^R_{s,t} - J$ have compact support uniformly contained in in $[-R_0 - 1, R_0 + 1] \times [0, 1]$. Hence by Gromov-Floer compactness (as in [F88, Theorem 1],[Oh15a, Theorem 11.2.8]), a further subsequence converges in the Gromov-Hausdorff sense to a cusp trajectory consisting *a priori* of an element $\bar{u} \in \mathcal{M}(\bar{R}; \hat{J})$; a collection of J-holomorphic strips $(\mathbb{R} \times [0,1], \mathbb{R} \times \{0\}, \mathbb{R} \times \{1\}) \to (\mathbb{R} \times Y, P, \mathbb{R} \times \Lambda)$; and a collection of $\hat{J}^R_{s,t}$-holomorphic spheres or J-holomorphic disks with boundary on either P or $\mathbb{R} \times \Lambda$. (The disks are J-holomorphic because $\hat{J}^R_{s,0} = \hat{J}^R_{s,1} = J$ for all s by the definition of $\hat{\mathcal{J}}$.) Moreover the energies of the various components of this cusp trajectory are each positive and their sum is, like the individual $E(\tilde{u}_m)$, bounded above by c.

But since our symplectic form $\omega = d(e^r\alpha)$ is exact there can be no non-constant $\hat{J}^R_{s,t}$-holomorphic spheres, and since $e^r\alpha$ vanishes on $\mathbb{R} \times Y$ there can be no J-holomorphic disks with boundary on $\mathbb{R} \times \Lambda$. Also by Proposition 3.8 any strips or disks with boundary on P that might otherwise appear have energy greater than c, which is impossible given the bound on $E(\tilde{u}_m)$. So the limiting cusp trajectory in fact has only the single element $\bar{u}$, and then by the last clause of [Oh15a, Theorem 11.2.8] the sequence $\{\tilde{u}_m\}$ converges to $\bar{u}$ smoothly. □

3.2.4. Transversality. As a technical device related to the fact that our functions β_R and almost complex structures $\hat{J}^R$ do not depend differentiably on R at $R = 1$, throughout this subsection we will use a smooth, strictly increasing function $\gamma\colon [0, \infty) \to [0, \infty)$ with $\gamma(R) = R$ for $R \notin (1/2, 2)$, such that $\gamma(1) = 1$ and the derivative γ' vanishes to infinite order at $R = 1$. These conditions ensure that $\beta_{\gamma(R)}$ and $\hat{J}^{\gamma(R)}$ have smooth dependence on R everywhere.

This subsection will be devoted to proving the following result, which when combined with Proposition 3.14 has Proposition 3.11(B) as an easy corollary as we will see at the start of Section 3.2.5.

Proposition 3.15. There are $\hat{J} \in \hat{\mathcal{J}}$ such that

$$\tilde{\mathcal{M}}_\gamma([0,\infty);\hat{J}) := \{(R,\tilde{u})|R \in [0,\infty), \tilde{u} \in \mathcal{M}(\gamma(R);\hat{J})\}$$

is a smooth manifold with boundary. Moreover $\partial\tilde{\mathcal{M}}_\gamma([0,\infty);\hat{J}) = \{0\} \times \mathcal{M}(0;\hat{J})$, which is a singleton.

This proposition will follow by a more-or-less standard argument that exhibits $\tilde{\mathcal{M}}_\gamma([0,\infty);\hat{J})$ as the zero-locus of a section of a Banach bundle which (for suitable $\hat{J}$) is cut out transversely. For the basic analytic setup we can use the spaces that were already introduced by Floer in [F88, Section 3] and also form the basis for the treatment in [Oh15b, Chapter 15]. Recall the notation 0_p for the homotopy class in $\pi_2(p,p)$ represented by the elements of our spaces $\mathcal{M}(R;\hat{J})$. For an integer $k \geq 2$, an exponent $q > 2$, and a pair of exponential weights $\delta = (a,b) \in (0,\infty) \times (-\infty,0)$ with $|a|$ and $|b|$ sufficiently small, the constructions in [Oh15b, Section 15.1] give a Banach manifold $\mathcal{P}^{k,q}_\delta(p,p;0_p)$ and a Banach bundle $\mathcal{L}^{k-1,q}_\delta$ over $\mathcal{P}^{k,q}_\delta(p,p;0_p)$; the elements of $\mathcal{P}^{k,q}_\delta(p,p;0_p)$ are Sobolev-class-(k,q) maps $\tilde{u}\colon \mathbb{R}\times[0,1] \to \mathbb{R}\times Y$ that represent the trivial homotopy class 0_p and have decay properties near $\{\pm\infty\}\times[0,1]$ dictated by δ, and the fiber of $\mathcal{L}^{k-1,q}_\delta$ over $\tilde{u}$ consists of Sobolev-class-$(k-1,q)$ sections of $\tilde{u}^*T(\mathbb{R}\times Y)$ that are tangent to P along $\mathbb{R}\times\{0\}$ and to $\mathbb{R}\times\Lambda$ along $\mathbb{R}\times\{1\}$, again with exponential decay properties dictated by δ. By [Oh15b, Theorems 15.1.2(4) and 15.1.3], provided that $|a|$ and $|b|$ are sufficiently small the Dolbeault operator $\bar{\partial}_{\tilde{J}}$ associated to a family $\tilde{J} = \{\tilde{J}_{s,t}\}_{(s,t)\in\mathbb{R}\times[0,1]}$ of compatible almost complex structures, defined by $\bar{\partial}_{\tilde{J}}\tilde{u}(s,t) = \frac{\partial\tilde{u}}{\partial s} + \tilde{J}_{s,t}(\tilde{u}(s,t))\frac{\partial\tilde{u}}{\partial t}$, gives a Fredholm section of the bundle $\mathcal{L}^{k-1,q}_\delta \to \mathcal{P}^{k,q}_\delta(p,p;0_p)$, and the index of the linearization at any zero of $\bar{\partial}_J$ is the Maslov-Viterbo index of 0_p which (as is evident from [Oh15b, Definition 13.6.2]) is zero because we are working with the trivial homotopy class. Of course these properties depend on the fact that P is transverse to $\mathbb{R}\times\Lambda$.

Proposition 3.16. The unique zero of the section $\bar{\partial}_{\tilde{J}}\colon \mathcal{P}^{k,q}_\delta(p,p;0_p) \to \mathcal{L}^{k-1,q}_\delta$ is the constant map $\tilde{u}_p(s,t) = p$. Moreover the linearization of $\bar{\partial}_J$ at $\tilde{u}_p$ is an isomorphism.

Proof. If $\tilde{u} \in \mathcal{P}^{k,q}_{\delta}(p,p;0_p)$ then $\tilde{u}|_{(-\infty,\infty)\times\{0\}}$ and $\tilde{u}|_{(-\infty,\infty)\times\{1\}}$ compactify to contractible loops in the Lagrangian submanifolds P and $\mathbb{R}\times\Lambda$ respectively, and then Stokes' theorem readily yields that $\int_{\mathbb{R}\times[0,1]}\tilde{u}^*\omega = 0$. If $\tilde{u}$ is additionally a zero of $\bar{\partial}_{\tilde{J}}$ (*i.e.*, if $\tilde{J}_{s,t}\frac{\partial u}{\partial s} = \frac{\partial u}{\partial t}$), we thus obtain that

$$0 = \int_{\mathbb{R}\times[0,1]}\tilde{u}^*\omega = \int_{-\infty}^{\infty}\int_0^1 \omega\left(\frac{\partial \tilde{u}}{\partial s}, J_{s,t}\frac{\partial \tilde{u}}{\partial s}\right) dtds.$$

The integrand in the final expression above is everywhere nonnegative and vanishes only where $\frac{\partial\tilde{u}}{\partial s} = 0$, and so we conclude that $\frac{\partial \tilde{u}}{\partial s} = 0$ everywhere (note that $k \geq 2$ and $q > 2$, so $\tilde{u}$ is C^1). But then the fact that $\bar{\partial}_J\tilde{u} = 0$ implies that $\frac{\partial\tilde{u}}{\partial t} = 0$ everywhere as well, whence $\tilde{u}$ is constant. Since by definition of $\mathcal{P}^{k,q}_{\delta}(p,p;0_p)$ we have $\tilde{u}(s,0)\to p$ as $s\to\pm\infty$ it follows that $\tilde{u} = \tilde{u}_p$.

Now the tangent space to $\mathcal{P}^{k,q}_{\delta}(p,p;0_p)$ at the constant map $\tilde{u}_p$ consists of Sobolev-class-(k,q) maps $\xi\colon \mathbb{R}\times[0,1]\to T_p(\mathbb{R}\times Y)$ with exponential decay properties given by δ, and the linearization $D_{\tilde{u}_p}$ of $\bar{\partial}_{\tilde{J}}$ at $\tilde{u}_p$ is given by

$$D_{\tilde{u}_p}\xi = \frac{\partial \xi}{\partial s} + (\tilde{J}_{s,t})_p\frac{\partial\xi}{\partial t}.$$

(This follows from [Oh15b, Theorem 15.1.2], noting that the fact that $\tilde{u}_p$ is constant simplifies some formulas.) A solution ξ to $D_{\tilde{u}_p}\xi = 0$ would extend to a map $[-\infty,\infty]\times[0,1]\to T_p(\mathbb{R}\times Y)$ that maps $\{\pm\infty\}\times[0,1]$ to 0 and $\{0\}\times\mathbb{R}$ and $\{1\}\times\mathbb{R}$ to the Lagrangian subspaces T_pP and $T_p(\mathbb{R}\times\Lambda)$, and which obeys $\frac{\partial\xi}{\partial t} = (\tilde{J}_{s,t})_p\frac{\partial\xi}{\partial s}$. Just the same reasoning as in the first paragraph (but now with the target symplectic manifold $(\mathbb{R}\times Y,\omega)$ replaced by the symplectic vector space $(T_p(\mathbb{R}\times Y),\omega_p)$) then shows that ξ vanishes identically. This proves that $D_{\tilde{u}_p}$ has trivial kernel. Since the index of $D_{\tilde{u}_p}$ is zero it follows that $D_{\tilde{u}_p}$ is an isomorphism. □

Given a smooth compactly supported function $G\colon \mathbb{R}\times[0,1]\times\mathbb{R}\times Y\to\mathbb{R}$, denote

$$(\bar{\partial}_{\tilde{J},G}\tilde{u})(s,t) = \frac{\partial\tilde{u}}{\partial s} + \tilde{J}_{s,t}\left(\frac{\partial\tilde{u}}{\partial t} - Z_{G_{s,t}}(\tilde{u}(s,t))\right)$$

where $G_{s,t} = G(s,t,\cdot)$. Then $\bar{\partial}_{\tilde{J},G}$ likewise defines a section of the bundle $\mathcal{L}^{k-1,q}_{\delta}$ over $\mathcal{P}^{k,q}_{\delta}(p,p;0_p)$; of course $\bar{\partial}_{\tilde{J},0} = \bar{\partial}_{\tilde{J}}$.

At any $\tilde{u}\in\mathcal{P}^{k,q}_{\delta}(p,p;0_p)$, the linearization $D_u\bar{\partial}_{\tilde{J},G}$ differs from $D_u\bar{\partial}_{\tilde{J}}$ by the addition of a compactly-supported zeroth order term; thus $D_u\bar{\partial}_{\tilde{J},G} -$

$D_u\bar{\partial}_{\tilde{J}}$ is a compact operator and so the fact that $D_u\bar{\partial}_{\tilde{J}}$ is an index-zero Fredholm operator implies the same property for $D_u\bar{\partial}_{\tilde{J},G}$.

Now any one of the moduli spaces $\mathcal{M}(R;\hat{J})$ is the zero locus of the operator $\bar{\partial}_{\tilde{J},G}$ given by setting $\tilde{J}=\hat{J}^R$ and $G(s,t,\cdot)=\beta_R(s)K(t,\cdot)$. (We abbreviate the latter formula by writing $G=\beta_R K$.) The space $\tilde{\mathcal{M}}_\gamma([0,\infty);\hat{J})$ of Proposition 3.15 is in turn the zero locus of the map

$$\begin{aligned}\mathcal{F}_{\hat{J}}\colon\ [0,\infty)\times\mathcal{P}^{k,q}_\delta(p,p;0_p)&\to\mathcal{L}^{k-1,q}_\delta\\ (R,\tilde{u})&\mapsto\bar{\partial}_{\hat{J}^{\gamma(R)},\beta_{\gamma(R)}K}\tilde{u};\end{aligned}\tag{3.6}$$

this is a C^∞ map since $\hat{J}$ is C^∞, and since the linearization of each $\mathcal{F}_{\hat{J}}|_{\{R\}\times\mathcal{P}^{k,q}_\delta(p,p;0_p)}$ is Fredholm of index zero it follows that the linearization of $\mathcal{F}_{\hat{J}}$ is Fredholm of index one.

Our space of almost complex structures $\hat{\mathcal{J}}$ is defined in (3.2) to be parametrized by an open subset of a Banach space and so is a Banach manifold. Each $\hat{J}=\{\hat{J}_{s,t}\}\in\hat{\mathcal{J}}$ is given by $\hat{J}_{s,t}=J\exp(S_{s,t})$ where $S=\{S_{s,t}\}\in\mathcal{U}^\epsilon\subset\mathcal{S}^\epsilon_J$, with notation as defined above (3.2). So the tangent space $T_{\hat{J}}\hat{\mathcal{J}}$ is the image of $\mathcal{S}^\epsilon_J$ under the derivative of the exponential map $\{S_{s,t}\}\mapsto\{J\exp(S_{s,t})\}$; in particular $T_{\hat{J}}\hat{\mathcal{J}}$ consists of certain smooth families $X=\{X_{s,t}\}_{(s,t)\in\mathbb{R}\times[0,1]}$ of smooth sections of the bundle $End(T(\mathbb{R}\times Y))$ such that $X_{s,t}\equiv 0$ when $(s,t)\notin(1,2)\times(0,1)$ and every $X_{s,t}$ vanishes outside $(-d,d)\times Y$. For $X\in T_{\hat{J}}\hat{\mathcal{J}}$ and $R\geq 0$ let $X^R_{s,t}=\begin{cases}X_{s-R+1,t} & \text{if } R\geq 1\\ X_{s,t} & \text{if } 0\leq R\leq 1\end{cases}$

Proposition 3.17. Define $\tilde{\mathcal{F}}\colon\ [0,\infty)\times\mathcal{P}^{k,q}_\delta(p,p;0_p)\times\hat{\mathcal{J}}\to\mathcal{L}^{k-1,q}_\delta$ by $\tilde{F}(R,\tilde{u},\hat{J})=\mathcal{F}_{\hat{J}}(R,\tilde{u})$. Then $\tilde{\mathcal{F}}$ is transverse to the zero section of the bundle $\mathcal{L}^{k-1,q}_\delta\to\mathcal{P}^{k,q}_\delta(p,p;0_p)$.

Proof. We follow the proof of [Al08, Claim 4.32]. Given $(R,\tilde{u},\hat{J})$ with $\tilde{\mathcal{F}}(R,\tilde{u},\hat{J})=0$ we will in fact show that the restriction of the linearization of $\tilde{\mathcal{F}}$ to $\{0\}\times T_{\tilde{u}}\mathcal{P}^{k,q}_\delta(p,p;0_p)\times T_{\hat{J}}\hat{\mathcal{J}}$ is surjective. This map is given by

$$(0,\xi,X)\mapsto D_u\bar{\partial}_{\hat{J}^{\gamma(R)},\beta_{\gamma(R)}K}\xi+X^{\gamma(R)}(u)\circ\left(\frac{\partial\tilde{u}}{\partial t}-\beta_{\gamma(R)}(s)Z_{K_t}(u)\right).\tag{3.7}$$

If $R=0$ (so $\beta_{\gamma(R)}=0$) then Proposition 3.16 shows that $D_u\bar{\partial}_{\hat{J}^{\gamma(R)},\beta_{\gamma(R)}K}$ is already surjective and we are done, so assume for the rest of the proof that $R>0$ and hence that $\gamma(R)>0$.

Now the image of (3.7) contains the image of the the Fredholm operator $D_u\bar{\partial}_{\hat{J}^{\gamma(R)},\beta_{\gamma(R)}K}$ (as we see by setting $X=0$ in (3.7)), so since the image of $D_u\bar{\partial}_{\hat{J}^{\gamma(R)},\beta_{\gamma(R)}K}$ is closed and of finite codimension the same is true of the image of (3.7). So in order to show that (3.7) is surjective it suffices to show that the only continuous linear functional $\hat{\eta}$ on $\mathcal{L}^{k-1,q}_{\delta}$ that annihilates the image of (3.7) is the zero functional. Supposing $\hat{\eta}$ to be such a linear functional and treating $\hat{\eta}$ as a distribution, the fact that $\hat{\eta}$ annihilates the image of $D_u\bar{\partial}_{\hat{J}^{\gamma(R)},\beta_{\gamma(R)}K}$ implies that $\hat{\eta}$ is in the kernel of the formal adjoint of $D_u\bar{\partial}_{\hat{J}^{\gamma(R)},\beta_{\gamma(R)}K}$ and hence, by elliptic regularity, is given by L^2-inner product with a smooth section that we denote by η. Now since $\tilde{u}$ obeys $(*_{\hat{J},\gamma(R)})$ we have $\left(\frac{\partial \tilde{u}}{\partial t} - \beta_{\gamma(R)}(s)Z_{K_t}(\tilde{u})\right) = \hat{J}^{\gamma(R)}_{s,t}\frac{\partial \tilde{u}}{\partial s}$, so

$$(D_{R,\tilde{u},\hat{J}}\tilde{\mathcal{F}}(0,0,X))(s,t) = X^{\gamma(R)}_{s,t}(\tilde{u}(s,t))\hat{J}^{\gamma(R)}_{s,t}(\tilde{u}(s,t))\frac{\partial \tilde{u}}{\partial s}.$$

Let $\mathcal{I}_{\gamma(R)}$ be as in Proposition 3.10. That proposition and the smoothness of $\tilde{u}$ imply that there is a nonempty open subset V of $\mathcal{I}_{\gamma(R)} \times (0,1) \cap \tilde{u}^{-1}((-d,d)\times Y)$ throughout which $\hat{J}^{\gamma(R)}_{s,t}(\tilde{u}(s,t))\frac{\partial \tilde{u}}{\partial s}$ is nonzero. If there were some point $(s_0,t_0) \in V$ at which $\eta(s_0,t_0) \neq 0$, then we could choose a suitable $X \in T_{\hat{J}}\hat{\mathcal{J}}$, with $X^{\gamma(R)}$ supported in a small neighborhood of (s_0,t_0), so as to make

$$\int_{\mathbb{R}\times[0,1]} \left\langle \eta(s,t), X^{\gamma(R)}_{s,t}(\tilde{u}(s,t))J^{\gamma(R)}(\tilde{u}(s,t))\frac{\partial \tilde{u}}{\partial s}\right\rangle dsdt \neq 0$$

(see [SZ92, pp. 1346-1347] for an explicit construction). Since the left hand side above is the pairing of $\hat{\eta}$ with $D_{R,\tilde{u},\hat{J}}\tilde{\mathcal{F}}(0,0,X))$ this contradicts the assumption that $\hat{\eta}$ annihilates the image of (3.7). So η must in fact vanish throughout the open set V. As in the penultimate paragraph of [Al08, Proof of Claim 4.32], the unique continuation result [FHS95, Proposition 3.1] then implies that η vanishes identically, *i.e.* that $\hat{\eta} = 0$, as desired. □

Proof of Proposition 3.15. The Proposition now follows by a standard application of the Sard-Smale theorem: Propositions 3.16 and 3.17 imply that $\tilde{\mathcal{M}}^{\text{univ}} := \{(R,\tilde{u},\hat{J}) | (R,\tilde{u}) \in \tilde{\mathcal{M}}_{\gamma}([0,\infty);\hat{J})\}$ is a Banach manifold with boundary the subset where $R = 0$ (that there are local collar neighborhoods around the points where $R = 0$ follows from Proposition 3.16 and the implicit function theorem), and then $\tilde{\mathcal{M}}_{\gamma}([0,\infty);\hat{J})$ will be a manifold with boundary $\{0\} \times \mathcal{M}(0;\hat{J})$ for any $\hat{\mathcal{J}}$ which is a regular value of the projection $\tilde{\mathcal{M}}^{\text{univ}} \to \hat{\mathcal{J}}$. This boundary is a singleton by Proposition 3.16. □

3.2.5. Final ingredients for Lemma 3.6.

Proof of Proposition 3.11(B). Let $\hat{J}$ be as in Proposition 3.15. Since our smooth function $\gamma\colon [0,\infty) \to [0,\infty)$ is surjective it suffices to show that $\mathcal{M}(\gamma(R);\hat{J}) \neq \varnothing$ for every $R > 0$. (The case $R = 0$ is already covered by the fact that $\mathcal{M}(0;\hat{J})$ is a singleton.)

For any $R > 0$ which is a regular value of the projection $\pi\colon \tilde{\mathcal{M}}_\gamma([0,\infty);\hat{J}) \to \mathbb{R}$ (given by $(R,\tilde{u}) \mapsto R$), the space $\tilde{\mathcal{M}}([0,\gamma(R)];\hat{J})$ from Propostion 3.14 is a one-dimensional manifold with boundary in bijection with $\mathcal{M}(0;\hat{J}) \sqcup \mathcal{M}(\gamma(R);\hat{J})$. Moreover Proposition 3.14 asserts that $\tilde{\mathcal{M}}([0,\gamma(R)];\hat{J})$ is compact, so $\mathcal{M}(0;\hat{J}) \sqcup \mathcal{M}(\gamma(R);\hat{J})$ contains an even number of elements whenever R is a regular value of π. Since $\mathcal{M}(0;\hat{J})$ is a singleton it follows that $\mathcal{M}(\gamma(R);\hat{J})$ has an odd number of elements whenever R is a regular value of π. One case in which R would be a regular value of π would be if $\mathcal{M}(\gamma(R);\hat{J}) = \varnothing$ (*i.e.*, if $\pi^{-1}(\{R\}) = \varnothing$), and so we have shown that if $\mathcal{M}(\gamma(R);\hat{J}) = \varnothing$ then the empty set has an odd number of elements, a contradiction which proves the desired result. □

Proof of Lemma 3.6. We argue very similarly to the end of the proof of [U14, Theorem 4.9]. We will prove the contrapositive of the lemma:

$$\text{if } \int_0^1 \max|K(t,\cdot)|dt < \hbar \text{ then } \sigma_K^1(P) \cap (\mathbb{R}\times\Lambda) \neq \varnothing.$$

Given any integer $m \geq 1$ we have according to Proposition 3.11(B) an element $\tilde{u}_m \in \mathcal{M}(m;\hat{J})$. Noting that $\beta_m(s) = 1$ and $\hat{J}^m_{s,t} = J$ whenever $|s| \leq m$, we see that

$$\frac{\partial \tilde{u}_m}{\partial s}(s,t) + J\left(\frac{\partial \tilde{u}_m}{\partial t}(s,t) - Z_{K_t}(u(s,t))\right) = 0 \quad \text{for } (s,t) \in [-m,m]\times[0,1].$$

Thus, writing in general $|v|_J^2 = \omega(v,Jv)$, we have

$$\left|\frac{\partial \tilde{u}_m}{\partial s}(s,t)\right|_J^2 = \left|\frac{\partial \tilde{u}_m}{\partial t}(s,t) - Z_{K_t}(\tilde{u}_m(s,t))\right|_J^2 \quad \text{for } (s,t) \in [-m,m]\times[0,1].$$

So by Proposition 3.12 and the assumption that $\int_0^1 \max|K(t,\cdot)|dt < \hbar$ we have, for each m,

$$\int_{-m}^m \int_0^1 \left|\frac{\partial \tilde{u}_m}{\partial t}(s,t) - Z_{K_t}(\tilde{u}_m(s,t))\right|_J^2 dtds < 2\hbar.$$

Hence there is $s_m \in [-m, m]$ so that, defining $\gamma_m\colon [0,1] \to \mathbb{R} \times Y$ by $\gamma_m(t) = \tilde{u}_m(s_m, t)$, it holds that

$$\int_0^1 \left|\gamma_m'(t) - Z_{K_t}(\gamma_m(t))\right|_J^2 dt < \frac{\hbar}{m}. \tag{3.8}$$

Just as at the end of the proof of [U14, Theorem 4.9], Morrey's inequality then gives a bound on the $C^{1/2}$ norms of the γ_m, whereupon the Arzelà-Ascoli theorem (which applies here because Proposition 3.11(A) ensures that the γ_m are uniformly bounded) shows that a subsequence $\{\gamma_{m_k}\}$ of $\{\gamma_m\}_{m=1}^{\infty}$ converges uniformly, say to $\gamma\colon [0,1] \to \mathcal{K} \subset \mathbb{R} \times Y$. Hence the functions $t \mapsto Z_{K_t}(\gamma_{m_k}(t))$ likewise converge uniformly to $t \mapsto Z_{K_t}(\gamma)$, and so $\{\gamma_{m_k}'\}$ is a Cauchy sequence in L^2 by another application of (3.8), in view of which γ is in fact the limit of γ_{m_k} in the Sobolev space $W^{1,2}$ with $\gamma'(t) = Z_{K_t}(\gamma(t))$ at least in the sense of weak derivatives. But given that γ (and hence also $t \mapsto Z_{K_t}(\gamma(t))$) is continuous this implies that $\gamma'(t) = Z_{K_t}(\gamma(t))$ in the usual sense. In particular $\gamma(1) = \sigma_K^1(\gamma(0))$.

Since the γ_{m_k} converge uniformly to γ and have $\gamma_{m_k}(0) \in P$ and $\gamma_{m_k}(1) \in \mathbb{R} \times \Lambda$ we likewise have $\gamma(0) \in P$ and $\gamma(1) \in \mathbb{R} \times \Lambda$. Thus $\gamma(1) \in \sigma_K^1(P) \cap (\mathbb{R} \times \Lambda)$. □

4. Coisotropic submanifolds

The proof of Theorem 1.2 is completed at the end of this section, after we establish some basic results about coisotropic submanifolds of contact manifolds and their connection to local rigidity. The literature is somewhat inconsistent as to the definition of a coisotropic submanifold of a contact manifold; our convention in this paper is:

Definition 4.1. [H15] Let (Y, ξ) be a contact manifold, $C \subset Y$ a submanifold, and $p \in C$. We say that C is **coisotropic at** p if, for one and hence every contact form α for ξ defined on a neighborhood of p, $T_pC \cap \xi_p$ is a coisotropic subspace of the symplectic vector space $(\xi_p, d\alpha_p)$ (*i.e.*, if the $d\alpha_p$-orthogonal complement to $T_pC \cap \xi_p$ is contained in $T_pC \cap \xi_p$).

We say the submanifold $C \subset Y$ is a coisotropic submanifold if it is coisotropic at p for every $p \in C$.

Assuming that ξ is coorientable, [RZ18, Proposition 3.1] shows that C is coisotropic if and only if $\mathbb{R} \times C$ is a coisotropic submanifold of the symplectization of (Y, ξ). See [RZ18, Proposition 1.2], as well as Corollaries 4.10 and 4.12 below, for other conditions equivalent to coisotropy.

We quickly observe:

Proposition 4.2. Let C be a submanifold of codimension k in a $(2n+1)$-dimensional contact manifold (Y, ξ), and $p \in C$. If $k > n+1$ then C is not coisotropic at p, and if $k = n+1$ then C is coisotropic if and only if C is Legendrian.

Proof. For each $p \in C$ the subspace $T_pC \cap \xi_p$ of the $2n$-dimensional vector space ξ_p has codimension $k-1$ if $T_pC \subset \xi_p$, and codimension k otherwise. Since a coisotropic subspace of ξ_p would have codimension at most n this shows that C can never be coisotropic at p if $k > n+1$, and that if $k = n+1$ then C is coisotropic at p if and only if T_pC is a Lagrangian subspace of ξ_p with respect to the form $(d\alpha)_p$ (where α is a contact form for ξ defined near p). If C is Legendrian (and hence has codimension $n+1$) then α and $d\alpha$ both vanish on T_pC for all p and hence each T_pC is indeed a Lagrangian subspace of $\xi_p = \ker \alpha_p$. Conversely if the codimension-$(n+1)$ submanifold C is coisotropic then the above discussion shows that $T_pC \subset \xi_p$ for all $p \in C$ and hence that C is Legendrian. $\square$

In general if (V, ω) is a symplectic vector space and $W \leq V$ is a subspace we write W^ω for the ω-orthogonal complement: $W^\omega = \{v \in V | (\forall w \in W)(\omega(v, w) = 0)\}$. Of course $\dim V = \dim W + \dim W^\omega$, and W is coisotropic iff $W^\omega \leq W$.

Lemma 4.3. Let (V, ω) be a $2n$-dimensional symplectic vector space, and let $W \leq V$ be a subspace of codimension $c \leq n$. Then $(\omega|_W)^{\wedge(n-c)} \neq 0$, and $(\omega|_W)^{\wedge(n-c+1)} = 0$ if and only if W is a coisotropic subspace.

Proof. Choose any subspace $X \leq W$ such that $W = (W \cap W^\omega) \oplus X$. It is then straightforward to see that ω restricts nondegenerately to X, and that if $\pi\colon W \to X$ is the projection with kernel $W \cap W^\omega$ then $\omega|_W = \pi^*(\omega|_X)$. So X has some even dimension $2j$, in which case $(\omega|_X)^{\wedge j} \neq 0$ while $(\omega|_X)^{\wedge(j+1)} = 0$, and hence $(\omega|_W)^{\wedge j} \neq 0$ while $(\omega|_W)^{\wedge(j+1)} = 0$.

Now

$$2j = \dim W - \dim(W \cap W^\omega) \geq \dim W - \dim W^\omega = 2n - 2c,$$

with equality holding iff $W \cap W^\omega = W^\omega$, *i.e.* iff W is coisotropic. Since in any event $j \geq n - c$ and, as already noted, $(\omega|_W)^{\wedge j} \neq 0$, this shows that we have $(\omega|_W)^{\wedge(n-c)} \neq 0$ for arbitrary W. If W is not coisotropic then $j \geq n - c + 1$ and so likewise $(\omega|_W)^{\wedge(n-c+1)} \neq 0$, while if W is coisotropic then $n - c + 1 = j + 1$ and hence $(\omega|_W)^{\wedge(n-c+1)} = (\omega|_W)^{\wedge(j+1)} = 0$. $\square$

Proposition 4.4. Let C be a submanifold of codimension $k \leq n$ in a $(2n+1)$-dimensional contact manifold (Y, ξ), let $p \in C$, let U be a neighborhood of p and $\alpha \in \Omega^1(U)$ a contact form for $\xi|_U$, and write $\lambda = \alpha|_{C\cap U}$.

- If $\lambda_p = 0$, then $(d\lambda)_p^{\wedge(n-k+1)} \neq 0$, and $(d\lambda)_p^{\wedge(n-k+2)} = 0$ if and only if C is coisotropic at p.
- If $\lambda_p \neq 0$, then $\lambda_p \wedge (d\lambda)_p^{\wedge(n-k)} \neq 0$, and $\lambda_p \wedge (d\lambda)_p^{\wedge(n-k+1)} = 0$ if and only if C is coisotropic at p.

Proof. If $\lambda_p = 0$, then $T_pC = T_pC \cap \xi_p$ is a codimension-$(k-1)$ subspace of ξ_p, so the statement follows from Lemma 4.3.

If instead $\lambda_p \neq 0$, then since

$$\dim(T_pC) - \dim(T_pC \cap \xi_p) = \dim(T_pY) - \dim(\xi_p) = 1$$

we see that $T_pC \cap \xi_p$ has codimension k in ξ_p. So applying Lemma 4.3 shows that $\left((d\lambda)|_{T_pC\cap\xi_p}\right)^{\wedge(n-k)} \neq 0$, and that $\left((d\lambda)|_{T_pC\cap\xi_p}\right)^{\wedge(n-k+1)} = 0$ if and only if C is coisotropic at p. If we fix an an arbitrary element v of $T_pC \setminus \xi_p$ then, for $j \in \mathbb{N}$, the $(2j+1)$-form $\lambda_p \wedge (d\lambda)_p^j$ on T_pC is zero iff it evaluates to 0 on all tuples of form $(v, w_1, \ldots, w_{2j})$ where $w_1, \ldots, w_{2j} \in T_pC \cap \xi_p$. So what we have shown about powers of $(d\lambda)|_{T_pC\cap\xi_p}$ implies that indeed $\lambda_p \wedge (d\lambda)_p^{\wedge(n-k)} \neq 0$ and that $\lambda_p \wedge (d\lambda)_p^{\wedge(n-k+1)} = 0$ iff C is coisotropic at p. $\square$

Proposition 4.5. Let C be a submanifold of a contact manifold (Y, ξ) and $p \in C$, and suppose that there is a Legendrian submanifold Λ of Y such that $p \in \Lambda \subset C$. Then C is coisotropic at p.

Proof. Under the assumption we have $T_p\Lambda = T_p\Lambda \cap \xi_p \subset T_pC \cap \xi_p$ with $T_p\Lambda$ a Lagrangian subspace of ξ_p and hence, taking $(d\alpha_p)$-orthogonal complements within ξ_p where α is a contact form defined near p,

$$(T_pC \cap \xi_p)^{d\alpha_p} \subset (T_p\Lambda)^{d\alpha_p} = T_p\Lambda \subset T_pC \cap \xi_p.$$

$\square$

Below in Proposition 4.8 we will establish a partial converse to Proposition 4.5; we begin with observations concerning flows of certain contact vector fields. For this purpose it is convenient to identify contact vector fields with Hamiltonians, which requires choosing a contact form, so the next couple of lemmas will require the ambient contact manifold to be coorientable; while we ultimately want to prove certain statements that do not require a coorientability hypothesis, these statements are local so this does not pose a serious problem.

Recall that if α is a contact form on a smooth manifold Y and $\xi = \ker \alpha$ the Hamiltonian vector field of a smooth function $H\colon Y \to \mathbb{R}$ is the vector field X_H characterized uniquely by the properties that $\alpha(X_H) = H$ and $\iota_{X_H} d\alpha = dH(R_\alpha)\alpha - dH$ where R_α is the Reeb field of α.

Lemma 4.6. Let C be a submanifold of Y, let α be a contact form on Y with $\xi = \ker \alpha$, and let $H\colon Y \to \mathbb{R}$ be smooth. Then $(X_H)_q \in (T_qC \cap \xi_q)^{d\alpha|_{\xi_q}}$ for all $q \in C$ if and only if $H|_C = 0$.

Proof. The forward implication is trivial: if $(X_H)_q \in (T_qC \cap \xi_q)^{d\alpha|_{\xi_q}}$ for all $q \in C$ then in particular $(X_H)_q \in \xi_q = \ker \alpha_q$ and so, for all $q \in C$, $H(q) = \alpha_q(X_H) = 0$.

Conversely if $H|_C = 0$ then for each $q \in C$ we have $\alpha_q(X_H) = 0$ and so $(X_H)_q \in \xi_q$, and moreover, for each $v \in T_qC \cap \xi_q$,

$$d\alpha(X_H, v) = dH(R_\alpha)\alpha(v) - dH(v) = 0$$

where the first term vanishes because $v \in \xi_q$ and the second vanishes because $v \in T_qC$. □

Lemma 4.7. If C is a submanifold of a smooth manifold Y equipped with a contact form α and if $H\colon Y \to \mathbb{R}$ is smooth with $H|_C = 0$, then for any other smooth function $f\colon Y \to \mathbb{R}$ we have $(X_{fH})_q = f(q)(X_H)_q$ for all $q \in C$.

Proof. For any $q \in Y$, the tangent vector $(X_{fH})_q \in T_qY$ is uniquely characterized by the properties that $\alpha((X_{fH})_q) = f(q)H(q)$ and $\iota_{(X_{fH})_q} d\alpha|_{\xi_q} = -d(fH)|_{\xi_q}$ where $\xi_q = \ker \alpha_q$ so we just need to check that $f(q)(X_H)_q$ obeys the same properties when $q \in C$. This is clear since, due to the assumption that $H|_C = 0$, we have $d(fH)_q = f(q)dH_q$. □

Proposition 4.8. Let C be a coisotropic submanifold of a contact manifold (Y, ξ) and let $p \in C$ with $T_pC \not\subset \xi_p$. Then there is a Legendrian submanifold Λ of Y such that $p \in \Lambda \subset C$.

Proof. A concise summary of the proof is that, for a suitably small neighborhood W of p with α a contact form for $\xi|_W$, the neighborhood $C \cap W$ of p in C can be "coisotropically reduced," yielding a projection $\pi\colon C \cap W \to Z$ where Z comes equipped with a contact form β having $\pi^*\beta = \alpha|_{C\cap W}$, and then we can take $\Lambda = \pi^{-1}(\Lambda_0)$ for a Legendrian submanifold $\Lambda_0 \subset Z$ that passes through $\pi(p)$. (Below Z will be constructed as a local transversal to the foliation spanned by $(T(C\cap W)\cap\xi)^{d\alpha|_\xi}$ and β will just be $\alpha|_Z$. See [AM78, Theorem 5.3.30] for an analogous construction in the symplectic case, and for the contact case compare [LdL19, Theorem 13], though note that the definition of coisotropy therein is slightly different from ours.)

We now give full details. Choose a neighborhood W of p and smooth functions $H_1,\dots H_k\colon W\to\mathbb{R}$ such that $C\cap W$ is given as a regular level set $C\cap W=\{H_1=\cdots=H_k=0\}$. (In particular we are assuming the dH_j to be pointwise-linearly-independent along C, so $\dim C=\dim Y-k$.) Shrinking W if necessary, let $\alpha\in\Omega^1(W)$ have $\ker\alpha=\xi|_W$, and assume that $T_qC\not\subset\xi_q$ for all $q\in C\cap W$. Note that this implies that the restrictions $dH_j|_{\xi_q}$ are linearly independent at each $q\in C\cap W$: choosing $v\in T_qC\setminus\xi_q$, a linear combination $H=\sum_j c_jH_j$ automatically has $dH(v)=0$, so if $(dH)_q|_{\xi_q}=0$ then $(dH)_q$ vanishes identically on T_qY and hence the coefficients c_j are all zero.

By Lemma 4.6, we have $(X_{H_j})_q\in(T_qC\cap\xi_q)^{d\alpha|_{\xi_q}}$ for all $q\in C\cap W$ and each $j=1,\dots,k$. Because each of the $T_qC\cap\xi_q$ (for $q\in C\cap W$) has codimension k in ξ_q, each of the $(T_qC\cap\xi_q)^{d\alpha|_{\xi_q}}$ is a k-dimensional subspace of T_qC; thus we have a rank-k distribution $(TC\cap\xi)^{d\alpha|_\xi}$ on $C\cap W$, of which each $X_{H_j}|_{C\cap W}$ is a section. These sections $X_{H_1}|_{C\cap W},\dots,X_{H_k}|_{C\cap W}$ are moreover linearly independent, since $\iota_{X_{H_j}}d\alpha|_\xi=-dH_j|_\xi$ and as noted at the end of the previous paragraph the $dH_j|_\xi$ are linearly independent along $C\cap W$. Thus our distribution $\mathcal{F}:=(TC\cap\xi)^{d\alpha|_\xi}$ on $C\cap W$ is the pointwise-linearly-independent span of the restrictions of the vector fields $X_{H_1},\dots,X_{H_k}$ to $C\cap W$.

We next claim that this distribution $\mathcal{F}$ is involutive. Indeed letting $\{\cdot,\cdot\}$ denote the contact Poisson bracket as in [McSa17, Remark 3.5.18], one has $[X_{H_i},X_{H_j}]=X_{\{H_i,H_j\}}$ for any $i,j\in\{1,\dots,k\}$, and then by [RZ18, Proposition 1.2] $\{H_i,H_j\}|_{C\cap W}=0$, whence $X_{\{H_i,H_j\}}$ is a section of $(TC\cap\xi)^{d\alpha|_\xi}$ by Lemma 4.6.

By the Frobenius theorem, the involutivity of $\mathcal{F}$ implies that, after perhaps shrinking W, we can find vector fields $V_i=\sum_j f_{ij}X_{H_j}$ (for $i=1,\dots,k$ and some smooth functions $f_{ij}\colon W\to\mathbb{R}$) which continue to span $\mathcal{F}$ pointwise in $C\cap W$ and which obey $[V_i,V_j]=0$. (Specifically the V_i may be identified with coordinate vector fields for a flat chart for $(TC\cap\xi)^{d\alpha|_\xi}$ around p.) By Lemma 4.7, we have $V_i|_C=\sum_j X_{f_{ij}H_j}|_C$, so if $K_i=\sum_j f_{ij}H_j$ the functions $K_1,\dots,K_k$ vanish along $C\cap W$ and have the property that, along $C\cap W$, the X_{K_j} pairwise commute and span $\mathcal{F}=(TC\cap\xi)^{d\alpha|_\xi}$.

Now, possibly after shrinking W again, let Z be a codimension-k submanifold of $C\cap W$ that passes through our point p and is transverse (in $C\cap W$) to the k-dimensional foliation spanned by $\mathcal{F}$. In particular for each $q\in Z$,

$T_qZ \not\subset \xi_q$. Now by Proposition 4.4, the $(2n+1-2k)$-form $\alpha \wedge (d\alpha)^{\wedge(n-k)}$ has nowhere-vanishing restriction to $C \cap W$. So for $q \in Z$ we can find $v \in T_qZ \setminus \xi_q$ and $w_1, \ldots, w_{2(n-k)} \in T_qC \cap \xi_q$ such that

$$0 \neq \alpha \wedge (d\alpha)^{\wedge(n-k)}(v, w_1, \ldots, w_{2n-2k}) = \alpha(v)(d\alpha)^{\wedge(n-k)}(w_1, \ldots, w_{2n-2k}).$$

But since $\mathcal{F}$ is contained in and $d\alpha$-orthogonal to $TC \cap \xi$, and since $T_qC \cap \xi_q = (TZ \cap \xi_q) \oplus \mathcal{F}_q$, replacing the w_i above by their projections to $T_qZ \cap \xi_q$ will not change the property that $\alpha(v)(d\alpha)^{\wedge(n-k)}(w_1, \ldots, w_{2n-2k}) \neq 0$. This proves that $\left(\alpha \wedge (d\alpha)^{\wedge(n-k)}\right)|_Z$ is a nowhere-vanishing $(2n+1-2k)$-form on Z. But $\dim Z = \dim C - k = 2n+1-2k$, so what we have just shown is that $\alpha|_Z$ is a contact form.

Now (for instance by the contact Darboux theorem), within the contact manifold $(Z, \ker \alpha|_Z)$ we can take a Legendrian submanifold Λ_Z of Z that passes through the point p. (Thus $\dim \Lambda_Z = n-k$.) In general letting ϕ_H^t denote the Hamiltonian flow of the contact Hamiltonian H with respect to the contact form α on W we now take

$$\Lambda = \left\{\phi_{K_1}^{t_1} \circ \cdots \circ \phi_{K_k}^{t_k}(x) | x \in \Lambda_Z \cap W', (t_1, \ldots, t_k) \in U\right\}$$

for neighborhoods W' of p in C and U of the origin in $\mathbb{R}^k$ that are sufficiently small for the relevant Hamiltonian flows to be defined and for Λ as given above to be an embedded submanifold. The tangent space to Λ at $\phi_{K_1}^{t_1} \circ \cdots \circ \phi_{K_k}^{t_k}(x)$ is spanned by the vector fields X_{K_k} (which lie in $\ker \alpha$) together with the image under the linearization of $\phi_{K_1}^{t_1} \circ \cdots \circ \phi_{K_k}^{t_k}$ of the tangent space $T_x\Lambda_Z$, and this image is annihilated by α because $T_x\Lambda_Z \subset \xi_x$ while the $\phi_{K_k}^{t_k}$ are contactomorphisms. So Λ is an n-dimensional submanifold of C containing p with $\alpha|_{T\Lambda} = 0$, as desired. □

Remark 4.9. The assumption that $T_pC \not\subset \xi_p$ in Lemma 4.8 cannot be completely discarded, as can already be seen in the case that $\dim Y = 3$ and $\dim C = 2$. In this case the Legendrian submanifolds of Y that are contained in C coincide away from the singular set $\{p \in C | T_pC \subset \xi_p\}$ with the leaves of the characteristic foliation (*i.e.* the foliation tangent to $TC \cap \xi$). If p is an isolated point of this singular set then it may not be possible to find a one-dimensional smooth submanifold passing through p that coincides away from the singular set with a union of such leaves—for example if the foliation has a spiral source at p then any smoothly embedded arc through p will have infinitely many transverse intersections with each leaf that approaches p.

Corollary 4.10. A submanifold C of a contact manifold (Y, ξ) is coisotropic if and only if there is a dense, relatively open subset $U \subset C$ such that for each $p \in U$ there exists a Legendrian submanifold Λ of Y such that $p \in \Lambda \subset C$.

Proof. As before write $\dim Y = 2n + 1$ and $k = \dim Y - \dim C$. If $k > n + 1$ the statement of the corollary is vacuous since a nonempty submanifold of codimension greater than $n + 1$ can neither be coisotropic nor contain a nonempty Legendrian submanifold. If $k = n + 1$ and C is coisotropic then C is Legendrian by Proposition 4.2 so we can take $U = \Lambda = C$. Conversely if $k = n + 1$ and $p \in \Lambda \subset C$ with Λ Legendrian then by dimensional considerations Λ contains an open-in-C neighborhood of p, so that $T_pC \subset \xi_p$. So if the set of points p admitting such a Legendrian is dense in C then for any open set V on which ξ can be written as $\ker \alpha$ it holds that $\alpha|_{T(C \cap V)}$ vanishes on a dense subset of $C \cap V$ and hence on all of $C \cap V$, whence C is Legendrian and thus coisotropic. So assume for the rest of the proof that $k \leq n$.

In this case, we claim that the set of points $p \in C$ such that $T_pC \subset \xi_p$ has empty interior. If this were false there would be a nonempty open subset $V \subset Y$ intersecting C on which $\xi|_V = \ker \alpha$ for some $\alpha \in \Omega^1(V)$ such that $\lambda := \alpha|_{C \cap V}$ vanished throughout $C \cap V$, in which case $d\lambda$ would also vanish throughout $C \cap V$. But by Proposition 4.4 we have $(d\lambda)^{\wedge(n-k+1)} \neq 0$ and so (since $k \leq n$) $d\lambda \neq 0$. So indeed $U = \{p \in C | T_pC \not\subset \xi_p\}$ is open and dense in C (regardless of whether C is coisotropic), and by Proposition 4.8 if C is coisotropic then for each $p \in U$ there is a Legendrian Λ with $p \in \Lambda \subset C$.

Conversely, if $W \subset C$ is an open and dense subset such that each $p \in W$ admits a Legendrian Λ with $p \in \Lambda \subset C$, then C is coisotropic at p for each $p \in W$ by Proposition 4.5. So letting $U = \{p \in C | T_pC \not\subset \xi_p\}$ as above and considering any sufficiently small open V and $\alpha \in \Omega^1(V)$ with $\xi|_V = \ker \alpha$, for each $p \in U \cap V \cap W \subset C$, Proposition 4.4 shows that $\lambda_p \wedge (d\lambda)_p^{\wedge(n-k+1)} = 0$ where $\lambda = \alpha|_{C \cap V}$. But since $U \cap V \cap W$ is dense in $C \cap V$ (being the intersection of two open dense sets $U \cap V$ and $W \cap V$) this implies that $\lambda \wedge (d\lambda)^{\wedge(n-k+1)} = 0$ everywhere on $C \cap V$, and hence also that $(d\lambda)^{\wedge(n-k+2)} = 0$ everywhere on $C \cap V$. Another appeal to Proposition 4.4 thus shows that $C \cap V$ is coisotropic at p for every $p \in C \cap V$. Allowing V to vary through open subsets on which $\xi|_V$ is coorientable thus shows that C is coisotropic. □

Proposition 4.11. Let C be a submanifold of a contact manifold (Y, ξ) and suppose that $p \in C$ is locally rigid with respect to C. Then C is coisotropic at p.

Proof. Let W be a neighborhood of p that is sufficiently small for $C \cap W$ to be closed as a subset of W and for there to be a contact form α for $\xi|_W$. Suppose that C is not coisotropic at p, so that there is $v \in \xi_p$ such that $v \in (T_pC \cap \xi_p)^{d\alpha|_{\xi_p}} \subset \xi_p$ while $v \notin T_pC$. We will find a neighborhood U of p with $\bar{U} \subset W$ such that $e^W_\alpha(U, C) = 0$; in view of Proposition 2.2(iv) and the fact that W is arbitrary subject to being sufficiently small, this will prove that p is not locally rigid with respect to C.

To do this, following the strategy of [U14, Lemma 4.3] and [RZ18, Proposition 7.3], let $H\colon Y \to \mathbb{R}$ be a smooth function having compact support contained in W such that $H|_C = 0$ and $dH_p(v) > 0$, as is possible since v is not tangent to C. The contact Hamiltonian vector field X_H of H on W with respect to α will then obey $\alpha(X_H) = 0$ at all points of $C \cap W$ and, using that $v \in \xi_p$, $d\alpha(X_H, v) = -dH_p(v) \neq 0$. Thus $(X_H)_p \in \xi_p$ but $(X_H)_p \notin T_pC \cap \xi_p$, since $d\alpha(\cdot, v)$ restricts to zero on $T_pC \cap \xi_p$. Thus for sufficiently small positive t we will have $\phi^t_H(p) \notin C$; replacing H by H/t if necessary we may as well assume that $\phi^1_H(p) \notin C$. Since $C \cap W$ is closed as a subset of W this implies that there is an open set U around p, which we can assume to obey $\bar{U} \subset W$, such that $\phi^1_H(\overline{U}) \cap C = \varnothing$. The proof will be complete when we show that $e^W_\alpha(U, C) = 0$.

Choose a sequence of smooth functions $\beta_k\colon \mathbb{R} \to \mathbb{R}$ such that:

- $\beta_k(s) = s$ whenever $|s| \geq 2/k$;
- $\beta_k(s) = 0$ whenever $|s| \leq 1/k$; and
- $0 \leq \beta_k'(s) \leq 3$ for all s.

The functions $\beta_k \circ H$ are supported in W and each vanish throughout a (k-dependent) neighborhood of C, and so $\phi^t_{\beta_k \circ H}$ will restrict to the identity on C for each t and k. So the fact that $\phi^1_H(\overline{U}) \cap C = \varnothing$ implies that

$$(\phi^1_{\beta_k \circ H})^{-1} \circ \phi^1_H(\overline{U}) \cap C = (\phi^1_{\beta_k \circ H})^{-1}(\phi^1_H(\overline{U}) \cap C) = \varnothing.$$

If we write $f_{k,t}\colon W \to \mathbb{R}$ for the smooth functions such that $\phi^{t*}_{\beta_k \circ H}\alpha = f_{k,t}\alpha$, then by a standard calculation as in [MüSp15, Lemma 2.2] the isotopy $\{(\phi^t_{\beta_k \circ H})^{-1} \circ \phi^t_H\}_{t \in [0,1]}$ is generated by the unique contact vector field $(V_{k,t})_{t \in [0,1]}$ that obeys

$$\alpha(V_{k,t}) = \frac{1}{f_{k,t}} (H - \beta_k \circ H) \circ \phi^t_{\beta_k \circ H}. \tag{4.1}$$

This is slightly more complicated than the situation in [RZ18, Proof of Proposition 7.3] because the order in which we need to compose our diffeomorphisms is opposite to theirs, leading to a factor $\frac{1}{f_{k,t}}$ in (4.1) that depends on k, but these factors can be estimated as follows. The Lie derivative of α along the Hamiltonian vector field $X_{\beta_k \circ H}$ is given by

$$L_{X_{\beta_k \circ H}} \alpha = d(\beta_k \circ H) + (\iota_{R_\alpha} d(\beta_k \circ H))\alpha - d(\beta_k \circ H) = \big((\beta_k' \circ H)\iota_{R_\alpha} dH\big)\,\alpha$$

where R_α is the Reeb vector field of α, and thus we have

$$\log f_{k,t} = \int_0^t \big((\beta_k' \circ H)\iota_{R_\alpha} dH\big) \circ \phi^s_{\beta_k \circ H} ds.$$

So choosing $M > 0$ such that $|\iota_{R_\alpha} dH| \leq M$ everywhere on W, our assumption that $0 \leq \beta_k' \leq 3$ shows that we have

$$|\log f_{k,t}(x)| \leq 3M \text{ for all } k \in \mathbb{Z}_+, t \in [0,1], x \in Y.$$

Moreover our construction of β_k also ensures that $|H - \beta_k \circ H| \leq \frac{2}{k}$ everywhere. Hence (4.1) yields

$$|\alpha(V_{k,t})| \leq \frac{2e^{3M}}{k}$$

everywhere, where the constant M depends on H and α but not on k. Since the time-one flow $(\phi^1_{\beta_k \circ H})^{-1} \circ \phi^1_H$ disjoins $\bar{U}$ from N this proves that $e^W_\alpha(U, C) \leq \frac{2e^{3M}}{k}$ for all positive integers k, and hence that $e^W_\alpha(U, C) = 0$, as desired. □

Corollary 4.12. Let C be a submanifold of a contact manifold (Y, ξ). Then C is coisotropic if and only if there is a relatively open and dense subset $U \subset C$ such that every point $p \in U$ is locally rigid with respect to C.

Proof. If C is coisotropic then all of the points in the relatively open and dense subset from Corollary 4.10 will be locally rigid by Corollary 3.4 and Proposition 2.4(i). On the other hand if C is not coisotropic we claim that the set of points at which it fails to be coisotropic contains a nonempty open set. Let V be an open subset of Y such that $\xi|_V = \ker \alpha$ with $\alpha \in \Omega^1(V)$ and such that $C \cap V$ contains a point at which C is not coisotropic. Write $\lambda = \alpha|_{C \cap V}$ and $k = \dim Y - \dim C$. If we had $\lambda_p \wedge (d\lambda)_p^{\wedge(n-k+1)} = 0$ at every point of $C \cap V$ then taking a derivative would show $(d\lambda)^{\wedge(n-k+2)} = 0$ throughout $C \cap V$ which is impossible by Proposition 4.4 and our assumption on V. So there must be some point $p \in C \cap V$ at which $\lambda_p \wedge (d\lambda)_p^{\wedge(n-k+1)} \neq 0$. But then $\lambda \wedge (d\lambda)^{(n-k+1)}$ (and hence also λ) is nowhere vanishing on a neighborhood W of p in C, and so for each $q \in W$, C is not coisotropic at q by another application of Proposition 4.4. By Proposition 4.11 this implies that, for each q in the nonempty relatively open set W, q is not locally rigid with respect to C; thus C cannot contain a dense set of points each of which is locally rigid. □

Proof of Theorem 1.2. Let $\psi\colon Y \to Y$ be a contact homeomorphism and $C \subset Y$ a coisotropic submanifold such that $\psi(C)$ is a smooth submanifold and ψ is bounded below near every point of C. By Corollary 4.12, there is a dense and relatively open subset $U \subset C$ such that each point of U is locally rigid with respect to C. Then Proposition 2.6 shows that each point of $\psi(U) \subset \psi(C)$ (which is open and dense since $\psi|_C$ is a homeomorphism) is likewise locally rigid with respect to $\psi(C)$. But then $\psi(C)$ is coisotropic by Corollary 4.12. □

5. Instability of coisotropy at a point

This section contains the examples which prove Theorem 1.4, showing that a contact homeomorphism ψ can map a submanifold that is not coisotropic at some point p to one which is coisotropic at $\psi(p)$. Our constructions are local in nature, taking place in an open subset of $\mathbb{R}^{2n+1}$ in Section 5.1 and in an open subset of the one-jet bundle of the n-torus in Section 5.2; we always use the contact form

$$\alpha = dz - \sum_{j=1}^{n} y_j dx_j$$

in either case (with x_j valued in $\mathbb{R}$ in Section 5.1 and in $\mathbb{R}/\mathbb{Z}$ in Section 5.2). The Hamiltonian vector field X_H of a smooth function H with respect to this contact form α is then given by
(5.1)

$$X_H = -\sum_j \frac{\partial H}{\partial y_j}\partial_{x_j} + \sum_j \left(\frac{\partial H}{\partial x_j} + y_j \frac{\partial H}{\partial z}\right)\partial_{y_j} + \left(H - \sum_j y_j \frac{\partial H}{\partial y_j}\right)\partial_z.$$

One then has $L_{X_H}\alpha = \frac{\partial H}{\partial z}\alpha$, and so if ϕ_H^t is the time-t map of the Hamiltonian flow of H then the function f obeying $\phi_H^{1*}\alpha = f\alpha$ is given by

$$f(p) = \exp\left(\int_0^1 \frac{\partial H}{\partial z}(\phi_H^t(p))dt\right). \tag{5.2}$$

5.1. The Buhovsky-Opshtein construction

[BO16, Corollary 4.4] exhibits compactly supported symplectic homeomorphisms of $\mathbb{R}^{2n}$ that map the symplectic subspace $\{(0,0)\} \times \mathbb{R}^{2n-2}$ to a smooth, non-symplectic submanifold—more specifically, to $\{(F(\vec{z}), 0, \vec{z}) | \vec{z} \in \mathbb{R}^{2n-2}\}$ where $F\colon \mathbb{R}^{2n-2} \to \mathbb{R}$ is a continuous function whose graph is smooth and has vertical tangencies. As we now show, Buhovsky and Opshtein's construction can be adapted to the contact context.

Proposition 5.1. Let $U \subset \mathbb{R}^{2n-1}$ be an open ball, and $F\colon U \to \mathbb{R}$ a continuous function with compact support such that $\max_U |F| < 1$. Then for any $\delta > 0$ there is a sequence of uniformly compactly supported contactomorphisms $\psi_m\colon (-1,1) \times (-\delta,\delta) \times U \to (-1,1) \times (-\delta,\delta) \times U$ that converges uniformly to a homeomorphism ψ of $(-1,1) \times (-\delta,\delta) \times U$ such that, for all $w \in U$, we have

$$\psi(0,0,w) = (F(w), 0, w).$$

(Here the contact structure on $(-1,1) \times (-\delta,\delta) \times U$ is the kernel of $\alpha = dz - \sum_{j=1}^n y_j dx_j$, with (x_1, y_1) the coordinates on $(-1,1) \times (-\delta,\delta)$ and $(x_2, y_2, \ldots, x_n, y_n, z)$ the coordinates on U.) Moreover there is a constant $C > 1$ such that the functions f_m characterized by $\psi_m^*\alpha = f_m\alpha$ obey $\frac{1}{C} < \max |f_m| < C$.

Proof. We closely follow [BO16, Proof of Lemma 4.3]. First construct a sequence of smooth functions $\{F_k\}_{k=0}^\infty$ on U such that;

- For some compact subset $K \subset U$, each F_k has support contained in K;
- For some $\epsilon > 0$, $\max_k \max_U |F_k| < 1 - \epsilon$;
- $F_k \to F$ uniformly; and
- $F_0 \equiv 0$, and $\max_U |F_k - F_{k-1}| < \frac{1}{2^k}$.

Also let us abbreviate

$$G_k = F_k - F_{k-1}, \quad \text{so } F = \sum_{k=1}^{\infty} G_k.$$

Now choose smooth functions $u, v\colon \mathbb{R} \to \mathbb{R}$, with u having compact support in $(-1, 1)$ and v having compact support in $(-\delta, \delta)$, such that:

$$u|_{[-1+\epsilon, 1-\epsilon]} \equiv 1, \quad v(0) = 0, \ v'(0) = -1$$

and, for all positive integers k, ℓ, define

$$H_{k\ell}(x_1, y_1, x_2, \ldots, y_n, z) = u(x_1)\frac{v(\ell y_1)}{\ell} G_k(x_2, \ldots, y_n, z).$$

Let V_{G_k} denote the Hamiltonian vector field of the function G_k on U with respect to the contact form $dz - \sum_{j=2}^{n} y_j dx_j$. Then the Hamiltonian vector field of $H_{k\ell}$ on $(-1, 1) \times (-\delta, \delta) \times U$ is

$$\begin{aligned} X_{H_{k\ell}} = {} & u(x_1)\frac{v(\ell y_1)}{\ell} V_{G_k} - u(x_1)v'(\ell y_1)y_1 G_k \partial_z \\ & - u(x_1)v'(\ell y_1)G_k \partial_{x_1} + \frac{v(\ell y_1)}{\ell}\left(u'(x_1)G_k + u(x_1)y_1\frac{\partial G_k}{\partial z}\right)\partial_{y_1}. \end{aligned}$$

In particular this vector field is tangent to the hypersurface $\{y_1 = 0\}$, and restricts to that hypersurface as $u(x_1)G_k\partial_{x_1}$. As in [BO16] the desired contactomorphisms ψ_m will be given by

$$\psi_m = \phi^1_{H_{m\ell_m}} \circ \cdots \circ \phi^1_{H_{1\ell_1}} \tag{5.3}$$

for a suitably chosen sequence $\{\ell_k\}_{k=1}^{\infty}$. To describe the inductive procedure for choosing the ℓ_k, note first that because all terms in the formula for $X_{H_{k\ell}}$ except the coefficient of ∂_{x_1} are bounded by a k-dependent constant times $\frac{1}{\ell}$, and since $\max|G_k| < 2^{-k}$, for all sufficiently large values of ℓ_k it will hold that $\max\|X_{H_{k\ell_k}}\| < C2^{-k}$ where the constant C depends only on the auxiliary functions u and v. Also for all sufficiently large values of ℓ_k it will hold that

$$\max\left|\frac{\partial H_{k\ell_k}}{\partial z}\right| \leq \frac{1}{\ell_k}\max\left|\frac{\partial G_k}{\partial z}\right| < \frac{1}{k^2}.$$

Moreover since $H_{k\ell}$ has support contained in the region $\{|y_1| < \frac{\delta}{\ell}\}$ and since $X_{H_{k\ell}}$ is tangent to $\{y_1 = 0\}$ we can inductively choose the ℓ_k sufficiently large that, in addition to having $\max\|X_{H_{k\ell_k}}\| < C2^{-k}$ and $\max\left|\frac{\partial H_{k\ell_k}}{\partial z}\right| < \frac{1}{k^2}$, we have

$$(5.4)\qquad \operatorname{supp}(\phi^1_{H_{k\ell_k}}) \subset \left(\phi^1_{H_{k-1\ell_{k-1}}} \circ \cdots \circ \phi^1_{H_{1\ell_1}}\right)\left(\left\{|y_1| < \frac{\delta}{k}\right\}\right).$$

For such a choice of $\{\ell_k\}_{k=1}^\infty$, if we define ψ_m as in (5.3) then (5.4) implies that $\psi_k(x_1, y_1, w) = \psi_{k-1}(x_1, y_1, w)$ whenever $|y_1| \geq \frac{\delta}{k}$. Hence if $y_1 \neq 0$ then $\psi_m(x_1, y_1, w)$ is independent of m once m is sufficiently large. On the other hand since $u|_{[-1+\epsilon,1-\epsilon]} \equiv 1$ and the restriction of $X_{H_{k\ell_k}}$ to $\{y_1 = 0\}$ is $u(x_1)G_k\partial_{x_1}$ we have, for all $w \in U$,

$$\psi_m(0,0,w) = \left(\sum_{k=1}^m G_k(w), 0, w\right) = (F_m(w), 0, w)\,.$$

The estimate $\max\left|\frac{\partial H_{k\ell_k}}{\partial z}\right| < \frac{1}{k^2}$ implies, as in (5.2), that the conformal factor of the contactomorphism $\phi^1_{H_{k\ell_k}}$ is bounded between $e^{-\frac{1}{k^2}}$ and $e^{\frac{1}{k^2}}$, implying an m-independent bound between $e^{-\frac{\pi^2}{6}}$ and $e^{\frac{\pi^2}{6}}$ for the conformal factors of the ψ_m. Finally, the bound $\max\|X_{H_{k\ell_k}}\| < C2^{-k}$ implies that the sequence $\{\psi_m\}_{m=1}^\infty$ is uniformly Cauchy, and so uniformly converges to a map $\psi\colon (-1,1)\times(-\delta,\delta)\times U \to (-1,1)\times(-\delta,\delta)\times U$. Since $\psi_m(0,0,w) = (F_m(w), 0, w)$ we indeed have $\psi(0,0,w) = (F(w),0,w)$. That ψ is injective (from which it easily follows that it is a homeomorphism since it is continuous and is the identity outside a compact subset of $(-1,1)\times(-\delta,\delta)\times U$ where U is a ball) follows by the same argument that is used in [BO16, Proof of Lemma 4.3]. □

Corollary 5.2. For any $k \in \{2, \ldots, n+1\}$ and any $(2n+1)$-dimensional contact manifold (Y, ξ) there exist a contact homeomorphism $\psi\colon Y \to Y$, a codimension-k submanifold $N \subset Y$, and a point $p \in N$ such that $T_pN \not\subset \xi_p$ and N is not coisotropic at p, but $\psi(N)$ is smooth and $T_{\psi(p)}\psi(N)$ is a coisotropic subspace of $\xi_{\psi(p)}$. Moreover ψ can be arranged to be bounded both above and below near every point of Y.

Proof. Choose a Darboux chart $\phi\colon V \to \mathbb{R}^{2n+1}$ sending some point p of Y to the origin such that $\phi(V)$ contains $(-1,1)\times(-1,1)\times U$ for some open ball $U \subset \mathbb{R}^{2n-1}$, and let N be a submanifold whose intersection with V is identified by ϕ with

$$\{(x_1, y_1, x_2, \ldots, y_n, z) \in (-1,1) \times (-1,1) \times \mathbb{R}^{2n-1} \\ \mid x_1 = y_1 = 0,\, y_{n-k+3} = \cdots = y_n = 0\}.$$

(If $k = 2$ this should just be interpreted as $\{(x_1, y_1, x_2, \ldots, y_n, z) \in (-1,1) \times (-1,1) \times \mathbb{R}^{2n-1} | x_1 = y_1 = 0\}$.) In particular $\partial_z \in T_pN$, so $T_pN \not\subset \xi_p$. Also the $d\alpha$-orthogonal complement of $T_pN \cap \xi_p$ inside ξ_p contains the tangent vector ∂_{y_1}, which is not contained in the tangent space to N, so N is not coisotropic at p.

Similarly to the proof of [BO16, Corollary 4.4], apply Proposition 5.1 with $\delta = 1$ and with a compactly supported function $F\colon U \to (-1,1)$ whose graph is smooth and which restricts to a neighborhood of the origin in U as a function f of the single variable z with $f(0) = 0$, such that f is invertible on a neighborhood of 0 on which f^{-1} is smooth with $(f^{-1})'(0) = 0$. (So f itself has derivative tending to $\pm\infty$ at 0.) The resulting contact homeomorphism ψ will have $\psi(p) = p$ and will send N to a smooth submanifold whose intersection with V is contained in the hypersurface $\{y_1 = 0\}$ and coincides there with the graph of F. The tangent space $T_p\psi(N)$ will be spanned by $\partial_{x_1}, \ldots, \partial_{x_n}$ together with some subset (depending on k) of the ∂_{y_j} with $j \geq 2$; in particular this tangent space will be a coisotropic subspace of $\xi_{\psi(p)}$, and so $\psi(N)$ is coisotropic at $p = \psi(p)$. That ψ is bounded both above and below follows directly from the last sentence of Proposition 5.1. □

5.2. Collapsing toward a Legendrian torus

In this section we describe a family of examples of contact homeomorphisms ψ of a neighborhood of a Legendrian torus which are not bounded below near points on the torus, and which can be arranged to send a nowhere-Legendrian submanifold N to a smooth submanifold that is tangent, possibly (depending one one's choice of parameters) even to infinite order, to the Legendrian torus; moreover unlike in Section 5.1 the restriction $\psi|_N$ can be arranged to be a smooth map. The section concludes with the proof of Theorem 1.4.

We work throughout this section in a smooth manifold of the form $\mathcal{B}_V = (\mathbb{R}^n/\mathbb{Z}^n) \times V$ where V is a neighborhood of the origin in $\mathbb{R}^{n+1}$ (which will be specified more precisely in particular examples), with coordinates $\vec{x} = (x_1, \ldots, x_n)$ on $\mathbb{R}^n/\mathbb{Z}^n$ and $(\vec{y}, z) = (y_1, \ldots, y_n, z)$ on V. We continue to use the contact form $\alpha = dz - \sum_j y_j dx_j$ on $\mathcal{B}_V$. By the Legendrian neighborhood theorem any Legendrian torus T in a contact manifold has a neighborhood contactomorphic to such a contact manifold $(\mathcal{B}_V, \ker\alpha)$, so the constructions of this section can be exported to other contact manifolds.

Let

$$\mathcal{Z} = \{(x_1, \dots, x_n, y_1, \dots, y_n, z) \in \mathcal{B}_V | y_1 = \cdots = y_n = z = 0\}$$

and

$$\mathcal{B}_V^* = \mathcal{B}_V \setminus \mathcal{Z}.$$

We will study flows of certain types of autonomous contact Hamiltonians $H\colon \mathcal{B}_V^* \to \mathbb{R}$ that extend continuously to all of $\mathcal{B}_V$. The open sets V and Hamiltonians $H = H_{F,\rho}$ that we consider are required to take the following general forms, and will satisfy additional constraints to be specified later:

- There exist even integers d_y, d_z with $d_y \geq d_z \geq 2$ and a real number $c > 0$ such that $V = \rho^{-1}([0, c))$, where $\rho\colon \mathbb{R}^{n+1} \to [0, \infty)$ is given by

$$\rho(\vec{y}, z) = \sum_{j=1}^{n} y_j^{d_y} + z^{d_z}.$$

- There is a smooth function $F\colon (-\log c, \infty) \to (-\infty, 0]$ such that $H : \mathcal{B}_V^* \to \mathbb{R}$ is given by

$$H_{F,\rho}(\vec{x}, \vec{y}, z) = zF(-\log\rho(\vec{y}, z)).$$

We will show:

Proposition 5.3. There are choices of F, d_y, d_z, c as above such that the Hamiltonian flow $\phi^1_{H_{F,\rho}}$ is a diffeomorphism of $\mathcal{B}_V^*$ that extends by the identity on $\mathcal{Z}$ to a contact homeomorphism $\bar{\phi}^1_{H_{F,\rho}}$ of $\mathcal{B}_V$ that is a uniform limit of uniformly compactly supported contactomorphisms of $\mathcal{B}_V$ and has the following properties:

(a) The nowhere-Legendrian codimension-$(n+1)$ submanifold $N = \{(\vec{x}, \vec{0}, x_n)\} \subset \mathcal{B}_V$ is mapped by $\bar{\phi}^1_{H_{F,\rho}}$ to a smooth submanifold having an infinite-order tangency to the contact distribution $\ker\alpha$ at the origin.

(b) The contact homeomorphism $\bar{\phi}^1_{H_{F,\rho}}$ is bounded above but not below at the origin.

(c) The restriction of $\bar{\phi}^1_{H_{F,\rho}}$ to the codimension-n submanifold $\{(\vec{x},\vec{0},z)\}$ is a self-homeomorphism having the form $(\vec{x},\vec{0},z)\mapsto(\vec{x},\vec{0},g(z))$ for a certain continuous function $g:\mathbb{R}\to\mathbb{R}$.

Proposition 5.3 is an immediate consequence of the following ingredients from later in this section:

- Proposition 5.9, which shows that under a general set of assumptions on F, d_y, d_z, c (namely Assumptions 5.4 below) the diffeomorphism $\phi^1_{H_{F,\rho}}$ extends by the identity on $\mathcal{Z}$ to a contact homeomorphism;
- Corollary 5.10, which shows that if additionally $\lim_{u\to\infty} F(u) = -\infty$ then this contact homeomorphism satisfies property (b) in Proposition 5.3.
- Proposition 5.11, which gives a formula for the restriction of $\bar{\phi}^1_{H_{F,\rho}}$ to $\{(\vec{x},\vec{0},z)\}$ that implies property (c) of Proposition 5.3.
- Example 5.14, which shows that that if we take $\rho(\vec{y},z)=\sum y_j^4+z^2$ and $F(u)=-u\log u$ for sufficiently large u then property (a) in Proposition 5.3 will hold.

Examples 5.12 and 5.13 are analogous to but simpler than Example 5.14, and lead to contact homeomorphisms satisfying weaker versions of property (a).

The additional conditions on F, d_y, d_z that we will assume throughout what follows are:

Assumptions 5.4.

(i) $F' \le 0$ everywhere.

(ii) There is $u_0 > -\log c$ such that $F(u) = 0$ if and only if $u \le u_0$.

(iii) For one and hence every $u_1 > u_0$ we have

$$\int_{u_1}^{\infty}\frac{du}{F(u)} = -\infty. \tag{5.5}$$

(iv)

$$\lim_{u\to\infty} e^{(\frac{1}{d_y}-\frac{1}{d_z})u}F'(u) = 0.$$

(v)

$$\lim_{u\to\infty}\frac{F'(u)}{F(u)} = 0.$$

We now begin to study the Hamiltonian flows of functions of the form $H_{F_\rho}(\vec{x},\vec{y},z) = zF(-\log\rho(\vec{y},z))$ where F and ρ are as above. The trajectories of this flow are, in view of (5.1), solutions to the following system:

$$\begin{aligned} x_j' &= \frac{d_y y_j^{d_y-1} z}{\rho(\vec{y},z)} F'(-\log\rho(\vec{y},z)) \\ y_j' &= y_j F(-\log\rho(\vec{y},z)) - \frac{d_z y_j z^{d_z}}{\rho(\vec{y},z)} F'(-\log\rho(\vec{y},z)) \qquad (5.6) \\ z' &= zF(-\log\rho(\vec{y},z)) + \sum_j \frac{d_y y_j^{d_y} z}{\rho(\vec{y},z)} F'(-\log\rho(\vec{y},z)). \end{aligned}$$

In particular, such solutions always obey

$$\begin{aligned} \frac{d}{dt}\left(\rho(\vec{y}(t),z(t))\right) &= \sum_j \frac{\partial\rho}{\partial y_j} y_j' + \frac{\partial \rho}{\partial z} z' \\ &= \left(\sum_j d_y y_j^{d_y} + d_z z^{d_z}\right) F(-\log\rho(\vec{y},z)). \end{aligned}$$

(Note the convenient cancellation of the terms involving F'.) Bearing in mind that $F \le 0$ and $d_y \ge d_z$, this yields

$$d_y \rho(\vec{y},z) F(-\log\rho(\vec{y},z)) \le \frac{d}{dt}\left(\rho(\vec{y},z)\right) \le d_z\rho(\vec{y},z)F(-\log\rho(\vec{y},z)),$$

i.e.,

$$(5.7) \qquad -d_z F(-\log\rho(\vec{y},z)) \le \frac{d}{dt}\left(-\log(\rho(\vec{y},z))\right) \le -d_y F(-\log\rho(\vec{y},z))$$

for any flowline $t \mapsto (\vec{x}(t),\vec{y}(t),z(t))$ of the Hamiltonian flow of $H_{F,\rho}$.

We will see presently that Assumptions 5.4 together with (5.7) imply that Hamiltonian flowlines for $H_{F,\rho}$ which begin in $\mathcal{B}_V^*$ at $t=0$ exist (within $\mathcal{B}_V^*$) for all positive t. By (5.5), the map $G\colon (u_0,\infty) \to (-\infty,\infty)$ defined by

$$G(u) = \int_{u_1}^{u} \frac{dv}{F(v)}$$

(for an arbitrary choice of $u_1 > u_0$) is a diffeomorphism, with $G'(u) = \frac{1}{F(u)}$. (That $G(u) \to \infty$ as $u \to u_0^+$ follows from the fact that F vanishes to infinite order at u_0.) If $r \in \mathbb{R}$ and $v_0 > u_0$, the unique solution to the equation $u'(t) =$

$-rF(u(t))$ obeying an initial condition $u(0) = v_0$ is then $u(t) = G^{-1}(G(v_0) - rt)$. In particular this solution exists and remains in the interval (u_0, ∞) for all time. Of course if we instead have $v_0 \le u_0$ the unique solution to $u' = -rF(u)$ with $u(0) = v_0$ is constant.

In the case that $d_y = d_z$, then based on (5.7) the above considerations allow one to compute $\rho(\vec{y}(t), z(t))$ as a function of t directly from F. More generally we have the following:

Proposition 5.5. With F and G as above, suppose that I is an open interval around zero and $u\colon I \to \mathbb{R}$ obeys the differential inequalities

$$-d_zF(u(t)) \le u'(t) \le -d_yF(u(t)), \tag{5.8}$$

and that $u(0) > u_0$. Then for all $t \in I$ with $t \ge 0$,

$$G^{-1}(G(u(0)) - d_zt) \le u(t) \le G^{-1}(G(u(0)) - d_yt).$$

Proof. Since $-d_zF(v) \ge 0$ for all v the hypothesis implies that u is a monotone increasing function and hence in particular that $u(t) > u_0$ and hence $F(u(t)) < 0$ for all $t \ge 0$. We have

$$\frac{d}{dt}G(u(t)) = G'(u(t))u'(t) = \frac{u'(t)}{F(u(t))} \in [-d_y, -d_z] \text{ for all } t$$

based on (5.8) and the fact that $F(u(t)) < 0$. Integrating with respect to t shows that, if $t \ge 0$, then

$$G(u(0)) - d_yt \le G(u(t)) \le G(u(0)) - d_zt.$$

Since G and hence also G^{-1} is a decreasing function, the above inequalities directly imply that $G^{-1}(G(u(0)) - d_zt) \le u(t) \le G^{-1}(G(u(0)) - d_yt)$. □

Corollary 5.6. If $F\colon (-\log c, \infty) \to (-\infty, 0]$ satisfies Assumptions 5.4 then the contact Hamiltonian flow $\phi^t_{H_{F,\rho}}$ of $H_{F,\rho}\colon \mathcal{B}_V^* \to \mathbb{R}$ is well-defined as a diffeomorphism of $\mathcal{B}_V^*$ for all $t \in \mathbb{R}$, and is the identity on the subset of $\mathcal{B}_V^*$ on which $\rho(\vec{y}, z) > e^{-u_0}$.

Proof. Since $H_{F,\rho}$ is smooth throughout $\mathcal{B}_V^*$, standard results in ODE theory imply that in order for the corollary to be false there would need to be an integral curve $\gamma\colon (T_-, T_+) \to \mathcal{B}_V^*$ (with T_-, T_+ both finite and $T_- < 0 < T_+$) of $X_{H_{F,\rho}}$ whose image is not contained in any compact subset of $\mathcal{B}_V^*$. Now since $H_{F,\rho}$ vanishes everywhere that $\rho(\vec{y}, z) \in [e^{-u_0}, c)$, an integral curve of $X_{H_{F,\rho}}$ must either be constant or be contained in the region $\{0 < \rho(\vec{y}, z) \le e^{-u_0}\}$. By (5.7), the function $(\vec{x}, \vec{y}, z) \mapsto -\log \rho(\vec{y}, z)$ is monototone increasing along the integral curve γ, and by Proposition 5.5 if the value of this function at time zero is $v_0 > -\log c$ then it will never take a value larger than $G^{-1}(G(v_0) - d_y T_+)$ for $t \in [T_-, T_+]$. So in this case $\rho(\vec{y}, z) \ge e^{-G^{-1}(G(v_0) - d_y T_+)}$ everywhere along γ. Thus every integral curve of $X_{H_{F,\rho}}$ defined on a bounded time interval remains inside a compact subset of $\mathcal{B}_V^*$, as desired. □

We now begin calculations directed toward showing that $\phi^t_{H_{F,\rho}}$ extends to a contact homeomorphism of $\mathcal{B}_V$.

Proposition 5.7. Let $u(\vec{x}, \vec{y}, z) = -\log \rho(\vec{y}, z)$, let F, d_y, d_z obey Assumptions 5.4, let $u_1 > u_0$ where u_0 is as in Assumption 5.4(ii), and denote $K = \{(\vec{x}, \vec{y}, z) \in \mathcal{B}_V | u(\vec{x}, \vec{y}, z) \ge u_1\}$. Then there is a constant C such that the Hamiltonian vector field $X_{H_{F,\rho}}$ obeys

$$\|X_{H_{F,\rho}}\| \le C \left(e^{-u(\vec{x},\vec{y},z)/d_y} F(u(\vec{x}, \vec{y}, z)) + e^{\left(\frac{1}{d_y} - \frac{1}{d_z}\right) u(\vec{x},\vec{y},z)} F'(u(\vec{x}, \vec{y}, z)) \right)$$

for all $(\vec{x}, \vec{y}, \vec{z}) \in \mathcal{B}_V^* \cap K$.

Proof. Consulting (5.6), we see that the y_j component of $X_{H_{F,\rho}}$ has norm bounded above by

$$|y_j| \left(|F(u(\vec{x}, \vec{y}, \vec{z}))| + \left| \frac{d_z z^{d_z}}{\rho(\vec{y}, z)} F'(u(\vec{x}, \vec{y}, z)) \right| \right)$$

which in turn can be bounded above by (for an appropriate constant C_1)

$$C_1 e^{-u(\vec{x},\vec{y},z)/d_y} |F(u(\vec{x}, \vec{y}, z))|$$

using Assumption 5.4(v) and the facts that $|y_j| \le \rho(\vec{y}, z)^{1/d_y} = e^{-u(\vec{x},\vec{y},z)/d_y}$ and that $|z|^{d_z} \le \rho(\vec{y}, z)$. Identical reasoning shows that the z component of $X_{H_{F,\rho}}$ is bounded above by $C_1 e^{-u(\vec{x},\vec{y},z)/d_z} |F(u(\vec{x}, \vec{y}, z))|$; note moreover that since $d_y \ge d_z$ we have $e^{-u(\vec{x},\vec{y},z)/d_z} \le C_2 e^{-u(\vec{x},\vec{y},z)/d_y}$ for all $(\vec{x}, \vec{y}, z) \in K$ and an appropriate constant C_2. So the y_j and z components are bounded as indicated in the proposition (in fact with only the first term needed).

To analyze the x_j component of $X_{H_{F,\rho}}$, namely $\frac{d_y y_j^{d_y - 1} z}{\rho(\vec{y}, z)} F'(-\log \rho(\vec{y}, z))$, let us write $\vec{v} = \rho(\vec{y}, z)^{-1/d_y} \vec{y}$ and $w = \rho(\vec{y}, z)^{-1/d_z} z$. Then $\rho(\vec{v}, w) = 1$ (so in particular each $|v_j| \leq 1$ and $|w| \leq 1$), and

$$\left|\frac{y_j^{d_y-1}z}{\rho(\vec{y},z)}\right| = \left|\frac{\rho(\vec{y},z)^{\frac{d_y-1}{d_y}}v_j^{d_y-1}\rho(\vec{y},z)^{1/d_z}w}{\rho(\vec{y},z)}\right|$$
$$= \rho(\vec{y},z)^{\frac{1}{d_z}-\frac{1}{d_y}}|v_j^{d_y-1}w| \leq e^{\left(\frac{1}{d_y}-\frac{1}{d_z}\right)u(\vec{x},\vec{y},z)}.$$

Thus the x_j component of $X_{H_{F,\rho}}$ has norm bounded above by

$$|d_y e^{\left(\frac{1}{d_y}-\frac{1}{d_z}\right)u(\vec{x},\vec{y},z)}F'(u(\vec{x},\vec{y},z))|;$$

combining this fact with our earlier bounds on the y_j and z components completes the proof. □

To set up notation for the following two propositions, if F and G are as in Proposition 5.5, for all sufficiently large $m \in \mathbb{N}$ let us choose a smooth function $\beta_m\colon \mathbb{R} \to \mathbb{R}$ such that:

- $\beta_m(u) = u$ for all $u \leq G^{-1}(G(m) - d_y)$;
- $0 \leq \beta_m'(u) \leq 1$ for all u; and
- $\beta_m'(u) = 0$ for all $u \geq 1 + G^{-1}(G(m) - d_y)$.

Moreover, set

$$F_m = F \circ \beta_m.$$

Then F_m also satisfies Assumptions 5.4, and it has the additional property that there are constants u_m, c_m such that $F_m(u) = c_m$ for all $u > u_m$. The latter property immediately implies that $H_{F_m,\rho}(\vec{x},\vec{y},z) = zF_m(-\log\rho(\vec{y},z))$ extends smoothly across the zero section, and hence so too does its Hamiltonian flow $\phi^t_{H_{F_m,\rho}}$ (specifically this flow is given on a neighborhood of the zero section by $\phi^t_{H_{F_m,\rho}}(\vec{x},\vec{y},z) = (\vec{x}, e^{c_m t}\vec{y}, e^{c_m t}z)$).

Proposition 5.8. With notation as above, the sequence $\{\phi^1_{H_{F_m,\rho}}\}$ converges uniformly to the map $\bar{\phi}^1_{H_{F,\rho}}\colon \mathcal{B}_V \to \mathcal{B}_V$ given by extending $\phi^1_{H_{F,\rho}} : \mathcal{B}_V^* \to \mathcal{B}_V^*$ by the identity on $\mathcal{Z}$.

Proof. As before let $u(\vec{x},\vec{y},z) = -\log\rho(\vec{y},z)$. By (5.7) and Proposition 5.5, if $u(\vec{x},\vec{y},z) \leq m$ then for all $t \in [0,1]$, writing $(\vec{x}(t),\vec{y}(t),z(t)) = \phi^t_{H_{F,\rho}}(\vec{x},\vec{y},z)$, we will have $u(\vec{x},\vec{y}(t),z(t)) \leq G^{-1}(G(m) - d_y)$. Since $H_{F,\rho}$ coincides with

$H_{F_m,\rho}$ everywhere that $u(\vec{x},\vec{y},z) \leq G^{-1}(G(m)-d_y)$ it follows that the restriction of $\phi^1_{H_{F_m,\rho}}$ to $\{\rho(\vec{y},z) \geq e^{-m}\}$ coincides with that of $\overline{\phi}^1_{H_{F,\rho}}$. Of course $\overline{\phi}^1_{H_{F,\rho}}$ and $\phi^1_{H_{F_m,\rho}}$ also coincide on $\mathcal{Z}$.

The definition of F_m obviously implies that $|F_m| \leq |F|$ everywhere, and it also implies that

$$|F'_m(u)| \leq \max_{v\in[u-1,u]} |F'(v)|.$$

(Indeed, if $u \notin [G^{-1}(G(m)-d_y), 1+G^{-1}(G(m)-d_y)]$ then $F'_m(u)$ is equal either to $F'(u)$ or to 0, while if $u \in [G^{-1}(G(m)-d_y), 1+G^{-1}(G(m)-d_y)]$ then $|F'_m(u)| = |F'(\beta_m(u))||\beta'_m(u)| \leq |F'(\beta_m(u))|$ where $\beta_m(u) \in [u-1,u]$.)

Assumptions 5.4(i) and (iii) imply that $e^{-u/d_y}F(u) \to 0$ as $u \to \infty$, and Assumption 5.4(iv) states that $e^{\left(\frac{1}{d_y}-\frac{1}{d_z}\right)u}F'(u) \to 0$, so the preceding paragraph and Proposition 5.7 imply that for every $\epsilon > 0$ there is a number A_ϵ (independent of m) such that, whenever $u(\vec{x},\vec{y},z) > A_\epsilon$ we have both $\|X_{H_{F,\rho}}(\vec{x},\vec{y},z)\| < \frac{\epsilon}{2}$ and $\|X_{H_{F_m,\rho}}(\vec{x},\vec{y},z)\| < \frac{\epsilon}{2}$ for every m. Since (5.7) implies that the function u increases along the flowlines both of $X_{H_{F,\rho}}$ and $X_{H_{F_m,\rho}}$, so that the locus where $u > A_\epsilon$ is preserved by each of these flows, it follows that the C^0 distance between the restrictions to $\{u \geq A_\epsilon\}$ of $\bar{\phi}^1_{H_{F,\rho}}$ and $\phi^1_{H_{F_m,\rho}}$ is less than ϵ. Since as already noted the maps $\bar{\phi}^1_{H_{F,\rho}}$ and $\phi^1_{H_{F_m,\rho}}$ coincide on the set $\{\rho(\vec{y},z) \geq e^{-m}\} = \{u \leq m\}$ and on the set $\mathcal{Z}$ (where u is undefined), this shows that we will have

$$\operatorname{dist}\left(\bar{\phi}^1_{H_{F,\rho}}(\vec{x},\vec{y},z), \phi^1_{H_{F_m,\rho}}(\vec{x},\vec{y},z)\right) < \epsilon$$

for *all* $(\vec{x},\vec{y},z)$ provided that m is larger than A_ϵ. □

Proposition 5.9. Let F satisfy Assumptions 5.4. Then for all $t \in \mathbb{R}$ the time-t map $\phi^t_{H_{F,\rho}} \colon \mathcal{B}^*_V \to \mathcal{B}^*_V$ extends by the identity along the zero section $\mathcal{Z} = \{y_1 = \cdots = y_n = z = 0\}$ to a contact homeomorphism of $\mathcal{B}_V$, which we denote by $\overline{\phi}^t_{H_{F,\rho}}$.

Moreover, assuming that $t > 0$ and that $\lim_{u\to\infty} F(u) = -\infty$, and denoting $\psi_m = \phi^t_{H_{F_m,\rho}}$ where F_m is defined just above Proposition 5.8, $\overline{\phi}^t_{H_{F,\rho}}$ is the C^0-limit of contactomorphisms $\psi_m \colon \mathcal{B}_V \to \mathcal{B}_V$ having uniform compact support, and such that $\psi_m^*\alpha = f_m\alpha$ where the smooth functions f_m uniformly converge to a continuous function $f \colon \mathcal{B}_V \to [0,\infty)$ with $\phi^{t*}_{H_{F,\rho}}\alpha = f\alpha$ on $\mathcal{B}^*_V$ and $f|_{\mathcal{Z}} = 0$.

Proof. Since the inverse of a contact homeomorphism is a contact homeomorphism it suffices to prove the result for $t > 0$. Since $\phi^1_{tH} = \phi^t_H$, by replacing F by tF (which does not affect whether F obeys the hypotheses for $t > 0$) we may as well assume that $t = 1$.

Let us first show that the map $\overline{\phi}^1_{H_{F,\rho}}$ given by extending $\phi^1_{H_{F,\rho}}$ by the identity over the zero section is a homeomorphism. Proposition 5.5 shows that if $\epsilon > 0$ there is $\delta > 0$ such that $\overline{\phi}^1_{H_{F,\rho}}$ and its inverse each map the region $\{\rho(\vec{y}, z) < \delta\}$ inside the region $\{\rho(\vec{y}, z) < \epsilon\}$, so in order to establish the continuity of $\overline{\phi}^1_{H_{F,\rho}}$ and of its inverse it suffices to check that the Hamiltonian vector field $X_{H_{F,\rho}}$ can be bounded in terms of a function of $\rho(\vec{y}, z)$ that approaches zero as $\rho(\vec{y}, z) \to 0$. But this follows from Proposition 5.7, since (as noted in the proof of Proposition 5.8) Assumptions 5.4 imply that $e^{-u/d_y} F(u) \to 0$ and $e^{\left(\frac{1}{d_y} - \frac{1}{d_z}\right)u} F'(u) \to 0$ as $u \to \infty$.

The contactomorphisms $\psi_m = \phi^t_{H_{F_m,\rho}}$ are all supported in the compact set on which $\rho(\vec{y}, z) \leq e^{-u_0}$, and they converge uniformly to $\bar{\phi}^1_{H_{F,\rho}}$ by Proposition 5.8. It remains only to prove the statement at the end of the proposition about the conformal factors f_m of the ψ_m. These conformal factors are related by (5.2) to the functions $\frac{\partial H_{F_m,\rho}}{\partial z}$. By construction, the maps $\frac{\partial H_{F,\rho}}{\partial z} \circ \phi^t_{H_{F,\rho}}$ and $\frac{\partial H_{F_m,\rho}}{\partial z} \circ \phi^t_{H_{F_m,\rho}}$ coincide on the set $\{\rho(\vec{y}, z) \geq e^{-m}\}$ for all $t \in [0, 1]$, so we have

$$f_m|_{\{\rho(\vec{y},z) \geq e^{-m}\}} = f|_{\{\rho(\vec{y},z) \geq e^{-m}\}} \tag{5.9}$$

where as is the statement of the proposition $f\colon \mathcal{B}_V \to \mathbb{R}$ restricts to $\mathcal{B}^*_V$ as the conformal factor of $\phi^1_{H_{F,\rho}}$ and to $\mathcal{Z}$ as zero.

Now

$$\begin{aligned}\frac{\partial H_{F_m,\rho}}{\partial z} &= F(\beta_m(-\log \rho(\vec{y}, z))) \\ &\quad - \frac{d_z z^{d_z}}{\rho(\vec{y}, z)} \beta'_m(-\log \rho(\vec{y}, z)) F'(\beta_m(-\log \rho(\vec{y}, z))).\end{aligned}$$

So if m is large enough Assumptions 5.4(v) and (i) imply that, if $\rho(\vec{y}, z) \leq e^{-m}$, then

$$\frac{\partial H_{F_m,\rho}}{\partial z}(\vec{y}, z) \leq \frac{1}{2} F(\beta_m(-\log \rho(\vec{y}, z))) \leq \frac{1}{2} F(m).$$

So since the set $\{\rho(\vec{y}, z) \leq e^{-m}\}$ is preserved by $\phi^t_{H_{F_m,\rho}}$ for $t \geq 0$ it follows from (5.2) that

$$f_m|_{\{\rho(\vec{y},z) \leq e^{-m}\}} \leq e^{\frac{1}{2} F(m)}.$$

The same reasoning applied to F in place of F_m shows that $f|_{\{\rho(\vec{y},z)\leq e^{-m}\}} \leq e^{\frac{1}{2}F(m)}$. Of course f_m and f are both nonnegative, so in view of (5.9) we see that

$$\sup_{\mathcal{B}_V} |f_m - f| \leq e^{\frac{1}{2}F(m)},$$

which converges to zero based on our assumption that $\lim_{u\to\infty} F(u) = -\infty$. So indeed $f_m \to f$ uniformly, and hence f is continuous. □

Corollary 5.10. Assuming that $t > 0$ and $\lim_{u\to\infty} F(u) = -\infty$, the contact homeomorphism $\overline{\phi}^t_{H_{F,\rho}}$ is bounded above near every point of $\mathcal{B}_V$, but for every $p \in \mathcal{Z}$ it is not bounded below near p.

Proof. The functions f_m in Proposition 5.9 are positive and uniformly bounded above (since they converge uniformly to the function f, which is bounded above since it is continuous on $\mathcal{B}_V$ and equal to 1 outside a compact subset of $\mathcal{B}_V$); this suffices to prove that $\overline{\phi}^t_{H_{F,\rho}}$ is bounded above near every point.

If $p \in \mathcal{Z}$ we can see that $\overline{\phi}^t_{H_{F,\rho}}$ is not bounded below near p by using Propositions 2.7 and 5.9. Indeed the former implies that if $\overline{\phi}^t_{H_{F,\rho}}$ were bounded below near p there would be $\delta > 0$ so that for every sufficiently small neighborhood W of p we would have

$$\int_{\overline{\phi}^t_{H_{F,\rho}}(W)} \alpha \wedge (d\alpha)^{\wedge n} \geq \delta \int_W \alpha \wedge (d\alpha)^{\wedge n}.$$

But given any $\delta > 0$, if we choose W so small that the function f in Proposition 5.9 has $\sup_W |f|^{n+1} < \delta$ we see, using that $\overline{\phi}^t_{H_{F,\rho}}$ is smooth on the full-measure subset $\mathcal{B}^*_V \subset \mathcal{B}_V$ (allowing us to apply the change of variables theorem),

$$\begin{aligned}\int_{\overline{\phi}^t_{H_{F,\rho}}(W)} \alpha \wedge (d\alpha)^{\wedge n} &= \int_{\overline{\phi}^t_{H_{F,\rho}}(W\cap\mathcal{B}^*_V)} \alpha \wedge (d\alpha)^{\wedge n} \\ &= \int_{W\cap\mathcal{B}^*_V} f^{n+1}\alpha \wedge (d\alpha)^{\wedge n} < \delta \int_W \alpha \wedge (d\alpha)^{\wedge n},\end{aligned}$$

a contradiction. □

Proposition 5.11. The contact homeomorphism $\overline{\phi}^t_{H_{F,\rho}}\colon \mathcal{B}_V \to \mathcal{B}_V$ from Proposition 5.9 has restriction to the locus

$$\mathcal{W} := \{(\vec{x},\vec{0},z)|0<|z|<e^{-u_0/d_z}\} \subset \mathcal{B}_V^*$$

given by

$$\overline{\phi}^t_{H_{F,\rho}}(\vec{x},\vec{0},z) = \left(\vec{x},\vec{0},e^{-\frac{1}{d_z}G^{-1}(G(-d_z\log|z|)-d_zt)}\mathrm{sgn}(z)\right)$$

where $G\colon (u_0,\infty)\to(-\infty,\infty)$ is an antiderivative of $\frac{1}{F}$. Moreover if $t>0$ and $\lim_{u\to\infty}F(u)=-\infty$, the contactomorphisms ψ_m from Proposition 5.9 have the property that $\psi_m|_{\mathcal{W}} \to \overline{\phi}^t_{H_{F,\rho}}|_{\mathcal{W}}$ in the C^1 topology.

Proof. Examining (5.6) and recalling that $\rho(\vec{0},z)=z^{d_z}$ where d_z is an even integer, we see that one obtains integral curves of $X_{H_{F,\rho}}$ by taking each x_j equal to an arbitrary constant, each y_j equal to zero, and z equal to a solution of

$$z' = zF(-\log z^{d_z}) = zF(-d_z\log|z|).$$

The latter equation can be rewritten as $\frac{d}{dt}\log|z(t)| = F(-d_z\log|z(t)|)$, which has general solution $-\log|z(t)| = \frac{1}{d_z}G^{-1}(G(-d_z\log|z(0)|)-d_zt)$. Since $\overline{\phi}^t_{H_{F,\rho}}$ is given on $\mathcal{B}_V^*$ as the time-t flow of $X_{H_{F,\rho}}$, the formula in the statement of the proposition follows directly by exponentiating this formula for $\log|z(t)|$.

For the second statement, recall that the ψ_m are taken in Proposition 5.9 to be of the form $\psi_m = \phi^t_{H_{F_m,\rho}}$ where the F_m satisfy Assumptions 5.4 and additionally have $F_m|_{[u_m,\infty)}$ constant for suitable u_m (so that $H_{F_m,\rho}$ can be seen as a smooth function on all of $\mathcal{B}_V$). So, as an instance of the first statement of this proposition, we have $\psi_m(\vec{x},\vec{0},z)=(\vec{x},0,g_m(z))$ for a certain smooth function g_m. Let us write $g(z)=e^{-\frac{1}{d_z}G^{-1}(G(-d_z\log|z|)-d_zt)}$ for the third component of $\overline{\phi}^t_{H_{F,\rho}}|_{\mathcal{W}}$. Since $\psi_m\to\overline{\phi}^t_{H_{F,\rho}}$ uniformly we evidently have $g_m\to g$ uniformly. Furthermore, notice that the contact form $\alpha = dz-\sum_j y_jdx_j$ restricts to $\mathcal{W}$ as dz. So $\psi_m^*(\alpha|_{\mathcal{W}}) = g_m'(z)dz = g_m'(z)\alpha|_{\mathcal{W}}$ and likewise $\overline{\phi}^{t*}_{H_{F,\rho}}(\alpha|_{\mathcal{W}}) = g'(z)\alpha|_{\mathcal{W}}$. So the last clause of Proposition 5.9 implies that $g_m'\to g'$ uniformly, and hence $g_m\to g$ in C^1. □

Now let us consider specific examples of the contact homeomorphisms supplied by Proposition 5.9 for particular choices of F, d_y, d_z.

Example 5.12. If $d_y=d_z=2$, so that $\rho(\vec{y},z)=\sum_j y_j^2+z^2$, then it turns out that one can give an explicit formula for $\phi^t_{H_{F,\rho}}$ on all of $\mathcal{B}_V^*$, not just on the locus where $\vec{y} = \vec{0}$. If $F(-\log \rho(\vec{y}, z)) = 0$ then $X_{H_{F,\rho}}$ vanishes at $(\vec{x}, \vec{y}, z)$, so we restrict attention to those $(\vec{x}, \vec{y}, z)$ where $F(-\log \rho(\vec{y}, z)) < 0$. Then letting as before G be an antiderivative of $\frac{1}{F}$, and also abbreviating

$$u_t(\vec{y}, z) = G^{-1}(G(-\log \rho(\vec{y}, z)) - 2t),$$

one has

$$\phi^t_{H_{F,\rho}}(\vec{x}, \vec{y}, 0) = \left(\vec{x}, e^{-u_t(\vec{y},0)/2}\frac{\vec{y}}{\|\vec{y}\|}, 0\right);$$
$$\phi^t_{H_{F,\rho}}(\vec{x}, \vec{0}, z) = \left(\vec{x}, \vec{0}, e^{-u_t(\vec{0},z)/2}\mathrm{sgn}(z)\right);$$

and if both $\vec{y}$ and z are nonzero then $\phi^t_{H_{F,\rho}}(\vec{x}, \vec{y}, z) = (\vec{X}(t), \vec{Y}(t), Z(t))$ where:

$$\vec{X}(t) = \vec{x} + \left(\arctan\left(\frac{\|\vec{y}\|}{z}\right) - \arctan\left(\frac{F(u_t(\vec{y}, z))}{F(u_0(\vec{y}, z))}\frac{\|\vec{y}\|}{z}\right)\right)\frac{\vec{y}}{\|\vec{y}\|},$$
$$\vec{Y}(t) = -\frac{e^{-u_t(\vec{y},z)/2}F(u_t(\vec{y}, z))\vec{y}}{\sqrt{\|F(u_t(\vec{y}, z)\vec{y}\|^2 + (F(u_0(\vec{y}, z))z)^2}},$$
$$Z(t) = -\frac{e^{-u_t(\vec{y},z)/2}F(u_0(\vec{y}, z))z}{\sqrt{\|F(u_t(\vec{y}, z)\vec{y}\|^2 + (F(u_0(\vec{y}, z))z)^2}}.$$

(To derive such a formula from scratch, one can observe from (5.6) that, along the Hamiltonian flow of $H_{F,\rho}$, one has $-\log \rho(\vec{Y}(t), Z(t)) = u_t(\vec{y}, z)$ by Proposition 5.5 since $d_y = d_z = 2$, and moreover that if $\gamma(t) = \frac{\|\vec{Y}(t)\|^2 - Z(t)^2}{\|\vec{Y}(t)\|^2 + Z(t)^2}$ then a routine calculation yields $\gamma'(t) = -2(1 - \gamma(t)^2)F'(u_t(\vec{y}, z))$, which one can then solve easily for γ, hence determining $\|\vec{Y}(t)\|$ and $Z(t)$. Of course if one has been given the above formulas one can also simply confirm by direct substitution that they satisfy the ODEs (5.6).)

Specializing this example further, we could choose F so that $F(v) = -\sqrt{v}$ for all $v \geq v_1$ (for an arbitrary v_1 which is greater than $-\log c$ in the notation of Assumption 5.4). This yields, for all $\vec{y}, z$ with $\rho(\vec{y}, z) \leq e^{-v_1}$,

$$u_t(\vec{y}, z) = \left(\sqrt{-\log(\|\vec{y}\|^2 + z^2)} + t\right)^2.$$

We find in particular that

$$\overline{\phi}^t_{H_{F,\rho}}(\vec{x}, \vec{0}, z) = \left(\vec{x}, \vec{0}, e^{-t\sqrt{-2\log|z|} - t^2/2}z\right)$$

for all sufficiently small z. For any fixed $t > 0$ the third component above is a C^1 function of z which vanishes together with its derivative at $z = 0$; however its second derivative at $z = 0$ does not exist. Thus, for $t > 0$, $\overline{\phi}^t_{H_{F,\rho}}$ maps a neighborhood of the origin in the nowhere-Legendrian submanifold $\{(\vec{x}, \vec{0}, x_n)\}$ of $\mathcal{B}_V$ to a neighborhood of the origin in the C^1-submanifold $\{(\vec{x}, \vec{0}, e^{-t\sqrt{-2\log|x_n|-t^2/2}}x_n)\}$, which is tangent to the contact distribution at the origin.

One obtains similar behavior if one takes $F(v) = -v^\beta$ with $0 < \beta < 1$. Note that the condition $\beta < 1$ is forced by Assumption 5.4(iv) because in this example $d_y = d_z$.

Example 5.13. If we instead take $\rho(\vec{y}, z) = \sum_j y_j^4 + z^2$ then Assumption 5.4(iv) only requires that $\lim_{u\to\infty} e^{-u/4}F'(u) = 0$, allowing more freedom in the choice of F and ultimately leading to examples that improve on the C^1-smoothness in Example 5.12. (In Example 5.12, Assumption 5.4(iv) required $\lim_{u\to\infty} F'(u) = 0$.) With this new choice of ρ, the author does not know an explicit formula for the maps $\phi^t_{H_{F,\rho}}$ on all of $\mathcal{B}^*_V$ as in Example 5.12, but Proposition 5.11 still applies to compute their restrictions to the locus $\{\vec{y} = \vec{0}\}$.

More concretely, if $F(v) = -v$ for all sufficiently large v, one finds (for $|z|$ sufficiently small)

$$G^{-1}(G(-2\log|z|) - 2t) = e^{2t+\log(-2\log|z|)} = -2e^{2t}\log|z|,$$

so that Proposition 5.11 gives

$$\phi^t_H(\vec{x}, \vec{0}, z) = \left(\vec{x}, \vec{0}, \mathrm{sgn}(z)|z|^{e^{2t}}\right).$$

So for any odd integer $m > 1$, the contact homeomorphism $\overline{\phi}^{\frac{1}{2}\log m}_{H_{F,\rho}}$ maps a neighborhood of the origin in the nowhere-Legendrian submanifold $\{(\vec{x}, \vec{0}, x_n)\} \subset \mathcal{B}_V$ to a neighborhood of the origin in $\{(\vec{x}, 0, x_n^m)\}$, which is of course smooth and has an order-m tangency to the contact distribution at the origin.

Example 5.14. To get an infinite-order tangency, we can again take $\rho(\vec{y}, z) = \sum_j y_j^4 + z^2$ and now set $F(v) = -v\log v$ for all sufficiently large v, so that $\frac{1}{F}$ has antiderivative $G(v) = -\log(\log v)$ for all large v. One then computes that $G^{-1}(G(-2\log|z|) - 2t) = (-2\log|z|)^{e^{2t}}$ and hence that, by

Proposition 5.11,

$$\phi^t_{H_{F,\rho}}(\vec{x},\vec{0},z) = \left(\vec{x},\vec{0},\operatorname{sgn}(z)e^{-\frac{1}{2}\left(\log\frac{1}{z^2}\right)^{e^{2t}}}\right)$$

for all sufficiently small z. For any fixed $t > 0$ and any positive integer m the third component above approaches zero as $z \to 0$ faster than $|z|^m = e^{-\frac{m}{2}\log\frac{1}{z^2}}$. Consequently a neighborhood of the origin in the nowhere-Legendrian submanifold $\{\vec{y} = \vec{0}, z = x_n\}$ is sent by $\overline{\phi}^t_{H_{F,\rho}}$ to a smooth submanifold with an infinite-order tangency to the contact disribution at the origin.

Proof of Theorem 1.4. Corollary 5.2, specialized to the case $k = n + 1$, gives examples for variation (i) of the theorem. Proposition 5.3 gives examples for variation (iii) (in fact one could use either Example 5.13 or Example 5.14 here), bearing in mind that an arbitrary contact manifold contains Legendrian tori (contained in Darboux charts, for instance) which have tubular neighborhoods contactomorphic to $\mathcal{B}_V$, and that our examples are limits of contactomorphisms that are uniformly compactly supported in $\mathcal{B}_V$ which can thus be exported to any contact manifold.

We now explain how to combine appropriate examples for variations (iii) and (i) in order to provide examples for variation (ii). First take a standard neighborhood $\mathcal{N} \cong \mathcal{B}_V$ of a Legendrian torus in (Y, ξ) and let $\psi_3\colon Y \to Y$ be a contact homeomorphism which coincides on $\mathcal{N}$ with a contact homeomorphism produced by Proposition 5.3 and is given by the identity outside $\mathcal{N}$. Next we will let ψ_1 be a contact homeomorphism of Y given by the identity outside of a small neighborhood $W = (-\delta, \delta)^{2n+1} \subset \mathbb{R}^n/\mathbb{Z}^n \times \mathbb{R}^{n+1}$ of the origin under the identification $\mathcal{B}_V \cong \mathcal{N} \subset Y$ and, inside W, by a contact homeomorphism as in Proposition 5.1 for a suitable function F similar to that used in Corollary 5.2. Specifically we require F to be supported inside $(-\delta, \delta)^{2n-1}$, to obey $\max|F| < \delta$, and to have a smooth graph with a vertical tangency at the origin, with F depending only on z on a small neighborhood of the origin in $(-\delta, \delta)^{2n-1}$. Thus on this small neighborhood ψ_1 is given on the submanifold $\{x_1 = y_1 = 0\}$ by

$$\psi_1(0, 0, x_2, y_2, \ldots, x_n, y_n, z) = (f(z), 0, x_2, y_2, \ldots, x_n, y_n, z)$$

where f is a local homeomorphism with smooth local inverse and $f(0) = (f^{-1})'(0) = 0$.

Examples for variation (ii) of Theorem 1.4 are then provided by taking $\psi_2 = \psi_1 \circ \psi_3^{-1}$. Indeed by Proposition 5.3(c), ψ_3 maps the nowhere-Legendrian submanifold $\Lambda = \{x_1 = y_1 = y_2 = \cdots = y_n = 0\} \subset \mathcal{N}$ homeomorphically to itself, fixing the origin, and ψ_1 maps Λ to a submanifold that is tangent to the contact distribution at the origin. Moreover the fact that ψ_3 is bounded above but not below near the origin while ψ_1 is bounded both above and below near the origin readily implies that $\psi_2 = \psi_1 \circ \psi_3^{-1}$ is bounded below but not above near the origin. □

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