On contact invariants of non-simply connected Gorenstein toric contact manifolds

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The first two authors showed in [1] how the Conley-Zehnder index of any contractible periodic Reeb orbit of a non-degenerate toric contact form on a good toric contact manifold with zero first Chern class, i.e. a Gorenstein toric contact manifold, can be explicitly computed using moment map data. In this paper we show that the same explicit method can be used to compute Conley-Zehnder indices of non-contractible periodic Reeb orbits. Under appropriate conditions, the (finite) number of such orbits in a given free homotopy class and with a given index is a contact invariant of the underlying contact manifold. We compute these invariants for two sets of examples that satisfy these conditions: 5-dimensional contact manifolds that arise as unit cosphere bundles of 3-dimensional lens spaces, and $2n + 1$-dimensional Gorenstein contact lens spaces. As applications, we will see that these invariants can be used to show that diffeomorphic lens spaces might not be contactomorphic and that there are homotopy classes of diffeomorphisms of some lens spaces that do not contain any contactomorphism. Following a suggestion by one referee, we will also see that this type of applications can be proved alternatively by looking at the total Chern class of these canonical contact structures on lens spaces.

1. Introduction

The first two authors showed in [1] how the Conley-Zehnder index of any contractible periodic Reeb orbit of a non-degenerate toric contact form on a good toric contact manifold with zero first Chern class, i.e. a Gorenstein toric contact manifold, can be explicitly computed using moment map data. In this paper we show that the same explicit method can be used to compute Conley-Zehnder indices of non-contractible periodic Reeb orbits, discuss several examples and give applications.

Recall that the contact homology degree, or Symplectic Field Theory (SFT) degree, of a non-degenerate closed Reeb orbit is equal to the Conley-Zehnder index plus $n - 2$, where throughout this paper contact manifolds have dimension $2n + 1$. As shown in [1, 2], this degree is always even for the closed Reeb orbits of any non-degenerate toric contact form. Hence, given a non-degenerate toric contact form α on a Gorenstein toric contact manifold (M, ξ), with corresponding Reeb vector field R_α, each *contact Betti number* $cb_j(M, \alpha)$, $j \in \mathbb{Z}$, defined by

$$cb_j(M, \alpha) = \text{number of closed } R_\alpha\text{-orbits with contact homology degree } j,$$

should be a contact invariant of (M, ξ), the rank of its degree j cylindrical contact homology, Unfortunately, and despite recent foundational developments (e.g. [16, 17]), cylindrical contact homology has not been proved to

be a well defined invariant in the presence of contractible closed Reeb orbits, even in this restricted context of Gorenstein toric contact manifolds.

There are however at least two particular contexts that allow us to conclude that the contact Betti numbers are indeed contact invariants:

(i) Gorenstein toric contact manifolds that have crepant (i.e. with zero first Chern class) toric symplectic fillings.

(ii) Gorenstein toric contact manifolds that have a non-degenerate toric contact form with all of its closed contractible Reeb orbits having Conley-Zehnder index strictly greater than $3 - n$, i.e. SFT degree strictly greater than 1.

Indeed, in both of these contexts we can use positive equivariant symplectic homology to conclude that the contact Betti numbers are contact invariants, which will be denoted by $cb_j(M, \xi)$ or just $cb_j(M)$ when the contact structure is clear from the context. For (i), one considers the positive equivariant symplectic homology of the filling and recent work of McLean and Ritter [14], which uses previous work of Kwon and van Koert [8] (see [2, Remark 1.4] and [3, section 2]). For (ii), one considers the positive equivariant symplectic homology of the symplectization and the work of Bourgeois and Oancea [5, section 4.1.2]. In this case, we can also consider the decomposition of positive equivariant symplectic homology induced by homotopy classes $[\gamma]$ in the (cyclic) fundamental group of these Gorenstein toric manifolds to get possibly finer invariants of the their (co-oriented) contact structures:

$$cb_j^{[\gamma]}(M, \xi) = \text{number of closed } R_\alpha\text{-orbits in } [\gamma]$$
$$\text{with contact homology degree } j.$$

The examples discussed in this paper are all within context (ii) above, with some being also within context (i).

The first set of examples consists of the 5-dimensional contact manifolds that arise as unit cosphere bundles of 3-dimensional lens spaces. We will show that they are Gorenstein toric contact manifolds, fit within context (ii) and compute their contact Betti numbers, including the decomposition induced by homotopy classes in the fundamental group. Note that since any 5-dimensional Gorenstein toric contact manifold has a crepant toric symplectic filling, these examples also fit within context (i).

The second set of examples consists of $2n + 1$-dimensional lens spaces

$$L_p^{2n+1}(\ell_0, \ell_1, \ldots, \ell_n)$$

with canonical contact structure, arising as the quotient of the standard contact sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ by the $\mathbb{Z}_p$-action with weights $\ell_0, \ell_1, \ldots, \ell_n \in \mathbb{Z}$. Such an action is free when the weights are coprime with p and the resulting smooth contact lens space has zero first Chern class if and only if the sum of the weights is zero mod p. Being finite quotients of the standard contact sphere, all these Gorenstein toric contact lens spaces can easily be seen to fit context (ii), although most of them do not fit context (i). We will give an explicit method to compute their contact Betti numbers and their decomposition by homotopy classes of the fundamental group (see Proposition 5.1), and illustrate how these invariants can be used to prove the following type of results.

Proposition 1.1. *The diffeomorphic lens spaces*

$$L_5^{13}(1,1,1,1,2,-2,1) \quad and \quad L_5^{13}(1,-1,-1,-1,-2,-2,1)$$

have inequivalent canonical contact structures, both with zero first Chern class.

Proposition 1.2. *The lens space $L_5^{15}(1,1,1,2,-2,-2,-2,1)$ has an homotopy class of orientation preserving diffeomorphisms that does not contain any contactomorphism of its canonical contact structure. In fact, it has an orientation preserving diffeomorphism that acts by multiplication by $2 \in (\mathbb{Z}_5)^*$ on its fundamental group $\mathbb{Z}_5$, while any contactomorphism of its canonical contact structure acts by multiplication by $\pm 1 \in (\mathbb{Z}_5)^*$.*

Remark 1.3. *Following a very relevant suggestion by one referee, it turns out that this type of results can also be proved by looking at the total Chern class of these canonical contact structures on lens spaces. In fact, we have that*

$$H^{2j}(L_p^{2n+1}(\ell_0, \ell_1, \ldots, \ell_n); \mathbb{Z}) \cong \mathbb{Z}_p, \ for \ j = 1, \ldots, n,$$

and the Chern classes of the canonical contact structure are given by (cf. Proposition A.1 in appendix A)

$$c_j = \sigma_j \bmod p, j = 1, \ldots n,$$

where $\sigma_j = j$-th elementary symmetric polynomial of $\ell_0, \ell_1, \ldots, \ell_n$. For example,

$$c_1 = \sum_{k=0}^n \ell_k \mod p \quad and \quad c_n = \sum_{k=0}^{n+1} \frac{\ell_0 \cdot \ldots \cdot \ell_n}{\ell_k} \mod p.$$

In particular, this gives

$$c_6(L_5^{13}(1,1,1,1,2,-2,1)) = 1+1+1+1-2+2+1 = 5 = 0 \mod 5$$

and

$$\begin{aligned} c_6(L_5^{13}(1,-1,-1,-1,-2,-2,1)) \\ = 1-1-1-1+2+2+1 = 3 \neq 0 \mod 5\,, \end{aligned}$$

which implies Proposition 1.1. Regarding Proposition 1.2, we have that

$$\begin{aligned} c_7(L_5^{15}(1,1,1,2,-2,-2,-2,1)) \\ = -1-1-1+2-2-2-2-1 = 2 \neq 0 \mod 5\,. \end{aligned}$$

Moreover, if a diffeomorphism of a lens space acts by multiplication by $k \in \mathbb{Z}_k^*$ on its fundamental group, then it acts by multiplication by $k^j \in \mathbb{Z}_k^*$ on its degree $2j$ cohomology. Hence, the orientation preserving diffeomorphism considered in Proposition 1.2 does not fix $c_7(L_5^{15}(1,1,1,2,-2,-2,-2,1))$, since it acts on it by multiplication by $2^7 = 128 \equiv 3 \mod 5$, which immediately implies that there is no contactomorphism in its homotopy class.

Although there are lens spaces with canonical contact structure having total Chern class equal to zero, they do not provide relevant examples for the type of results considered in Propositions 1.1 and 1.2. In fact, as we will see in appendix A, at least for prime p there are no examples of lens spaces where the type of results considered in Propositions 1.1 and 1.2 do not follow from total Chern class considerations, the reason being that the canonical contact structure of a lens space $L_p^{2n+1}(\ell_0, \ell_1, \ldots, \ell_n)$ is completely determined by the diffeomorphism type of the lens space and the total Chern class of the contact structure.

The paper is organized as follows.

- In section 2 we recall fundamental facts about Gorenstein toric contact manifolds, including some (perhaps new) observations relating their fundamental groups with combinatorial properties of their toric diagrams. We will also present some examples, describing in detail the ones mentioned above.

- Section 3 is the main technical section of the paper, where we show that the explicit method developed in [1], to compute the Conley-Zehnder index of any contractible periodic Reeb orbit of a non-degenerate toric

contact form on a Gorenstein toric contact manifold, can also be used for non-contractible periodic Reeb orbits.

- In section 4 we compute the contact Betti numbers of unit cosphere bundles of 3-dimensional lens spaces, including their decomposition induced by homotopy classes in the fundamental group.

- In section 5, we give an explicit method to compute the contact Betti numbers of Gorenstein contact lens spaces and their decomposition by homotopy class of the fundamental group (Proposition 5.1), illustrate how it can be explicitly used in some examples, including Gorenstein prequantizations of $\mathbb{C}P^n$ (Example 5.5), and prove Propositions 1.1 and 1.2.

- Finally, appendix A is devoted to considerations regarding the total Chern class of the canonical contact structure of a lens space and results of the type considered in Propositions 1.1 and 1.2.

2. Gorenstein toric contact manifolds

In this section we recall fundamental facts about Gorenstein toric contact manifolds that will be needed in the paper, including some (perhaps new) observations relating their fundamental groups with combinatorial properties of their toric diagrams. We will also present some examples, describing in detail the ones that were already mentioned in the Introduction, i.e. 5-dimensional contact manifolds that arise as unit cosphere bundles of 3-dimensional lens spaces and $2n + 1$-dimensional lens spaces.

2.1. Toric diagrams

In this subsection we closely follow the presentation in [2] to recall the 1-1 correspondence between Gorenstein toric contact manifolds, i.e. good toric contact manifolds (in the sense of Lerman [9]) with zero first Chern class, and toric diagrams (defined below).

Via symplectization, there is a 1-1 correspondence between co-oriented contact manifolds and symplectic cones, i.e. triples (W, ω, X) where (W, ω) is a connected symplectic manifold and X is a vector field, the Liouville vector field, generating a proper $\mathbb{R}$-action $\rho_t : W \to W$, $t \in \mathbb{R}$, such that $\rho_t^*(\omega) = e^t \omega$. A closed symplectic cone is a symplectic cone (W, ω, X) for which the corresponding contact manifold $M = W/\mathbb{R}$ is closed.

A toric contact manifold is a contact manifold of dimension $2n+1$ equipped with an effective Hamiltonian action of the standard torus of dimension $n+1$: $\mathbb{T}^{n+1} = \mathbb{R}^{n+1}/2\pi\mathbb{Z}^{n+1}$. Also via symplectization, toric contact manifolds are in 1-1 correspondence with toric symplectic cones, i.e. symplectic cones (W, ω, X) of dimension $2(n+1)$ equipped with an effective X-preserving Hamiltonian $\mathbb{T}^{n+1}$-action, with moment map $\mu : W \to \mathbb{R}^{n+1}$ such that $\mu(\rho_t(w)) = e^t\mu(w)$, for all $w \in W$ and $t \in \mathbb{R}$. Its moment cone is defined to be $C := \mu(W) \cup \{0\} \subset \mathbb{R}^{n+1}$.

A toric contact manifold is *good* if its toric symplectic cone has a moment cone with the following properties.

Definition 2.1. *A cone $C \subset \mathbb{R}^{n+1}$ is* good *if it is strictly convex and there exists a minimal set of primitive vectors $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$, with $d \geq n+1$, such that*

(i) $C = \bigcap_{j=1}^{d}\{x \in \mathbb{R}^{n+1} \mid \ell_j(x) := \langle x, \nu_j \rangle \geq 0\}$.

(ii) *Any codimension-k face of C, $1 \leq k \leq n$, is the intersection of exactly k facets whose set of normals can be completed to an integral basis of $\mathbb{Z}^{n+1}$.*

The primitive vectors $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$ are called the defining normals of the good cone $C \subset \mathbb{R}^{n+1}$.

The analogue for good toric contact manifolds of Delzant's classification theorem for closed toric symplectic manifolds is the following result (see [9]).

Theorem 2.2. *For each good cone $C \subset \mathbb{R}^{n+1}$ there exists a unique closed toric symplectic cone $(W_C, \omega_C, X_C, \mu_C)$ with moment cone C.*

The existence part of this theorem follows from an explicit symplectic reduction of the standard euclidean symplectic cone $(\mathbb{R}^{2d} \setminus \{0\}, \omega_{\mathrm{st}}, X_{\mathrm{st}})$, where d is the number of defining normals of the good cone $C \subset \mathbb{R}^{n+1}$, with respect to the action of a subgroup $K \subset \mathbb{T}^d$ induced by the standard action of $\mathbb{T}^d$ on $\mathbb{R}^{2d} \setminus \{0\} \cong \mathbb{C}^d \setminus \{0\}$. More precisely,

$$(1) \qquad K := \left\{ [y] \in \mathbb{T}^d \mid \sum_{j=1}^{d} y_j \nu_j \in 2\pi\mathbb{Z}^{n+1} \right\},$$

where $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$ are the defining normals of C, i.e. $K := \ker(\beta)$ where $\beta : \mathbb{T}^d \to \mathbb{T}^{n+1}$ is represented by the matrix

$$(2) \qquad\qquad [\, \nu_1 \mid \cdots \mid \nu_d \,] \,.$$

Depending on the context, which will be clear in each case, we will also denote by β the map from $\mathbb{Z}^d$ to $\mathbb{Z}^{n+1}$ represented by this matrix.

The Chern classes of a co-oriented contact manifold can be canonically identified with the Chern classes of the tangent bundle of the associated symplectic cone. The following proposition gives a moment cone characterization of the vanishing of the first Chern class that is commonly used in toric Algebraic Geometry (see, e.g., §4 of [4]).

Proposition 2.3. *Let (W_C, ω_C, X_C) be a good toric symplectic cone. Let $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$ be the defining normals of the corresponding moment cone $C \subset \mathbb{R}^{n+1}$. Then $c_1(TW_C) = 0$ if and only if there exists $\nu^* \in (\mathbb{Z}^{n+1})^*$ such that*

$$\nu^*(\nu_j) = 1\,, \ \forall\, j = 1, \ldots, d\,.$$

By an appropriate change of basis of the torus $\mathbb{T}^{n+1}$, i.e. an appropriate $SL(n+1, \mathbb{Z})$ transformation of $\mathbb{R}^{n+1}$, this implies the following.

Corollary 2.4. *Let (W_C, ω_C, X_C) be a good toric symplectic cone with $c_1(TW_C) = 0$. Then there exists an integral basis of $\mathbb{T}^{n+1}$ for which the defining normals $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$ of the corresponding moment cone $C \subset \mathbb{R}^{n+1}$ are of the form*

$$\nu_j = (v_j, 1)\,, \ v_j \in \mathbb{Z}^n\,, \ j = 1, \ldots, d\,.$$

The next definition and theorem are then the natural analogues for Gorenstein toric contact manifolds of Definition 2.1 and Theorem 2.2.

Definition 2.5. *A toric diagram $D \subset \mathbb{R}^n$ is an integral simplicial polytope with all of its facets $\mathrm{Aff}(n, \mathbb{Z})$-equivalent to $\mathrm{conv}(e_1, \ldots, e_n)$, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.*

Remark 2.6. *The group $\mathrm{Aff}(n, \mathbb{Z})$ of integral affine transformations of $\mathbb{R}^n$ can be naturally identified with the elements of $SL(n+1, \mathbb{Z})$ that preserve the hyperplane $\{(v, 1) \mid v \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$.*

Theorem 2.7. *For each toric diagram $D \subset \mathbb{R}^n$ there exists a unique Gorenstein toric contact manifold (M_D, ξ_D) of dimension $2n+1$.*

The $\mathbb{T}^{n+1}$-action associates to every vector $\nu \in \mathbb{R}^{n+1}$ a contact vector field $R_\nu \in \mathcal{X}(M_D, \xi_D)$. We will say that a contact form $\alpha_\nu \in \Omega^1(M_D, \xi_D)$ is *toric* if its Reeb vector field R_{α_ν} satisfies

$$R_{\alpha_\nu} = R_\nu \quad \text{for some } \nu \in \mathbb{R}^{n+1}.$$

In this case we will say that $\nu \in \mathbb{R}^{n+1}$ is a *toric Reeb vector* and that R_ν is a *toric Reeb vector field*.

Definition 2.8. *A normalized toric Reeb vector is a toric Reeb vector $\nu \in \mathbb{R}^{n+1}$ of the form*

$$\nu = (v, 1) \quad \text{with } v \in \mathbb{R}^n.$$

Proposition 2.9 ([12] or [2, Corollary 2.15]). *The interior of a toric diagram $D \subset \mathbb{R}^n$ parametrizes the set of normalized toric Reeb vectors on the Gorenstein toric contact manifold (M_D, ξ_D), i.e. $\nu = (v, 1)$ is a normalized toric Reeb vector iff $v \in \mathrm{int}(D)$.*

2.2. Fundamental group

Let W_C be a good toric symplectic cone and $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$ the defining normals of the corresponding moment cone $C \subset \mathbb{R}^{n+1}$. By a result of Lerman [10], the fundamental group $\pi_1(W_C)$ is isomorphic (canonically with respect to base points) to $\mathbb{Z}^{n+1}/\mathcal{N}$, where $\mathcal{N}$ is the $\mathbb{Z}$-span of $\{\nu_1, \ldots, \nu_d\}$. Any toric Reeb vector field R_ν, $\nu \in \mathbb{R}^{n+1}$, has at least m simple closed orbits $\gamma_1, \ldots, \gamma_m$, corresponding to the m edges $E_1, \ldots, E_m$ of the cone C, and we will now determine how these simple closed orbits can be seen and related as elements of $\pi_1(W_C)$.

Consider an edge E_ℓ of C obtained as the intersection of n facets with normals $\nu_{\ell_1}, \ldots, \nu_{\ell_n}$. We can complete $\nu_{\ell_1}, \ldots, \nu_{\ell_n}$ to a $\mathbb{Z}^{n+1}$-basis $\nu_{\ell_1}, \ldots, \nu_{\ell_n}, \eta_\ell$. Given a toric Reeb vector field R_ν, we can write $\nu \in \mathbb{R}^{n+1}$ uniquely as

$$\nu = \sum_{i=1}^n b_{\ell_i} \nu_{\ell_i} + b_\ell \eta_\ell .$$

Then, the simple closed orbit γ_ℓ of R_ν corresponding to the edge E_ℓ represents the element $[\mathrm{sgn}(b_\ell)\eta_\ell] \in \mathbb{Z}^{n+1}/\mathcal{N}$ in the fundamental group of W_C.

Consider the homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}^{n+1}/\mathcal{N}$ defined by $\phi(1) = [\eta_\ell]$. Since

$$\eta_\ell \mathbb{Z} + \mathcal{N} \supseteq \eta_\ell \mathbb{Z} + \nu_{\ell_1} \mathbb{Z} + \ldots + \nu_{\ell_n} \mathbb{Z} = \mathbb{Z}^{n+1} ,$$

the homomorphism ϕ is surjective. Thus $\mathbb{Z}^{n+1}/\mathcal{N} \cong \mathbb{Z}/\ker\phi = \mathbb{Z}_{N_\ell}$ where

$$N_\ell = \min\{k \in \mathbb{Z}^+ : k\eta_\ell \in \mathcal{N}\} .$$

This shows that $\pi_1(W_C)$ is cyclic and is generated by any of the simple closed orbits γ_ℓ of R_ν. Moreover it follows that $N_\ell = N$ is the same for each edge E_ℓ and is the order of the fundamental group of W_C. A way to compute this order N is the following (see also [11]).

Proposition 2.10. *The fundamental group of W_C is a cyclic group of order N where*

$$N = \gcd\left\{ \left| \begin{matrix} | & & | \\ \nu_{i_1} & \ldots & \nu_{i_{n+1}} \\ | & & | \end{matrix} \right| : 1 \le i_1 < \ldots < i_{n+1} \le d \right\} .$$

In particular, if C is determined by a toric diagram $D = \mathrm{conv}(v_1, \ldots, v_d)$ (i.e. $\nu_j = (v_j, 1)$) then

$$N = \gcd\left\{ \frac{1}{n!}\mathrm{Vol}(\mathrm{conv}(v_{i_1}, \ldots, v_{i_{n+1}})) : 1 \le i_1 < \ldots < i_{n+1} \le d \right\} .$$

Proof. Let

$$A = \begin{bmatrix} | & & | \\ \nu_1 & \ldots & \nu_d \\ | & & | \end{bmatrix} ,$$

which is a $(n+1) \times d$ matrix. By the Smith normal form we have

$$\pi_1(W_C) \cong \mathbb{Z}^{n+1}/\mathcal{N} = \mathbb{Z}^{n+1}/\mathrm{im}(A) \cong \mathbb{Z}_{\alpha_1} \oplus \ldots \oplus \mathbb{Z}_{\alpha_{n+1}} ,$$

where $\alpha_1, \ldots, \alpha_{n+1}$ are the invariant factors of A given by

$$\alpha_i = \frac{d_i(A)}{d_{i-1}(A)} ,$$

where $d_i(A)$ is the greatest common divisor of all the $i \times i$ minors of A ($d_0(A)$ is set to be 1). We claim that $d_n(A) = 1$, from which follows that $\alpha_1 = \ldots = \alpha_n = 1$.

Indeed let $\nu_{\ell_1}, \ldots, \nu_{\ell_n}$ be the normals corresponding to facets intersecting at an edge E_ℓ of C and let $\eta_\ell \in \mathbb{Z}^{n+1}$ be such that $\nu_{\ell_1}, \ldots, \nu_{\ell_n}, \eta_\ell$ is a basis of $\mathbb{Z}^{n+1}$. Then

$$\begin{vmatrix} | & & | & | \\ \nu_{\ell_1} & \cdots & \nu_{\ell_n} & \eta_\ell \\ | & & | & | \end{vmatrix} = \pm 1 \,.$$

By the Laplace rule on the last column we get an integral combination of the $n \times n$ minors of

$$\begin{bmatrix} | & & | \\ \nu_{\ell_1} & \cdots & \nu_{\ell_n} \\ | & & | \end{bmatrix}$$

giving 1, hence their greatest common divisor is 1, showing that $d_n(A) = 1$.

The result now follows since $\pi_1(W_C) \cong \mathbb{Z}_{\alpha_{n+1}}$ and $\alpha_{n+1} = d_{n+1}(A) = N$. $\qquad\square$

If we now consider two different edges, say E_ℓ and E_k, how can we relate the corresponding simple closed Reeb orbits, γ_ℓ and γ_k, as elements in the fundamental group $\pi_1(W_C)$? Let $\nu_{\ell_1}, \ldots, \nu_{\ell_n}, \eta_\ell$ and $\nu_{k_1}, \ldots, \nu_{k_n}, \eta_k$ be integral basis of $\mathbb{Z}^{n+1}$ corresponding to the edges E_ℓ and E_k, respectively. Consider the change of $\mathbb{Z}$-basis

$$(3) \qquad\qquad \eta_\ell = \sum_{j=1}^{n} b_j \nu_{k_j} + b \eta_k$$

$$(4) \qquad\qquad \nu_{\ell_i} = \sum_{j=1}^{n} a_{ij} \nu_{k_j} + a_i \eta_k$$

with $b_j, b, a_{ij}, a \in \mathbb{Z}$. Then

$$\begin{vmatrix} b_1 & \cdots & b_n & b \\ a_{11} & \cdots & a_{1n} & a_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & a_n \end{vmatrix} = \pm 1 \,.$$

If follows from (4) that $a_i \eta_k \in \mathcal{N}$, and so we have that $N | a_i$ for $i = 1, \ldots, n$. Thus, the determinant being ± 1 gives

$$b \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \equiv \pm 1 \mod N.$$

Equation (3) gives $[\eta_\ell] = b[\eta_k]$, that is, $[\eta_k] = c[\eta_\ell]$ where

$$c \equiv b^{-1} \equiv \pm \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} \quad \mod N \,.$$

Moreover, this is still valid if a_{ij} satisfy the equations

$$\nu_{\ell_i} \equiv \sum_{j=1}^{n} a_{ij} \nu_{k_j} \quad \mod N \,,$$

which allows the computation of c without determining η_ℓ, η_k.

2.3. Examples

2.3.1. Prequantizations.   One source for examples of good toric contact manifolds is the prequantization construction over integral closed toric symplectic manifolds, i.e. (M, ξ) with M given by the S^1-bundle over (B, ω) with Euler class $-[\omega]/2\pi$ and ξ being the horizontal distribution of a connection with curvature ω. The corresponding good cones have the form

$$C := \{z(x,1) \in \mathbb{R}^n \times \mathbb{R} \mid x \in P, \; z \geq 0\} \subset \mathbb{R}^{n+1}$$

where $P \subset \mathbb{R}^n$ is a Delzant polytope with vertices in the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Note that if

$$P = \bigcap_{j=1}^{d} \{x \in \mathbb{R}^n \mid \langle x, v_j \rangle + \lambda_j \geq 0\} \,,$$

with integral $\lambda_1, \dots, \lambda_d \in \mathbb{Z}$ and primitive $v_1, \dots, v_d \in \mathbb{Z}^n$, then the defining normals of $C \subset \mathbb{R}^{n+1}$ are

$$\nu_j = (v_j, \lambda_j) \,, \; j = 1, \dots, d \,.$$

When $\lambda_1 = \cdots \lambda_d = 1$ we have that $c_1(TB, \omega) = [\omega]/2\pi$ and the prequantization is a Gorenstein toric contact manifold with toric diagram $D := \mathrm{conv}(v_1, \dots, v_d) \subset \mathbb{R}^n$.

2.3.2. 3-dimensional lens spaces and their unit cosphere bundles.
A 3-dimensional lens space $L_p^3(q) := L_p^3(1, q)$, where $p, q \in \mathbb{N}$ are coprime, is

the quotient of $S^3 \subset \mathbb{C}^2 \setminus \{0\}$ by the free action of $\mathbb{Z}_p$ generated by

$$[1].(z_0, z_1) = \left(e^{\frac{2\pi i}{p}} z_0, e^{\frac{2\pi iq}{p}} z_1 \right) .$$

Recall that the unit cosphere bundle S^*N of any smooth (oriented) manifold N has a canonical (co-oriented) contact structure with symplectization $T^*N \setminus \{0\text{-section}\}$ with its canonical symplectic structure. We will now show that any $S^*L_p^3(q)$ is a Gorenstein toric contact manifold with toric diagram given by a parallelogram in $\mathbb{R}^2$.

Let us start by characterizing the smooth 5-manifolds that can be obtained from parallelograms in $\mathbb{R}^2$.

Proposition 2.11. *Let $D = \mathrm{conv}((0,0), (1,0), (q,p), (q+1,p))$ be a toric diagram where $p, q \in \mathbb{N}$ are coprime. Then the corresponding manifold $M = M_D$ is diffeomorphic to $S^*L \cong L \times S^2$ where $L = L_p^3(q)$.*

Remark 2.12. *Note that any parallelogram is $\mathrm{Aff}(2, \mathbb{Z})$-equivalent to one of the diagrams considered in the theorem, so this shows that any toric diagram which is a parallelogram corresponds to S^*L for some lens space L. The order p of the fundamental group p is given by the area of the toric diagram.*

Proof. The symplectization W of M is the toric symplectic cone determined by the good moment cone $C \subset \mathbb{R}^3$ with normals

$$\nu_1 = (q+1, p, 1) , \quad \nu_2 = (0, 0, 1) , \quad \nu_3 = (1, 0, 1) \quad \text{and} \quad \nu_4 = (q, p, 1) .$$

More precisely, it is the symplectic reduction of $(\mathbb{C}^4 \setminus \{0\}, \omega_{\mathrm{st}}, X_{\mathrm{st}})$ with respect to the naturally induced action of $K \subset \mathbb{T}^4$, with $K := \ker \beta$ where $\beta : \mathbb{T}^4 \to \mathbb{T}^3$ is represented by the matrix (cf. (2))

$$\begin{bmatrix} q+1 & 0 & 1 & q \\ p & 0 & 0 & p \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Hence we have that

$$K \cong \left\{ \left(e^{it}, e^{it + \frac{2\pi i(q-1)k}{p}}, e^{-it - \frac{2\pi iqk}{p}}, e^{-it + \frac{2\pi ik}{p}} \right) : t \in \mathbb{R}, k \in \mathbb{Z} \right\} = S^1 \times \langle \xi \rangle$$

where $\xi = \left(1, e^{\frac{2\pi i(q-1)}{p}}, e^{-\frac{2\pi iq}{p}}, e^{\frac{2\pi i}{p}} \right)$ (notice that $\langle \xi \rangle \cong \mathbb{Z}_p$). This means that

$$W = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \setminus \{0\} : |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = 0\}/(S^1 \times \langle \xi \rangle)$$

and $M = (S^3 \times S^3)/(S^1 \times \langle \xi \rangle)$ where

$$S^3 \times S^3 = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : |z_1|^2 + |z_2|^2 = |z_3|^2 + |z_4|^2 = 1\} \subset \mathbb{C}^4$$

and

$$S^1 = \{\left(e^{it}, e^{it}, e^{-it}, e^{-it}\right) : t \in \mathbb{R}\}$$

acts naturally on $S^3 \times S^3 \subset \mathbb{C}^2 \times \mathbb{C}^2$. By identifying $\mathbb{C}^2$ with the quaternions $\mathbb{H}$ via $(z_1, z_2) \to z_1 + jz_2$ we can also look at S^3 as the unit quaternions, giving S^3 a Lie group structure. Notice that the maps $(z_1, z_2, z_3, z_4) \to (z_1 : z_2) \in \mathbb{C}P^1 \cong S^2$ and

$$(z_1, z_2, z_3, z_4) \to (z_1 + jz_2)(z_3 + z_4 j)$$
$$= (z_1 z_3 - \overline{z}_2 \overline{z}_4) + (z_1 z_4 + \overline{z}_2 \overline{z}_3)j \in S^3 \subseteq \mathbb{H}$$

are both invariant under the action of S^1.

Lemma 2.13. *The map $\psi : S^3 \times S^3 \to S^3 \times \mathbb{C}P^1 \cong S^3 \times S^2$ defined by*

$$(z_1, z_2, z_3, z_4) \to ((z_1 z_3 - \overline{z}_2 \overline{z}_4, z_1 z_4 + \overline{z}_2 \overline{z}_3), (z_1 : z_2))$$

induces a diffeomorphism $(S^3 \times S^3)/S^1 \cong S^3 \times \mathbb{C}P^1$.

Proof. It suffices to show that ψ is surjective with S^1-orbits as fibers.
Surjectivity. Let $(x, z) \in S^3 \times \mathbb{C}P^1$. If we choose z_1, z_2 such that $|z_1|^2 + |z_2|^2 = 1$ and $z_1/z_2 = z$ and define z_3, z_4 by $z_3 + z_4 j = (z_1 + jz_2)^{-1}x$ we get $\psi(z_1, z_2, z_3, z_4) = (x, z)$.
Fibers. Suppose that $\psi(z) = \psi(z')$. From $z_1/z_2 = z_1'/z_2'$ follows that there is $t \in \mathbb{R}$ such that $z_1' = e^{it}z_1$ and $z_2' = e^{it}z_2$. Moreover,

$$(z_1 + jz_2)(z_3 + z_4 j) = (z_1' + jz_2')(z_3' + z_4'j) = (z_1 + jz_2)\left(e^{it}z_3' + e^{it}z_4'j\right).$$

Thus $z_3 = e^{it}z_3'$ and $z_4 = e^{it}z_4'$; that is,

$$(z_1', z_2', z_3', z_4') = (e^{it}z_1, e^{it}z_2, e^{-it}z_3, e^{-it}z_4),$$

so z and z' are in the same S^1-orbit. $\qquad \square$

One can easily see that the $\langle \xi \rangle$-action on $S^3 \times S^3$ descends via ψ to a $\langle \tilde{\xi} \rangle$-action on $S^3 \times \mathbb{C}P^1$ with

$$\tilde{\xi} \cdot (z_1, z_2, z) = \left(e^{-\frac{2\pi iq}{p}} z_1, e^{\frac{2\pi i}{p}} z_2, e^{-\frac{2\pi i(q-1)}{p}} z\right)$$

for $(z_1, z_2) \in S^3$ and $z \in \mathbb{C}P^1$.

Since S^3 is a Lie group, the unit sphere bundle SS^3 admits a trivialization $\varphi : SS^3 \to S^3 \times S^2$ and $S^3 \times S^2$ can be identified with $S^3 \times \mathbb{C}P^1$ via $\mathrm{id}_{S^3} \times \mathrm{st}$, where $\mathrm{st} : S^2 \to \mathbb{C}P^1$ is the standard identification between $S^2 \cong \mathbb{R}^2 \cup \{\infty\}$ and $\mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$. We will show that, under these identifications, the $\langle \tilde{\xi} \rangle$-action on $S^3 \times \mathbb{C}P^1$ can be seen as a $\langle dg \rangle$-action on SS^3, where dg is the natural lift to SS^3 of the $\langle g \rangle$-action on S^3 determined by

$$g \cdot (z_1, z_2) = \left(e^{-\frac{2\pi i q}{p}} z_1, e^{\frac{2\pi i}{p}} z_2 \right) .$$

More precisely, we will prove that the diagram

$$
\begin{array}{ccc}
S^3 \times \mathbb{C}P^1 & \xrightarrow{\tilde{\xi}} & S^3 \times \mathbb{C}P^1 \\
{\scriptstyle \mathrm{id}\times\mathrm{st}} \uparrow & & {\scriptstyle \mathrm{id}\times\mathrm{st}} \uparrow \\
S^3 \times S^2 & \xrightarrow{g \times r_{2\alpha}} & S^3 \times S^2 \\
{\scriptstyle \varphi} \uparrow & & {\scriptstyle \varphi} \uparrow \\
SS^3 & \xrightarrow{\ \ dg\ \ } & SS^3
\end{array}
$$

commutes, where $r_{2\alpha}$ is a certain rotation of S^2 with angle 2α (see below). It will then follow that

$$M = (S^3 \times S^2)/\langle \tilde{\xi} \rangle \cong SS^3/\langle dg \rangle \cong S\left(S^3/\langle g \rangle \right) \cong SL_p^3(q) \cong L_p^3(q) \times S^2$$

as desired.

To show that the diagram commutes, consider again $S^3 \subset \mathbb{H}$ as the unit quaternions. Since

$$e^{-\frac{2\pi i q}{p}} z_1 + e^{\frac{2\pi i}{p}} z_2 j = e^{-\frac{\pi i (q-1)}{p}} (z_1 + z_2 j) e^{-\frac{\pi i (q+1)}{p}}$$

the $\langle g \rangle$-action on S^3 can be written as

$$g \cdot x = e^{i\alpha} x e^{i\beta} , \ x \in S^3 \subset \mathbb{H} ,$$

where $\alpha = -\frac{\pi(q-1)}{p}$, $\beta = -\frac{\pi(q+1)}{p}$. We can now think of the tangent space $S_x S^3$ at x as the set of unit quaternions $v \in \mathbb{H}$ perpendicular to x. Thus SS^3 consists of pairs of orthogonal quaternions $(x, v) \in \mathbb{H} \times \mathbb{H}$ such that

$|x| = |v| = 1$. We have a trivialization

(5)　　$\varphi : SS^3 \to S^3 \times S_e S^3 \cong S^3 \times S^2$　given by　$\varphi(x, v) = \left(x, vx^{-1}\right).$

Here $S_e S^3$ is the unit sphere bundle at the identity of S^3, that is,

$$S_e S^3 = \{ai + bj + ck : a^2 + b^2 + c^2 = 1\} \cong S^2.$$

For $(x, v) \in SS^3$ we have that

$$\begin{aligned}
\varphi((dg) \cdot (x, v)) &= \varphi \left(e^{i\alpha} x e^{i\beta}, e^{i\alpha} v e^{i\beta} \right) \\
&= \left(e^{i\alpha} x e^{i\beta}, e^{i\alpha} v x^{-1} e^{-i\alpha} \right) = \left(g \cdot x, r_{2\alpha}(vx^{-1}) \right),
\end{aligned}$$

where $r_{2\alpha}(w) = e^{i\alpha} w e^{-i\alpha}$ is the rotation of $S^2 = S_e S^3$ with angle 2α about the i-axis. This shows that the down square of the diagram commutes.

Consider now the $\langle \tilde{\xi} \rangle$-action on $S^3 \times S^2$. It is clear that on the first coordinate $\tilde{\xi}$ acts as g. If we give spherical coordinates (θ, ϕ) to S^2 with respect to the i-axis, then the stereographic projection st $: S^2 \to \mathbb{C}P^1$ from i is given by $\mathrm{st}(\theta, \phi) = \cot(\theta/2) e^{i\phi}$ and the rotation around the i-axis with angle 2α is given by $r_{2\alpha}(\theta, \phi) = (\theta, \phi + 2\alpha)$. Therefore

$$e^{2i\alpha} \cdot \mathrm{st}(\theta, \phi) = \cot(\theta/2) e^{i(\phi + 2\alpha)} = \mathrm{st}(\theta, \phi + 2\alpha) = \mathrm{st}(r_{2\alpha}(\theta, \phi)).$$

From this we can conclude that the top square commutes.　　　　□

We will now prove that the toric contact structures on $L_p^3(q) \times S^2$ are indeed the standard contact structures on the unit cosphere bundles $S^* L_p^3(q)$. Let us start with the case $p = 1$ and $q = 0$, i.e. when the toric diagram is

$$D = \mathrm{conv}((0,0), (1,0), (0,1), (1,1)).$$

The corresponding moment cone in $\mathbb{R}^3$ has normals

$$\nu_1 = (1,1,1), \ \nu_2 = (0,0,1), \ \nu_3 = (1,0,1) \quad \text{and} \quad \nu_4 = (0,1,1).$$

Via the $SL(3, \mathbb{Z})$ transformation

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

this cone is equivalent to the one with normals

$$\tilde{\nu}_1 = (1,0,0)\,, \ \tilde{\nu}_2 = (-1,0,1)\,, \ \tilde{\nu}_3 = (0,-1,1) \quad \text{and} \quad \tilde{\nu}_4 = (0,1,0)\,,$$

which is the cone over the Delzant polytope $[0,1] \times [0,1]$. This means that (M_D, ξ_D) is the prequantization of $(B = S^2 \times S^2, \omega = \sigma_1 + \sigma_2)$, where $\sigma_i = \pi_i^*(\sigma)$ with $\pi_i : S^2 \times S^2 \to S^2$, $i = 1,2$, the factor projections and $\int_{S^2} \sigma = 2\pi$. This prequantization is well known to be contactomorphic to S^*S^3 with its canonical contact structure, but we present here a proof for completeness.

Proposition 2.14. *The prequantization of $(S^2 \times S^2, \sigma_1 + \sigma_2)$ is S^*S^3 with its canonical contact structure.*

Proof. We can use the bi-invariant metric $\langle \cdot, \cdot \rangle$ on S^3 to identify S^*S^3 with SS^3. This provides SS^3 with a contact form α which is given by $\alpha_{(x,v)}(U) = \langle pr_*U, v \rangle_x$ for $(x,v) \in SS^3$ and $U \in T_{(x,v)}SS^3$ where $pr : SS^3 \to S^3$ is the canonical projection. The Reeb flow for this contact form is the geodesic flow on SS^3.

Consider the diffeomorphism $\varphi : SS^3 \to S^3 \times S^2$ defined by (5). Since the usual Riemannian metric of S^3 is bi-invariant for the Lie group structure on S^3, under this diffeomorphism the geodesic flow is given by $(t, (x,w)) \mapsto (x \exp(tw), w)$; note that $t \mapsto \exp(tw)$ has period 2π for every $w \in S^2$, so the geodesic flow gives a S^1-action on SS^3. We can check that this action gives a principal S^1-bundle $\pi : SS^3 \to S^2 \times S^2$ where π is defined by $\pi(x,v) = (vx^{-1}, x^{-1}v)$.

We want to show that the curvature form $\omega \in \Omega^2(S^2 \times S^2)$ of this bundle for the connection determined by the contact form α is $\sigma_1 + \sigma_2$; recall that ω is determined by the formula $d\alpha = \pi^*\omega$.

We begin by showing that ω is invariant for the $SO(3) \times SO(3)$-action on $S^2 \times S^2$. First, we note that $S^3 \times S^3$ acts on SS^3 via left and right multiplication, that is,

$$(g_1, g_2) \cdot (x, v) = (dL_{g_1})(dR_{g_2})^{-1}(x, v) = (g_1 x g_2^{-1}, g_1 v g_2^{-1})\,.$$

Moreover, it acts on $S^2 \times S^2$ via conjugation on both coordinates:

$$(g_1, g_2) \cdot (w_1, w_2) = (g_1 w_1 g_1^{-1}, g_2 w_2 g_2^{-1})\,.$$

We can easily check that $\pi : SS^3 \to S^2 \times S^2$ is equivariant with respect to these actions. Thus, if we show that α is invariant with respect to the $S^3 \times S^3$ action on SS^3 it follows that ω is invariant with respect to the $S^3 \times S^3$ action

on $S^2 \times S^2$ which descends to the usual $SO(3) \times SO(3)$-action by rotations on each component. Indeed, we have:

$$[(dL_{g_1})^*\alpha]_{(x,v)}(U) = \alpha_{(g_1 x, g_1 v)}((dL_{g_1})_* U) = \langle g_1 v, pr_*(dL_{g_1})_* U \rangle_{g_1 x}$$
$$= \langle g_1 v, g_1 pr_* U \rangle_{g_1 x} = \langle v, pr_* U \rangle_x = \alpha_{(x,v)}(U).$$

Hence $(dL_{g_1})^*\alpha = \alpha$. Note that we used that $pr \circ dL_{g_1} = L_{g_1} \circ pr$ and that the metric $\langle \cdot, \cdot \rangle$ is left-invariant. Similarly $(dR_{g_1})^*\alpha = \alpha$, so the claim follows.

Now, since the $SO(3) \times SO(3)$-action is transitive, it is clear that this action preserves ω if and only if ω has the form $\lambda_1 \sigma_1 + \lambda_2 \sigma_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. We compute λ_1 by computing the integral of ω on the sphere $S^2 \times \{i\}$. We can check that π maps the disk

$$D = \{(x, xi) \in SS^3 : x = a + bj + ck \text{ with } a^2 + b^2 + c^2 = 1, a \geq 0\} \subseteq SS^3$$

onto $S^2 \times \{i\}$ and is injective in $D \setminus \partial D$ (and collapses the boundary to the point $(-i, i)$). Hence

$$\lambda_1 = \frac{1}{2\pi} \int_{S^2 \times \{i\}} \omega = \frac{1}{2\pi} \int_D \pi^* \omega = \frac{1}{2\pi} \int_{\partial D} \alpha$$

by Stokes' theorem and $\pi^*\omega = d\alpha$. But ∂D is the closed orbit of the Reeb flow (since it is mapped by π to a point) and the closed orbits of the Reeb flow have period 2π, hence $\lambda_1 = \frac{1}{2\pi} \int_{\partial D} \alpha = 1$. Similarly $\lambda_2 = 1$ and thus $\omega = \sigma_1 + \sigma_2$. $\qquad\square$

Now we prove that in general the contact structure on $M_D \cong SL$ is the canonical one. The inclusion $\ker \beta_D \subseteq \ker \beta_Q$ induces a projection

$$S^3 \times S^2 \cong M_Q \to M_D \cong S^3 \times S^2 / \langle \tilde{\bar{\xi}} \rangle$$

which by naturality of the Delzant construction is a contact transformation (and it is also a local diffeomorphism). This projection fits in the following commutative diagram:

$$
\begin{array}{ccc}
SS^3 & \xrightarrow{\;d\bar{p}\;} & SL_p^3(q) \\
\downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\bar{\varphi}} \\
M_Q & \longrightarrow & M_D
\end{array}
$$

where $\bar{p} : S^3 \to L_p^3(q)$ is the quotient projection. We already showed that φ is a contactomorphism, $d\bar{p}$ is also a contact transformation and a local

diffeomorphism. Hence it follows that $\overline{\varphi}$ is a contactomorphism. That is, we just showed the following:

Theorem 2.15. *The contact manifold $S^* L_p^3(q)$ is a Gorenstein toric contact manifold with corresponding toric diagram*

$$\text{conv}((0,0), (1,0), (q,p), (q+1,p)).$$

Moreover, we can understand the action of T^3 on $S^* L_p^3(q)$. For $L_1^3(0) = S^3$, this action is the action of $T^3 = T^2 \times S^1$ where S^1 acts on $S^* S^3$ by the geodesic flow and the action of T^2 is the lift of the usual action of T^2 on $S^2 \times S^2$ via $\pi : S^* S^3 \to S^2 \times S^2$. But the later is easily seen to be induced by the action of T^2 on S^3 (the same giving S^3 a toric structure). Explicitly, identifying $S^* S^3$ with $S^3 \times S^2$ as before, the action is given by

$$\left(e^{i\alpha}, e^{i\beta}, e^{i\gamma} \right) \cdot (x, w) = \left(e^{i\beta} e^{\gamma w} x e^{i\alpha}, e^{i\beta} w e^{-i\beta} \right).$$

In general, we have an action of $\widetilde{T}^2 \times S^1$ on $S^* L_p^3(q)$ where $\widetilde{T}^2 = T^2/(\mathbb{Z}_p)$ still comes from the usual toric structure on $L_p^3(q)$ and S^1 still is the geodesic flow. Note that when $p > 1$ the geodesic flow action is no longer free.

Remark 2.16. *The contact manifold $S^* S^{n+1}$, or more generally $S^* L^{n+1}$, is not toric when $n > 2$. In fact, $H^2(S^* S^{n+1}, \mathbb{R}) = 0$ for $n > 2$ and, by [10], this implies that if $S^* S^{n+1}$ was toric then its moment cone $C \subset \mathbb{R}^{n+1}$ would have $d = n + 1$ facets. Any such moment cone gives rise to a toric contact manifold diffeomorphic to $S^{2n+1}/\ker \beta$, where $\ker \beta$ is a finite group (cf. (2)). Since $S^* S^{n+1}$ is simply connected, this would be possible only if $S^* S^{n+1}$ was diffeomorphic to S^{2n+1}, which clearly is not the case since they have different cohomology types.*

2.3.3. Higher dimensional lens spaces. We will now consider higher dimensional lens spaces, i.e. quotients of odd dimensional spheres by cyclic groups. More concretely, regarding S^{2n+1} as a subset of $\mathbb{C}^{n+1} \setminus \{0\}$ we have an action of $\mathbb{Z}_p$ generated by

$$[1].(z_0, \ldots, z_n) = \left(e^{\frac{2\pi i \ell_0}{p}} z_0, e^{\frac{2\pi i \ell_1}{p}} z_1, \ldots, e^{\frac{2\pi i \ell_n}{p}} z_n \right)$$

where $\ell_0, \ldots, \ell_n$ are integers called the weights of the action. Such an action is free when the weights are coprime with p and in that case we have a lens

space obtained as the quotient of S^{2n+1} by the action of $\mathbb{Z}_p$. We denote such a lens space by

$$L_p^{2n+1}(\ell_0, \ell_1, \ldots, \ell_n).$$

Note that different sequences of weights might produce diffeomorphic lens spaces, for instance by permuting the weights, multiplying every weight by some k coprime with p and changing the sign of some weights. Moreover these are the only possibilities leading to diffeomorphic lens spaces (see section 12 of [15]).

Suppose now that L^{2n+1} is a lens space admitting a toric contact structure. We have that $H^2(L, \mathbb{R}) = 0$, since $H^2(L; \mathbb{Z}) \cong \mathbb{Z}_p$, which again implies that the moment cone of L has $d = n + 1$ facets (see Remark 2.16). Restricting ourselves to Gorenstein lens spaces, Proposition 2.3 implies that the moment cone of the symplectization of L must be $SL(n + 1, \mathbb{Z})$ equivalent to a cone $C \subset \mathbb{R}^{n+1}$ with defining normals

$$\nu_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ \nu_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \ \text{for } j = 1, \ldots, n - 1,$$

$$\nu_n = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \\ p \\ 1 \end{bmatrix}, \ \alpha_1, \ldots, \alpha_{n-1} \in \mathbb{Z}, \ p \in \mathbb{N}.$$

The system $\beta(x) = 0$, with β given by (2), can be written as

$$\begin{cases} x_1 + \alpha_1 x_n = 0 \\ \quad \vdots \\ x_{n-1} + \alpha_{n-1} x_n = 0 \\ p x_n = 0 \\ \sum_{i=0}^n x_i = 0 \end{cases}$$

which has p solutions in $\mathbb{T}^{n+1}$, generated by

$$\left(\frac{\alpha_0}{p}, -\frac{\alpha_1}{p}, \ldots, -\frac{\alpha_{n-1}}{p}, \frac{1}{p}\right) \in \mathbb{T}^{n+1}, \text{with } \alpha_0 := \alpha_1 + \ldots + \alpha_{n-1} - 1.$$

Hence the corresponding contact manifold is

$$L_p^{2n+1}(\alpha_0, -\alpha_1, \ldots, -\alpha_{n-1}, 1).$$

It is not hard to see that C is a good cone if and only if $\gcd(\alpha_j, p) = 1$ for $j = 0, 1, \ldots, n-1$. Note that when n is even this implies in particular that p must be odd.

3. Conley-Zehnder index of non-contractible closed orbits

Let (M^{2n+1}, ξ) be a good toric contact manifold and $(W^{2(n+1)}, \omega, X)$ its associated toric symplectic cone, obtained via symplectic reduction of $(\mathbb{C}^d \setminus \{0\} \cong \mathbb{R}^{2d} \setminus \{0\}, \omega_{\mathrm{st}}, X_{\mathrm{st}})$ by the linear action of a subtorus $K \subset \mathbb{T}^d$ (see (1)). Denote by $F : \mathbb{C}^d \setminus \{0\} \to \mathfrak{k}^*$ the associated moment map, so that $W = Z/K$ with $Z := F^{-1}(0)$.

Let α be a toric contact form supporting ξ. As discussed in [2, section 3] (c.f. [6, 13]) and assuming that $c_1(\xi) = 0$, the index of every closed orbit of α can be defined using a trivialization of the determinant line bundle $\Lambda_{\mathbb{C}}^n \xi$. In this section, we will show how to compute this index for non-contractible orbits using the above symplectic reduction description.

In order to do this, let us first explain how we dealt with contractible orbits in [1]. Consider a periodic orbit γ of α. If γ is contractible then it admits a lift to a closed orbit $\hat{\gamma}$ in Z of a suitable linear unitary Hamiltonian flow on $\mathbb{C}^d$ and [1, Lemma 3.14] establishes that the Robbin-Salamon index of $\hat{\gamma}$ with respect to the usual (constant) symplectic trivialization of $T\mathbb{C}^d$ equals the Robbin-Salamon index of γ with respect to the trivialization induced by a capping disk. (The Robbin-Salamon index [18] coincides with the Conley-Zehnder whenever the periodic orbit is non-degenerate.)

Since Z is simply connected (c.f. [1, section 3]), this lift exists if, and only if, γ is contractible. Thus, if γ is not contractible we do not have a closed lift and need to consider an arc of trajectory in $\mathbb{C}^d$. More precisely, let m be the smallest positive integer such that γ^m is contractible (recall that the fundamental group of M is finite cyclic) and let $\hat{\gamma}$ be a closed lift of γ^m as before. Suppose, without loss of generality, that γ^m has period one so that γ has period $1/m$. Consider the arc $\bar{\gamma} := \hat{\gamma}|_{[0,1/m]}$. We can compute the index of $\bar{\gamma}$ using the constant trivialization of $T\mathbb{C}^d$ but it turns out that,

in general, this index is not equal to the index of γ if $m > 1$. We need to "correct" the symplectic path along $\bar{\gamma}$ as stated in the next proposition.

Before we give the precise statement, let us introduce some terminology. Let g be an element of K such that

$$(6) \qquad \bar{\gamma}(1/m) = g\bar{\gamma}(0) \quad \text{and} \quad g^m = id \, .$$

Since $c_1(\xi) = 0$ every element of K is in $\mathrm{SU}(d)$. Let $\psi : [0, 1/m] \to \mathrm{SU}(d)$ be a smooth map such that $\psi(0) = id$ and $\psi(1/m) = g$. Denote by φ_t^H the Hamiltonian flow on $\mathbb{C}^d$ for which $\bar{\gamma}$ is an arc of trajectory. As mentioned before, φ_t^H is a linear unitary flow.

Proposition 3.1. *The Robbin-Salamon index of $\psi(t)^{-1} \circ \varphi_t^H$ with respect to the constant symplectic trivialization of $T\mathbb{C}^d$ is equal to the Robbin-Salamon index of γ defined using a trivialization of the determinant line bundle $\Lambda_{\mathbb{C}}^n \xi$.*

Proof. Let $\pi : Z \to W$ be the quotient projection. Choose a basis of the dual Lie algebra $\mathfrak{k}^*$ of K and let $F_1, \ldots, F_{d-n-1}$ be the corresponding moment map components. Let $\mathfrak{D}$ be the distribution on Z given by

$$\mathfrak{D} = \mathrm{span}\{X_{F_1}, \ldots, X_{F_{d-n-1}}, \nabla F_1, \ldots, \nabla F_{d-n-1}\},$$

where the gradient is taken with respect to the Euclidean metric. Clearly $\mathfrak{D}$ is a symplectic distribution and we can choose the basis of $\mathfrak{k}^*$ such that

$$\{X_{F_1}, \ldots, X_{F_{d-n-1}}, \nabla F_1, \ldots, \nabla F_{d-n-1}\}$$

defines a symplectic basis of $\mathfrak{D}$. Denote by $\mathfrak{D}^\omega$ the symplectic orthogonal of $\mathfrak{D}$ and note that $d\pi|_{\mathfrak{D}^\omega} : \mathfrak{D}^\omega \to TW$ is a symplectic isomorphism at every point of Z.

Choose a symplectic frame $(u_1, \ldots, u_{2n+2})$ of $\gamma^* TW$ obtained via a global trivialization of $\Lambda_{\mathbb{C}}^n \xi$ (c.f. [2, section 3]). Let $(\hat{u}_i) = (d\pi|_{\mathfrak{D}^\omega})^{-1}(u_i)$ be the frame of $\bar{\gamma}^* \mathfrak{D}^\omega$ given by the lift of (u_i) and define the (ordered) frame $(v_1, \ldots, v_{2d})$ of $\bar{\gamma}^* T\mathbb{C}^d$ as

$$(\hat{u}_1, \ldots, \hat{u}_{2n+2}, X_{F_1}, \ldots, X_{F_{d-n-1}}, \nabla F_1, \ldots, \nabla F_{d-n-1}).$$

Denote by Φ the trivialization of $\bar{\gamma}^* T\mathbb{C}^d$ induced by (v_i). Since $d\varphi_t^H(X_{F_i}) = X_{F_i}$ and $d\varphi_t^H(\nabla F_i) = \nabla F_i$ for every $i \in \{1, \ldots, d-n-1\}$ and $t \in \mathbb{R}$, we

have, by construction, that

$$\mu_{\mathrm{RS}}(\gamma) = \mu_{\mathrm{RS}}(\bar{\gamma}; \Phi)$$

where the index on the left hand side is computed using the frame (u_i).

Now, consider the symplectic frame (w_i) of $\bar{\gamma}^* T\mathbb{C}^d$ given by $w_i(t) = \psi(t)(v_i(0))$ and let Ψ be the trivialization of $\bar{\gamma}^* T\mathbb{C}^d$ induced by (w_i). It is easy to see that

$$\mu_{\mathrm{RS}}(\bar{\gamma}; \Psi) = \mu_{\mathrm{RS}}(\psi(t)^{-1} \circ \varphi_t^H),$$

where the index on the right hand side is computed using the constant trivialization of $T\mathbb{C}^d$. Therefore, we have to show that

$$(7) \qquad\qquad \mu_{\mathrm{RS}}(\bar{\gamma}; \Phi) = \mu_{\mathrm{RS}}(\bar{\gamma}; \Psi).$$

To accomplish this equality, consider the extensions of the frames (v_i) and (w_i) to $\hat{\gamma}^* T\mathbb{C}^d$ given by

$$v_i'(t + j/m) = dg^j(v_i(t)) \text{ and } w_i'(t + j/m) = dg^j(w_i(t))$$

for every $t \in [0, 1/m]$ and $j \in \{0, \dots, m-1\}$. Since $g^j \circ \pi = \pi$, $dg^j(X_{F_i}) = X_{F_i}$ and $dg^j(\nabla F_i) = \nabla F_i$, we have that

$$(8) \qquad\qquad (v_1', \dots, v_{2n+2}', v_{2n+3}', \dots, v_{2d}')$$

$$= (\hat{u}_1', \dots, \hat{u}_{2n+2}', X_{F_1}, \dots, X_{F_{d-n-1}}, \nabla F_1, \dots, \nabla F_{d-n-1}),$$

where $(\hat{u}_i') = (d\pi|_{\mathfrak{D}^\omega})^{-1}(u_i')$ is the lift of the obvious extension (u_i') of the frame (u_i) to $(\gamma^m)^* TW$. It follows from this that

$$(9) \qquad\qquad \mu_{\mathrm{RS}}(\hat{\gamma}; \Phi') = \mu_{\mathrm{RS}}(\gamma^m),$$

where Φ' is the trivialization of $\hat{\gamma}^* T\mathbb{C}^d$ induced by (v_i') and the index on the right hand side is computed using the frame (u_i'). We also have that

$$(10) \qquad\qquad w_i'(t) = \psi'(t)(v_i(0))$$

where $\psi' : [0, 1] \to \mathrm{SU}(d)$ is the extension of ψ given by $\psi'(t + j/m) = g^j \circ \psi(t)$ for every $t \in [0, 1/m]$ and $j \in \{0, \dots, m-1\}$. Note that ψ' is a loop in

$\mathrm{SU}(d)$ based at the identity and therefore,

$$(11) \qquad \mu_{\mathrm{RS}}(\hat{\gamma}; \Psi') = \mu_{\mathrm{RS}}(\hat{\gamma})$$

where Ψ' is the trivialization of $\hat{\gamma}^* T\mathbb{C}^d$ induced by (w_i') and the index on the right hand side is computed using the constant trivialization of $T\mathbb{C}^d$. We have from [1, Lemma 3.4] that the right hand sides of (9) and (11) are equal. Hence, we arrive at

$$(12) \qquad \mu_{\mathrm{RS}}(\hat{\gamma}; \Phi') = \mu_{\mathrm{RS}}(\hat{\gamma}; \Psi').$$

Consider the map $A' : [0,1] \to \mathrm{Sp}(2d)$ uniquely defined by the property that $A_t' v_i'(t) = w_i'(t)$ for every i and t. Using (8), (10) and the fact that $g^j \circ \pi = \pi$, $dg^j(X_{F_i}) = X_{F_i}$ and $dg^j(\nabla F_i) = \nabla F_i$ we conclude that

$$v_i'(j/m) = w_i'(j/m) \text{ for every } i \implies A_{j/m}' = id$$

for every $j \in \{0, \ldots, m\}$. We have that

$$(13) \qquad \mu_{\mathrm{RS}}(\hat{\gamma}; \Phi') = \mu_{\mathrm{RS}}(\hat{\gamma}; \Psi') + 2\mu_{\mathrm{Maslov}}(A')$$

and

$$(14) \qquad \mu_{\mathrm{RS}}(\bar{\gamma}; \Phi) = \mu_{\mathrm{RS}}(\bar{\gamma}; \Psi) + 2\mu_{\mathrm{Maslov}}(A)$$

where $A = A'|_{[0,1/m]}$. We claim that

$$(15) \qquad \mu_{\mathrm{Maslov}}(A') = m\mu_{\mathrm{Maslov}}(A).$$

Indeed,

$$\begin{aligned}
g^j A_t v_i'(t) &= g^j w_i'(t) \\
&= w_i'(t + j/m) \\
&= A_{t+j/m}' v_i'(t + j/m) \\
&= A_{t+j/m}' g^j v_i'(t)
\end{aligned}$$

for every i and consequently

$$(16) \qquad A_{t+j/m}' = g^j A_t g^{-j}$$

for every $t \in [0, 1/m]$ and $j \in \{0, \ldots, m-1\}$. But the conjugation map $P \mapsto g^j P g^{-j}$ is clearly homotopic to the identity and therefore induces the identity

map on $\pi_1(\mathrm{Sp}(2d))$. Thus, all the loops $t \in [0, 1/m] \mapsto A'_{t+j/m}$, $j \in \{0, \ldots, m-1\}$, are homotopic to the loop $t \in [0, 1/m] \mapsto A_t$. Since A' is a concatenation of these loops, we conclude the equality (15).

Finally, we deduce from (12) and (13) that $\mu_{\mathrm{Maslov}}(A') = 0$. Thus, (7) follows immediately from (14) and (15), proving the proposition. $\qquad\square$

As we will now explain, Proposition 3.1 implies in particular that the explicit method to compute the Conley-Zehnder index of any contractible non-degenerate closed toric Reeb orbit described in [1, section 5] also applies to non-contractible orbits.

Given a toric diagram $D = \mathrm{conv}(v_1, \ldots, v_d) \subset \mathbb{R}^n$ and corresponding Gorenstein toric contact manifold (M_D, ξ_D), consider a toric Reeb vector field $R_\nu \in \mathcal{X}(M_D, \xi_D)$ determined by the normalized toric Reeb vector (cf. Proposition 2.9)

$$\nu = (v, 1) \quad \text{with} \quad v = \sum_{j=1}^{d} a_j v_j, \ a_j \in \mathbb{R}^+, \ j = 1, \ldots, d, \ \text{and} \ \sum_{j=1}^{d} a_j = 1.$$

By a small abuse of notation, we will also write

$$R_\nu = \sum_{j=1}^{d} a_j \nu_j,$$

where $\nu_j = (v_j, 1)$, $j = 1, \ldots, n$, are the defining normals of the associated good moment cone $C \subset \mathbb{R}^n$. Making a small perturbation of ν if necessary, we can assume that

the 1-parameter subgroup generated by R_ν is dense in $\mathbb{T}^{n+1}$,

which means that if $v = (r_1, \ldots, r_n)$ then $1, r_1, \ldots, r_n$'s are $\mathbb{Q}$-independent. This is equivalent to the corresponding toric contact form being non-degenerate. In fact, the toric Reeb flow of R_ν on (M_D, ξ_D) has exactly m simple closed orbits $\gamma_1, \ldots, \gamma_m$, all non-degenerate, corresponding to the m edges $E_1, \ldots, E_m$ of the cone C, i.e. one non-degenerate closed simple toric R_ν-orbit for each S^1-orbit of the $\mathbb{T}^{n+1}$-action on (M_D, ξ_D). Equivalently, there is *one non-degenerate closed simple toric R_ν-orbit for each facet of the toric diagram D*. Let γ denote one of those non-degenerate closed simple toric R_ν-orbits and assume without loss of generality that the vertices of the corresponding facet, necessarily a simplex, are $v_1, \ldots, v_n \in \mathbb{Z}^n$. Let

$h \in \mathbb{Z}^n$ be such that

$$\{\nu_1 = (v_1, 1), \ldots, \nu_n = (v_n, 1), \eta = (h, 1)\} \text{ is a } \mathbb{Z}\text{-basis of } \mathbb{Z}^{n+1}.$$

Then R_ν can be uniquely written as

$$R_\nu = \sum_{j=1}^n b_j \nu_j + b\eta, \text{ with } b_1, \ldots, b_n \in \mathbb{R} \text{ and } b = 1 - \sum_{j=1}^n b_j \neq 0,$$

and for contractible γ, as shown in [1, section 5], the Conley-Zehnder index of γ^N, for any $N \in \mathbb{N}$, is given by

$$(17) \qquad \mu_{\mathrm{CZ}}(\gamma^N) = 2 \left(\sum_{j=1}^n \left\lfloor N \frac{b_j}{|b|} \right\rfloor + N \frac{b}{|b|} \sum_{j=1}^d \tilde{\eta}_j \right) + n,$$

where $\tilde{\eta} \in \mathbb{Z}^d$ is an integral lift of $\eta \in \mathbb{Z}^{n+1}$ under the map $\beta : \mathbb{Z}^d \to \mathbb{Z}^{n+1}$ defined by (2). The value in the last coordinate of $\nu_1 = (v_1, 1), \ldots, \nu_d = (v_d, 1)$ and $\eta = (h, 1)$ implies that $\sum_j \tilde{\eta}_j = 1$ and so

$$(18) \qquad \mu_{\mathrm{CZ}}(\gamma^N) = 2 \left(\sum_{j=1}^n \left\lfloor N \frac{b_j}{|b|} \right\rfloor + N \frac{b}{|b|} \right) + n.$$

When γ is not contractible $(\eta, 1) \in \mathbb{Z}^{n+1}$ does not have an integral lift under the map β, but it does have an integral lift modulo $\frac{1}{2\pi}\beta(K) \subset \mathbb{Z}^{n+1}$. In other words, there is $\tilde{\eta} \in \mathbb{Z}^d$ such that

$$\beta(\tilde{\eta}) = \eta - \frac{\beta(g)}{2\pi}, \text{ where } g \in K \cap SU(d) \text{ is characterized by (6)}.$$

It follows from Proposition 3.1 that the index of γ^N is then given by (17) for this $\tilde{\eta}$. Since $g \in SU(d)$ we still have that $\sum_j \tilde{\eta}_j = 1$ and so formula (18) remains valid when γ is not contractible.

4. Contact Betti numbers of $S^*(L_p^3(q))$

In this section we will use formula (18) for the Conley-Zehnder index to compute the contact Betti numbers of the unit cosphere bundle of 3-dimensional lens spaces $(L_p^3(q))$, which by Theorem 2.15 are the Gorenstein toric contact

5-manifolds determined by toric diagrams of the form

$$\mathrm{conv}(v_0 = (0,0), v_1 = (1,0), v_2 = (q,p), v_3 = (q+1,p)) \subset \mathbb{R}^2$$
with $p, q \in \mathbb{N}$ coprime.

Let E_1, E_2, E_3, E_4 be the edges of the toric diagram with endpoints $\{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}$. Consider a normalized toric Reeb vector $R_\varepsilon = (\varepsilon_1, \varepsilon_2, 1)$, with $0 < \varepsilon_1, \varepsilon_2 << 1$, and denote by γ_j its simple closed Reeb orbit corresponding to E_j, $j = 1, 2, 3, 4$. By taking $\varepsilon_1, \varepsilon_2$ such that $1, \varepsilon_1, \varepsilon_2$ are $\mathbb{Q}$-independent, we know that the Reeb flow of R_ε has exactly these four simple closed Reeb orbits and they are all non-degenerate. Either by direct computation using (18) or by using [2, Theorem 1.6], we also know that by considering ε_1 and ε_2 arbitrarily small we will have that $\mu_{CZ}(\gamma_1^N)$ and $\mu_{CZ}(\gamma_2^N)$ are arbitrarily large for any $N \in \mathbb{N}$. Hence, we just need to compute $\mu_{CZ}(\gamma_3^N)$ and $\mu_{CZ}(\gamma_4^N)$ up to arbitrarily large $N \in \mathbb{N}$ and for arbitrarily small $0 < \varepsilon_1, \varepsilon_2 << 1$.

For γ_3, take a vector $\eta_3 = (a_1, a_2, 1) \in \mathbb{Z}^3$ such that $\{\nu_1 = (v_1, 1), \nu_3 = (v_3, 1), \eta_3\}$ is a $\mathbb{Z}$-basis. This condition is written as

$$1 = \det(\nu_1, \nu_3, \eta_3) = (1 - a_1)p + a_2 q.$$

Solving the corresponding system, we find out that

$$R_\varepsilon = b_1 \nu_1 + b_2 \nu_3 + b \eta_3$$

where

$$b_1 = (1 - p + a_2)\frac{b}{p} + \varepsilon_1 - \frac{(q+1)\varepsilon_2}{p},$$
$$b_2 = -a_2 \frac{b}{p} + \frac{\varepsilon_2}{p} \quad \text{and} \quad b = p(1 - \varepsilon_1) + q\varepsilon_2.$$

Therefore, considering also $0 < \varepsilon_2 << \varepsilon_1$ so that $\varepsilon_2/\varepsilon_1$ is arbitrarily small, we can apply (18) to get that

$$\mu_{CZ}\left(\gamma_3^N\right) = 2 + 2\left(\left\lfloor \frac{Nb_1}{b} \right\rfloor + \left\lfloor \frac{Nb_2}{b} \right\rfloor + N\right)$$
$$= 2 + 2\left(\left\lfloor -N + \frac{N(1+a_2)}{p} + \varepsilon' \right\rfloor + \left\lfloor \frac{-Na_2}{p} + \varepsilon'' \right\rfloor + N\right)$$
$$= 2 + 2\left(\left\lfloor \frac{N(1+a_2)}{p} \right\rfloor + \left\lfloor \frac{-Na_2}{p} \right\rfloor\right)$$

up to an arbitrarily large N.

For γ_4, one can check easily that the vector $\eta_4 = (q, p-1, 1)$ is such that $\{\nu_2 = (v_2, 1), \nu_3 = (v_3, 1), \eta_4\}$ is a $\mathbb{Z}$-basis. Solving the system we get

$$R_\varepsilon = (-q + \varepsilon_1)\nu_2 + (-(p-q-1+\varepsilon_1 - \varepsilon_2))\nu_3 + (p - \varepsilon_2)\eta_4.$$

Therefore, considering again $0 < \varepsilon_2 << \varepsilon_1$, we can apply (18) to get that

$$\mu_{CZ}(\gamma_4^N) = 2 + 2\left(\left\lfloor \frac{-Nq}{p} \right\rfloor + \left\lfloor \frac{N(q+1)}{p} - \varepsilon \right\rfloor\right)$$

up to an arbitrarily large N.

The fundamental group $\pi_1(S^*(L_p^3(q)))$ is isomorphic to $\mathbb{Z}^3/\mathcal{N} \cong \mathbb{Z}_p$ where

$$\mathcal{N} = \mathrm{span}_{\mathbb{Z}}\{\nu_0 = (v_0, 1), \nu_1 = (v_1, 1), \nu_2 = (v_2, 1), \nu_3 = (v_3, 1)\}$$

and the isomorphism $\mathbb{Z}^3/\mathcal{N} \cong \mathbb{Z}_p$ is induced by projection on the second coordinate to $\mathbb{Z}$ composed with the projection $\mathbb{Z} \to \mathbb{Z}_p$, $m \mapsto \underline{m}$. Via this isomorphism the classes $[\gamma_3], [\gamma_4]$ in the fundamental group correspond respectively to $\eta_3 + \mathcal{N}, \eta_4 + \mathcal{N}$, which correspond to $\underline{a_2} \in \mathbb{Z}_p$ and $\underline{p-1} \in \mathbb{Z}_p$, respectively. Let $k \in \{1, \ldots, p-1\}$ and take $r \in \{1, \ldots, p-1\}$ such that $r \equiv kq \mod p$. Since $(1 - a_1)p + a_2 q = 1$, $q \equiv a_2^{-1} \mod p$, hence $ra_2 \equiv k \mod p$. Thus $[\gamma_3^{mp+r}] = [\gamma_4^{(m+1)p-k}]$, for $m \in \mathbb{Z}_{\geq 0}$, are both the class corresponding to $\underline{k} \in \mathbb{Z}_p \cong \pi_1(M)$. We can compute

$$\mu_{CZ}\left(\gamma_3^{mp+r}\right) = 2 + 2m + 2\left(\left\lfloor \frac{k+r}{p} \right\rfloor + \left\lfloor -\frac{k}{p} \right\rfloor\right) = \begin{cases} 2m & \text{if } k+r < p \\ 2m+2 & \text{if } k+r \geq p \end{cases}$$

and similarly

$$\mu_{CZ}\left(\gamma_4^{(m+1)p-k}\right) = 4 + 2m + 2\left(\left\lfloor \frac{r}{p} \right\rfloor + \left\lfloor -\frac{r+k}{p} - \varepsilon \right\rfloor\right)$$
$$= \begin{cases} 2m+2 & \text{if } k+r < p \\ 2m & \text{if } k+r \geq p \end{cases}$$

up to an arbitrarily large m. Since for contact 5-manifolds, i.e. $n = 2$, the contact homology degree is equal to the Conley-Zehnder index, for any nontrivial class $\underline{k} \in (\mathbb{Z}_p) \setminus \{0\} \cong \pi_1(S^*(L_p^3(q))) \setminus \{0\}$ we get that

$$cb_*^k(S^*(L_p^3(q))) = \begin{cases} 1 & \text{if } * = 0 \\ 2 & \text{if } * \geq 2 \text{ is even} \\ 0 & \text{otherwise} \end{cases}.$$

For the trivial class we note that $\deg(\gamma_3^{mp}) = 2 + 2m$ and $\deg(\gamma_4^{mp}) = 2m$, up to an arbitrarily large $m \in \mathbb{Z}_{>0}$, and so

$$
cb_*^0(S^*(L_p^3(q))) = \begin{cases} 1 & \text{if } * = 2 \\ 2 & \text{if } * > 2 \text{ is even} \\ 0 & \text{otherwise} \end{cases}.
$$

Hence, we get the following contact Betti numbers:

$$
cb_*(S^*(L_p^3(q))) = \begin{cases} p - 1 & \text{if } * = 0 \\ 2p - 1 & \text{if } * = 2 \text{ is even} \\ 2p & \text{if } * \geq 2 \text{ is even} \\ 0 & \text{otherwise} \end{cases}.
$$

Note that these contact Betti numbers do not detect the value of q and do not distinguish any non-trivial class in $\pi_1(S^*(L_p^3(q)))$.

5. Contact Betti numbers of higher dimensional lens spaces and applications

In this section we will use formula (18) for the Conley-Zehnder index to compute the contact Betti numbers cb_* of Gorenstein contact lens spaces

$$
L_p^{2n+1}(\alpha_0, -\alpha_1, \ldots, -\alpha_{n-1}, 1)
$$

with $n, p \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{Z}$, $\alpha_0 := \alpha_1 + \cdots + \alpha_{n-1} - 1$ and $\gcd(\alpha_j, p) = 1$ for $j = 0, \ldots, n-1$. As illustrative contact topology applications, we will then prove Propositions 1.1 and 1.2. To simplify notation, we will denote these lens spaces by $L_p^{2n+1}(\bar{\alpha})$, with $\bar{\alpha} = (-\alpha_1, \ldots, -\alpha_{n-1}) \in \mathbb{Z}^{n-1}$, and their contact Betti numbers by $cb_*(n, p, \bar{\alpha})$ and $cb_*^{\underline{k}}(n, p, \bar{\alpha})$, where $\underline{k} \in \mathbb{Z}_p \cong \pi_1(L_p^{2n+1}(\bar{\alpha}))$.

5.1. Contact Betti numbers

As we saw in § 2.3.3, the moment cone $C \subset \mathbb{R}^{n+1}$ of $L_p^{2n+1}(\bar{\alpha})$ has defining normals

$$
\nu_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}, \; \nu_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \text{ for } j = 1, \ldots, n-1, \; \nu_n = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \\ p \\ 1 \end{bmatrix}.
$$

Let E_j denote the edge of C given by the intersection of all its facets except the one with normal ν_j. Consider a normalized toric Reeb vector $R_\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-1}, \varepsilon_n, 1)$, with $0 < \varepsilon_1, \ldots, \varepsilon_n << 1$, and denote by γ_j its simple closed Reeb orbit corresponding to E_j, $j = 0, \ldots, n$. By taking $\varepsilon_1, \ldots, \varepsilon_n$ such that $1, \varepsilon_1, \ldots, \varepsilon_n$ are $\mathbb{Q}$-independent, we know that the Reeb flow of R_ε has exactly these $n + 1$ simple closed Reeb orbits and they are all non-degenerate. Again, either by direct computation using (18) or by using [2, Theorem 1.6], we also know that by considering $\varepsilon_1, \ldots \varepsilon_n$ arbitrarily small we will have that $\mu_{CZ}(\gamma_j^N)$, $j = 1, \ldots, n$, are arbitrarily large for any $N \in \mathbb{N}$. Hence, we just need to compute $\mu_{CZ}(\gamma_0^N)$ up to arbitrarily large $N \in \mathbb{N}$ and for arbitrarily small $0 < \varepsilon_1, \ldots, \varepsilon_n << 1$.

Take a vector $\eta_0 = (a_1, \ldots, a_{n-1}, a_n, 1) \in \mathbb{Z}^{n+1}$ that completes $\nu_1, \ldots, \nu_n$ to a $\mathbb{Z}^{n+1}$-basis, that is, such that the following equation holds:

$$
1 = \begin{vmatrix} 1 & 0 & & 0 & \alpha_1 & a_1 \\ 0 & 1 & \ldots & 0 & \alpha_2 & a_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & & 1 & \alpha_{n-1} & a_{n-1} \\ 0 & 0 & \ldots & 0 & p & a_n \\ 1 & 1 & & 1 & 1 & 1 \end{vmatrix}
$$

$$
= p \begin{vmatrix} 1 & 0 & & 0 & a_1 \\ 0 & 1 & \ldots & 0 & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & a_{n-1} \\ 1 & 1 & & 1 & 1 \end{vmatrix} - a_n \begin{vmatrix} 1 & 0 & & 0 & \alpha_1 \\ 0 & 1 & \ldots & 0 & \alpha_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & \alpha_{n-1} \\ 1 & 1 & & 1 & 1 \end{vmatrix}
$$

$$= -p(1 - a_1 - \ldots - a_{n-1}) - a_n(1 - \alpha_1 - \ldots - \alpha_{n-1})$$

$$= \left(1 - \sum_{j=1}^{n-1} a_j\right) p + \alpha_0 a_n \, .$$

Thus we can take $a_n \equiv \alpha_0^{-1} \mod p$ and a_j, $j = 1, \ldots, n-1$, in a way that the above equality holds.

We now want to write R_ε in the basis $\{\nu_1, \ldots, \nu_n, \eta_0\}$, i.e. we want to find b_j, $j = 1, \ldots, n$, and b such that

$$R_\varepsilon = b_1 \nu_1 + \ldots + b_n \nu_n + b \eta_0 \, .$$

We get the following system

$$\begin{cases} \varepsilon_j = b_j + \alpha_j b_n + a_j b & j = 1, \ldots, n-1 \\ \varepsilon_n = p b_n + a_n b \\ \sum_{j=1}^{n} b_j + b = 1 \, . \end{cases}$$

The solution for this system is given by

$$b_n = -\frac{b a_n}{p} + \frac{\varepsilon_n}{p} \, , \quad b_j = \frac{\alpha_j a_n b}{p} - a_j b + \varepsilon_j - \frac{\alpha_j \varepsilon_n}{p} \, , 1 \leq j \leq n-1 \, ,$$

$$\text{and} \quad b = p\left(1 - \sum_{j=1}^{n-1} \varepsilon_j\right) + \alpha_0 \varepsilon_n.$$

Therefore, considering also $0 < \varepsilon_n << \varepsilon_j$ so that $\varepsilon_n / \varepsilon_j$ is arbitrarily small, for all $j = 1, \ldots, n-1$, we can apply (18) to get that

$$\mu_{CZ}\left(\gamma_0^N\right) = n + 2\left(\sum_{j=1}^{n-1}\left\lfloor \frac{N b_j}{b} \right\rfloor + \left\lfloor \frac{N b_n}{b} \right\rfloor + N\right)$$

$$= n + 2\left(\sum_{j=1}^{n-1}\left\lfloor \frac{N \alpha_j a_n}{p} - N a_j + \varepsilon_j' \right\rfloor + \left\lfloor -\frac{N a_n}{p} + \varepsilon_n' \right\rfloor + N\right)$$

$$= n + 2\left(\sum_{j=1}^{n-1}\left\lfloor \frac{N \alpha_j a_n}{p} \right\rfloor + \left\lfloor -\frac{N a_n}{p} \right\rfloor + N\left(1 - \sum_{j=1}^{n-1} a_j\right)\right)$$

$$= n + 2 \left(\sum_{j=1}^{n-1} \left\lfloor \frac{N\alpha_j a_n}{p} \right\rfloor + \left\lfloor -\frac{N a_n}{p} \right\rfloor + N \left(\frac{1 - \alpha_0 a_n}{p} \right) \right)$$

$$= n + 2 \left(\frac{N}{p} - \sum_{j=1}^{n-1} \left\{ \frac{N\alpha_j a_n}{p} \right\} - \left\{ -\frac{N a_n}{p} \right\} \right)$$

up to an arbitrarily large N. Recall that $\{x\} = x - \lfloor x \rfloor$ for any $x \in \mathbb{R}$.

Let us define the (SFT) degree function $g : \mathbb{N} \to \mathbb{Z}$ by

$$(19) \qquad g(N) \equiv \mu_{CZ}(\gamma_0^N) + n - 2$$

$$= 2 \left(\frac{N}{p} + (n-1) - \sum_{j=1}^{n-1} \left\{ \frac{N\alpha_j a_n}{p} \right\} - \left\{ -\frac{N a_n}{p} \right\} \right).$$

We can observe the following:

- $g(N) \geq 0$ for every $N > 0$;
- $g(p) = 2n$;
- $g(N) \leq 2(n-1)$ for $N < p$;
- $g(N+p) = g(N) + 2$ for every $N > 0$.

By the last property, it is clear that

$$cb_{2j}(n, p, \bar{\alpha}) = \# \{ N \in \{1, \ldots, p\} : g(N) \leq 2j \}$$

since the congruence class $\{kp + N : k \in \mathbb{Z}_0^+\}$ contributes to the homology of degree $2j$ with 1 if $g(N) \leq 2j$ and with 0 otherwise. In particular

$$cb_{2(n-1)}(n, p, \bar{\alpha}) = p - 1 \quad \text{and} \quad cb_{2j}(n, p, \bar{\alpha}) = p \text{ if } j \geq n.$$

Note also that, since $\{x\} + \{-x\} = 1$ if $x \notin \mathbb{Z}$, we have $g(N) + g(p - N) = 2(n-1)$ for $N = 1, \ldots, p - 1$. Thus

$$g(N) \leq 2j \quad \text{if and only if} \quad g(p - N) > 2(n - 2 - j)$$

and the following symmetry property follows:

$$cb_{2j}(n, p, \bar{\alpha}) + cb_{2(n-2-j)}(n, p, \bar{\alpha}) = p - 1 \quad \text{for } j = 0, 1, \ldots, n - 2.$$

For information on the decomposition induced by homotopy classes in the cyclic fundamental group of these Gorenstein contact lens spaces, note that

γ_0 represents a generator and we get that

$$cb_{2j}^{[\gamma_0^N]}(n, p, \bar{\alpha}) = \begin{cases} 1 & \text{if } 2j \geq \widetilde{g}(N) \\ 0 & \text{otherwise,} \end{cases}$$

where $\widetilde{g} : \mathbb{N} \to \mathbb{Z}$ is defined by

(20)
$$\begin{aligned} & \widetilde{g}(N) = g(N) \quad \text{if } N = 1, \ldots, p, \\ & \text{and} \quad \widetilde{g}(N + p) = \widetilde{g}(N) \quad \text{for all } N \in \mathbb{N}. \end{aligned}$$

We summarize these facts in the following proposition.

Proposition 5.1. *We have that* $cb_j(n, p, \bar{\alpha}) = 0$ *whenever j is odd or $j < 0$ and*

$$cb_{2j}(n, p, \bar{\alpha}) = \# \{N \in \{1, \ldots, p\} : g(N) \leq 2j\} \text{ for all } j \geq 0,$$

where g is the degree function defined by (19). In particular:

(i) $cb_{2j}(n, p, \bar{\alpha}) = p$ *for all $j \geq n$;*

(ii) $cb_{2(n-1)}(n, p, \bar{\alpha}) = p - 1$;

(iii) $cb_{2j}(n, p, \bar{\alpha}) + cb_{2(n-2-j)}(n, p, \bar{\alpha}) = p - 1$ *for $j = 0, 1, \ldots, n - 2$.*

Moreover,

$$cb_{2j}^{[\gamma_0^N]}(n, p, \bar{\alpha}) = \begin{cases} 1 & \text{if } 2j \geq \widetilde{g}(N) \\ 0 & \text{otherwise,} \end{cases}$$

where $\widetilde{g}$ is defined by (20).

When the decomposition induced by homotopy classes in the cyclic fundamental group is not relevant, the following definition provides a useful compact way of presenting the relevant information provided by the contact Betti numbers of a Gorenstein contact lens space.

Definition 5.2. *The* contact Betti numbers sequence *of a Gorenstein contact lens space $L_p^{2n+1}(\bar{\alpha})$ is defined as*

$$cbs(n, p, \bar{\alpha}) := \big(cb_0(n, p, \bar{\alpha}), cb_2(n, p, \bar{\alpha}), \ldots, cb_{2(n-1)}(n, p, \bar{\alpha})\big) \in \mathbb{N}_0^n.$$

Example 5.3. *When $n = 2$ we have that p must be odd and*

$$cbs(2, p, \bar{\alpha}) = \left(\frac{p-1}{2}, p - 1 \right) ,$$

i.e. for 5-dimensional Gorenstein contact lens spaces $L_p^5(\bar{\alpha})$ the contact Betti numbers sequence is independent of the weight $\bar{\alpha} = (\alpha_1)$.

Example 5.4. *To see that when $n \geq 3$ the contact Betti numbers sequence does depend on $\bar{\alpha}$, let us compute it for $L_5^7(-3, 1, 1, 1)$ and $L_5^7(1, -1, -1, 1)$.*

For $L_5^7(-3, 1, 1, 1)$ we have that $n = 3$, $p = 5$, $\alpha_1 = \alpha_2 = -1$. Then $\alpha_0 = -3$ and we can take $a_n = 3 \equiv_5 (-3)^{-1}$ in (19) to get that the degree function is given by

$$g(N) = 2 \left(\frac{N}{5} + 2 - 3 \left\{ \frac{-3N}{5} \right\} \right) .$$

This implies that $g(1) = g(4) = 2$, $g(2) = 0$, $g(3) = 4$ and $g(5) = 6$. It follows that the contact Betti numbers sequence is

$$cbs(3, 5, (-1, -1)) = (1, 3, 4) .$$

For $L_5^7(1, -1, -1, 1)$ we have that $n = 3$, $p = 5$, $\alpha_1 = \alpha_2 = 1$. Then $\alpha_0 = 1$ and we can take $a_n = 1 \equiv_5 1^{-1}$ in (19) to get that the degree function is given by

$$g(N) = 2 \left(\frac{N}{5} + 2 - 2 \left\{ \frac{N}{5} \right\} - \left\{ -\frac{N}{5} \right\} \right) .$$

This implies that $g(1) = g(2) = g(3) = g(4) = 2$ and $g(5) = 6$. It follows that the contact Betti numbers sequence is

$$cbs(3, 5, (1, 1)) = (0, 4, 4) .$$

Example 5.5. *When $n + 1 = kp$ for some $k \in \mathbb{N}$, the contact lens space*

$$L_p^{2n+1}(1, \ldots, 1)$$

is Gorenstein. Indeed, we can take $\alpha_j = -1$, $j = 1, \ldots, n - 1$, $\alpha_0 = -n = 1 - (n + 1) \equiv_p 1$ and $a_n = 1$. The degree function (19) is given by

$$g(N) = 2 \left(\frac{N}{p} + (n - 1) - n \left\{ -\frac{N}{p} \right\} \right) .$$

When $N \in \{1, 2, \ldots, p-1\}$ we get

$$g(N) = 2\left(\frac{N}{p} + (n-1) - n\left(1 - \frac{N}{p}\right)\right) = 2\left(\frac{N(n+1)}{p} - 1\right) = 2(kN - 1).$$

It follows that

$$cb_{2*}(kp - 1, p, (-1, \ldots, -1)) = j$$

whenever $j = 1, \ldots, p-1$ and $kj - 1 \leq * < k(j+1) - 1$. For example, this implies that the contact Betti numbers sequence of Gorenstein real projective spaces, i.e. $\mathbb{RP}^{2n+1}$ with n odd, is

$$cbs(n, 2, (-1, \ldots, -1)) = (0, \ldots, 0, 1, \ldots, 1)$$
$$\text{with } (n-1)/2 \text{ zeros and } (n+1)/2 \text{ ones.}$$

It also implies that the contact Betti numbers sequence of the prequantization of $(\mathbb{CP}^n, [\omega] = 2\pi c_1(\mathbb{CP}^n))$, i.e. $L_p^{2n+1}(1, \ldots, 1)$ with $p = n + 1$ (hence $k = 1$), is

$$cbs(n, n+1, (-1, \ldots, -1)) = (1, 2, \ldots, n).$$

5.2. Applications

We will now prove Propositions 1.1 and 1.2.

To prove Proposition 1.1, we will show that the Gorenstein contact lens spaces

$$L_5^{13}(1, 1, 1, 1, 2, -2, 1) \quad \text{and} \quad L_5^{13}(1, -1, -1, -1, -2, -2, 1),$$

which are diffeomorphic as naturally oriented manifolds (they have the same weights up to an even number of sign changes), have different contact Betti numbers sequences.

For $L_5^{13}(1, 1, 1, 1, 2, -2, 1)$ we have that $n = 6$, $p = 5$, $\bar{\alpha} = (-1, -1, -1, -2, 2)$ and $\alpha_0 = -4 \equiv_5 1$. We can take $a_6 = 1 \equiv_5 1^{-1}$ in (19) to get that the degree function is given by

$$g(N) = 2\left(\frac{N}{5} + 5 - 4\left\{\frac{-N}{5}\right\} - \left\{\frac{-2N}{5}\right\} - \left\{\frac{2N}{5}\right\}\right).$$

This implies that $g(1) = 2$, $g(2) = 4$, $g(3) = 6$, $g(4) = 8$ and $g(5) = 12$. It follows that the contact Betti numbers sequence is

$$cbs(6, 5, (-1, -1, -1, -2, 2)) = (0, 1, 2, 3, 4, 4).$$

For $L_5^{13}(1, -1, -1, -1, -2, -2, 1)$ we have that $n = 6$, $p = 5$, $\bar{\alpha} = (1, 1, 1, 2, 2)$ and $\alpha_0 = 6 \equiv_5 1$. We can take $a_6 = 1 \equiv_5 1^{-1}$ in (19) to get that the degree function is given by

$$g(N) = 2 \left(\frac{N}{5} + 5 - 3 \left\{ \frac{N}{5} \right\} - 2 \left\{ \frac{2N}{5} \right\} - \left\{ \frac{-N}{5} \right\} \right) .$$

This implies that $g(1) = g(3) = 6$, $g(2) = g(4) = 4$ and $g(5) = 12$. It follows that the contact Betti numbers sequence is

$$cbs(6, 5, (1, 1, 1, 2, 2)) = (0, 0, 2, 4, 4, 4) .$$

To prove Proposition 1.2, we will use the decomposition of the contact Betti numbers sequence by homotopy classes in the fundamental group to show that

$$L_5^{15}(1, 1, 1, 2, -2, -2, -2, 1)$$

is an example of a Gorenstein contact lens space with an homotopy class of orientation preserving diffeomorphisms that does not contain any contactomorphism. Before analysing this particular example in detail, let us make some more general considerations.

Let L_p denote a lens space with $\pi_1(L_p) \cong \mathbb{Z}_p$. Any diffeomorphism $f : L_p \to L_p$ induces an automorphism $f_* : \mathbb{Z}_p \to \mathbb{Z}_p$. Such an automorphism must be multiplication by k for some $k \in (\mathbb{Z}_p)^*$. It follows from the last part of Proposition 5.1 that if such an automorphism is induced by a co-orientation preserving contactomorphism of L_p then $\widetilde{g}(kN) = \widetilde{g}(N)$ for every $N \in \mathbb{N}$, where $\widetilde{g}$ is defined by (20). This is a restrictive condition. As the examples above and in the previous subsection illustrate well, it often implies that $k = 1$. However, it is also somewhat restrictive to find lens spaces L_p with non-trivial automorphisms of $\mathbb{Z}_p$ induced by orientation preserving diffeomorphisms (see [15] and [7]). In fact, for p odd, the automorphisms induced by orientation preserving homotopy equivalences correspond to multiplication by $k \in (\mathbb{Z}_p)^*$ such that $k^{n+1} \equiv 1 \mod p$, where $\dim L_p = 2n + 1$. To characterize the homotopy equivalences that can be represented by diffeomorphisms, let $\ell_0, \ell_1, \ldots, \ell_n \in \mathbb{Z}$ be the weights of L_p and consider $\Delta(L_p) \in \mathbb{Q}[\mathbb{Z}_p]$ defined by

$$\Delta(L_p)(t) = \prod_{j=0}^{n} (t^{r_j} - 1) \in \mathbb{Q}[\mathbb{Z}_p] ,$$

where t is a (multiplicative) generator of $\mathbb{Z}_p$ and $r_j \ell_j \equiv 1 \mod p$. Any automorphism of $\mathbb{Z}_p$ extends to an automorphism of $\mathbb{Q}[\mathbb{Z}_p]$ and the class $[f]$

of homotopy equivalences can be represented by an orientation preserving diffeomorphism if and only if $f_*\Delta(L_p) = t^u\Delta(L_p)$ for some $u \in \mathbb{Z}$ (see [7, Proposition 3.3]).

Consider now the Gorenstein contact lens space $L_5^{15}(1, 1, 1, 2, -2, -2, -2, 1)$. We have that $n = 7$, $p = 5$, $\bar{\alpha} = (-1, -1, -2, 2, 2, 2)$ and $\alpha_0 = 1$. We can take $a_7 = 1 \equiv_5 1^{-1}$ in (19) to get that the degree function is given by

$$g(N) = 2\left(\frac{N}{5} + 6 - 3\left\{\frac{2N}{5}\right\} - 3\left\{\frac{-N}{5}\right\} - \left\{\frac{-2N}{5}\right\}\right).$$

This implies that $g(1) = g(2) = 4$, $g(3) = g(4) = 8$ and $g(5) = 14$, which means that there is no $1 \neq k \in (\mathbb{Z}_5)^*$ such that $\tilde{g}(kN) = \tilde{g}(N)$ for every $N \in \mathbb{N}$. Hence, any co-orientation preserving contactomorphism of $L_5^{15}(1, 1, 1, , 2, -2, -2, -2, 1)$ acts trivially on its fundamental group. Since the contactomorphism induced by complex conjugation on all coordinates reverses the co-orientation and acts by -1 on the fundamental group, we conclude that any contactomorphism acts by multiplication by $\pm 1 \in (\mathbb{Z}_5)^*$. However, this lens space admits orientation preserving diffeomorphisms that act by multiplication by $2 \in (\mathbb{Z}_5)^*$ on its fundamental group. In fact note that $2^8 = 256 \equiv_5 1$, while

$$\Delta(t) = (t - 1)^4(t^3 - 1)(t^{-3} - 1)^3 = \frac{(-1)^3}{t^4}(t - 1)^4(t^3 - 1)^4$$

and

$$\Delta(t^2) = (t^2 - 1)^4(t - 1)(t^{-1} - 1)^3 = (-1)^7(t - 1)^4(t^3 - 1)^4 = t^4\Delta(t).$$

An explicit example of such a diffeomorphism is the one induced by the following linear automorphism of $\mathbb{C}^8$:

$$(z_0, z_1, z_2, z_3, z_4, z_5, z_6, z_7) \mapsto (\bar{z}_3, z_4, z_5, z_7, \bar{z}_0, \bar{z}_1, \bar{z}_2, z_6).$$

Appendix A. Total Chern class of lens spaces

The following proposition is elementary and certainly well known, although we could not find an explicit reference to it in the literature.

Proposition A.1. *Under natural isomorphisms $H^{2j}(L_p^{2n+1}(\ell_0, \ell_1, \ldots, \ell_n);$ $\mathbb{Z}) \cong \mathbb{Z}_p$, for $j = 1, \ldots, n$, the Chern classes of the canonical contact structure ξ on*

$$L_p^{2n+1}(\ell_0, \ell_1, \ldots, \ell_n)$$

are given by

$$c_j = \sigma_j \bmod p, \, j = 1, \ldots n,$$

where $\sigma_j = j$-th elementary symmetric polynomial of $\ell_0, \ell_1, \ldots, \ell_n$.

Proof. This is a particular case of Proposition 2.16 in [1] and its proof, together with the fact that the j-th Chern class of a sum of line bundles is the j-th elementary symmetric polynomial of their first Chern classes. The main points are the following:

(i) The quotient map from S^{2n+1} to $L_p^{2n+1}(\ell_0, \ell_1, \ldots, \ell_n)$ is a principal $\mathbb{Z}_p$-bundle and its classifying map $f : L_p^{2n+1}(\ell_0, \ell_1, \ldots, \ell_n) \to B\mathbb{Z}_p$ induces an isomorphism $f^* : H^2(B\mathbb{Z}_p; \mathbb{Z}) \to H^2(L_p^{2n+1}(\ell_0, \ell_1, \ldots, \ell_n); \mathbb{Z})$.

(ii) The $\mathbb{Z}_p$-action on $\mathbb{C}^{n+1}$ gives rise to an associated vector bundle $S^{2n+1} \times_{\mathbb{Z}_p} \mathbb{C}^{n+1}$ over $L_p^{2n+1}(\ell_0, \ell_1, \ldots, \ell_n)$, and

$$c_1(\xi) = c_1(S^{2n+1} \times_{\mathbb{Z}_p} \mathbb{C}^{n+1}) = f^* c_1(E\mathbb{Z}_p \times_{\mathbb{Z}_p} \mathbb{C}^{n+1}).$$

(iii) $H^2(B\mathbb{Z}_p; \mathbb{Z}) \cong \mathbb{Z}_p \cong$ character group of $\mathbb{Z}_p$ and $c_1(E\mathbb{Z}_p \times_{\mathbb{Z}_p} \mathbb{C}^{n+1})$ is the sum of the characters that determine the $\mathbb{Z}_p$-action on $\mathbb{C}^{n+1}$.

$\square$

Consider the polynomial $c(\xi) \in \mathbb{Z}_p[x]$ given by

$$c(\xi) = \prod_{i=0}^{n} (1 + \ell_i x) = 1 + \sum_{i=1}^{n+1} \sigma_i x^i \in \mathbb{Z}_p[x].$$

The total Chern class of ξ is given by $[c(\xi)] \in \mathbb{Z}_p[x]/x^{n+1}$ and Proposition A.1 says that ξ has total Chern class equal to zero if and only if we have that

$$\prod_{i=0}^{n} (1 + \ell_i x) = 1 + \sigma_{n+1} x^{n+1} = 1 + \left(\prod_{i=0}^{n} \ell_i \right) x^{n+1} \in \mathbb{Z}_p[x].$$

This equation is quite restrictive but it does have some solutions that provide a few interesting examples of lens spaces with canonical contact structure

with zero total Chern class. For lens spaces of the form

$$L_p^{2n+1}(1, 1, \ldots, 1) \,, \text{ i.e. } \ell_0 = \ell_1 = \cdots = \ell_n = 1 \,,$$

we have that

$$[c(\xi)] = 0 \Leftrightarrow (1 + x)^{n+1} = 1 + x^{n+1} \in \mathbb{Z}_p[x]$$
$$\Leftrightarrow n + 1 = p^k \text{ for } p \text{ prime and some } k \in \mathbb{N}.$$

Hence, any lens space of the form $L_p^{2p^k-1}(1, 1, \ldots, 1)$, with p prime, has zero total Chern class. Another set of examples arises from the fact that, for prime p, we have

$$1 - x^{p-1} = \prod_{i=0}^{p-2}(1 + (i + 1)x) \in \mathbb{Z}_p[x] \,.$$

Hence, any lens space of the form $L_p^{2p-3}(1, 2, \ldots, p-1)$, with p prime, also has zero total Chern class.

Despite these and some other few examples of lens spaces with canonical contact structure with zero total Chern class, they do not provide relevant examples for the type of results considered in Propositions 1.1 and 1.2. In fact, at least for prime p, there are no examples of lens spaces where the type of results considered in Propositions 1.1 and 1.2 do not follow from total Chern class considerations, the reason being that, as we will now see, the canonical contact structure of a lens space $L_p^{2n+1}(\ell_0, \ell_1, \ldots, \ell_n)$ is completely determined by the diffeomorphism type of the lens space and the total Chern class of the contact structure.

Proposition A.2. *Let p be a prime and f be an orientation preserving diffeomorphism of $L_p = L_p^{2n+1}(\ell_0, \ldots, \ell_n)$. Suppose that $f_* : \pi_1(L_p) \to \pi_1(L_p)$ is multiplication by $k \in \mathbb{Z}_p^*$. If f preserves the total Chern class of the canonical contact structure of L_p, then there is a permutation τ of the set $\{0, 1, \ldots, n\}$ such that*

$$\ell_{\tau(i)} \equiv k\ell_i \mod p.$$

In particular, $\tilde{g}$ defined by (20) satisfies $\tilde{g}(kN) = \tilde{g}(N)$ for every N, so the contact Betti numbers and their decomposition by homotopy classes of the fundamental group do not impose any restriction for f to be homotopic to a contactomorphism of the canonical contact structure of L_p.

Proof. The condition that f preserves the total Chern class is equivalent to $k^i\sigma_i \equiv \sigma_i \mod p$ for $i = 1, \ldots, n$. Moreover, since f is an orientation preserving diffeomorphism we also have $k^{n+1} \equiv 1 \mod p$. Hence the congruence above also holds for $i = n + 1$. Therefore

$$\prod_{i=0}^{n} (1 + \ell_i x) = 1 + \prod_{i=1}^{n+1} \sigma_i x^i = 1 + \prod_{i=1}^{n+1} k^i \sigma_i x^i = \prod_{i=0}^{n} (1 + k\ell_i x) \ .$$

The existence of the permutation τ follows from $\mathbb{Z}_p$ being a field. $\square$

Proposition A.3. *Let p be a prime. Let f be an orientation preserving diffeomorphism*

$$f \colon L_p = L_p^{2n+1}(\ell_0, \ldots, \ell_n) \to L_p^{2n+1}(\ell'_0, \ldots, \ell'_n) = L'_p.$$

If f preserves the total Chern class of the corresponding canonical contact structures ξ and ξ', i.e. $f^*[c(\xi')] = [c(\xi)] \in \mathbb{Z}_p[x]/x^{n+1}$, then there is $k \in \mathbb{Z}_p^*$ and a permutation τ of the set $\{0, 1, \ldots, n\}$ such that

$$\ell'_{\tau(i)} \equiv k\ell_i \mod p.$$

Proof. Exactly the same as the proof of the previous proposition once we fix isomorphisms $\pi_1(L_p) \cong \mathbb{Z}_p \cong \pi_1(L'_p)$. $\square$

Corollary A.4. *Let p be a prime. Let f be an orientation preserving diffeomorphism*

$$f \colon L_p = L_p^{2n+1}(\ell_0, \ldots, \ell_n) \to L_p^{2n+1}(\ell'_0, \ldots, \ell'_n) = L'_p.$$

Then f is homotopic to a contactomorphism of the corresponding canonical contact structures ξ and ξ' if and only if $f^*[c(\xi')] = [c(\xi)] \in \mathbb{Z}_p[x]/x^{n+1}$.

Proof. This follows from the previous proposition together with the fact that diffeomorphisms of lens spaces are homotopic if and only if they induce the same map on the fundamental group (because inducing the same map on π_1 implies they induce the same map on cohomology due to its ring structure; see also Proposition 3.2 in [7]). $\square$

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answer is given in Theorem 2.15 and provides one of the two sets of examples discussed in detail in this paper. We are also grateful to one referee for making us look at the total Chern class of the canonical contact structures on lens spaces and to Gustavo Granja for helping us with that.

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