Math. Res. Lett.
Volume 29, Number 1, 43–58, 2022

Four manifolds with no smooth spines

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Let W be a compact smooth orientable 4-manifold that deformation retract to a PL embedded closed surface. One can arrange the embedding to have at most one non-locally-flat point, and near the point the topology of the embedding is encoded in the singularity knot K. If K is slice, then W has a smooth spine, i.e., deformation retracts onto a smoothly embedded surface. Using the obstructions from the Heegaard Floer homology and the high-dimensional surgery theory, we show that W has no smooth spine if K is a knot with nonzero Arf invariant, a nontrivial L-space knot, the connected sum of nontrivial L-space knots, or an alternating knot of signature < -4. We also discuss examples where the interior of W is negatively curved.

1. Introduction

A *spine* is a topological (not necessarily locally flat), compact, boundaryless submanifold that is a strong deformation retract of the ambient manifold. A spine is *smooth or* PL if the submanifold has this property.

Examples of 4-manifolds that are homotopy equivalent to closed sufaces but have no PL spines can be found in [Mat75, MV79, LL19, HP]. It is shown in [Ven98] that an example in [MV79] does not even have a topological spine. Some 4-manifolds with topological spines and no PL spines can be found in [KR20]. The present paper constructs 4-manifolds with PL spines and no smooth spines.

In this section W denotes a compact oriented smooth 4-manifold with a PL spine S homeomorphic to a closed oriented connected surface. By a standard argument S can be moved by a PL homeomorphism to a spine with at most one non-locally-flat point; henceforth we assume that S has this property. If S is locally flat, then the submanifold S is smoothable [RS68, Corollary 6.8]. Otherwise, S intersects the link of the non-locally-flat point in a *singularity knot* K. If K is smoothly slice, then replacing the cone on K in S with a smoothly embedded disk in W gives a smooth spine of W.

Belegradek was partially supported by the Simons Foundation grant 524838.

Conversely, if Σ is an oriented connected surface with one boundary component, then attaching $\Sigma \times D^2$ to the 4-ball along the knot K in its boundary with framing r gives a compact oriented 4-manifold with a PL spine homeomorphic to $\Sigma/\partial\Sigma$, which has normal Euler number r and singularity knot K. If $\Sigma = D^2$, the 4-manifold is denoted by K^r and called a *knot trace.*

Examples of non-slice singularity knot such that W has a smooth spine come from exotic knot traces. Namely [Akb93, Theorem A] describes knots K_1, K_2 such that K_1 is slice, K_2 is not slice, and K_1^r, K_2^r are diffeomorphic for some r. We refer to [HMP, HP, FMN$^+$] for a recent study of relations between invariants of knot traces and knot concordance.

Cappell and Shaneson [CS76] developed a surgery-theoretic criterion that can help decide when a manifold with a PL spine of dimension ≥ 3 and codimension 2 also admits a locally flat spine. Applying the criterion to $W \times S^1$ we prove

Theorem 1.1. *If W is a compact oriented smooth 4-manifold that has a* PL *spine whose singularity knot has nonzero Arf invariant, then W contains no smooth spine.*

Tye Lidman and Daniel Ruberman asked us if the generalized Rokhlin invariant can be used to give a purely 4-dimensional proof of Theorem 1.1. This was done in [Sae92, Theorem 3.1] in the case when W is a homotopy S^2 with finite $H_1(\partial W)$. We leave the question to an interested reader.

The criterion of [CS76] also gives a weak converse of Theorem 1.1: If K has zero Arf invariant, then $W \times S^1$ has a smooth spine, see Remark 3.2.

If W has two PL spines with regular neighborhoods R_1, R_2 in the interior of W, then there is a homology cobordism between the boundaries ∂R_1, ∂R_2 obtained by gluing $W \setminus \mathrm{Int}(R_1)$ and $W \setminus \mathrm{Int}(R_2)$ along ∂W. The Heegaard-Floer d-invariants d_{top}, d_{bot} are preserved under homology cobordisms. Furthermore, one can express the d-invariants of ∂R_1, ∂R_2 via singularity knots of their spines, and for some knots the d-invariants can be explicitly computed, which gives the following.

Theorem 1.2. *If W is a compact oriented smooth 4-manifold that has a* PL *spine whose singularity knot is a nontrivial L-space knot, the nontrivial connected sum of nontrivial L-space knots, or an alternating knot of signature < -4, then W contains no smooth spine.*

Recall that a knot $K \subset S^3$ is an *L-space knot* if there is an integer $n > 0$ such that the n-framed surgery on the knot is an L-space [OS05, Definition 1.1]. For example, torus knots are L-space knots [OS05, p.1285].

The d-invariants obstruction applies to some topologically slice knots in [HKL16], which gives

Corollary 1.3. *For any $g, e \in \mathbb{Z}$ with $g \geq 0$ there exists a compact smooth oriented 4-manifold with no smooth spine and a topological locally flat spine that is an oriented closed genus g surface with normal Euler number e.*

We were led to the subject of this paper while thinking of examples in [GLT88, Kui88] of oriented hyperbolic 4-manifolds with PL spines. Each of these manifolds is a quotient of the hyperbolic space $\mathbb{H}^4$ by a Kleinian group Γ_0 which is a torsion-free finite index subgroup in a certain discrete group Γ of orientation preserving isometries of $\mathbb{H}^4$. The group Γ is described via face-pairings of its fundamental domain F, which is obtained by removing from $\mathbb{H}^4$ a neighborhood of a nontrivial torus knot T in the ideal boundary of $\mathbb{H}^4$. In turn, the fundamental domain F_0 for Γ_0 is obtained by gluing k copies of F, where k is the index of Γ_0 in Γ, and one can describe F_0 as the result of removing from $\mathbb{H}^4$ a neighborhood of the k-fold connected sum of T. This k-fold connected sum is the singularity knot in a PL spine of $\mathbb{H}^4/\Gamma_0$, and hence $\mathbb{H}^4/\Gamma_0$ has no smooth spine by Theorem 1.2.

A related construction in [GLT88, Section 6] replaces the torus knot T by an arbitrary nontrivial knot K but the group Γ is now generated by reflections in the codimension one faces of F. The resulting singularity knot of the PL spine of $\mathbb{H}^4/\Gamma_0$ is the $\frac{k}{2}$-fold connected sum of $K\#r\bar{K}$, where $r\bar{K}$ is the reverse of the mirror image of K. (Here k is even because $\mathbb{H}^4/\Gamma_0$ is orientable and Γ does not preserve orientation). Since $K\#r\bar{K}$ is slice, the singularity knot is slice, and $\mathbb{H}^4/\Gamma_0$ has a smooth spine.

An analog of these examples with variable pinched negative curvature is discussed in [Bel], which is based on Ontaneda's Riemannian hyperbolization [Ont20]. Here there is no need to pass to a finite index torsion-free subgroup, and for any knot K one gets pinched negatively curved 4-manifolds whose PL spine has K as a singularity knot. In particular, if K satisfies the assumptions of Theorems 1.1 or 1.2, the negatively pinched 4-manifold has no smooth spine, while in the setting of Corollary 1.3 there exists a topologically flat spine.

The structure of the paper is as follows. In Section 2 we review results on the Kervaire invariant of compact oriented manifolds with codimension 2 spines. In Section 3 we specialize to dimension 4, relate the Kervaire invariant of W and the Arf invariant of the singularity knot, and prove Theorem 1.1. Section 4 is a review of Heegaard Floer d-invariants, whose relationship to V-functions is explored in Section 5. In Section 6 we investigate how the assumption "W has a smooth spine" affects the V-function of the singularity knot. Section 7 contains a proof of Theorem 1.2 and Corollary 1.3.

2. Kervaire invariant of codimension two thickenings

Let W be a compact oriented PL manifold with a PL embedded spine S, a closed connected oriented manifold of $\dim(S) = \dim(W) - 2$. Let ξ be an oriented plane bundle over S whose Euler class is the normal Euler class of S in W, and let $p_\xi \colon D_\xi \to S$ be the associated 2-disk bundle. Then [CS76, Proposition 1.6] gives a homology isomorphism $h\colon (W, \partial W) \to (D_\xi, \partial D_\xi)$ such that h preserves the orientation class in the relative second cohomology, and $p_\xi \circ h|_S$ is homotopic to the identity of S. The map h pulls $\alpha := p_\xi^*(\nu_W|_S)$ to the stable normal bundle ν_W of W because $h^*\alpha$ and ν_W are isomorphic over S to which W deformation retracts. This gives a normal map (h, b_W) where $b_W \colon \nu_W \to \alpha$ is the above bundle map.

Assuming, as we can, that h is transverse regular to the zero section S of D_ξ, we see that $N := h^{-1}(S)$ is a closed surface which is locally flat in W with normal bundle $h^*\xi$. The stable normal bundle to N is $\nu_N = \nu_W|_S \oplus h^*\xi = h^*(\alpha|_S \oplus \xi)$. Thus $h|_N \colon N \to S$ is covered by the bundle map $b_N \colon \nu_N \to \alpha|_S \oplus \xi$. The orientation on ξ and W defines an orientation on N for which $h|_N \colon N \to S$ has degree one, and hence $(h|_N, b_N)$ is a normal map.

The normal invariant of $(h|_N, b_N)$ is the image of the normal invariant of (h, b_W) under the inclusion-induced map $[W, G/PL] \to [S, G/PL]$, which is a bijection because $S \hookrightarrow W$ is a homotopy equivalence. This standard fact is stated on [CS76, p.195] and in the appendix of [KR08], and the proof amounts to comparing various definitions of the normal invariant.

By [RS71, Section 1] the Kervaire invariant of the normal map $(h|_N, b_N)$ is the Arf invariant of a certain quadratic form on the kernel of $h|_{N*} \colon H_1(N; \mathbb{Z}_2) \to H_1(S; \mathbb{Z}_2)$. A normal map with nontrivial Kervaire invariant represents a nontrivial class in $[S, G/PL]$, see [RS71, Theorem 1.4(ii)], and in fact, the Kervaire invariant defines a group homomorphism $[S, G/PL] \to \mathbb{Z}_2$ [RS71, Corollary 4.5].

3. Kervaire and Arf invariants in dimension four

Let us adopt notations of Section 2 and suppose $\dim(W) = 4$. Then the group $[S, G/PL]$ is isomorphic to $H^2(S; \mathbb{Z}_2) \cong \mathbb{Z}_2$, see e.g. [KT01, Section 2], and hence the Kervaire invariant defines an isomorphism $[S, G/PL] \to \mathbb{Z}_2$.

Fix a triangulation of W for which S is a full subcomplex with only one non-locally flat point. Its star is an embedded 4-ball B, and $C := S \cap B$ is the cone on the knot $K = S \cap \partial B$, the singularity knot of $S \subset W$.

Lemma 3.1. *The Kervaire invariant of W in $[S, G/PL]$ is the Arf invariant of the knot K.*

Proof. Let V be the smallest subcomplex that contains a neighborhood of S in W. Since S is a full subcomplex, V is a regular neighborhood of S in W, to which W deformation retracts. Denote the relative interiors of B, C, V by $\mathring{B}$, $\mathring{C}$, $\mathring{V}$. Then $V \setminus \mathring{B}$ is a trivial 2-disk bundle over $S \setminus \mathring{C}$. Give B the structure of a trivial 2-disk bundle over a 2-disk, whose zero section Z intersects ∂B in an unknot U. Glue $V \setminus \mathring{B}$ and B by an orientation-preserving 2-disk bundle automorphism identifying ∂C with $\partial Z = U$ so that the resulting 2-disk bundle $D_\xi \to S$ has the same Euler class as $S \subset W$. Denote the regular neighborhoods of K and U in ∂B by R_K and R_U, respectively.

Then the above-mentioned map $h\colon W \to D_\xi$ can be chosen so that $h|_{W \setminus \mathring{V}}$ is a deformation retraction onto ∂V, the map $h|_{V \setminus B}$ is the identity, h takes $(B, \partial B, K)$ to $(B, \partial B, U)$, and maps R_K homeomorphically onto R_U. To define $h|_B$ apply [CS76, Proposition 1.6] to the thickening B of C and use the fact that any homology equivalence $(R_K, \partial R_K) \to (R_U, \partial R_U)$ is homotopic to a homeomorphism.

Isotope the zero section Z to a 2-disk $Z_0 \subset \partial B$ rel boundary, and perturb h near B to be transverse regular to Z_0. Then $\Sigma := h^{-1}(Z_0)$ is a Seifert surface of K and $N := (S \setminus B) \cup \Sigma$ is a closed surface such that $h\colon N \setminus \Sigma \to S \setminus Z_0$ is the identity. Since the surgery obstruction is additive [Bro72, Theorem III.4.14], the Kervaire invariants of the normal maps $(h|_\Sigma, b_N|_\Sigma)$ and $(h|_N, b_N)$ are equal. Finally, the Kervaire invariant of $(h|_\Sigma, b_N|_\Sigma)$ equals the Arf invariant of K, as stated on [Ran98, page XXXIII] and proved in [Lev66, Proposition 3.3]. □

Proof of Theorem 1.1. The above thickening V of S is classified by the homotopy class of a map $f\colon S \to BSRN_2$. Let $\eta\colon BSRN_2 \to G/PL$ be the normal invariant map, see [CS76, p.182]. Then $\eta \circ f$ is the normal invariant of $S \hookrightarrow V$. Since K has nonzero Arf invariant, by the above discussion the Kervaire invariant of $\eta \circ f$ is nonzero.

It is easy to check that the thickening $V' := V \times S^1$ of $S' := S \times S^1$ is the pullback of $S \hookrightarrow V$ under the coordinate projection $p\colon\ S \times S^1 \to S$, see [CS76, pp.173–175]. Let $i\colon\ S \to S \times S^1$ be a section of p, say, given by $i(m) = (m, 1)$. Since $\eta \circ f = \eta \circ f \circ p \circ i$ is homotopically nontrivial, so is $\eta \circ f \circ p$. Hence $S' \hookrightarrow V'$ is a thickening with nontrivial normal invariant.

Arguing by contradiction suppose that W has a locally flat spine L. Then $L' := L \times S^1$ is a locally flat spine of W'. The restriction to L' of the deformation retraction $W' \to S'$ is homotopic to a diffeomorphism $g\colon\ L' \to S'$, see e.g. [Lau74, p.5]. Hence the normal invariant of g is trivial.

As we explain in [Bel, Appendix C], the pullback via g of the Poincaré embedding given by the inclusion $S' \subset W'$ is isomorphic to the Poincaré embedding of the locally flat inclusion $L' \subset W'$. Since $\dim(S')$ is odd and ≥ 3, Theorem 6.2 of [CS76] implies that the Poincaré embedding for $L' \subset W'$ can be realized by a locally flat embedding if and only if the normal invariants of g equals the normal invariant of the Poincaré embedding $S' \subset W'$. This is a contradiction because these normal invariants are different and $L' \subset W'$ is locally flat. $\square$

Remark 3.2. The above argument can be reversed, namely, if K has zero Arf invariant, then the Poincaré embedding induced by the inclusion $S \hookrightarrow W$ has trivial normal invariant, and hence so does its product with a circle or more generally, with any closed manifold L, and if $\dim(L)$ is odd, then $W \times L$ has a locally flat spine [CS76, Theorem 6.2].

4. Heegaard Floer d-invariants and V-functions of knots

Ozsváth and Szabó introduced [OS03, OS04c, OS04b] Heegaard-Floer homology theories $HF^o(M, \mathfrak{t})$ associated with a Spin^c structure $\mathfrak{t}$ on a closed oriented 3-manifold M. Here o is a decoration indicating the flavor of a Heegaard-Floer theory, and in this paper o is ∞ or $-$. The homology groups $HF^-(M, \mathfrak{t})$ and $HF^\infty(M, \mathfrak{t})$ are modules over $\mathbb{Z}[U]$ and $\mathbb{Z}[U, U^{-1}]$, respectively, where U is a formal variable whose action lowers the relative homological degree by 2. Related invariants for knots and links in 3-manifolds were developed in [Ras03, OS04a, OS08a]. We refer to these papers for background.

Henceforth, we assume that M has standard HF^∞ [OS03, p.240], and the Spin^c structure $\mathfrak{t}$ is torsion, i.e., its first Chern class has finite order in $H^2(M)$.

According to [OS04c, Section 4.2.5] the group $H_1^T(M) := H_1(M)/\mathrm{Tors}$ acts on the Heegaard-Floer chain complex $CF^o(M, \mathfrak{t})$, and on the corresponding homology group $HF^o(M, \mathfrak{t})$. Let $HF^o(M, \mathfrak{t})_{\mathrm{bot}}$ and $HF^o(M, \mathfrak{t})_{\mathrm{top}}$ denote the kernel and the cokernel of the $H_1^T(M)$-action on $HF^o(M, \mathfrak{t})$. The *d-invariants* $d_{\mathrm{top}}(M, \mathfrak{t})$ and $d_{\mathrm{bot}}(M, \mathfrak{t})$ are the maximal homological degrees of a non-torsion class in $HF^-_{\mathrm{top}}(M, \mathfrak{t})$ and $HF^-_{\mathrm{bot}}(M, \mathfrak{t})$, respectively, see [OS03, Section 9] and [LR14, Section 3]. If M is a rational homology sphere, the $H_1^T(M)$-action is trivial, so that $HF^-_{\mathrm{top}}(M, \mathfrak{t}) = HF^-_{\mathrm{bot}}(M, \mathfrak{t}) = HF^-(M, \mathfrak{t})$, and $d_{\mathrm{top}}(M, \mathfrak{t}) = d_{\mathrm{bot}}(M, \mathfrak{t})$ is the usual d-invariant for rational homology spheres, as in [OS03]. The invariants $d_{\mathrm{top}}(M, \mathfrak{t})$, $d_{\mathrm{bot}}(M, \mathfrak{t})$ are preserved under rational homology cobordisms [LR14, Proposition 4.5].

A null-homologous knot K in M gives rise to a $\mathbb{Z} \oplus \mathbb{Z}$ filtered chain complex $CFK^\infty(M, K, \mathfrak{s})$, which is a $\mathbb{F}[U, U^{-1}]$-module, see [OS04a, Ras03]. The filtration is indexed by the pair of integers (i, j), where i keeps track of the power of U, and j records the so called *Alexander filtration*. For $s \in \mathbb{Z}$, let $A_s^-(K) := A_s^-(M, K, \mathfrak{s})$ be the subcomplex of $CFK^\infty(M, K, \mathfrak{s})$ corresponding to $\max(i, j - s) \leq 0$ [MO, Remarks 3.7–3.8]. By the large surgery formula the homology of $A_s^-(K)$ is the sum of one copy of $\mathbb{F}[U]$ and a U-torsion submodule.

Following [NW15] we define the *V-function* $V_s(K)$ of an oriented knot $K \subset S^3$ so that $-2V_s(K)$ is the maximal homological degree of the free part of $H_*(A_s^-(K))$. For one-component links the H-function for links of [BG18, Liu17] is the V-function of knots. For example, the V-function of the unknot U is given by $V_s(U) = 0$ for $s \geq 0$ and $V_s(U) = -s$ for $s < 0$. The V-function takes values in nonnegative integers [BG18, Proposition 3.10], and furthermore, [BG18, Proposition 3.10] and [Liu17, Lemma 5.5] give

Proposition 4.1. *The V-function of an oriented knot $K \subset S^3$ satisfies*

$$V_{-s}(K) = V_s(K) + s \qquad \textit{and} \qquad V_{s-1}(K) - V_s(K) \in \{0, 1\}.$$

Example 4.2. Let K be an alternating knot of signature σ; recall that $\sigma \in 2\mathbb{Z}$. By [HM17, Theorem 1.7] if $\sigma > 0$, then $V_s(K) = 0$ for all s, and if $\sigma \leq 0$, the values $V_0(K)$ are given in the table below:

σ	$V_0(K)$
$-8k$	$2k$
$-8k-2$	$2k+1$
$-8k-4$	$2k+1$
$-8k-6$	$2k+2$

5. Surgeries on knots and d-invariants

For a positive integer g let $C^g := \#^{2g} S^2 \times S^1$, the connected sum of $2g$ copies of $S^2 \times S^1$. As usual $M_n(K)$ denotes the n-framed surgery on a closed oriented 3-manifold M along a knot $K \subset M$; in what follows M is S^3 or C^g.

If $B \subset C^g$ is the Borromean knot, as defined e.g. in [Par14, Figure 4.1], then $C^g_n(B)$ has the structure of an oriented circle bundle over the genus g oriented surface with Euler number n [OS08b, Section 5.2]. For the unknot $U \subset S^3$ it is well-known that $S^3_n(U)$ is an oriented circle bundle over S^2 with Euler number n.

It follows from [OS03, Propositions 9.3–9.4], cf. [Par14, Proposition 4.0.5], that the manifold $C^g_n(K\#B)$ has standard HF^∞ for any oriented knot $K \subset S^3$. The same is true for $S^3_n(K)$ [OS04b, Theorem 10.1]. Thus the d-invariants d_{top}, d_{bot} are defined for $C^g_n(K\#B)$ and $S^3_n(K)$, and moreover, for $S^3_n(K)$ they reduce to the usual d-invariants. They were computed by Ni-Wu [NW15, Proposition 1.6] for $S^3_n(K)$, and by Park [Par14, Theorem 4.2.3] for $C^g_n(B)$, $n \neq 0$. Park's argument extends to $C^g_n(K\#B)$ as follows.

Theorem 5.1. *For $n > 0$, we have*

$$d_{\mathrm{top}}(C^g_n(K\#B),k)) = g + \frac{(2k-n)^2 - n}{4n} - 2 \min_{a=0,\cdots,g}\{a + V_{k-g+2a}(K)\}. \tag{5.2}$$

$$d_{\mathrm{bot}}(C^g_n(K\#B,k)) = g + \frac{(2k-n)^2 - n}{4n} - 2 \max_{a=0,\cdots,g}\{a + V_{k-g+2a}(K)\} \tag{5.3}$$

where k labels the torsion Spin^c structures on $C^g_n(K\#B)$ with $-n/2 < k \leq n/2$. The d-invariant of $S^3_n(K)$ is given by

$$d(S^3_n(K),k) = \frac{(2k-n)^2 - n}{4n} - 2V_k(K). \tag{5.4}$$

Proof. As in the proof of [Par14, Theorem 4.1.1], a diagram chase in the surgery mapping cone formula [OS08b, Theorem 4.10] shows that the free part of $H_*(A^-_k)$ is isomorphic to the free part of $HF^-(C^g_n(K\#B),k)$. The grading of the free part of $H_*(A^-_k)$ can be found in [BHL17, Theorem 6.10]. □

Remark 5.5. Theorem 5.1 extends to $n < 0$ as follows. Since the Borromean knot is amphichiral, $C^g_n(K\#B) = -C^g_{-n}(\bar{K}\#B)$, where $\bar{K}$ is the mirror of K. Then [LR14, Proposition 3.7] gives

$$d_{\mathrm{bot}}(C^g_n(K\#B),k) = -d_{\mathrm{top}}(C^g_{-n}(\bar{K}\#B),k)$$
$$d_{\mathrm{top}}(C^g_n(K\#B),k) = -d_{\mathrm{bot}}(C^g_{-n}(\bar{K}\#B),k).$$

Remark 5.6. A similar argument also computes d_{top} and d_{bot} for rational surgeries, i.e., when $0 \neq n \in \mathbb{Q}$.

6. Spines, homology cobordisms, and *d*-invariants

Let W be a compact, oriented, smooth 4-manifold with a PL spine S_1, an oriented genus g surface with normal Euler number e. As before assume that S_1 has at most one non-locally-flat point with singularity knot $K \subset S^3$. If W also has a smooth spine S_2, then there is a homology cobordism C between the boundaries M_1, M_2 of the regular neighborhoods of S_1, S_2. Namely, C is obtained by removing the interiors of the regular neighborhoods from W and gluing the results along ∂W.

Here M_1 can be described as an e-surgery on $C^g = \#^{2g} S^2 \times S^1$ along the knot $K\#B$ where B is the Borromean knot [BHL17, Theorem 3.1], while M_2 is the circle bundle over S_2 with Euler number e, which is the e-surgery on C^g along B.

Since $H^2(C) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/e\mathbb{Z}$, every torsion Spin^c structure on C can be thought of an element of $\mathbb{Z}/e\mathbb{Z}$ indexed by $k \in (-e/2, e/2]$. Restricting the element to M_j, $j \in \{1,2\}$, gives a torsion Spin^c structure on M_j, which we denote $\mathfrak{t}_{kj}$. Thus

$$d_{\mathrm{top}}(M_1, \mathfrak{t}_{k1}) = d_{\mathrm{top}}(M_2, \mathfrak{t}_{k2}). \tag{6.1}$$

Theorem 6.2. *If $e \geq 0$, and W contains a smooth spine, then the singularity knot K satisfies*

$$\min_{a=0,\cdots,g} \{a + V_{-g+2a}(K)\} = \lceil g/2 \rceil, \tag{6.3}$$

where $\lceil g/2 \rceil$ is the smallest integer that is $\geq g/2$.

Proof. Since $V_s(U) = \dfrac{|s| - s}{2}$ we compute

$$\min_{a=0,\cdots,g} \{a + V_{-g+2a}(U)\} = \lceil g/2 \rceil. \tag{6.4}$$

If $e > 0$, by Theorem 5.1

$$d_{\mathrm{top}}(M_1, \mathfrak{t}_{k1}) = g + s - 2 \min_{a=0,\cdots,g} \{a + V_{k-g+2a}(K)\}$$

and

$$d_{\mathrm{top}}(M_2, \mathfrak{t}_{k2}) = g + s - 2 \min_{a=0,\cdots,g} \{a + V_{k-g+2a}(U)\}$$

where $s = \frac{(2k-e)^2 - e}{4e}$. Hence

$$\min_{a=0,\cdots,g} \{a + V_{k-g+2a}(K)\} = \min_{a=0,\cdots,g} \{a + V_{k-g+2a}(U)\}. \tag{6.5}$$

Combining (6.4) and (6.5) for $k = 0$ gives (6.3) in the case $e > 0$.

Assume $e = 0$. Then M_j is the 0-surgery on C^g, where $j = 1, 2$. Let M'_j denote the 1-surgery on C^g along the same knot as for M_j. By the equality part of [OS03, Corollary 9.14],

$$d_{\mathrm{top}}(M_j, \mathfrak{t}_{0j}) - \frac{1}{2} = d_{\mathrm{top}}(M'_j, \mathfrak{t}'_{0j}),$$

where $\mathfrak{t}_{0j}, \mathfrak{t}'_{0j}$ are the trivial Spin^c structures. Even though [OS03, Corollary 9.14] is stated for knots in S^3, it generalizes (with the same proof) to knots in 3-manifolds with standard HF^∞ and trivial HF_{red}, which is how we apply it.

By (6.1) M_1, M_2 have the same d_{top}, and hence

$$d_{\mathrm{top}}(M'_1, \mathfrak{t}'_{01}) = d_{\mathrm{top}}(M'_2, \mathfrak{t}'_{02}),$$

and as before (6.4)–(6.5) imply (6.3), now for $e = 0$. □

Corollary 6.6. *If W contains a smooth spine with normal Euler number $e \geq 0$, then the singularity knot K satisfies*

$$V_0(K) = 0 \text{ if } g \text{ is even and } V_1(K) = 0 \text{ if } g \text{ is odd.} \tag{6.7}$$

Proof. If $g = 2k$, then by Theorem 6.2

$$\min_k \{V_0(K) + k, V_2(K) + k + 1, \cdots, V_{2k}(K) + 2k\} = k.$$

Proposition 4.1 gives $V_{s-1}(K) \leq V_s(K) + 1$, and hence the minimum occurs for $V_0(K) + k = k$, which implies $V_0(K) = 0$. Similarly, if $g = 2k + 1$, we have

$$\min_k\{V_1(k)+k+1,\cdots,V_{2k+1}(K)+2k+1\}=k+1$$

which means that $V_1(K)+k+1=k+1$, and hence $V_1(K)=0$. □

7. Singularity knots and smooth spines

As in Section 6 let W a compact, oriented, smooth 4-manifold with a PL spine which is an oriented genus g surface with normal Euler number e, and at most one non-locally-flat point with singularity knot K. After changing the orientation of W, if needed, we can and will assume that $e \geq 0$.

Proof of Theorem 1.2. By Corollary 6.6 $V_0(K)=0$ or $V_1(K)=0$ depending on the parity of g, and hence $g(K)\leq 1$ [Liu, Lemma 2.11], where $g(K)$ is the genus of of K. If $g(K)=0$, then K is the unknot. A genus-one L-space knot is the right-handed trefoil [Ghi08, Corollary 1.5]. According to [NNU98] the Arf invariant for the torus knot $T(p,q)$ is $(p^2-1)(q^2-1)/24 \pmod 2$. Thus the Arf invariant of $T(2,3)$ is nonzero, which implies by Theorem 1.1 that W cannot contain a smooth spine. This completes the proof when K is an L-space knot.

Suppose K is an alternating knot of signature <-4. Hence $V_0(K)\geq 2$ by Example 4.2. Then Proposition 4.1 gives $V_1(K)\geq 1$, which by Corollary 6.6 shows that W does not have a smooth spine.

Finally, suppose that K is the connected sum of nontrivial L-space knots $K_1,\cdots,K_n$ with $n\geq 2$. Thus $g(K)=g(K_1)+\cdots+g(K_n)$. Since K_i is nontrivial, $g(K_i)\geq 1$, and hence $g(K)\geq n$. For $j\in\mathbb{Z}$ set $R_{K_i}(j):=V_{g(K_i)-j}(K_i)$ and

$$R_K(j):=\min_{j_1+\cdots+j_n=j} R_{K_1}(j_1)+\cdots+R_{K_n}(j_n).$$

By Proposition 4.1 the function R_{K_i} is nonnegative and nondecreasing, and combining the proposition with [Liu, Lemma 2.11] gives

$$R_{K_i}(1)=V_{g(K_i)-1}(K_i)=1.$$

Hence $R_{K_i}(j)\geq 1$ for every $j\geq 1$.

Propositions 5.1 and 5.6 and Lemma 6.2 of [BL14] imply $V_j(K)+j=R_K(g(K)+j)$; the notations in [BL14] are different. Again, by Corollary 6.6 if $V_0(K)$ and $V_1(K)$ are both nonzero, then W does not have a smooth spine.

To see that $V_0(K) = R_K(g(K)) \geq 1$ assume the minimum of $R_K(g(K))$ is attained for $j_1 + \cdots + j_n = g(K)$. Then $j_i \geq 1$ for some i, and $R_K(g(K)) \geq R_{K_i}(j_i) \geq 1$.

To show that $1 \leq V_1(K) = R_K(g(K)+1) - 1$ assume that the minimum of $R_K(g(K)+1)$ is attained for $j_1 + \cdots + j_n = g(K)+1$. If $j_i \geq g(K_i)+2$, then

$$R_K(g(K)+1) \geq R_{K_i}(j_i) \geq R_{K_i}(g(K_i)+2) = V_{-2}(K_i) = V_2(K_i) + 2 \geq 2$$

as claimed. Otherwise, there are indices with $j_i \geq g(K_i)$ and $j_l = g(K_l) + 1$. Then $R_{K_i}(j_i) \geq V_0(K_i) \geq 1$ and $R_{K_l}(j_l) = V_1(K_l) + 1 \geq 1$, and hence $R_K(g(K)+1) \geq 2$ as desired. □

Proof of Corollary 1.3. For any $m \geq 2$, there is a topologically slice knot K_m with $V_0(K_m) = m$ [HKL16, Proposition 6 and Theorem B.1]. The corresponding manifold W has a topologically flat spine. By Corollary 6.6 and Proposition 4.1 if W has a smooth spine, then $V_0(K) \in \{0, 1\}$. □

Acknowledgements

We are grateful to Lisa Piccirillo for leading us to [Akb93] and to Kyle Hayden for expository suggestions. Liu appreciates the support and hospitality of the Max Planck Institute for Mathematics in Bonn, where she was a member when this work began.

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Received March 5, 2021
Accepted September 13, 2021