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Resolvent estimates, wave decay, and resonance-free regions for star-shaped waveguides

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Using coordinates $(x, y) \in \mathbb{R} \times \mathbb{R}^{d-1}$, we introduce the notion that an unbounded domain in $\mathbb{R}^d$ is star shaped with respect to $x = \pm\infty$. For such domains, we prove estimates on the resolvent of the Dirichlet Laplacian near the continuous spectrum. When the domain has infinite cylindrical ends, this has consequences for wave decay and resonance-free regions. Our results also cover examples beyond the star-shaped case, including scattering by a strictly convex obstacle inside a straight planar waveguide.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be an open set of infinite volume, and equip the Laplacian Δ on Ω with Dirichlet boundary conditions. We wish to understand how the behavior of the resolvent of the Laplacian near the spectrum is related to the geometry of Ω, and to deduce consequences for wave evolution and decay, and for the distribution of resonances when these can be defined.

When $\mathbb{R}^d \setminus \Omega$ is bounded, this is the celebrated obstacle scattering problem. Then a particularly favorable geometric assumption, going back to the original work of Morawetz [Mo61], is that the obstacle is star shaped. In this paper we adapt this assumption to the study of waveguides, which are domains bounded in some directions and unbounded in others. We focus especially on domains with cylindrical ends (which have one infinite dimension), but our resolvent estimates hold for domains with more general ends. Our results also cover the problem of scattering by a strictly convex obstacle inside a straight planar waveguide (see Figure 1) for which we prove a resolvent estimate in Theorem 4, wave decay in Theorem 6, and a resonance-free region in Theorem 8.

Our analysis is based on the following definition, which applies to some but not all strictly convex obstacles inside a straight planar waveguide.

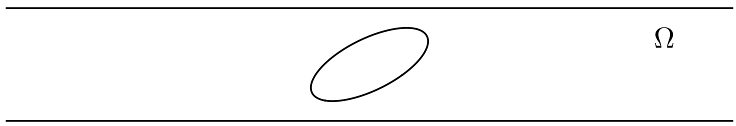

Figure 1: A strictly convex obstacle inside a straight planar waveguide.

Definition. Let $(x, y) = (x, y_1, \dots, y_{d-1})$ be Cartesian coordinates on $\mathbb{R}^d$ and let $\nu = (\nu_x, \nu_{y_1}, \dots \nu_{y_{d-1}})$ be the outward pointing unit normal vector to $\partial\Omega$. We say that Ω is *star shaped with respect to* $x = \pm\infty$ if

$$x\nu_x \leq 0 \text{ throughout } \partial\Omega. \tag{1.1}$$

In other words, (1.1) says that, if $p \in \partial\Omega$ does not lie in the coordinate hyperplane $x = 0$, then ν at p either points toward $x = 0$ or is parallel to $x = 0$. See Figures 2, 3, and 4 below for examples of such domains with $d = 2$; in Figure 2 the vertical axis $x = 0$ is to the left of the domain, and in Figures 3 and 4 the axis is in the center. Note that the example in Figure 1 does *not* obey (1.1).

This is to be compared with the analogous assumption in scattering by a compact obstacle, as in [Mo61, (11)], which says that

$$x\nu_x + y_1\nu_{y_1} + \cdots + y_{d-1}\nu_{y_{d-1}} \leq 0 \text{ throughout } \partial\Omega. \tag{1.2}$$

In other words, (1.2) says that the obstacle $\mathbb{R}^d \setminus \Omega$ is star shaped.

Note that (1.2) implies that there are no trapped billiard trajectories in Ω; that is, all billiard trajectories go to infinity forward and backward in time. By contrast, our assumption (1.1) allows trapping, and indeed all domains with cylindrical ends have trapped trajectories. In some sense, among domains with cylindrical ends, those obeying (1.1) have the least trapping possible. In Figures 2, 3, and 4 below, the trapped trajectories are the vertical bouncing ball orbits between points where the boundary is horizontal.

Fundamental to the proofs of the resolvent estimates we give under the assumption (1.1) are some integration by parts identities. By combining the assumption (1.1) on the boundary $\partial\Omega$ with the Dirichlet boundary condition, we obtain identities in which the boundary term has favorable sign. This part of our proofs is analogous to [Mo61, (16)].

We first present our resolvent estimates, which hold for domains with rather general infinite ends. Afterwards we give consequences for wave evolution and analytic continuation of the resolvent, under the additional assumption that Ω has cylindrical ends.

1.1. Resolvent estimates

The spectrum of $-\Delta$, with Dirichlet boundary conditions, is contained in $[0,\infty)$, and for $z \in \mathbb{C}$ not in the spectrum, let

$$(-\Delta - z)^{-1} \colon L^2(\Omega) \to L^2(\Omega),$$

be the corresponding resolvent. Throughout the paper we assume that $\partial\Omega$ is Lipschitz, and that every point p on $\partial\Omega$ has a neighborhood U_p such that either $U_p \cap \Omega$ is convex or $U_p \cap \partial\Omega$ is $C^{1,1}$.

Our strongest result holds in the case that the domain has only one infinite end:

Theorem 1. *Suppose that Ω satisfies the assumption* (1.1) *and that $x > 0$ throughout Ω. Then for any $\delta \in (0,1]$ and $z \in \mathbb{C} \setminus [0,\infty)$ we have*

$$\|(1+x)^{-\frac{3+\delta}{2}}(-\Delta - z)^{-1}(1+x)^{-\frac{3+\delta}{2}}\|_{L^2(\Omega)\to L^2(\Omega)} \le \frac{3}{\delta}(1+|z|^{1/2}). \tag{1.3}$$

Examples include cigar-shaped domains such as the union of the open ball $\{(x,y)\colon (x-1)^2 + |y|^2 < 1\}$ with the half-cylinder $\{(x,y)\colon x > 1$ and $|y| < 1\}$, the parabolic domain $\{(x,y)\colon x > |y|^2\}$, and more generally any epigraph $\{(x,y)\colon x > f(y)\}$ where $f \in C^{1,1}(\mathbb{R}^{d-1})$ is nonnegative. See also Figure 2.

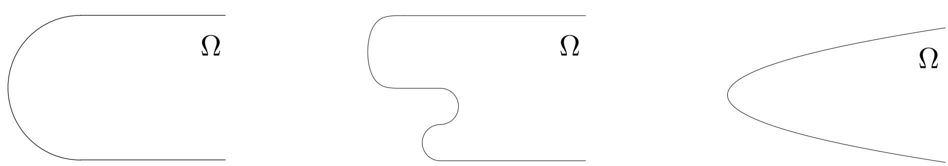

Figure 2: Some domains to which Theorem 1 applies. The first two have cylindrical ends, and the third does not.

A weaker version of Theorem 1 holds in the presence of multiple infinite ends, under a 'flaring' condition; namely when there is a suitable region where $x\nu_x$ is bounded away from zero. Note that some such condition is needed, because if $x\nu_x \equiv 0$ then by separation of variables and direct computation (as in Section 1.1 of [ChDa23]) one checks that there are infinitely many resonances embedded in the continuous spectrum.

Theorem 2. *Suppose that Ω satisfies the assumption* (1.1) *and there is an open interval I and a positive constant C_I such that*

$$x\nu_x \leq -C_I, \tag{1.4}$$

on the intersection of $\partial\Omega$ with $I \times \mathbb{R}^{d-1}$. Suppose further that the intersection of Ω with $I \times \mathbb{R}^{d-1}$ is bounded. Then for any $\delta > 0$ there are positive constants E_0 and C such that

$$\|(1+|x|)^{-\frac{3+\delta}{2}}(-\Delta - E - i\varepsilon)^{-1}(1+|x|)^{-\frac{3+\delta}{2}}\|_{L^2(\Omega)\to L^2(\Omega)} \leq CE^{1/2} \tag{1.5}$$

for all $E \geq E_0$ and $\varepsilon \in (0,1]$.

Examples include hourglass-shaped domains like $\{(x,y)\colon |y| < f(x)\}$ for some nonconstant $f \in C^{1,1}(\mathbb{R})$ satisfying $xf'(x) \geq 0$ for all x. See also Figure 3.

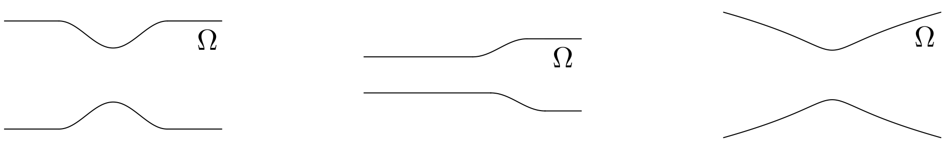

Figure 3: Some domains to which Theorem 2 applies. The first two have cylindrical ends and the third does not.

For planar domains, we only need the flaring requirement (1.4) on part of the intersection of $\partial\Omega$ with $I \times \mathbb{R}$:

Theorem 3. *Suppose that Ω satisfies the assumption* (1.1) *and that $d = 2$. Let I be an open interval and let C_I be a positive constant. Let Γ_F be part of the intersection of $\partial\Omega$ with $I \times \mathbb{R}$ on which the flaring requirement* (1.4) *holds. Suppose that the intersection of Ω with $I \times \mathbb{R}$ consists of bounded open sets $\Omega_1, \ldots, \Omega_K$ with mutually disjoint closures such that for each $k = 1, \ldots K$,*

$$(\partial\Omega \cap \partial\Omega_k) \setminus \Gamma_F \subset I \times \{a_k\},$$

for some real a_k. Then for any $\delta > 0$ there are positive constants E_0 and C such that (1.5) *holds for all $E \geq E_0$ and $\varepsilon \in (0,1]$.*

Some examples of domains for which Theorem 3 applies are shown in Figure 4. One class of such examples is that of straight planar waveguides with suitable convex obstacles $(\mathbb{R} \times (-1, 1)) \setminus \overline{\mathcal{O}}$, where $\overline{\mathcal{O}} \subset \mathbb{R} \times (-1, 1)$ is a convex closed set such that the maximum and minimum values of y on $\overline{\mathcal{O}}$ are both attained on the axis $x = 0$. In Theorem 4 we prove the corresponding result for more general convex $\overline{\mathcal{O}}$, as in Figure 1. These domains do not necessarily satisfy (1.1), but the proofs of Theorems 3 and 4 are similar.

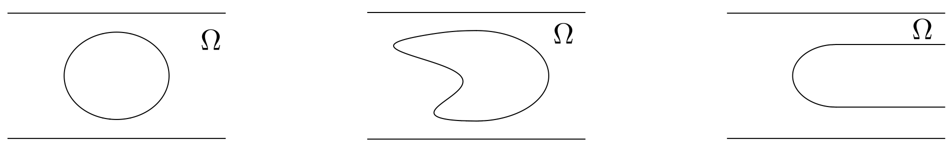

Figure 4: Some domains with cylindrical ends to which Theorem 3 applies.

Theorem 4. *Let* $\Omega = (\mathbb{R} \times (-1, 1)) \setminus \overline{\mathcal{O}}$, *where* $\mathcal{O}$ *is a non-empty open bounded strictly convex set with* $C^{1,1}$ *boundary and* $\overline{\mathcal{O}} \subset \mathbb{R} \times (-1, 1)$. *Then for any* $\delta > 0$ *there are positive constants* E_0 *and* C *such that (1.5) holds for all* $E \geq E_0$ *and* $\varepsilon \in (0, 1]$.

1.2. Wave asymptotics and absence of eigenvalues and embedded resonances

In this section we assume in addition that Ω has *cylindrical ends*. Specializing to our setting, by this we mean that there is $R_0 > 0$ such that $\Omega \cap ([-R_0, R_0] \times \mathbb{R}^{d-1})$ is bounded and

$$\begin{aligned} \Omega \cap ((-\infty, -R_0] \times \mathbb{R}^{d-1}) &= (-\infty, -R_0] \times Y_-, \\ \Omega \cap ([R_0, \infty) \times \mathbb{R}^{d-1}) &= [R_0, \infty) \times Y_+, \end{aligned} \tag{1.6}$$

where Y_- and Y_+ are (not necessarily connected) bounded open sets in $\mathbb{R}^{d-1}$. We allow the possibility that one, but not both, of $Y_\pm$ is the empty set.

Let $f_1,\ f_2 \in C_c^\infty(\Omega)$, and let u solve

$$(\partial_t^2 - \Delta)u = 0, \qquad (u, \partial_t u)|_{t=0} = (f_1, f_2), \qquad u|_{\partial\Omega} = 0. \tag{1.7}$$

We prove decay rates and asymptotics for such u, using results of [ChDa23]. We begin with a wave decay rate.

Theorem 5. *Suppose that Ω satisfies the assumptions of Theorem 1, Theorem 2, Theorem 3, or of Theorem 4. Suppose additionally that Ω has cylindrical ends. Let $f_1,\ f_2 \in C_c^\infty(\Omega)$ be given, and let $u(t)$ solve* (1.7)*. Then for any $\chi \in C_c^\infty(\Omega)$ and for any $m \in \mathbb{N}$ there is a constant C such that*

$$\|\chi(u(t) - u_p(t))\|_{H^m(\Omega)} \leq Ct^{-1} \text{ for } t \text{ sufficiently large,}$$

where $u_p(t)$ is a term corresponding to the projection of the initial data (f_1, f_2) onto any eigenvalues and embedded resonances of the Dirichlet Laplacian on Ω. If (1.1) *holds, then there are no such eigenvalues and embedded resonances, and $u_p(t) \equiv 0$.*

For a more detailed description of $u_p(t)$, see Theorem 1.1 of [ChDa23]. The only case of Theorem 5 in which we do not show $u_p(t) \equiv 0$ is that of a convex obstacle inside a straight planar waveguide such that (1.1) does not hold.

To state our next result, let Y be the disjoint union of Y_- and Y_+, let Δ_Y be the Dirichlet Laplacian on Y, and let $\{\phi_j\}_{j=0}^\infty$ be a complete orthonormal set of eigenfunctions of Δ_Y, with corresponding eigenvalues σ_j^2, so that

$$-\Delta_Y \phi_j = \sigma_j^2 \phi_j, \qquad 0 < \sigma_1 \leq \sigma_2 \leq \cdots . \tag{1.8}$$

We get an improvement of Theorem 5 under an additional assumption on the eigenvalues of $-\Delta_Y$. The assumption is that there are positive constants c_Y and N_Y, such that

$$\sigma_{j'} - \sigma_j \geq c_Y \sigma_j^{-N_Y}, \tag{1.9}$$

whenever $\sigma_{j'} > \sigma_j$. Note that this assumption allows the eigenvalues of $-\Delta_Y$ to have high multiplicities, but forbids *distinct* eigenvalues from clustering too closely together.

Theorem 6. *Suppose that the assumptions of Theorem 5 hold, and also that* (1.9) *holds. Then for each $k_0 \geq 2$ we can write*

$$u(t) = u_p(t) + \sum_{k=1}^{k_0-1} t^{-1/2-k} \sum_{j=1}^{\infty} (e^{it\sigma_j} b_{j,k,+} + e^{-it\sigma_j} b_{j,k,-}) + u_{r,k_0}(t),$$

for some $b_{j,k,\pm} \in C^\infty(\Omega)$, *where* $u_p(t)$ *is as in Theorem 5, and where for any* $\chi \in C_c^\infty(\Omega)$ *and* $m \in \mathbb{N}$ *there is a constant* C *so that*

$$\sum_{j=0}^{\infty} \|\chi b_{j,k,\pm}\|_{H^m(\Omega)} < +\infty, \; k = 1, 2, ..., k_0 - 1,$$

and

$$\|\chi u_{r,k_0}(t)\|_{H^m(\Omega)} \leq Ct^{-k_0} \text{ for } t \text{ sufficiently large.}$$

If (1.1) *holds, then* $u_p(t) \equiv 0$.

For a more detailed description of the $b_{j,k,\pm}$, see Theorem 1.2 and Lemma 4.7 of [ChDa23]. In particular, Theorem 6 shows that

$$\|\chi(u(t) - u_p(t))\| \leq Ct^{-3/2}.$$

This is sharp when $\Omega = (0, \infty) \times Y$, where $Y \subset \mathbb{R}^{d-1}$ is bounded, by the computation in Section 1.1 of [ChDa23], in particular equation (1.6) there.

The fact that if (1.1) holds, then $u_p(t) \equiv 0$ in Theorems 5 and 6, depends on the following result ruling out eigenvalues and real resonances. Although this is perhaps well known (see, for example, [MoWe87, Theorem 3.1] for a similar result), we include a proof both for completeness and because it uses an integration by parts identity similar to that used in the proofs of Theorems 1, 2, 3, and 4.

Theorem 7. *Suppose that* Ω *satisfies the assumption* (1.1) *and has cylindrical ends. Then the Dirichlet Laplacian on* Ω *has no eigenvalues. For such* Ω, *the Dirichlet Laplacian on* Ω *has resonances embedded in the continuous spectrum if and only if* Ω *is the product* $\mathbb{R} \times \widetilde{Y}$ *for some set* $\widetilde{Y} \subset \mathbb{R}^{d-1}$.

By separation of variables and direct computation (as in Section 1.1 of [ChDa23]) one checks that if $\Omega = \mathbb{R} \times \widetilde{Y}$ then the Dirichlet Laplacian on Ω has threshold resonances at every point in the Dirichlet spectrum of $-\Delta_{\widetilde{Y}}$. Theorem 7 shows that no other sufficiently regular domains Ω with cylindrical ends obeying (1.1) can have any poles of the Dirichlet resolvent on the real axis. Theorem 7 is a consequence of Theorem 1 when Ω has only one end; we prove the general case in Section 5.

The proofs of Theorems 5 and 6 depend upon results on resonance-free regions in a neighborhood of the spectrum. Because these are more complicated to state, we present them below in Theorem 8 in Section 6. Once Theorems 7 and 8 are established, Theorems 5 and 6 are direct consequences of Theorems 3.2 and 4.1 of [ChDa23] (see also Theorems 1.1 and 1.2 of [ChDa23]).

1.3. Background and context

A wave decay rate for star-shaped compact obstacles was proved by Morawetz in [Mo61], and the results there were refined and extended in many papers, including [LaMoPh63, Mo72, Ra78], and more recently revisited and adapted to hyperbolic scattering by Hintz and Zworski in [HiZw17, HiZw18].

Our results here build on those in [ChDa21, ChDa23] for manifolds with cylindrical ends, which in turn are based on the spectral and scattering theory of waveguides and manifolds with cylindrical ends developed in [Go74, Ly76, Gu89, Me93, Chr95, Pa95]. The main novelty in the present paper is the resolvent estimates in Theorems 1, 2, 3, and 4. These rely on integration by parts identities in the spirit of Morawetz [Mo61].

Waveguides appear in models of electron motion in semiconductors and of propagation of electromagnetic and sound waves; see for example [LoCaMu99, Ra00, RaBaBaHu12, ExKo15, BoGaWo17]. There are many results establishing the existence of eigenvalues for waveguides, under suitable geometric conditions. Something of a survey can be found in [KrKř05]. The result in [BuGeReSi97] holds in a setting in some sense opposite to ours, and shows in particular that if $\Omega \subset \mathbb{R}^2$ has cylindrical ends and obeys $x\nu_x \geq 0$, $x\nu_x \not\equiv 0$, then there is at least one eigenvalue. There are nonexistence results for eigenvalues in [MoWe87, DaPa98, BrDiKr20], and another for a resonance at the bottom of the spectrum in [GrJe09]. Some weaker wave decay results (expansions up to $o(1)$ as $t \to \infty$) for planar waveguides can be found in [Ly76, HeWe06].

The resonance-free region we establish in Theorem 8 is a close analogue of a corresponding region for manifolds with cylindrical ends established in [ChDa21], and relies on a resolvent identity due to Vodev [Vo14]. An existence result for resolvent poles (in the presence of appropriate quasimodes) on waveguides can be found in [Ed02]. Upper bounds on the number resonances for manifolds with cylindrical ends are given in [Ch02].

Previous work also shows that our results cannot carry over directly to the case of Neumann boundary conditions. For example, in [DaPa98], examples are given of domains Ω with cylindrical ends satisfying the hypotheses of Theorems 3 and 7 but whose Neumann Laplacians have eigenvalues. More simply, if $\Omega = (0,\infty) \times (-1,1)$, then Ω satisfies the hypotheses of Theorem 1, but the Neumann Laplacian has infinitely many embedded resonances (see Section 1.1 of [ChDa23]).

1.4. Outline

In Section 2 we review background regarding Sobolev spaces and establish notation. In Section 3 we prove Theorem 1. In Section 4 we prove Theorems 2, 3, and 4. In Section 5 we prove Theorem 7. In Section 6 we obtain a resonance-free region in a neighborhood of the spectrum.

2. Preliminaries and notation

Throughout the paper we assume that $\partial\Omega$ is Lipschitz, and that every point p on $\partial\Omega$ has a neighborhood U_p such that either $U_p \cap \Omega$ is convex or $U_p \cap \partial\Omega$ is $C^{1,1}$.

We denote by $C_c^\infty(\Omega)$ the space of functions in $C^\infty(\mathbb{R}^d)$ with compact support in Ω, and by $C_c^\infty(\overline{\Omega})$ the space of restrictions to $\overline{\Omega}$ of functions in $C^\infty(\mathbb{R}^d)$ with compact support in $\mathbb{R}^d$. We use three different kinds of Sobolev spaces on Ω. We denote by $H^k(\Omega)$ the Sobolev space of functions in $L^2(\Omega)$ whose partial derivatives up to kth order are in $L^2(\Omega)$, and by $H_0^k(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^k(\Omega)$. We denote by $H^k_{\text{comp}}(\Omega)$ the space of functions in $H^k(\Omega)$ with compact support in $\overline{\Omega}$ (by [Gr85, Theorem 1.4.3.1] this is the same as the space of restrictions to Ω of compactly supported functions in $H^k(\mathbb{R}^d)$), and similarly by $L^p_{\text{comp}}(\Omega)$ the space of functions in $L^p(\Omega)$ with compact support in $\overline{\Omega}$.

We integrate by parts using Green's theorem (see [Gr85, Theorem 1.5.3.1]). We use the fact that $C_c^\infty(\overline{\Omega})$ is dense in $H^k_{\text{comp}}(\Omega)$ (see [Gr85, Theorem 1.4.2.1]), and that the trace map $H^1_{\text{comp}}(\Omega) \to L^2(\partial\Omega)$ is continuous (see [Gr85, Theorem 1.5.1.3]).

We define the Dirichlet resolvent by taking the Friedrichs extension of Δ with domain $C_c^\infty(\Omega)$ (see pages 82 and 83 of [Ta96II, Chapter 8, Section 2]). For $z \notin [0,\infty)$ we have

$$(-\Delta - z)^{-1} \colon L^2(\Omega) \to \mathcal{D} := \{u \in H_0^1(\Omega) \colon \Delta u \in L^2(\Omega)\}. \tag{2.1}$$

We denote by $\mathcal{D}_{\text{comp}}$ the set of functions in $\mathcal{D}$ with compact support in $\overline{\Omega}$. The regularity assumption on $\partial\Omega$ is made so as to ensure that

$$\mathcal{D}_{\text{comp}} = H^2_{\text{comp}}(\Omega) \cap H_0^1(\Omega). \tag{2.2}$$

Near points on $\partial\Omega$ where $\partial\Omega$ is C^∞, (2.2) follows from [Ta96I, Chapter 5, Theorem 1.3]). Near points where $\partial\Omega$ is $C^{1,1}$, (2.2) follows from [Gr85, Corollary 2.2.2.4]. Near points where Ω is convex, (2.2) follows from [Gr85, Theorem 3.2.1.2] (see also [Ta96I, Chapter 5, Section 5, Exercise 7]).

For real E and ε we write for brevity

$$P = P(E, \varepsilon) = -\Delta - E - i\varepsilon.$$

We use $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ to denote the norm and inner product on $L^2(\Omega)$, and prime to denote differentiation with respect to x.

3. Domains with one end

We begin the proof of Theorem 1 with an integration by parts identity in the spirit of Morawetz and others. This identity, along with some variants of it, also plays a central role in the proofs of Theorems 2, 3, 4, and 7.

Lemma 1. *Let $\Omega \subset \mathbb{R}^d$ be an open set such that every point p on $\partial\Omega$ has a neighborhood U_p such that either $U_p \cap \Omega$ is convex or $U_p \cap \partial\Omega$ is $C^{1,1}$. Let $w \in C^3(\mathbb{R})$ be real valued, and suppose $w,\ w',\ w'',\ w'''$ are all bounded. Let $u \in \mathcal{D}$ and let $E,\ \varepsilon \in \mathbb{R}$. Then*

$$\begin{aligned} \langle w'u', u' \rangle = {} & \frac{1}{4}\langle w'''u, u\rangle + \frac{1}{2}\operatorname{Re}\langle Pu, (wu)'\rangle + \frac{1}{2}\operatorname{Re}\langle wu', Pu\rangle \\ & + \varepsilon \operatorname{Im}\langle wu', u\rangle + \frac{1}{2}\int_{\partial\Omega} w|\partial_\nu u|^2 \nu_x, \end{aligned} \tag{3.1}$$

where $w = w(x)$.

Proof. Let $u,\ v \in C_c^\infty(\overline{\Omega})$. We use a positive commutator argument with $w\partial_x$ as commutant. Computing this commutator two ways we have

$$\begin{aligned} \langle [w\partial_x, \partial_x^2]u, v\rangle &= -2\langle w'u'', v\rangle - \langle w''u', v\rangle \\ &= 2\langle w'u', v'\rangle + \langle w''u', v\rangle - 2\int_{\partial\Omega} w'u'\bar{v}\nu_x, \end{aligned} \tag{3.2}$$

and

$$\langle [w\partial_x, \partial_x^2]u, v\rangle = \langle [P, w\partial_x]u, v\rangle = \langle Pwu', v\rangle - \langle wPu', v\rangle. \tag{3.3}$$

We write the right hand side of (3.3) in terms of Pu and Pv by integrating by parts to obtain

$$\langle Pwu', v\rangle = \langle wu', (P + 2i\varepsilon)v\rangle + \int_{\partial\Omega} \left(wu'\partial_\nu \bar{v} - \partial_\nu(wu')\bar{v}\right), \tag{3.4}$$

and

$$-\langle wPu', v\rangle = \langle Pu, (wv)'\rangle - \int_{\partial\Omega} w(Pu)\bar{v}\nu_x. \tag{3.5}$$

Combining (3.3), (3.4), and (3.5) gives

$$\begin{aligned}\langle [w\partial_x, \partial_x^2]u, v\rangle = &\langle Pu, (wv)'\rangle + \langle wu', Pv\rangle - 2i\varepsilon\langle wu', v\rangle \\ &+ \int_{\partial\Omega} \left(wu'\partial_\nu\bar{v} - \partial_\nu(wu')\bar{v} - w(Pu)\bar{v}\nu_x\right),\end{aligned}$$

and combining also with (3.2) gives

$$\begin{aligned}2\langle w'u', v'\rangle = &- \langle w''u', v\rangle + \langle Pu, (wv)'\rangle + \langle wu', Pv\rangle - 2i\varepsilon\langle wu', v\rangle \\ &+ \int_{\partial\Omega} \left(wu'\partial_\nu\bar{v} - \partial_\nu(wu')\bar{v} - w(Pu)\bar{v}\nu_x + 2w'u'\bar{v}\nu_x\right),\end{aligned} \tag{3.6}$$

for all $u,\ v \in C_c^\infty(\overline{\Omega})$. By density, (3.6) also holds for all $v \in H^2_{\mathrm{comp}}(\Omega)$. Now let us specialize to the case that $v \in H^2_{\mathrm{comp}}(\Omega) \cap H^1_0(\Omega)$. Then (3.6) becomes

$$\begin{aligned}2\langle w'u', v'\rangle = &-\langle w''u', v\rangle + \langle Pu, (wv)'\rangle + \langle wu', Pv\rangle - 2i\varepsilon\langle wu', v\rangle \\ &+ \int_{\partial\Omega} wu'\partial_\nu\bar{v}.\end{aligned} \tag{3.7}$$

Again by density, we also have (3.7) for all $u \in H^2_{\mathrm{comp}}(\Omega)$ and $v \in H^2_{\mathrm{comp}}(\Omega) \cap H^1_0(\Omega)$.

Specializing further to the case that $u = v$, taking real parts of both sides, and using

$$-\operatorname{Re}\langle w''u', u\rangle = \frac{1}{2}\langle w'''u, u\rangle, \qquad u'|_{\partial\Omega} = \nu_x\partial_\nu u,$$

gives (3.1) for all $u \in \mathcal{D}_{\mathrm{comp}}$. To prove (3.1) for all $u \in \mathcal{D}$, use a partition of unity to write u as a locally finite sum of functions in $\mathcal{D}_{\mathrm{comp}}$. □

For the proof of Theorem 1 we will use

$$w(x) = 1 - (1+x)^{-\delta}, \qquad x \geq 0.$$

We will need the Hardy's inequality in the form of [HaLiPo34, Theorem 330]:

$$\|\sqrt{w'''}u\| \leq 2\frac{\sqrt{1+\delta}}{\sqrt{2+\delta}}\|\sqrt{w'}u'\|, \text{ for all } u \in H^1_0(\Omega), \tag{3.8}$$

which is proved by writing

$$\begin{aligned}\|\sqrt{w'''}u\|^2 &= -2\,\mathrm{Re}\langle w''u', u\rangle \\ &\le 2\|\sqrt{w'''}u\| \left\| \frac{w''}{\sqrt{w'''}}u' \right\| = 2\frac{\sqrt{1+\delta}}{\sqrt{2+\delta}}\|\sqrt{w'''}u\|\|\sqrt{w'}u'\|.\end{aligned}$$

Proof of Theorem 1. We simplify (3.1), assuming additionally that

$$(1+x)^{\frac{3+\delta}{2}} Pu \in L^2(\Omega). \tag{3.9}$$

Then, since $0 < w \le 1$ and the last term of (3.1) is nonpositive by (1.1), we have

$$\begin{aligned}\|\sqrt{w'}u'\|^2 \le \frac{1}{4}\|\sqrt{w'''}u\|^2 &+ \left\| \frac{Pu}{\sqrt{w'}} \right\| \|\sqrt{w'}u'\| \\ &+ \frac{1}{2}\left\| \frac{w'Pu}{\sqrt{w'''}} \right\| \|\sqrt{w'''}u\| + \varepsilon\|u'\|\|u\|.\end{aligned} \tag{3.10}$$

We first estimate the last term on the right using

$$\langle \Delta u, u\rangle = -\|\nabla u\|^2, \tag{3.11}$$

which gives

$$\|u'\|^2 \le \mathrm{Re}\langle Pu, u\rangle + E\|u\|^2 \le \left\| \frac{Pu}{\sqrt{w'''}} \right\| \|\sqrt{w'''}u\| + E\|u\|^2,$$

and

$$\varepsilon\|u\|^2 = -\,\mathrm{Im}\langle Pu, u\rangle \le \left\| \frac{Pu}{\sqrt{w'''}} \right\| \|\sqrt{w'''}u\|.$$

Combining these gives

$$\varepsilon^2\|u\|^2\|u'\|^2 \le (E+\varepsilon)\left\| \frac{Pu}{\sqrt{w'''}} \right\|^2 \|\sqrt{w'''}u\|^2,$$

and plugging into (3.10) gives

$$\begin{aligned}\|\sqrt{w'}u'\|^2 \le \frac{1}{4}\|\sqrt{w'''}u\|^2 + \left\| \frac{Pu}{\sqrt{w'}} \right\| \|\sqrt{w'}u'\| + \frac{1}{2}\left\| \frac{w'Pu}{\sqrt{w'''}} \right\| \|\sqrt{w'''}u\| \\ + \sqrt{E+\varepsilon}\left\| \frac{Pu}{\sqrt{w'''}} \right\| \|\sqrt{w'''}u\|.\end{aligned}$$

Now we use Hardy's inequality (3.8) to estimate all occurrences of $\|\sqrt{w'''}u\|$ on the right by $\|\sqrt{w'}u'\|$. We then cancel a factor of $\|\sqrt{w'}u'\|$ from all terms, and move the first term on the right over to the left. This gives

$$\frac{1}{2+\delta}\|\sqrt{w'}u'\| \le \left\|\frac{Pu}{\sqrt{w'}}\right\| + \frac{\sqrt{1+\delta}}{\sqrt{2+\delta}}\left\|\frac{w'Pu}{\sqrt{w'''}}\right\| + 2\frac{\sqrt{1+\delta}}{\sqrt{2+\delta}}\sqrt{E+\varepsilon}\left\|\frac{Pu}{\sqrt{w'''}}\right\|.$$

Now use $w' \le \delta$ and $w''' \le (1+\delta)(2+\delta)w'$ to combine terms:

$$\|\sqrt{w'}u'\| \le 2\sqrt{(1+\delta)(2+\delta)}\left(1+\delta+\sqrt{E+\varepsilon}\right)\left\|\frac{Pu}{\sqrt{w'''}}\right\|.$$

Using (3.8) again gives

$$\|\sqrt{w'''}u\| \le 4(1+\delta)\left(1+\delta+\sqrt{E+\varepsilon}\right)\left\|\frac{Pu}{\sqrt{w'''}}\right\|.$$

Plugging in the formula for w''' and using $\delta \le 1$ to simplify the constants gives

$$\|(1+x)^{-\frac{3+\delta}{2}}u\| \le \frac{3}{\delta}(1+\sqrt{E+\varepsilon})\left\|(1+x)^{\frac{3+\delta}{2}}Pu\right\|, \tag{3.12}$$

for all $u \in \mathcal{D}$ satisfying (3.9). For any $v \in L^2(\Omega)$, by (2.1) we may substitute $u = P^{-1}(1+x)^{-\frac{3+\delta}{2}}v$ into this last estimate to obtain

$$\|(1+x)^{-\frac{3+\delta}{2}}P^{-1}(1+x)^{-\frac{3+\delta}{2}}v\| \le \frac{3}{\delta}(1+\sqrt{E+\varepsilon})\|v\|,$$

for all $E \ge 0$ and $\varepsilon > 0$.

Applying the Phragmén–Lindelöf principle to the functions

$$z \mapsto \langle (1+x)^{-\frac{3+\delta}{2}}(-\Delta - z)^{-1}(1+x)^{-\frac{3+\delta}{2}}u, v\rangle/(1+\sqrt{-z}), \qquad u,\ v \in L^2(\Omega),$$

in the sectors

$$\{z \in \mathbb{C} \mid \alpha \operatorname{Re} z < |\operatorname{Im} z|\}, \qquad \alpha > 0,$$

(as in e.g. the end of the proof of (1.6) of [ChDa21]) gives the conclusion. □

4. Domains with multiple ends

The proofs of Theorems 2, 3, and 4 are more elaborate versions of the proof of Theorem 1. In comparison with the setting of Theorem 1, we still have the integration by parts identity (3.1), with which we control u'. However, we no longer have $x > 0$ throughout Ω, and hence can no longer use Hardy's inequality (3.8) to estimate u purely in terms of u'. To compensate for this, we apply (3.8) to a cut-off version of u in (4.3) below. For the proofs of Theorems 2 and 3, we choose the cut-off such that the resulting remainder term is supported where we have the flaring estimate (1.4). We then use variants of (3.1), proved in Lemmas 2 and 3 below, to control u in the flaring region by $\partial_\nu u$ on the boundary of the flaring region, and then the flaring estimate (1.4) and the original integration by parts identity (3.1) to control $\partial_\nu u$ on the boundary of the flaring region.

We begin by proving the new integration by parts identities that we will need.

Lemma 2. *Let $\mu = \mu(x)$ be C^1, real valued, and bounded with bounded derivative. Let $u \in \mathcal{D}$, and let $E,\ \varepsilon \in \mathbb{R}$. Then*

$$\begin{aligned}\langle \mu' u', u'\rangle + E\langle \mu' u, u\rangle = 2\operatorname{Re}\langle \mu P u, u'\rangle - 2\varepsilon \operatorname{Im}\langle \mu u, u'\rangle \\ + \sum_{j=1}^{d-1}\langle \mu' \partial_{y_j} u, \partial_{y_j} u\rangle + \int_{\partial\Omega} \mu |\partial_\nu u|^2 \nu_x.\end{aligned} \tag{4.1}$$

Proof. Let $u \in C_c^\infty(\overline{\Omega})$. Adding together the identities

$$\begin{aligned}\langle \mu' u', u'\rangle &= -2\operatorname{Re}\langle \mu u'', u'\rangle + \int_{\partial\Omega} \mu |u'|^2 \nu_x, \\ -\langle \mu' \partial_{y_j} u, \partial_{y_j} u\rangle &= 2\operatorname{Re}\langle \mu \partial_{y_j} u, \partial_{y_j} u'\rangle - \int_{\partial\Omega} \mu |\partial_{y_j} u|^2 \nu_x \\ &= -2\operatorname{Re}\langle \mu \partial_{y_j}^2 u, u'\rangle + \int_{\partial\Omega} \left(2\operatorname{Re} \mu \partial_{y_j} u \bar{u}' \nu_{y_j} - \mu |\partial_{y_j} u|^2 \nu_x\right), \\ E\langle \mu' u, u\rangle &= -2\operatorname{Re}\langle \mu E u, u'\rangle + E\int_{\partial\Omega} \mu |u|^2 \nu_x.\end{aligned}$$

gives

$$\begin{aligned}\langle \mu' u', u'\rangle + E\langle \mu' u, u\rangle = 2\operatorname{Re}\langle \mu P u, u'\rangle - 2\varepsilon \operatorname{Im}\langle \mu u, u'\rangle + \sum_{j=1}^{d-1}\langle \mu' \partial_{y_j} u, \partial_{y_j} u\rangle \\ + \int_{\partial\Omega} \Big(\mu |u'|^2 \nu_x + E\mu |u|^2 \nu_x + \sum_{j=1}^{d-1}\left(2\operatorname{Re} \mu \partial_{y_j} u \bar{u}' \nu_{y_j} - \mu |\partial_{y_j} u|^2 \nu_x\right)\Big).\end{aligned}$$

By density this holds for $u \in H^2_{\mathrm{comp}}$, and specializing to $u \in \mathcal{D}_{\mathrm{comp}}$ and using

$$u'|_{\partial\Omega} = \nu_x \partial_\nu u, \qquad \partial_{y_j} u|_{\partial\Omega} = \nu_{y_j} \partial_\nu u,$$

gives (4.1) for $u \in \mathcal{D}_{\text{comp}}$. To prove (4.1) for all $u \in \mathcal{D}$, use a partition of unity to write u as a locally finite sum of functions in $\mathcal{D}_{\text{comp}}$. □

Lemma 3. *Let $u,\ y_j u \in \mathcal{D}$, let $E,\ \varepsilon \in \mathbb{R}$, and let $j \in \{1, \dots, d-1\}$. Then*

$$\begin{aligned}\|\partial_{y_j} u\|^2 = \frac{1}{2}\operatorname{Re}\langle Pu, \partial_{y_j}(y_j u)\rangle + \frac{1}{2}\operatorname{Re}\langle y_j \partial_{y_j} u, Pu\rangle \\ + \varepsilon \operatorname{Im}\langle y_j \partial_{y_j} u, u\rangle + \frac{1}{2}\int_{\partial\Omega} y_j |\partial_\nu u|^2 \nu_{y_j}.\end{aligned} \tag{4.2}$$

Proof. This is proved in the same way as Lemma 1, but with the commutator $[w\partial_x, \partial_x^2]$ replaced by $[y_j\partial_{y_j}, \partial_{y_j}^2]$. □

Proof of Theorem 2. Fix $x_0 \in I$ and a closed interval $[x_0 - r, x_0 + r] \subset I$ and cut-off functions $\chi_0,\ \chi_1,\ \chi_2,\ \chi_3 \in C_c^\infty(I)$, taking values in $[0,1]$, such that $\chi_0 \equiv 1$ near $[x_0 - r, x_0 + r]$ and $\chi_{j+1} \equiv 1$ near $\operatorname{supp}\chi_j$ for $j = 0,\ 1,\ 2$. Below we abbreviate $\chi_j(x)$ as χ_j.

We claim that Hardy's inequality (3.8) implies

$$\|(1 + |x - x_0|)^{-\frac{3+\delta}{2}} u\| \le \frac{2}{2+\delta}\|(1 + |x - x_0|)^{-\frac{1+\delta}{2}} u'\| + C\|\chi_1 u\|. \tag{4.3}$$

Indeed, by (3.8) we have

$$\begin{aligned}&\|(1 + |x - x_0|)^{-\frac{3+\delta}{2}}(1 - \chi_0)u\| \\ &\quad \le \frac{2}{2+\delta}\|(1 + |x - x_0|)^{-\frac{1+\delta}{2}}(1 - \chi_0)u'\| \\ &\quad\quad + \frac{2}{2+\delta}\|(1 + |x - x_0|)^{-\frac{1+\delta}{2}}\chi_0' u\|,\end{aligned} \tag{4.4}$$

which combined with $\|(1 + |x - x_0|)^{-\frac{3+\delta}{2}}\chi_0 u\| \le \|\chi_1 u\|$ gives (4.3).

Now apply (3.1) with $w \in C^3(\mathbb{R})$ such that

- $w'(x) > 0$ and $xw(x) \ge 0$ for all x,
- and $w'(x) = \delta(1 + |x - x_0|)^{-1-\delta}$ when $|x - x_0| \ge r$.

Note that, with this choice of w, and for any fixed $\gamma > 0$, by (3.8) the first term on the right hand side of (3.1) obeys

$$
\begin{aligned}
\frac{1}{4}\langle w'''u, u\rangle &\leq \left(\frac{\delta(\delta+1)(\delta+2)}{4} + \gamma\right) \|(1+|x-x_0|)^{-\frac{3+\delta}{2}}(1-\chi_0)u\|^2 \\
&\quad + C\|\chi_0 u\|^2 \\
&\leq \left(\frac{\delta(\delta+1)}{2} + \frac{2\gamma}{\delta+2}\right) \|(1+|x-x_0|)^{-\frac{1+\delta}{2}}(1-\chi_0)u'\|^2 \\
&\quad + C\|\chi_1 u\|^2.
\end{aligned}
\tag{4.5}
$$

As long as $\delta < 1$ (which we may assume without loss of generality), we can choose γ small enough that

$$
\frac{\delta(\delta+1)}{2} + \frac{2\gamma}{\delta+2} < \delta.
$$

With γ so chosen, after plugging (4.5) into (3.1) we can subtract the first term on the right hand side of (4.5) to the left of (3.1), to obtain

$$
\begin{aligned}
&\|(1+|x|)^{-\frac{1+\delta}{2}}u'\|^2 - \int_{\partial\Omega} w|\partial_\nu u|^2\nu_x \\
&\qquad \lesssim \langle |Pu|, |u| + |u'|\rangle + \varepsilon\langle |u'|, |u|\rangle + \|\chi_1 u\|^2,
\end{aligned}
\tag{4.6}
$$

where, here and below, the implicit constants in $\lesssim$ are uniform for $u \in \mathcal{D}$, $\varepsilon \in (0,1]$, and $E \gg 1$ large enough. Then, using the fact that $w\nu_x \leq 0$ everywhere by (1.1), and that (1.4) implies $\nu_x w \leq -1/C < 0$ on $\operatorname{supp}\chi_3$, we obtain

$$
\begin{aligned}
&\|(1+|x|)^{-\frac{1+\delta}{2}}u'\|^2 + \int_{\partial\Omega} |\chi_3\partial_\nu u|^2 \\
&\qquad \lesssim \langle |Pu|, |u| + |u'|\rangle + \varepsilon\langle |u'|, |u|\rangle + \|\chi_1 u\|^2.
\end{aligned}
\tag{4.7}
$$

Adding (4.3) gives

$$
\begin{aligned}
&\|(1+|x|)^{-\frac{3+\delta}{2}}u\|^2 + \|(1+|x|)^{-\frac{1+\delta}{2}}u'\|^2 + \int_{\partial\Omega} |\chi_3\partial_\nu u|^2 \\
&\qquad \lesssim \langle |Pu|, |u| + |u'|\rangle + \varepsilon\langle |u'|, |u|\rangle + \|\chi_1 u\|^2.
\end{aligned}
\tag{4.8}
$$

The first two terms on the right side of (4.8) will be handled later as in the proof of Theorem 1. To handle the last term apply (4.1) with μ chosen to be nondecreasing such that $x\mu(x) \geq 0$, $\mu' = 1$ near $\operatorname{supp}\chi_1$, and

$\operatorname{supp}\mu' \subset \chi_2^{-1}(1)$. After discarding two terms with a favorable sign, that gives

$$E\|\chi_1 u\|^2 \le E\langle \mu' u, u\rangle \lesssim \langle |Pu|, |u'|\rangle + \varepsilon\langle |u|, |u'|\rangle + \sum_{j=1}^{d-1} \|\chi_2 \partial_{y_j} u\|^2, \tag{4.9}$$

which, combined with (4.8), gives

$$\begin{aligned} &\|(1+|x|)^{-\frac{3+\delta}{2}} u\|^2 + \|(1+|x|)^{-\frac{1+\delta}{2}} u'\|^2 + \int_{\partial\Omega} |\chi_3 \partial_\nu u|^2 \\ &\qquad \lesssim \langle |Pu|, |u| + |u'|\rangle + \varepsilon\langle |u'|, |u|\rangle + E^{-1} \sum_{j=1}^{d-1} \|\chi_2 \partial_{y_j} u\|^2. \end{aligned} \tag{4.10}$$

Now apply (4.2), with u replaced by $\chi_2 u$, and note that $[\partial_{y_j}, \chi_2] = 0$, to get, for each j,

$$\begin{aligned} \|\chi_2 \partial_{y_j} u\|^2 &\lesssim \langle |\chi_2 Pu| + |[P, \chi_2]u|, |\chi_2 u| + |\chi_2 \partial_{y_j} u|\rangle \\ &\quad + \varepsilon\langle |\chi_2 \partial_{y_j} u|, |\chi_2 u|\rangle + \int_{\partial\Omega} |\chi_2 \partial_\nu u|^2, \end{aligned} \tag{4.11}$$

which implies

$$\begin{aligned} \|\chi_2 \partial_{y_j} u\|^2 &\lesssim \langle |\chi_2 Pu|, |\chi_2 u| + |\chi_2 \partial_{y_j} u|\rangle + \varepsilon\langle |\chi_2 \partial_{y_j} u|, |\chi_2 u|\rangle \\ &\quad + \|\chi_3 u\|^2 + \|\chi_3 u'\|^2 + \int_{\partial\Omega} |\chi_2 \partial_\nu u|^2. \end{aligned} \tag{4.12}$$

This in turn implies, since $\varepsilon \in (0, 1]$,

$$\|\chi_2 \partial_{y_j} u\|^2 \lesssim \|\chi_2 Pu\|^2 + \|\chi_3 u\|^2 + \|\chi_3 u'\|^2 + \int_{\partial\Omega} |\chi_2 \partial_\nu u|^2. \tag{4.13}$$

Plugging (4.13) into (4.10) gives, for E large enough,

$$\begin{aligned} &\|(1+|x|)^{-\frac{3+\delta}{2}} u\|^2 + \|(1+|x|)^{-\frac{1+\delta}{2}} u'\|^2 + \int_{\partial\Omega} |\chi_3 \partial_\nu u|^2 \\ &\qquad \lesssim \langle |Pu|, |u| + |u'|\rangle + \varepsilon\langle |u'|, |u|\rangle + E^{-1}\|\chi_2 Pu\|^2. \end{aligned}$$

We now estimate the first two terms on the right in the same way that the last three terms in (3.10) were estimated in the proof of Theorem 1. Then dropping the last two terms on the left gives

$$\|(1+|x|)^{-\frac{3+\delta}{2}}u\|^2 \lesssim E\|(1+|x|)^{\frac{3+\delta}{2}}Pu\|^2.$$

□

Proof of Theorem 3. Here we use coordinates $(x, y) \in \mathbb{R}^2$. We begin as in the proof of Theorem 2, but when we get up to the analogue of (4.7) we have instead

$$\|(1+|x|)^{-\frac{1+\delta}{2}}u'\|^2 + \int_{\Gamma_F} |\chi_3 \partial_\nu u|^2 \lesssim \langle |Pu|, |u| + |u'|\rangle + \varepsilon\langle |u'|, |u|\rangle + \|\chi_1 u\|^2,$$

that is, the integral over $\partial\Omega$ is replaced by an integral over Γ_F. We then proceed as before, up to the analogue of (4.10), where we have instead

$$\begin{aligned}\|(1+|x|)^{-\frac{3+\delta}{2}}u\|^2 + \|(1+|x|)^{-\frac{1+\delta}{2}}u'\|^2 + \int_{\Gamma_F} |\chi_3\partial_\nu u|^2 \\ \lesssim \langle |Pu|, |u| + |u'|\rangle + \varepsilon\langle |u'|, |u|\rangle + E^{-1}\|\chi_2 \partial_y u\|^2.\end{aligned} \tag{4.14}$$

At this stage it does not work to apply (4.2) alone as in the proof of Theorem 2, with u replaced by $\chi_2 u$, as this produces a remainder $\int_{\partial\Omega} |\chi_2 \partial_\nu u|^2$ on the right which cannot be handled by the $\int_{\Gamma_F} |\chi_3 \partial_\nu u|^2$ we have on the left. To deal with this we will remove the part of the remainder over $\partial\Omega \setminus \Gamma_F$ using a multiple of the identity

$$0 = \operatorname{Re}\langle Pu, \partial_y u\rangle + \varepsilon \operatorname{Im}\langle \partial_y u, u\rangle + \frac{1}{2}\int_{\partial\Omega} |\partial_\nu u|^2 \nu_y, \tag{4.15}$$

which is just (4.2) with the commutator $[y\partial_y, \partial_y^2]$ replaced by $[\partial_y, \partial_y^2] = 0$.

More precisely, define cut-offs $\psi_k \in C_c^\infty(\overline{\Omega})$ such that $\psi_k = \chi_2$ on Ω_k and $\psi_k = 0$ otherwise. Multiply a_k by (4.15) applied to $\psi_k u$, and subtract the result from (4.2) applied to $\psi_k u$. Then use $[\psi_k, \partial_y] = 0$ as in (4.11) to get

$$\begin{aligned}\|\psi_k \partial_y u\|^2 \lesssim \langle |\psi_k Pu| + |[P, \psi_k]u|, |\psi_k u| + |\psi_k \partial_y u|\rangle + \varepsilon\langle |\psi_k \partial_y u|, |\psi_k u|\rangle \\ + \int_{\Gamma_F} |\psi_k \partial_\nu u|^2.\end{aligned}$$

Estimating as in (4.12) and (4.13) gives

$$\|\psi_k \partial_y u\|^2 \lesssim \|\psi_k Pu\|^2 + \|\chi_3 u\|^2 + \|\chi_3 u'\|^2 + \int_{\Gamma_F} |\psi_k \partial_\nu u|^2.$$

Summing in k, and plugging into (4.14), gives

$$\|(1+|x|)^{-\frac{3+\delta}{2}}u\|^2 + \|(1+|x|)^{-\frac{1+\delta}{2}}u'\|^2 + \int_{\Gamma_F} |\chi_3 \partial_\nu u|^2 \\ \lesssim \langle |Pu|, |u| + |u'| \rangle + \varepsilon \langle |u'|, |u| \rangle + E^{-1}\|\chi_2 Pu\|^2,$$

for E large enough, after which we conclude as in the end of the proof of Theorem 2. □

The proof of Theorem 4 is a further elaboration of the same ideas. The key point is that in the proofs of Theorems 2 and 3 above we did not use (1.1) directly, but rather used it to construct w such that $w\nu_x \geq 0$, $w' > 0$, and such that $w' = \delta(1+|x-x_0|)^{-1-\delta}$ away from the flaring set I. For a suitable (e.g. symmetric) convex obstacle in a straight planar waveguide this was done in Theorem 3. For a more general convex obstacle a more complicated construction of w is needed, and the set I will consist of three intervals chosen in the projection of the obstacle onto the x-axis and avoiding the points where $\nu_x = 0$: see Figure 5.

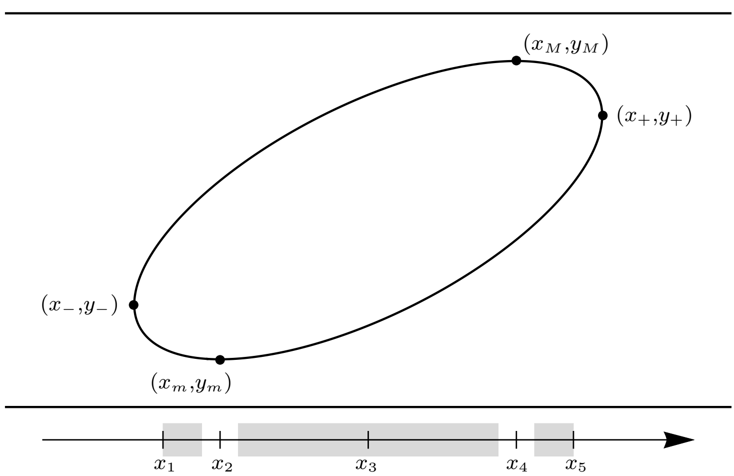

Figure 5: Notation used in the proof of Theorem 4. The shaded gray subset of $\mathbb{R}$ is K, and I is a small neighborhood of K.

The function w will be constructed from preliminary functions w_+ and w_- adapted to the upper and lower parts of Ω respectively.

Lemma 4. *Let $\delta > 0$, and let $-\infty < x_1 < x_2 < x_3 < x_4 < x_5 < \infty$. Then there are functions $w_\pm \in C^3(\mathbb{R})$ so that:*

- *For all* $x \in \mathbb{R}$, $w_\pm'(x) > 0$.
- *If* $x < x_1$ *or* $x > x_5$, *then* $w_+(x) = w_-(x)$ *and* $w_+'(x) = w_-'(x) = \delta(1 + |x - x_3|)^{-\delta-1}$.
- *Each of* x_2 *and* x_4 *is contained in an open interval on which* $w_+'(x) = w_-'(x) = \delta(1 + |x - x_3|)^{-\delta-1}$.
- *The equalities* $w_+(x_4) = 0$ *and* $w_-(x_2) = 0$ *hold.*

Proof. We will define $w_\pm$ via $w_+(x) = \int_{x_4}^x w_+'(t)dt$ and $w_-(x) = \int_{x_2}^x w_-'(t)dt$ in order to satisfy the last condition. Let $\rho_0 = \frac{1}{3}\min_{j=1,2,3,4}(x_{j+1} - x_j)$. Set

$$w_\pm'(x) = \begin{cases} \delta(1+|x-x_3|)^{-\delta-1} & \text{if } x < x_1 \text{ or } x > x_5 \\ \delta(1+|x-x_3|)^{-\delta-1} & \text{if } |x - x_2| < \rho_0 \\ & \text{or } \; |x - x_4| < \rho_0 \\ h_{\pm,1}(x) & \text{if } \; x_1 \le x \le x_2 - \rho_0 \\ h_{\pm,2}(x) & \text{if } \; x_2 + \rho_0 \le x \le x_4 - \rho_0 \\ h_{\pm,3}(x) & \text{if } \; x_4 + \rho_0 \le x \le x_5 \end{cases} \tag{4.16}$$

for some h's yet to be chosen. Now choose strictly positive $h_{\pm,1}$ $h_{\pm,2}$, $h_{\pm,3}$ so that the resulting functions $w_\pm'$ as defined above are C^2, $w_\pm'(x) > 0$ for all x, and so that $\int_{x_2}^{x_1} w_-'(t)dt = \int_{x_4}^{x_1} w_+'(t)dt$ and $\int_{x_2}^{x_5} w_-'(t)dt = \int_{x_4}^{x_5} w_+'(t)dt$. The conditions on the integrals guarantee that $w_+(x) = w_-(x)$ if $x < x_1$ or $x > x_5$. Satisfying this condition on the integrals may be accomplished by first choosing $h_{\pm,2}$, $h_{-,3}$, and $h_{+,1}$, and then choosing $h_{+,3}$ and $h_{-,1}$ so that the integral conditions are satisfied. □

Proof of Theorem 4. We begin by naming the coordinates of certain points on $\partial\mathcal{O}$ as in Figure 5. Set $y_M = \max\{y : (x, y) \in \overline{\mathcal{O}} \text{ for some } x \in \mathbb{R}\}$, $y_m = \min\{y : (x, y) \in \overline{\mathcal{O}} \text{ for some } x \in \mathbb{R}\}$, and let (x_M, y_M), (x_m, y_m) denote the corresponding points on $\partial\mathcal{O}$. Likewise, set $x_\pm = \pm\max\{\pm x : (x, y) \in \overline{\mathcal{O}}$ for some $y \in \mathbb{R}\}$, and let $(x_\pm, y_\pm)$ be the corresponding points in $\partial\mathcal{O}$. By the strict convexity of $\mathcal{O}$, each of these points is uniquely defined. Without loss of generality, we may assume $x_M \ge x_m$. Since the case $x_M = x_m$ is covered by Theorem 3, we assume for the remainder of the proof that $x_M > x_m$.

Let $r_1 = \frac{1}{3}\min(x_+ - x_M, x_M - x_m, x_m - x_-)$, and set

$$x_1 = x_- + r_1,\ x_2 = x_m,\ x_3 = \frac{x_m + x_M}{2},\ x_4 = x_M,\ x_5 = x_+ - r_1. \tag{4.17}$$

With these choices, set $w_\pm$ to be the functions given in Lemma 4. We shall use these $w_\pm$ to define a single function w on Ω, which is adapted to account for the fact that the "highest" and "lowest" points of $\partial\mathcal{O}$, (x_M, y_M) and (x_m, y_m), have different x coordinates. Otherwise, our function w will be very similar to the weights w we have used earlier.

We can write

$$\Omega \setminus \Big(\big((-\infty, x_-) \times \{y_-\}\big) \cup \big((x_+, \infty) \times \{y_+\}\big) \Big) = \Omega_+ \cup \Omega_-$$

where $\Omega_\pm$ are disjoint connected open sets. We label these so that $(-\infty, x_-) \times (y_-, 1) \subset \Omega_+$; that is, Ω_+ is the "upper" of the two components. Now define

$$w(x,y) = \begin{cases} w_+(x) & \text{if } (x,y) \in \Omega_+ \\ w_-(x) & \text{if } (x,y) \in \Omega_- \\ w_+(x) & \text{if } (x,y) \in (-\infty, x_-) \times \{y_-\} \\ & \text{or if } (x,y) \in (x_+, \infty) \times \{y_+\}. \end{cases} \tag{4.18}$$

By our choice of $w_\pm$, w is C^3 on Ω. Moreover, $w\nu_x \leq 0$ on $\partial\mathcal{O}$, with equality only at the points (x_m, y_m) and (x_M, y_M).

We now claim that, for $u \in \mathcal{D}$, (3.1) holds, even though our w is not independent of y. To see this, with $u, v \in C_c^\infty(\overline{\Omega})$ apply (3.6) on $\Omega_\pm$ with w replaced by $w_\pm$, and then add the resulting equalities. The boundary terms involving $\partial\Omega_\pm \setminus (\partial\Omega \cap \partial\Omega_\pm)$ sum to 0. The remainder of the proof follows as in the proof of Lemma 1. Alternatively, observe that the proof of Lemma 1 only used the fact that $\partial_y w \equiv 0$, and not that w is independent of y.

Set

$$K = \bigcup_\pm \operatorname{supp}(w'_\pm(x) - \delta(1 + |x - x_3|)^{-1-\delta}).$$

and note that by our choice of $w_\pm$, there is a $C_K > 0$ so that

$$w\nu_x \leq -C_K < 0 \text{ on } (K \times (-1,1)) \cap \partial\mathcal{O}.$$

Moreover, there is an open set I, $\overline{I} \subset (x_-, x_+) \subset \mathbb{R}$, $K \subset I$, and a $C_I > 0$ so that

$$w\nu_x \leq -C_I < 0 \text{ on } (I \times (-1,1)) \cap \partial\mathcal{O}.$$

The use of this I is very similar to the use of I of Theorem 3. Moreover, the union of curves $(I \times (-1,1)) \cap \partial\mathcal{O}$ plays a role similar to that of Γ_F from Theorem 3.

Now choose $\chi_0,\ \chi_1,\ \chi_2,\ \chi_3 \in C_c^\infty(I)$, taking values in $[0,1]$, so that $\chi_0 \equiv 1$ near K and $\chi_{j+1}\chi_j = \chi_j$ for $j = 0,\ 1,\ 2$.

Next, we note that (4.4) is valid for our χ_0, which in turn implies that (4.3) is valid for our χ_1. Then, just as in the proof of (4.6), if $\delta < 1$ we can show that

$$\|(1+|x|)^{-\frac{1+\delta}{2}}u'\|^2 - \int_{\partial\Omega} w|\partial_\nu u|^2\nu_x \lesssim \langle |Pu|, |u|+|u'|\rangle + \varepsilon\langle |u'|,|u|\rangle + \|\chi_1 u\|^2$$

and

$$-\int_{\partial\Omega} w|\partial_\nu u|^2\nu_x \geq C_I \int_{(I\times(-1,1))\cap\partial\mathcal{O}} |\partial_\nu u|^2.$$

Proceeding as in the proofs of Theorems 2 and 3, we get to the analogue of (4.8):

$$\begin{aligned}\|(1+|x|)^{-\frac{3+\delta}{2}}u\|^2 + \|(1+|x|)^{-\frac{1+\delta}{2}}u'\|^2 + \int_{(I\times(-1,1))\cap\partial\mathcal{O}} |\chi_3\partial_\nu u|^2 \\ \lesssim \langle |Pu|, |u|+|u'|\rangle + \varepsilon\langle |u'|,|u|\rangle + \|\chi_1 u\|^2.\end{aligned}$$

We now apply (4.1) with $\mu \in C^1(\Omega)$ chosen such that $\partial_y\mu \equiv 0$, $\mu(x_m,y_m) = \mu(x_M,y_M) = 0$, $\mu' \geq 0$, $\mu' = 1$ near $\operatorname{supp}\chi_1$, and $\operatorname{supp}\mu' \subset \chi_2^{-1}(1)$. The construction of such a μ follows along the same lines as (but is simpler than) the construction of w above. That gives (4.9), after which the proof proceeds just like the proof of Theorem 3. □

5. Absence of eigenvalues and embedded resonances

For the proof of Theorem 7, we use a variant of Lemma 1. For $R > 0$, let

$$\Omega_R := \{(x,y) \in \Omega : \ |x| < R\}.$$

Denote $\|v\|^2_{L^2(\Omega_R)} = \int_{\Omega_R} |v|^2$ and $\langle u,v\rangle_{\Omega_R} = \int_{\Omega_R} u\bar{v}$.

Lemma 5. *Let $u \in \mathcal{D}$, $R > 0$, and let E, $\varepsilon \in \mathbb{R}$. Then with ∇_y denoting the gradient in the y variables only,*

(5.1)
$$\begin{aligned}\|u'\|^2_{L^2(\Omega_R)} =& \frac{1}{2}\operatorname{Re}\langle Pu, (xu)'\rangle_{\Omega_R} + \frac{1}{2}\operatorname{Re}\langle xu', Pu\rangle_{\Omega_R} \\ &+ \varepsilon\operatorname{Im}\langle xu', u\rangle_{\Omega_R} + \frac{1}{2}\int_{\partial\Omega\cap\partial\Omega_R} x|\partial_\nu u|^2\nu_x \\ &+ \frac{1}{2}\sum_{\pm}\operatorname{Re}\int_{\partial\Omega_R\cap\{x=\pm R\}} \left(\pm u'\bar{u} + R(-|\nabla_y u|^2 + E|u|^2 + |u'|^2)\right).\end{aligned}$$

Moreover,

$$0 = \operatorname{Im}\langle Pu, (xu)'\rangle_{\Omega_R} + \operatorname{Im}\langle xu', Pu\rangle_{\Omega_R} - 2\varepsilon \operatorname{Re}\langle xu', u\rangle_{\Omega_R} + \sum_{\pm} \int_{\partial\Omega_R \cap \{x=\pm R\}} \varepsilon R|u|^2 \pm \operatorname{Im} u'\overline{u}. \tag{5.2}$$

Proof. The proof of (5.1) is essentially that of Lemma 1. In particular, we use (3.6), replacing Ω by Ω_R. In addition, we use $w(x) = x$. Then following the outline of Lemma 1 and taking real parts gives (5.1). The equality (5.2) follows from the same argument, but taking the imaginary part of the resulting equation, rather than the real part. □

Proof of Theorem 7. We give a proof by contradiction. Suppose the Dirichlet Laplacian on Ω has an eigenvalue E_1 or a resonance embedded in the continuous spectrum at E_1. Let u be an associated eigenfunction or outgoing resonance state. Then by separation of variables there are constants γ_j so that

$$u(x,y) = \sum_{\sigma_j^2 < E_1} \gamma_j e^{i|x|\sqrt{E_1-\sigma_j^2}} \phi_j(y) + \sum_{\sigma_j^2 \geq E_1} \gamma_j e^{-|x|\sqrt{\sigma_j^2 - E_1}} \phi_j(y), \quad |x| \geq R_0, \tag{5.3}$$

with notation as in (1.6) and (1.8). To see (5.3), recall that the outgoing condition says that

$$u|_{\{\pm x > R_0\}} = \left(\lim_{\varepsilon \downarrow 0} (-\Delta_\pm - E_1 - i\varepsilon)^{-1} f_\pm \right) \Big|_{\{\pm x > R_0\}}, \tag{5.4}$$

for some $f_\pm \in L^2_{\text{comp}}(\mathbb{R}_\pm \times Y_\pm)$, where $-\Delta_\pm$ is the Dirichlet resolvent on $\mathbb{R}_\pm \times Y_\pm$. (The choice of outgoing condition is not essential, and one could also use an incoming condition, replacing i by $-i$ in (5.4) and in (5.3)). Furthermore, by (5.4), the sum over $\sigma_j^2 \geq E_1$ in (5.3) converges in $L^2(\Omega \cap \{|x| > R_0\})$, and hence we have

$$\sum_{\sigma_j^2 > E_1} |\gamma_j|^2 e^{-2R_0\sqrt{\sigma_j^2 - E_1}} (\sigma_j^2 - E_1)^{-1/2} < +\infty. \tag{5.5}$$

We first prove that E_1 cannot be an eigenvalue, since this is easier.

Suppose E_1 is an eigenvalue, so that u is an associated eigenfunction. Since $u \in L^2$, we must have $\gamma_j = 0$ when $\sigma_j \leq E_1$. By (5.5), u and its derivatives tend to 0 exponentially in $|x|$.

Using Lemma 5 with $E = E_1$ and $\varepsilon = 0$ we find

(5.6)
$$\|u'\|^2_{L^2(\Omega_R)} = \frac{1}{2}\int_{\partial\Omega\cap\partial\Omega_R} x|\partial_\nu u|^2\nu_x + \frac{1}{2}\sum_{\pm}\mathrm{Re}\int_{\partial\Omega_R\cap\{x=\pm R\}} \left(\pm u'\overline{u} + R(-|\nabla_y u|^2 + E_1|u|^2 + |u'|^2)\right).$$

Taking the limit in as $R \to \infty$ and using the exponential decay of u and its derivatives gives
$$\|u'\|^2_{L^2(\Omega)} = \frac{1}{2}\int_{\partial\Omega} x|\partial_\nu u|^2\nu_x \le 0.$$
But this means that u is independent of x. Since $u \in L^2(\Omega)$ is nontrivial, this is a contradiction.

The argument for showing there are no resonances embedded in the continuous spectrum is similar, but requires some further computations.

Suppose u is a resonance state associated to $E_1 \in \mathbb{R}$. Applying (5.2) with $E = E_1$, $\varepsilon = 0$ and using (5.3) along with the orthonormality of $\{\phi_j\}$ gives, for sufficiently large R,

$$0 = \mathrm{Im}\sum_{\pm} \pm \int_{\partial\Omega_R\cap\{x=\pm R\}} u'\overline{u} = \sum_{\sigma_j^2 < E_1} \sqrt{E_1 - \sigma_j^2}|\gamma_j|^2,$$

which in turn implies that $\gamma_j = 0$ if $\sigma_j^2 < E_1$.

Returning to (5.3), note again that by (5.5) the terms with $\sigma_j^2 > E$ are exponentially decaying in $|x|$ along with their derivatives, while those with $\sigma_j^2 = E_1$ have x derivative 0. From (5.3) and using these observations,

$$\int_{\partial\Omega_R\cap\{x=\pm R\}} \left(\pm u'\overline{u} + R((-|\nabla_y u|^2 + E_1|u|^2 + |u'|^2)\right) = \int_{\partial\Omega_R\cap\{x=\pm R\}} \left(\pm u'\overline{u} + R((\Delta_y u)\overline{u} + E_1|u|^2 + |u'|^2)\right),$$

is exponentially decreasing in R. Again taking the limit of (5.6) as $R \to \infty$ we have
$$\|u'\|^2_{L^2(\Omega)} = \frac{1}{2}\int_{\partial\Omega} x|\partial_\nu u|^2\nu_x \le 0,$$
so that $u' \equiv 0$. But a nontrivial u with $u' \equiv 0$ and $-\Delta u = E_1 u$ can only satisfy Dirichlet boundary conditions on $\partial\Omega$ if $\partial\Omega$ is invariant under translation in the x direction; that is, $\Omega = \mathbb{R} \times \widetilde{Y}$ for some $\widetilde{Y} \subset \mathbb{R}^{d-1}$. □

6. Resonance-free regions

For a domain $\Omega \subset \mathbb{R}^d$ which has cylindrical ends, the resolvent of the Dirichlet Laplacian $(-\Delta - z)^{-1}$ has a meromorphic continuation to a Riemann surface $\hat{Z}$. Resolvent estimates of the type of Theorems 1, 2, 3, and 4 imply, essentially via an application of [ChDa21, Theorem 5.6], that there is a region near the continuous spectrum in which the meromorphic continuation of the resolvent is in fact analytic. To make a precise statement, we first introduce the space to which the resolvent continues.

The continuous spectrum of $-\Delta$ is given by $[\sigma_1^2, \infty)$, where σ_1^2 is the smallest Dirichlet eigenvalue of $-\Delta_Y$. For general domains or manifolds with cylindrical ends, there may, in addition, be eigenvalues of $-\Delta$, either in $(0, \sigma_1^2)$ or embedded in the continuous spectrum. For $z \in \mathbb{C}$ so that z is not in the spectrum of $-\Delta$ set $R(z) = (-\Delta - z)^{-1} : L^2(\Omega) \to L^2(\Omega)$. As an operator from $L^2_{\mathrm{comp}}(\Omega)$ into $L^2_{\mathrm{loc}}(\Omega)$, the resolvent $R(z)$ has a meromorphic continuation to the Riemann surface $\hat{Z}$ which we describe next.

The Riemann surface $\hat{Z}$ is determined by the set $\{\sigma_j^2\}$ of Dirichlet eigenvalues of $-\Delta_Y$. For $z \in \mathbb{C} \setminus [\sigma_1^2, \infty)$, define $\tau_j(z) = (z - \sigma_j^2)^{1/2}$, where we take the square root to have positive imaginary part. Then $\hat{Z}$ is the minimal Riemann surface so that for each $j \in \mathbb{N}$, $\tau_j(z)$ is an analytic, single-valued function on $\hat{Z}$. The Riemann surface $\hat{Z}$ forms a countable cover of $\mathbb{C}$, ramified at points corresponding to σ_j^2, $j \in \mathbb{N}$. For any $z \in \hat{Z}$, $\operatorname{Im} \tau_j(z) > 0$ for all but finitely many j. We call the "physical region" the portion of $\hat{Z}$ in which $\operatorname{Im} \tau_j(z) > 0$ for all $j \in \mathbb{N}$. In the physical region and away from eigenvalues of $-\Delta$, $R(z)$ is a bounded operator on $L^2(\Omega)$. For further details about the construction of $\hat{Z}$ and a proof that the resolvent of $-\Delta$ on Ω has a meromorphic continuation to $\hat{Z}$, see [Gu89], [Me93, Section 6.7], or [ChDa23, Section 2].

We define a distance on $\hat{Z}$ as follows: for $z,\ z' \in \hat{Z}$,

$$d(z, z') := \sup_j |\tau_j(z) - \tau_j(z')|. \tag{6.1}$$

That this is a metric is shown in [ChDa21, Section 5.1].

For $E > |\sigma_1|$, denote by $E \pm i0$ the points in $\hat{Z}$ which are on the boundary of the physical region and which are obtained as limits $\lim_{\pm\delta\downarrow 0} E + i\delta$. These points correspond to the continuous spectrum of $-\Delta$. If $E > \sigma_j^2$, then $\pm\tau_j(E \pm i0) > 0$, and if $\sigma_j^2 > E$ then $\tau_j(E \pm i0) \in i\mathbb{R}_+$.

The next theorem describes quantitatively a region near the boundary of the physical space in which the resolvent is guaranteed to be analytic.

Theorem 8. *Let $\Omega \subset \mathbb{R}^d$ be a domain with cylindrical ends which in addition satisfies the conditions of one of Theorem 1, 2, 3, or 4, and let $\chi \in L^\infty_{comp}(\Omega)$. Then there are positive constants C_1, C_2, and E_0 so that $\chi R(z)\chi$ is analytic in $\{z \in \hat{Z} : d(z, E \pm i0) < C_1(1+E)^{-1}\}$ for all $E \geq E_0$, and in this same region $\|\chi R(z)\chi\| \leq C_2(1+E)^{1/2}$.*

After a semiclassical rescaling, the proof of this theorem is the same as the proof of [ChDa21, Theorem 5.6]. More specifically, we write $(-\Delta - E) = h^{-2}(-h^2\Delta - 1)$ with $h = E^{-1/2}$. Then the $O(E^{1/2})$ resolvent bound implied by Theorem 1, 2, 3, or 4 corresponds to a $O(h^{-3})$ resolvent bound for the scaled operator.

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