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Duals of non-zero square

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In this short note, for each non-zero integer n, we construct a 4-manifold containing a smoothly concordant pair of spheres with a common dual of square n but no automorphism carrying one sphere to the other. Our examples, besides showing that the square zero assumption on the dual is necessary in Gabai's and Schneiderman-Teichner's versions of the 4D Light Bulb Theorem, have the interesting feature that both the Freedman-Quinn and Kervaire-Milnor invariant of the pair of spheres vanishes. The proof gives a surprising application of results due to Akbulut-Matveyev and Auckly-Kim-Melvin-Ruberman pertaining to the well-known Mazur cork.

1. Introduction and Motivation

We work throughout in the smooth, oriented category. Begin by considering a pair of homotopic 2-spheres S and T embedded in a smooth 4-manifold X, with an embedded 2-sphere $G \subset X$ intersecting both S and T transversally in a single point. Such a sphere is called a ***common dual*** of S and T. Recent work of Gabai [12] and Schneiderman-Teichner [25] has completely characterized the conditions under which the spheres S and T are isotopic, so long as their common dual G has square zero, i.e. a trivial normal bundle, in the 4-manifold X. We call such a dual ***standard***, and ***non-standard*** otherwise.

Although recent results about standard duals are numerous (see [12], [25], [13], and [18] for instance), general results about isotopy in the presence of non-standard duals are of interest but not currently known. For instance, a "homotopy implies isotopy" result for spheres with common dual of square $+1$ would imply, by work of Melvin [22], the longstanding conjecture that the Gluck twist [14] of any sphere in S^4 is standard. The Main Theorem of this note implies, however, that the assumption that there is a standard dual in the 4D light bulb theorems of [12] and [25] is a necessary one, in the absence of added assumptions.

Main Theorem. *For each $n \neq 0$, there exists a 4-manifold X_n (either closed or bounded) containing smoothly concordant embedded spheres S_n and T_n with a common dual of square n such that there is no automorphism of X_n carrying one sphere to the other.*

The proof of our Main Theorem gives a surprising application of well-studied 4-dimensional objects called ***corks***: compact contractible 4-manifolds C equipped with an orientation preserving diffeomorphism $h\colon \partial C \to \partial C$. The study of corks was initially motivated by the fact that the ***cork twist*** $X_{C,h} = (X - \text{int}(C)) \cup_h C$ of an embedded cork $C \subset X$ is homeomorphic to X by Freedman [9], but need not be diffeomorphic to X by Akbulut [1]. Such an embedding of a cork is called ***non-trivial***. Our construction builds upon examples given by Akbulut and Matveyev [3] of non-trivial embeddings of corks.

2. Warm-up

The first example of a cork with a non-trivial embedding was produced by Akbulut in [1] using as the cork a contractible manifold often called the "Mazur manifold" (named so for Mazur, who first built it using a single 1 and 2-handle in [21]). Now ubiquitous in the literature, the "Akbulut-Mazur cork" (W, τ) consists of the Mazur manifold W in Figure 1, together with the involution τ on its boundary induced by a rotation of π around the indicated axis of symmetry. Many 4-manifolds are now known to admit non-trivial embeddings of the Mazur cork; we outline one such embedding due to Akbulut and Matveyev [3] as a warm-up to the proof of the Main Theorem.

Let X denote the compact 4-manifold shown on the left in Figure 2, built from the Mazur manifold W by adding a single 2-handle. Note that X has a handlebody decomposition consisting of a single 1-handle, and two 2-handles each attached along knots in $S^1 \times S^2$ with framings less than their maximum Thurston-Bennequin numbers, as illustrated on the bottom right of Figure 3. Therefore X is a compact Stein domain, by a result of Eliashberg [7]; see also [15] for more exposition. For a precise definition of what we mean by "compact Stein domain", refer to [2].

On the other hand, the cork twist $X_{W,\tau}$ contains an embedded 2-sphere of square -1, seen in the diagram for $X_{W,\tau}$ in Figure 2 as the union of the shaded disk D and the core of the 2-handle attached along ∂D. Therefore $X_{W,\tau}$ must not be a compact Stein domain. This follows from a result due to Lisca and Matić [20] that compact Stein domains embed in minimal, closed

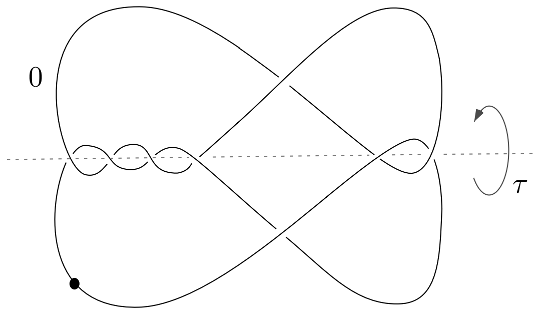

Figure 1. The Akbulut-Mazur cork (W, τ).

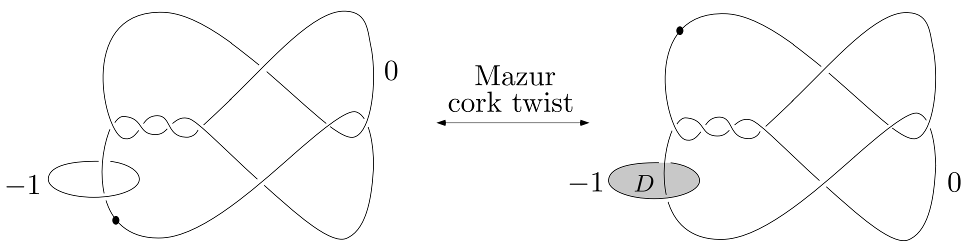

Figure 2. The manifold X (left) and the cork twist $X_{W,\tau}$ (right).

Kähler surfaces, which contain no smoothly embedded 2-spheres of square -1. This follows from results from [5], [8], and [11] that Kähler surfaces have non-vanishing Seiberg-Witten invariant, together with the fact due to [19] and [24] that surfaces in Kähler surfaces must satisfy the adjunction inequality. Therefore, X and $X_{W,\tau}$ are not diffeomorphic.

3. Main theorem

To contextualize our main result, we outline the previous results about common duals referred to in Section 1. By Gabai [12] and Schneiderman-Teichner [25], the existence of a common standard dual for homotopic spheres $S, T \subset X$ guarantees a smooth isotopy between S and T whenever the ***Freedman-Quinn invariant***, a concordance invariant defined in [10], of the pair (S, T)

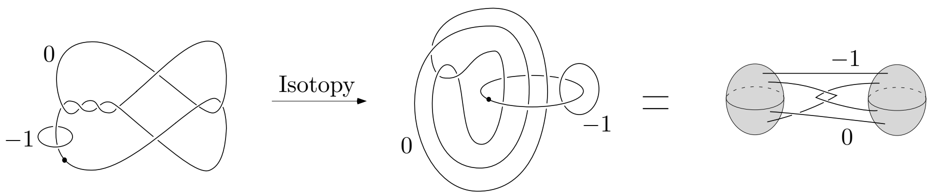

Figure 3. Identical handlebody structures for X, drawn with (left and middle) and without (right) the dotted circle notation for 1-handles from [16, Chapter I.2]. The Thurston-Bennequin framing of the attaching circle of each 2-handle is computed from the rightmost diagram using the usual formula (writhe) - (number of right cusps).

vanishes. Recent work of Gabai [13] and Kosanović - Teichner [18] (which extends to higher dimensional cases) shows that an analogous result holds for certain properly embedded disks with a common standard dual and vanishing ***Dax invariant***, an isotopy invariant of properly embedded disks recently formulated by Gabai in [13] using homotopy theoretic work of Dax [6] from the 70's. To guarantee even a smoothly embedded concordance between S and T when their common dual is non-standard, it is also required that their ***Kervaire-Milnor invariant***, defined by Stong in [28], vanishes.

3.1. Remark. Recently, Klug and Miller [17, Example 7.2] pointed out that it is necessary that the dual have square zero for Gabai [12] and Schneiderman-Teichner [25] to achieve an isotopy, by presenting a pair of spheres whose common dual of square +1, with vanishing Freedman-Quinn invariant but non-vanishing Kervaire-Milnor invariant. On the other hand, for each $n \neq 0$, the Main Theorem gives examples of pairs of spheres with dual of square n whose Freedman-Quinn invariant and Kervaire-Milnor invariants vanish, but that are not related by any automorphism of the ambient 4-manifold. Such an automorphism always exists for spheres when the common dual is standard by [26, Lemma 2.3], since in this case the common dual can be surgered (Gabai remarks after [13, Theorem 0.8] that by a similar proof, this also holds for properly embedded disks with a common standard dual).

Proof of Main Theorem. For $n \leq -1$, consider the 4-manifold X_n pictured in Figure 4. Since X_n is simply-connected, the spheres S_n and T_n are not

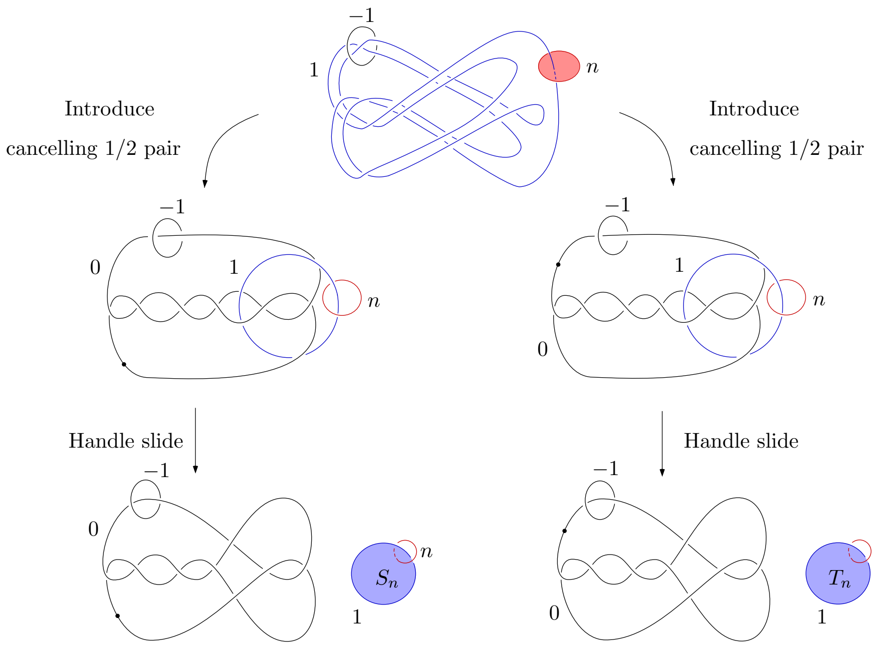

Figure 4. The spheres S_n and T_n (blue) in X_n, with their common dual (red).

only homologous, but also homotopic. It is also immediate that the both the Kervaire-Milnor and Freedman-Quinn invariants of the pair (S_n, T_n) vanish, since these invariants are elements of $H_1(X_n; \mathbb{Z}_2)$ and a quotient of $\mathbb{Z}[\pi_1(X)]$, respectively, which are both trivial in this case. Let R_n denote the sphere of square n gotten by capping off the red disk in Figure 4 with the core of the 2-handle attached with framing n along its boundary. The sphere R_n is dual to both S_n and T_n, since S_n and T_n each pass once (geometrically) over the 2-handle with framing 1 in the topmost diagram of Figure 4. Therefore, by [10] and [28], the spheres S_n and T_n are smoothly concordant in $X_n \times I$.

The manifold X_n contains Akbulut and Matveyev's manifold X [3] discussed in Section 2. To show that there is no automorphism of X_n carrying S_n to T_n, we use an argument similar to one of Auckly-Kim-Melvin-Ruberman [4, Theorem A]; see in particular Figure 18 of their paper. For, blowing down S_n gives the bottom left manifold of Figure 5, which is not Stein since it contains an embedded sphere of square -1, as in the argument from Section 2. On the other hand, blowing down T_n gives the bottom

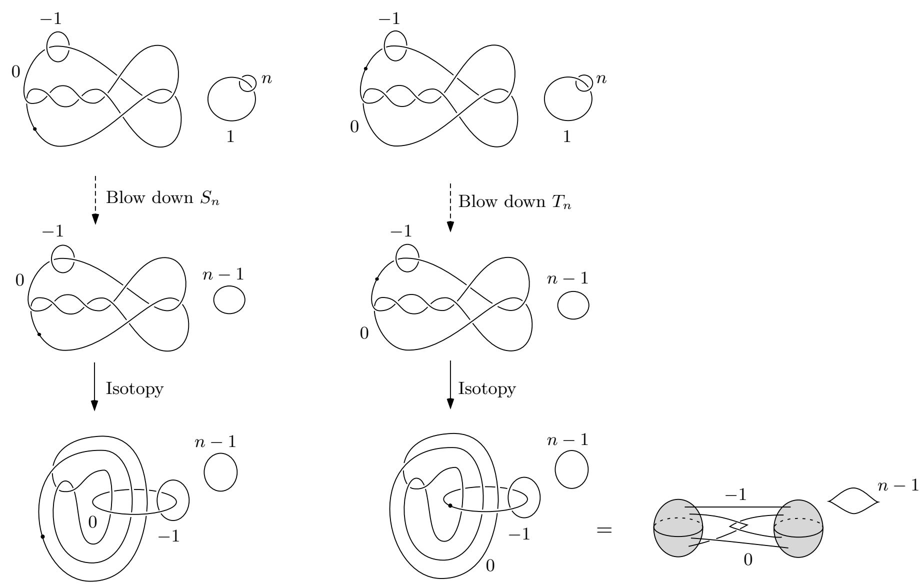

Figure 5. Blowing down the spheres S_n and T_n

right manifold of Figure 5, which is Stein whenever $n \leq -1$ by [7], since all 2-handles are attached along Legendrian knots whose framings are strictly less than their Thurston-Bennequin numbers.

As the manifolds that result from blowing down S_n and T_n are not diffeomorphic, there can be no automorphism of X_n carrying one sphere to the other when $n \leq -1$. The result therefore also holds for $n \geq 1$, setting $X_n = -X_{-n}$ and considering the spheres $S_n, T_n \subset X_n$ that are the images of the spheres $S_{-n}, T_{-n} \subset X_{-n}$ under the (orientation reversing) identity map.

To prove the result in the *closed* case, note that for each n, the Stein manifold on the bottom right of Figure 5 embeds in a closed Kähler manifold by Lisca and Matić [20], with complement K_n. Copies of S_n and T_n sit naturally in the union $X_n \cup -K_n$, and are not smoothly isotopic: blowing down T_n is Kähler, whereas blowing down S_n gives a manifold that is not Kähler (as it contains the sphere of square -1). □

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