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Instability of the solitary waves for the 1d NLS with an attractive delta potential in the degenerate case

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In this paper, we show the orbital instability of the solitary waves $Q_\Omega e^{i\Omega t}$ of the 1d NLS with an attractive delta potential ($\gamma > 0$)

$$\mathrm{i}\, u_t + u_{xx} + \gamma\delta u + |u|^{p-1}u = 0,\ p > 5,$$

where $\Omega = \Omega(p,\gamma) > \frac{\gamma^2}{4}$ is the critical oscillation number and determined by

$$\frac{p-5}{p-1}\int_{\operatorname{arctanh}\left(\frac{\gamma}{2\sqrt{\Omega}}\right)}^{+\infty} \operatorname{sech}^{\frac{4}{p-1}}(y)\,\mathrm{d}y = \frac{\gamma}{2\sqrt{\Omega}}\left(1-\frac{\gamma^2}{4\Omega}\right)^{-\frac{p-3}{p-1}}$$
$$\Longleftrightarrow \mathbf{d}''(\Omega) = 0.$$

The classical convex method and Grillakis-Shatah-Strauss's stability approach in [2, 10] doesn't work in this degenerate case, and the argument here is motivated by those in [5, 16, 17, 22, 23]. The main ingredients are to construct the unstable second order approximation near the solitary wave $Q_\Omega e^{i\Omega t}$ on the level set $\mathcal{M}(Q_\Omega)$ according to the degenerate structure of the Hamiltonian and to construct a refined Virial identity to show the orbital instability of the solitary waves $Q_\Omega e^{i\Omega t}$ in the energy space. Our result is the complement of the results in [8] in the degenerate case.

1. Introduction

In this paper, we consider the 1d nonlinear Schrödinger with a delta potential

$$\begin{cases} \mathrm{i}\, u_t + u_{xx} + \gamma\delta u + \mu|u|^{p-1}u = 0, & (t,x)\in\mathbb{R}_+\times\mathbb{R},\\ u(0,x) = u_0(x)\in H^1(\mathbb{R}), \end{cases} \tag{1.1}$$

where u is a complex-valued function of (t,x), $\gamma\in\mathbb{R}\backslash\{0\}$, δ is the Dirac delta distribution at the origin, $\mu = \pm 1$ and $1 < p < \infty$. For $\gamma \neq 0$, (1.1) appears in various physical models with a point defect on the line, for instance, nonlinear optics [9] and references therein. For the case $\gamma < 0$, it corresponds to the repulsive delta potential, while for the case $\gamma > 0$ it is attractive.

There are many results about (1.1). Local well-posedness for (1.1) in the energy space $H^1(\mathbb{R})$ is well understood by Cazenave in [4], Fukuizumi, Ohta and Ozawa in [8] and Masaki, Murphy and Segata in [20]. More precisely, we have

Proposition 1.1 (Local well-posedness in $H^1(\mathbb{R})$). *For any $u_0 \in H^1(\mathbb{R})$, there exists $T_{\max}$ with $0 < T_{\max} \leqslant +\infty$ and a unique solution $u \in \mathcal{C}\left([0, T_{\max}), H^1(\mathbb{R})\right) \cap \mathcal{C}^1\left([0, T_{\max}), H^{-1}(\mathbb{R})\right)$ for (1.1) satisfying*

$$\text{either } T_{\max} = +\infty, \text{ or } T_{\max} < +\infty \text{ and } \lim_{t \nearrow T_{\max}} \|\partial_x u(t)\|_2 = +\infty.$$

Moreover, the mass and the energy are conserved under the flow generated by (1.1), i.e., for any $t \in [0, T_{\max})$, we have

$$\text{(1.2)} \quad \mathcal{M}(u(t)) := \frac{1}{2}\int_{\mathbb{R}} |u(t,x)|^2 \mathrm{d}x = \mathcal{M}(u_0),$$

$$\text{(1.3)} \quad \mathcal{E}(u(t)) := \int_{\mathbb{R}} \left[\frac{1}{2}|u_x(t,x)|^2 - \frac{\gamma}{2}\delta(x)|u(t,x)|^2 - \frac{\mu}{p+1}|u(t,x)|^{p+1}\right]\mathrm{d}x = \mathcal{E}(u_0).$$

By the Gagliardo-Nirenberg inequality and the conservation laws, we have the global well-posedness of (1.1) in the energy space $H^1(\mathbb{R})$ for $1 < p < 5$.

In addition, for the repulsive potential case $\gamma < 0$, equation (1.1) is also studied from the point of view of scattering. Banica and Visciglia proved the global well-posedness and scattering result of the energy solution of (1.1) for the defocusing mass-supercritical nonlinearity $\mu < 0$, $p > 5$ in [3]. Ikeda and Inui obtained the scattering result of the energy solution of (1.1) below the ground state threshold for the focusing mass-supercritical nonlinearity $\mu > 0$, $p > 5$ in [13]. Masaki, Murphy and Segata showed the decay and modified scattering result of the solution of (1.1) with small initial data for $p = 3$ in a weighted space in [19], and recently established asymptotic stability for all solitary waves under a suitable spectral assumption in [18]. One can also refer the instability of the solitary waves of (1.1) for $p > 1$ to [7, 15, 25].

Such results are not expected for the attractive case $\gamma > 0$ because of the existence of the eigenvalue $-\frac{1}{4}\gamma^2$ of the Schrödinger operator $-\partial_x^2 - \gamma\delta$ (see [6, 12, 20] and the references therein). In this paper, we will focus on the attractive delta potential ($\gamma > 0$) and the focusing nonlinearity ($\mu = 1$) and consider the stability/instability of the nonlinear solitary wave solutions for (1.1) with the following form

$$u(t,x) = \mathrm{e}^{\,\mathrm{i}\,\omega t} Q_\omega(x).$$

It is easy to verify that Q_ω satisfies

$$-\partial_x^2 Q_\omega(x) + \omega Q_\omega(x) - \gamma\delta(x) Q_\omega(x) - |Q_\omega(x)|^{p-1} Q_\omega(x) = 0. \tag{1.4}$$

For the case $\omega > \frac{\gamma^2}{4}$, there exists a unique positive, radial symmetric solution to (1.4) which can be explicitly described as following (see [7–9, 15, 20])

$$Q_\omega(x) = \left[\frac{(p+1)\omega}{2} \operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}|x| + \operatorname{arctanh}\left(\frac{\gamma}{2\sqrt{\omega}}\right)\right)\right]^{\frac{1}{p-1}}. \tag{1.5}$$

The stability of Q_ω is a crucial problem during the study of the dynamics of the flow induced by (1.1). We firstly recall the definition of the orbital stability/instability in order to show the orbital stability/instability of the solitary waves in the energy space.

Definition 1.2. *The solitary wave* $e^{i\omega t} Q_\omega(x)$ *of* (1.1) *is said to be orbitally stable in* $H^1(\mathbb{R})$ *if for any* $\alpha > 0$*, there exists* $\beta = \beta(\alpha) > 0$ *such that for any solution* $u(t)$ *to* (1.1) *with initial data* $u_0 \in \mathcal{U}(Q_\omega\,,\ \beta)$*, we have*

$$u(t) \in \mathcal{U}(Q_\omega\,,\ \alpha), \quad \textit{for any}\ \ t \geqslant 0,$$

where

$$\mathcal{U}(Q_\omega\,,\ \alpha) = \left\{ u \in H^1(\mathbb{R}) \,\middle|\, \inf_{\theta\in\mathbb{R}} \|u(\cdot) - Q_\omega(\cdot)\,\mathrm{e}^{\mathrm{i}\theta}\|_{H^1} < \alpha \right\}. \tag{1.6}$$

Otherwise, the solitary wave $e^{i\omega t} Q_\omega(x)$ *is said to be orbitally unstable in* $H^1(\mathbb{R})$.

For (1.1) with the cubic nonlinearity, Goodman, Holmes and Weinstein showed the orbital stability of the solitary waves $\mathrm{e}^{\,\mathrm{i}\,\omega t} Q_\omega(x)$ with $4\omega > \gamma^2$ in the energy space $H^1(\mathbb{R})$ in [9]. Later, by the Vakhitov-Kolokolov stability criteria in [28] (see also [2, 10, 27]), Fukuizumi, Ohta and Ozawa generalized the result to the case $p > 1$ in [8] (see also [15]). More precisely, the following results hold:

1) for any $p \in (1,5]$, the solitary waves $\mathrm{e}^{\mathrm{i}\omega t} Q_\omega(x)$ with $\omega > \frac{\gamma^2}{4}$ are orbitally stable in $H^1(\mathbb{R})$;
2) for any $p > 5$, there exists $\Omega = \Omega(p,\gamma) > \frac{\gamma^2}{4}$, such that
   - the solitary waves $\mathrm{e}^{\mathrm{i}\omega t} Q_\omega(x)$ with $\omega \in (\frac{\gamma^2}{4}, \Omega)$ are orbitally stable in $H^1(\mathbb{R})$;
   - the solitary waves $\mathrm{e}^{\mathrm{i}\omega t} Q_\omega(x)$ with $\omega > \Omega$ are orbitally unstable in $H^1(\mathbb{R})$,

   where $\Omega(p,\gamma)$ is determined by

(1.7)
$$\frac{p-5}{p-1}\int_{\operatorname{arctanh}\left(\frac{\gamma}{2\sqrt{\Omega}}\right)}^{+\infty} \operatorname{sech}^{\frac{4}{p-1}}(y)\,\mathrm{d}y = \frac{\gamma}{2\sqrt{\Omega}}\left(1-\frac{\gamma^2}{4\Omega}\right)^{-\frac{p-3}{p-1}} \Longleftrightarrow \mathbf{d}''(\Omega)=0.$$

Above all, only the critical oscillation case $\omega = \Omega(p,\gamma)$ for $p > 5$ is left open, for which the Vakhitov-Kolokolov stability criteria breaks down because of the fact that $\mathbf{d}''(\Omega) = 0$, i.e., the degeneracy of the second order derivative of the function $\mathbf{d}(\omega) = S_\omega(Q_\omega)$ at $\omega = \Omega(p,\gamma)$. Fukuizumi, Ohta and Ozawa conjectured that the solitary wave $\mathrm{e}^{\mathrm{i}\omega t} Q_\omega(x)$ with $\omega = \Omega(p,\gamma)$ is orbitally unstable in [8]. The purpose of this paper is to prove this conjecture according to the observations in [5, 16, 17, 22, 23]. More precisely, we have the main result as following.

Theorem 1.3. *Let $\gamma > 0$, $\mu = 1$, $p > 5$ and $\Omega > \frac{\gamma^2}{4}$ satisfy* (1.7)*. The solitary wave $e^{i\Omega t} Q_\Omega(x)$ of* (1.1) *is orbitally unstable in the energy space $H^1(\mathbb{R})$. More precisely, there exist $\alpha_0 > 0$ and $\lambda_0 > 0$ such that if*

$$u_0(x) = Q_\Omega(x) + \lambda\, \varphi_\Omega(x) + \widetilde{\rho}(\lambda)\; Q_\Omega(x),$$

where $0 < \lambda < \lambda_0$, $\varphi_\Omega = \frac{\partial Q_\omega}{\partial \omega}|_{\omega=\Omega}$ and $\tilde{\rho}(\lambda) = -\frac{\|\varphi_\Omega\|_2^2}{2\|Q_\Omega\|_2^2}\lambda^2 + \mathrm{o}\left(\lambda^2\right)$ is chosen by the implicit function theorem such that

$$\mathcal{M}(u_0) = \mathcal{M}(Q_\Omega),$$

then there exists $t_0 = t_0(u_0)$ such that the solution $u(t)$ of (1.1) *with initial data u_0 satisfies*

$$\inf_{\theta\in\mathbb{R}} \|u(t_0,\cdot) - Q_\Omega(\cdot)\, e^{i\theta}\|_{H^1(\mathbb{R})} \geqslant \alpha_0.$$

As stated above, the classical modulation analysis and the Virial type identity doesn't work once again in [10, 11, 26, 27, 29, 30] because of the degenerate property of $\mathbf{d}''(\Omega)$, we now give more details about the refined modulation decomposition and the refined Virial identity.

Firstly, we use the following decomposition

$$u(x) = \mathrm{e}^{-\mathrm{i}\theta}\Big(Q_\Omega + \lambda\varphi_\Omega + \rho(\lambda)Q_\Omega + \varepsilon\Big)(x), \quad \rho(\lambda) = -\frac{\|\varphi_\Omega\|_2^2}{2\|Q_\Omega\|_2^2}\cdot\lambda^2 \tag{1.8}$$

for the function u in the η_0-tube $\mathcal{U}(Q_\Omega\,,\ \eta_0)$ of Q_Ω (see (1.6) for the definition of the η_0-tube of Q_Ω), the above refined decomposition is related with the landscape of the action functional $\mathcal{S}_\omega$ near Q_Ω.

1) By the variational characterization of Q_Ω, the action functional $\mathcal{S}_\omega(u) = \mathcal{E}(u) + \omega\mathcal{M}(u)$ has the following properties

$$\mathcal{S}'_\Omega(Q_\Omega) = 0, \ \ \mathcal{S}''_\Omega(Q_\Omega) = \mathcal{L},$$

where the null space of the linearized operator $\mathcal{L}$ is characterized by $\mathrm{Null}(\mathcal{L}) = \mathrm{span}\{\,\mathrm{i}\,Q_\Omega\}$.

By the finite degenerate property of the function $\mathbf{d}(\Omega) = \mathcal{S}_\Omega(Q_\Omega)$, we know that

$$\mathbf{d}''(\Omega) = 0, \ \text{ and } \ \mathbf{d}'''(\Omega) \neq 0,$$

where the first equality means that the mass conservation quantity $\mathcal{M}(u) = \mathcal{M}(u)$ has the local equilibrium point Q_Ω along the curve $\{Q_{\Omega+\lambda}\}_{\lambda\in\mathbb{R}}$.

2) Up to the phase rotation invariances, the first order approximation of u to Q_Ω comes from the tangent vector φ_Ω of the curve $\{Q_{\Omega+\lambda}\}_{\lambda\in\mathbb{R}}$ at Q_Ω, and we have the following degenerate result

$$\mathcal{S}''_\Omega(Q_\Omega)(\varphi_\Omega,\ \varphi_\Omega) = -\langle Q_\Omega, \varphi_\Omega\rangle = 0. \tag{1.9}$$

3) Up to the phase rotation invariances, the second order approximation of u to Q_Ω is the direction Q_Ω, which is the steepest descent direction of the quantity $\mathcal{M}(u)$ at Q_Ω along the curve $\{Q_{\Omega+\lambda}\}_{\lambda\in\mathbb{R}}$. At the same time, we have the algebraic relations

$$\mathcal{S}''_\Omega(Q_\Omega)\,\varphi_\Omega = -Q_\Omega, \quad \text{and} \quad \mathcal{S}'''_\Omega(Q_\Omega)(\varphi_\Omega,\varphi_\Omega,\varphi_\Omega) + 3\,(\varphi_\Omega,\varphi_\Omega) = \mathbf{d}'''(\Omega).$$

Now we take the approximation of the solution $u(t,x)$ as follows

$$Q_\Omega + \lambda\varphi_\Omega + \rho(\lambda)Q_\Omega, \tag{1.10}$$

up to the phase rotation invariances, where $\rho(\lambda)$ can be ensured by restriction of the solution on the level set $\mathcal{M}(Q_\Omega)$ and indeed can be determined by the implicit function theorem (see Lemma 2.6).

By the above approximation, we can characterize the landscape of the function $\mathcal{S}_\Omega$ at Q_Ω along the perturbation $\lambda\varphi_\Omega + \rho(\lambda)Q_\Omega$,

$$\mathcal{S}_\Omega\left(Q_\Omega + \lambda\varphi_\Omega + \rho(\lambda)Q_\Omega\right) = \mathcal{S}_\Omega\left(Q_\Omega\right) + \frac{1}{6}\mathbf{d}'''(\Omega)\cdot\lambda^3 + \mathrm{o}\left(|\lambda|^3\right),$$

$$\begin{aligned}\mathcal{S}_\Omega\left(Q_\Omega + \lambda\varphi_\Omega + \rho(\lambda)Q_\Omega + \varepsilon\right) = {} & \mathcal{S}_\Omega\left(Q_\Omega\right) + \frac{1}{6}\mathbf{d}'''(\Omega)\cdot\lambda^3 \\ & + \mathcal{S}''_\Omega\left(Q_\Omega\right)(\varepsilon,\ \varepsilon) + \mathrm{o}\left(|\lambda|^3 + \|\varepsilon\|^2_{H^1}\right),\end{aligned}$$

which means that if the small remainder term ε can be ignored, $\mathcal{S}_\Omega$ is a local monotone function with respect to λ under the special perturbation $\lambda\varphi_\Omega + \rho(\lambda)Q_\Omega$ near Q_Ω , that is to say, the perturbation in the direction φ_Ω can play the dominant role under this special perturbation. This definite property of $\mathcal{S}_\Omega$ helps us to show the orbital instability of the solitary waves of (1.1) with a Virial argument in the degenerate case.

4) The remainder ε in (1.8) is not only small, but also has some orthogonal structures, which makes the linearized operator $\mathcal{L} = \mathcal{S}''_\Omega(Q_\Omega)$ possess almost coercivity to ensure the control of the remainder term ε, see Lemma 2.12.

Secondly, in order to show the orbital instability of the solitary waves $Q_\Omega(x)\,\mathrm{e}^{\mathrm{i}\Omega t}$ of (1.1), we now turn to the effective monotonicity formula. By introducing the perturbation of φ_Ω in the subspace $\mathrm{Null}(\mathcal{L})$ to obtain the cancelation effect in the quadratic term of (4.20) in λ, we can construct the refined Virial type quantity in the remainder term $\varepsilon(t)$

$$\mathscr{I}(t) = \left\langle \mathrm{i}\,\varepsilon(t,x)\ ,\ \varphi_\Omega(x) - \lambda(t)\frac{\langle\varphi_\Omega,\varphi_\Omega\rangle}{\langle Q_\Omega, Q_\Omega\rangle}Q_\Omega(x)\right\rangle, \tag{1.11}$$

which has the monotone property in some sense (see (4.34)), to show the orbital instability of the solitary wave $Q_\Omega(x)\,\mathrm{e}^{\mathrm{i}\Omega t}$ of (1.1).

At last, the paper is organized as follows. In Section 2, we recall some properties of the linear Schrödinger operator with the dirac potential, the landscape of the action functional $\mathcal{S}_\omega$ at Q_Ω along the unstable direction φ_Ω, and the refined modulation decomposition of the functions in the η-tube of Q_Ω, and the coercivity property of the linearized operator $\mathcal{L} = \mathcal{S}''_\Omega(Q_\Omega)$ on the subspace with the finite co-dimensions; In Section 3, we deduce the equation obeyed by the remainder term $\varepsilon(t,x)$, and show the dynamical estimates of the parameters $\lambda(t)$ and $\theta(t)$ by the geometric structures of the remainder term. In Section 4, we first construct the solutions of (1.1) near the solitary wave with the refined geometric structures, then show the orbital instability of the solitary wave of (1.1) in the degenerate case by the dynamical behaviors of the remainder term and the parameters, and the refined Virial identity. In Appendix A, we calculate the third order derivative $\mathbf{d}'''(\Omega)$ of $\mathbf{d}(\omega) = \mathcal{S}_\omega(Q_\omega)$ at Ω.

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2. Preliminaries

We make some preparations in this section. From now on, we fix $p > 5$ and $\Omega = \Omega(p,\gamma) > \frac{\gamma^2}{4}$ is determined by (1.7). The Hilbert spaces $L^2(\mathbb{R},\mathbb{C})$ and $H^1(\mathbb{R},\mathbb{C})$ will be denoted by $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ respectively. We denote

$$\langle u, v\rangle = \Re \int u(x)\,\bar{v}(x)\,\mathrm{d}x, \qquad \text{for all } u,\ v \in L^2(\mathbb{R})$$

be the inner product on the space $L^2(\mathbb{R})$. For the simplification, we denote the following functions:

$$f(z) = |z|^{p-1}z, \quad F(z) = \frac{1}{p+1}|z|^{p+1}.$$

A direct computation implies that for any $z_0,\ z_1,\ z_2,\ z_3 \in \mathbb{C}$, the following estimates hold:

$$|f(z_0+z_1) - f'(z_0)\,z_1| \leqslant C(|z_0|)\left(|z_1|^2 + |z_1|^p\right), \tag{2.1}$$

$$\left|f(z_0+z_1) - f'(z_0)\,z_1 - \frac{1}{2}f''(z_0)\,z_1z_1\right| \leqslant C(|z_0|)\left(|z_1|^3 + |z_1|^p\right), \tag{2.2}$$

where $C(|z_0|)$ is a constant which only depends on $|z_0|$, and

$$f'(z_0)\,z_1 = |z_0|^{p-1} z_1 + (p-1)\,|z_0|^{p-3}\Re(z_0\overline{z_1})\,z_0, \tag{2.3}$$

$$\begin{aligned} f''(z_0)\,z_1 z_2 = &\ (p-1)\,|z_0|^{p-3}\left[\Re(z_0\overline{z_2})\,z_1 + \Re(z_0\overline{z_1})\,z_2 + \Re(z_2\overline{z_1})\,z_0\right] \\ &+ (p-1)(p-3)\,|z_0|^{p-5}\Re(z_0\overline{z_1})\,\Re(z_0\overline{z_2})\,z_0, \end{aligned} \tag{2.4}$$

and

$$F'(z_0)(z_1) = \Re\left[f(z_0)\,\bar{z}_1\right], \tag{2.5}$$

$$F''(z_0)(z_1,z_2) = |z_0|^{p-3}\Big[p\Re(z_0\bar{z}_1)\,\Re(z_0\bar{z}_2) + \Im(z_0\bar{z}_1)\,\Im(z_0\bar{z}_2)\Big], \tag{2.6}$$

$$\begin{aligned} F'''(z_0)(z_1,z_2,z_3) = &\ \frac{p^2-1}{4}|z_0|^{p-3}\Re\left[z_0\left(\bar{z}_1\bar{z}_2 z_3 + \bar{z}_1 z_2\bar{z}_3 + z_1\bar{z}_2\bar{z}_3\right)\right] \\ &+ \frac{(p-1)(p-3)}{4}|z_0|^{p-5}\Re\left(z_0\bar{z}_1 z_0\bar{z}_2 z_0\bar{z}_3\right). \end{aligned} \tag{2.7}$$

2.1. Linear Schrödinger operator with a delta potential

We now recall some well-known properties for the linear Schrödinger operator $-\frac{\partial^2}{\partial x^2} - \gamma\delta$ with $\gamma \in [-\infty, +\infty)$, which were used in the physics literature. In fact, the following self-adjoint operator:

$$\begin{aligned} &-\Delta_\gamma = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \quad \text{with} \\ &\mathrm{D}(-\Delta_\gamma) = \left\{ \psi \in H^1 \cap H^2(\mathbb{R}\setminus\{0\}) \,\middle|\, \frac{\mathrm{d}\psi}{\mathrm{d}x}(0+) - \frac{\mathrm{d}\psi}{\mathrm{d}x}(0-) = -\gamma\psi(0) \right\}. \end{aligned}$$

gives the precise formulation of $-\frac{\partial^2}{\partial x^2} - \gamma\delta$, see for instance [1]. Moreover, the essential spectrum of $-\Delta_\gamma$ coincides with $[0, +\infty)$. In addition, if $\gamma > 0$, $-\Delta_\gamma$ has exactly one negative, simple eigenvalue, i.e. $-\frac{\gamma^2}{4}$ with the positive normalized eigenfunction $\sqrt{\frac{\gamma}{2}}\mathrm{e}^{-\frac{\gamma}{2}|x|}$. Therefore, for any $\psi \in H^1(\mathbb{R})$ and any $\gamma > 0$, we have

$$-\frac{\gamma^2}{4}\int_{\mathbb{R}} |\psi(x)|^2\,\mathrm{d}x \leqslant \int_{\mathbb{R}} |\psi'(x)|^2\,\mathrm{d}x - \gamma\int_{\mathbb{R}} \delta(x)|\psi(x)|^2\,\mathrm{d}x. \tag{2.8}$$

As a consequence of the above inequality, we have

Lemma 2.1. *For any $\psi \in H^1(\mathbb{R})$, the following inequality holds,*

$$|\psi(0)|^2 \leqslant \|\psi'\|_2\|\psi\|_2. \tag{2.9}$$

Proof. For any $\psi \in H^1(\mathbb{R})$, since (2.8) holds for all $\gamma > 0$, one can rewrite (2.8) as

$$\int \delta(x)|\psi(x)|^2 \, \mathrm{d}x \leqslant \frac{1}{\gamma} \int |\psi'(x)|^2 \, \mathrm{d}x + \frac{\gamma}{4} \int |\psi(x)|^2 \, \mathrm{d}x, \qquad \text{for all } \gamma > 0,$$

which implies (2.9) by optimizing γ. □

2.2. Basic properties of the action functional $\mathcal{S}_\omega$ and $d(\omega)$

For any $u \in H^1(\mathbb{R})$, we define the action functional $\mathcal{S}_\omega$ as follows:

$$\mathcal{S}_\omega(u) = \mathcal{E}(u) + \omega\mathcal{M}(u), \tag{2.10}$$

where $\mathcal{M}(u)$ and $\mathcal{E}(u)$ are the mass and energy of u defined by (1.2) and (1.3) respectively. Since $p > 5$, we have that $\mathcal{S}_\omega$ is a $\mathcal{C}^3$ functional on $H^1(\mathbb{R})$ by (2.5), (2.6) and (2.7). In addition, we can obtain the following variational characterization of Q_ω by the concentration-compactness argument in [7, 8, 15].

Proposition 2.2. *Let ω satisfy $4\omega > \gamma^2$. Then the function defined by* (1.5) *is the unique positive, radial symmetric solution to* (1.4). *Moreover, the set $\{ Q_\omega e^{i\theta} \mid \theta \in \mathbb{R} \}$ coincides with all minimizers of the following minimization problem:*

$$\inf \{ \mathcal{S}_\omega(\psi) \mid \psi \in H^1(\mathbb{R}) \setminus \{0\}, \ \ \mathcal{K}_\omega(\psi) = 0 \}, \tag{2.11}$$

where $\mathcal{S}_\omega$ is the action functional defined by (2.10), *and $\mathcal{K}_\omega$ is the scaling derivative of $\mathcal{S}_\omega$ defined by*

$$\begin{aligned}\mathcal{K}_\omega(\psi) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{S}_\omega(\lambda\psi)\Big|_{\lambda=1} &= \int |\psi'(x)|^2 + \omega \int |\psi(x)|^2 \\ &\quad - \gamma \int \delta(x)|\psi(x)|^2 - \int |\psi(x)|^{p+1}.\end{aligned}$$

By the classical Weyl theorem in [24] and Proposition 2.2, one can give a precise description of the spectrum of the linearized operator $\mathcal{S}''_\omega(Q_\omega)$, which is self-adjoint operator and has the following quadratic form

$$\begin{aligned} \mathcal{S}''_\omega(Q_\omega)(f,g) &= \int \left(f_1'g_1' + \omega f_1 g_1 - \gamma\delta f_1 g_1 - pQ_\omega^{p-1} f_1 g_1\right) \\ &\quad + \int \left(f_2' g_2' + \omega f_2 g_2 - \gamma\delta f_2 g_2 - Q_\omega^{p-1} f_2 g_2\right), \end{aligned} \tag{2.12}$$

where f_1, g_1 are the real parts of $f \in H^1(\mathbb{R})$, and $g \in H^1(\mathbb{R})$ respectively, and f_2, g_2 are the imaginary parts of f, and g respectively. By a variational argument, Fukuizumi and Jeanjean obtained the following orthogonal decomposition about $H^1(\mathbb{R})$ according to the spectrum of the linearized operator $\mathcal{S}''_\omega(Q_\omega)$ in [7] (See also [15]).

Proposition 2.3. *Let $\gamma > 0$ and ω satisfy $4\omega > \gamma^2$. Then the space $H^1(\mathbb{R})$ can be decomposed as the following direct sum*

$$H^1 = N \oplus K \oplus P, \tag{2.13}$$

according to the spectrum of the operator $\mathcal{S}''_\omega(Q_\omega)$, where

(i) *the subspace N, which is spanned by the eigenvector corresponding to the negative eigenvalue $-\mu^2$ of the operator $\mathcal{S}''_\omega(Q_\omega)$, and is one-dimensional, i.e. for any $f \in N$ with $f \neq 0$, we have*

$$\mathcal{S}''_\omega(Q_\omega)(f,f) = -\mu^2 \langle f, f\rangle;$$

(ii) *the subspace K is the kernel (null) space for the operator $\mathcal{S}''_\omega(Q_\omega)$, which is*

$$K = \operatorname{span}\{\, iQ_\omega \,\};$$

(iii) *the subspace P where the operator $\mathcal{S}''_\omega(Q_\omega)$ has the coercivity, that is, for any $f \in P$, we have*

$$\mathcal{S}''_\omega(Q_\omega)(f,f) \geqslant c\|f\|^2_{H^1},$$

where c is a positive constant which does not depend on f.

Next, we turn to investigate some properties of

$$\mathbf{d}(\omega) = \mathcal{S}_\omega(Q_\omega)$$

which is related to the landscape of the action functional $\mathcal{S}_\omega$ around Q_ω. By Proposition 2.2 and (1.4), we have for all ω satisfying $4\omega > \gamma^2$,

$$\mathbf{d}'(\omega) = \mathcal{M}(Q_\omega)\,. \tag{2.14}$$

Moreover, let

$$\varphi_\omega(x) = \frac{\partial Q_\omega}{\partial \omega}(x)\,, \tag{2.15}$$

we have

$$\begin{aligned}\mathbf{d}''(\omega) &= \frac{\partial^2}{\partial\omega^2}\mathcal{S}_\omega(Q_\omega)\\ &= \langle \mathcal{S}'_\omega(Q_\omega)\,, \frac{\partial^2}{\partial\omega^2}Q_\omega\rangle + \mathcal{S}''_\omega(Q_\omega)\left(\frac{\partial}{\partial\omega}Q_\omega, \frac{\partial}{\partial\omega}Q_\omega\right) + 2\,\langle Q_\omega, \frac{\partial}{\partial\omega}Q_\omega\rangle\\ &= \mathcal{S}''_\omega(Q_\omega)(\varphi_\omega, \varphi_\omega) + 2\,\langle Q_\omega, \varphi_\omega\rangle\,,\end{aligned} \tag{2.16}$$

where we used the fact that $\mathcal{S}'_\omega(Q_\omega) = 0$. Furthermore, we have

Lemma 2.4. *Let $p > 5$, $4\omega > \gamma^2$ and φ_ω defined by* (2.15)*, the following result holds*

$$\mathcal{S}''_\omega(Q_\omega)(\varphi_\omega, \psi) = -\langle Q_\omega, \psi\rangle\,, \quad \textit{for any}\;\; \psi \in H^1(\mathbb{R})\,. \tag{2.17}$$

Proof. It is a well-known result and we can also refer to Lemma 2.7 in [7]. In fact, it suffices to check that the following facts hold

$$\left(\frac{\partial\varphi_\omega}{\partial x}\right)(0+) - \left(\frac{\partial\varphi_\omega}{\partial x}\right)(0-) = -\gamma\varphi_\omega(0)\,, \tag{2.18}$$

and

$$-\frac{\partial^2\varphi_\omega}{\partial x^2}(x) + \omega\varphi_\omega(x) - pQ_\omega(x)^{p-1}\varphi_\omega(x) = -Q_\omega(x), \qquad \text{for all } x \neq 0. \tag{2.19}$$

On the one hand, since Q_ω can be explicitly expressed by (1.5), a direct computation implies that (2.18) holds. On the other hand, since Q_ω satisfies (1.4), we can obtain (2.19) by taking derivative with respect to ω in (1.4). □

As a consequence of (2.16) and (2.17), we know that

$$\mathbf{d}''(\Omega) = 0 \iff \langle\varphi_\Omega, Q_\Omega\rangle = 0. \tag{2.20}$$

It corresponds to the degenerate case, and the Vakhitov-Kolokolov stability criterion in [28] (see also [2, 10, 27]) breaks down in this case. In fact, one can still consider the stability (or instability) of the solitary waves through the non-degenerate behavior of higher order derivative of $\mathbf{d}(\omega)$ as those in [17] (see also [5, 16, 22, 23]). For this purpose, we first characterize the behavior of $\mathbf{d}(\omega)$ at the critical value Ω.

Lemma 2.5. *Let $p > 5$ and Ω be defined by (1.7). Then we have the following results.*

1) *For the case $\frac{\gamma^2}{4} < \omega < \Omega$, we have $\mathbf{d}''(\omega) > 0$;*
2) *For the case $\omega = \Omega$, we have $\mathbf{d}''(\Omega) = 0$, and $\mathbf{d}'''(\Omega) < 0$;*
3) *For the case $\omega > \Omega$, we have $\mathbf{d}''(\omega) < 0$.*

Furthermore, $\mathbf{d}'''(\Omega)$ can be explicitly expressed as following

$$\mathbf{d}'''(\Omega) = \mathcal{S}_\Omega'''(Q_\Omega)(\varphi_\Omega, \varphi_\Omega, \varphi_\Omega) + 3\langle\varphi_\Omega, \varphi_\Omega\rangle, \tag{2.21}$$

where φ_Ω is defined by (2.15).

The proof of Lemma 2.5 is postponed in Appendix A.

2.3. Geometric decomposition of u and landscape of $\mathcal{S}_\Omega$ near Q_Ω

For the non-degenerate case, i.e. $\mathbf{d}''(\omega) < 0$ with $\omega > \Omega$, the first order approximation of the solitary wave in the unstable direction is enough to show the instability of the solitary waves, while for the degenerate case $\mathbf{d}''(\Omega) = 0$ and $\mathbf{d}'''(\Omega) \neq 0$, we are going to consider the second order approximation of the solitary waves $Q_\Omega e^{i\Omega t}$, up to the phase rotation invariance, on the level set $\mathcal{M}(Q_\Omega)$ to show its instability in the energy space.

Lemma 2.6. *There exist a constant $0 < \tilde{\lambda}_0 \ll 1$ and a $\mathcal{C}^2$ function $\tilde{\rho}: \left(-\tilde{\lambda}_0, \tilde{\lambda}_0\right) \mapsto \mathbb{R}$ such that for any $\lambda \in \left(-\tilde{\lambda}_0, \tilde{\lambda}_0\right)$, we have*

$$\mathcal{M}(Q_\Omega + \lambda\varphi_\Omega + \tilde{\rho}(\lambda) Q_\Omega) = \mathcal{M}(Q_\Omega),$$

where the function $\tilde{\rho}(\lambda)$ can be expressed as following:

$$\tilde{\rho}(\lambda) = -\frac{\|\varphi_\Omega\|_2^2}{2\|Q_\Omega\|_2^2}\lambda^2 + o\left(\lambda^2\right), \quad \text{for any } \lambda \in \left(-\tilde{\lambda}_0, \tilde{\lambda}_0\right). \tag{2.22}$$

Proof. Essentially, the result is a consequence of the Implicit Function Theorem. Let us define the function $G(\lambda,\rho)$ as following:

$$G(\lambda,\rho) = \mathcal{M}(Q_\Omega + \lambda\varphi_\Omega + \rho Q_\Omega) - \mathcal{M}(Q_\Omega).$$

By the simple calculations, one can obtain that $G(0,0)=0$, and

$$\frac{\partial G}{\partial \lambda}(0,0) = 0, \quad \frac{\partial G}{\partial \rho}(0,0) = \|Q_\Omega\|_2^2, \tag{2.23}$$

and

$$\frac{\partial^2 G}{\partial \lambda^2}(0,0) = \|\varphi_\Omega\|_2^2, \quad \frac{\partial^2 G}{\partial \lambda \partial \rho}(0,0) = 0, \quad \frac{\partial^2 G}{\partial \rho^2}(0,0) = \|Q_\Omega\|_2^2, \tag{2.24}$$

then by the Implicit Function Theorem, there exist a $\tilde{\lambda}_0$ with $0 < \tilde{\lambda}_0 \ll 1$ and a $\mathcal{C}^2$ function $\tilde{\rho} : \left(-\tilde{\lambda}_0, \tilde{\lambda}_0\right) \mapsto \mathbb{R}$ such that

$$g(\lambda) = G(\lambda, \tilde{\rho}(\lambda)) = 0, \quad \text{for all} \quad \lambda \in \left(-\tilde{\lambda}_0, \tilde{\lambda}_0\right). \tag{2.25}$$

Therefore, it follows from (2.25) that

$$0 = \frac{\mathrm{d}g}{\mathrm{d}\lambda}(0) = \frac{\partial G}{\partial \lambda}(0,0) + \frac{\partial G}{\partial \rho}(0,0)\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\lambda}(0), \tag{2.26}$$

then by (2.23), we obtain

$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\lambda}(0) = 0. \tag{2.27}$$

Again, by taking the second order derivative of the function g with respect to λ at 0, we have

$$0 = \frac{\mathrm{d}^2 g}{\mathrm{d}\lambda^2}(0) = \frac{\partial^2 G}{\partial \lambda^2}(0,0) + 2\frac{\partial^2 G}{\partial \lambda \partial \rho}(0,0)\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\lambda}(0) + \frac{\partial G}{\partial \rho}(0,0)\frac{\mathrm{d}^2\tilde{\rho}}{\mathrm{d}\lambda^2}(0) \tag{2.28}$$

By (2.24) and (2.27), we get

$$\frac{\mathrm{d}^2\tilde{\rho}}{\mathrm{d}\lambda^2}(0) = -\frac{\|\varphi_\Omega\|_2^2}{\|Q_\Omega\|_2^2}. \tag{2.29}$$

By the fundamental theorem of calculus, one can obtain the result. □

From now on, we will take the function $\rho(\lambda)$ as the main part of $\tilde{\rho}(\lambda)$, i.e.

$$\rho\,(\lambda) = -\frac{\|\varphi_\Omega\|_2^2}{2\|Q_\Omega\|_2^2}\lambda^2. \tag{2.30}$$

Now, we can show the refined modulational decomposition of the functions around the solitary waves Q_Ω.

Lemma 2.7. *There exists* $0<\tilde{\eta}_0\ll 1$ *and a unique* $\mathcal{C}^1$ *map* $(\theta,\lambda):\mathcal{U}\,(Q_\Omega,\tilde{\eta}_0) \mapsto \mathbb{R}$ *such that if* $u\in\mathcal{U}\,(Q_\Omega,\tilde{\eta}_0)$ *and* $\varepsilon_{\theta,\lambda}(x)$ *is defined by*

$$\varepsilon_{\theta,\lambda}\,(x) = u\,(x)\,e^{i\theta} - (Q_\Omega + \lambda\varphi_\Omega + \rho\,(\lambda)\,Q_\Omega)(x)$$

where $\rho(\lambda)$ *is define by* (2.30)*, then we have the following orthogonal structure*

$$\varepsilon_{\theta,\lambda}\perp\, iQ_\Omega \quad \text{and} \quad \varepsilon_{\theta,\lambda}\perp\varphi_\Omega.$$

Moreover, there exists a constant C *which is independent of* θ, λ *and* u*, such that if* $u\in\mathcal{U}\,(Q_\Omega,\eta)$ *with* $\eta<\tilde{\eta}_0$*, then we have*

$$\|\varepsilon_{\theta,\lambda}\|_{H^1} + |\theta| + |\lambda| \leqslant C\eta.$$

Proof. It is also a consequence of the Implicit Function Theorem for the functional

$$\mathbf{F}(u;\theta,\lambda) = (F^1(u;\theta,\lambda), F^2(u;\theta,\lambda)),$$

where

$$F^1(u;\theta,\lambda) = \Re\int\varepsilon_{\theta,\lambda}\,\overline{\mathrm{i}\,Q_\Omega},\quad F^2(u;\theta,\lambda) = \Re\int\varepsilon_{\theta,\lambda}\,\varphi_\Omega.$$

It suffices to verify the non-degeneracy of the following Jacobian matrix:

$$\begin{aligned}\det\frac{\partial\mathbf{F}}{\partial(\theta,\lambda)}\,(Q_\Omega,0,0) &= \det\begin{pmatrix}\langle \mathrm{i}\,Q_\Omega,\,\mathrm{i}\,Q_\Omega\rangle & \langle \mathrm{i}\,Q_\Omega,\varphi_\Omega\rangle\\ -\langle\varphi_\Omega,\,\mathrm{i}\,Q_\Omega\rangle & -\langle\varphi_\Omega,\varphi_\Omega\rangle\end{pmatrix}\\ &= -\|Q_\Omega\|_2^2\|\varphi_\Omega\|_2^2\neq 0.\end{aligned}$$

We omit the details here. One can refer to [22, Lemma 2.6] for the analogue proof as the derivative NLS case. □

The next lemma shows that one can obtain a refined estimate of the remainder term $\varepsilon_{\theta,\lambda}$ along the direction Q_Ω under the above refined modulational decomposition.

Lemma 2.8. *There exist* $0 < \tilde{\eta}_1 \ll 1$ *and* $0 < \tilde{\lambda}_1 \ll 1$*, such that if* $|\lambda| \leqslant \tilde{\lambda}_1$ *and any* $\varepsilon \in H^1(\mathbb{R})$ *with* $\|\varepsilon\|_{H^1} \leqslant \tilde{\eta}_1$*, satisfy*

$$\mathcal{M}\left(Q_\Omega + \lambda\varphi_\Omega + \rho\left(\lambda\right)Q_\Omega + \varepsilon\right) = \mathcal{M}\left(Q_\Omega\right),$$

where $\rho(\lambda)$ *is defined by* (2.30)*, then we have*

$$|\langle\varepsilon, Q_\Omega\rangle| \leqslant C\left(\|\varepsilon\|_{H^1}^2 + |\lambda|\|\varepsilon\|_{H^1} + \lambda^4\right), \tag{2.31}$$

where C *is a constant independent of* λ *and* ε*.*

Proof. By $\mathcal{M}\left(Q_\Omega + \lambda\varphi_\Omega + \rho\left(\lambda\right)Q_\Omega + \varepsilon\right) = \mathcal{M}\left(Q_\Omega\right)$, we have

$$\begin{aligned} 0 &= \mathcal{M}\left(Q_\Omega + \lambda\varphi_\Omega + \rho\left(\lambda\right)Q_\Omega + \varepsilon\right) - \mathcal{M}\left(Q_\Omega\right) \\ &= \left\langle Q_\Omega, \lambda\varphi_\Omega + \rho\left(\lambda\right)Q_\Omega + \varepsilon\right\rangle \\ &\quad + \frac{1}{2}\left\langle \lambda\varphi_\Omega + \rho\left(\lambda\right)Q_\Omega + \varepsilon, \lambda\varphi_\Omega + \rho\left(\lambda\right)Q_\Omega + \varepsilon\right\rangle. \end{aligned} \tag{2.32}$$

which together with (2.20) and (2.30) implies that

$$\begin{aligned} \langle\varepsilon, Q_\Omega\rangle &= -\lambda\left\langle\varphi_\Omega, \varepsilon\right\rangle - \frac{1}{2}\left\langle\rho\left(\lambda\right)Q_\Omega + \varepsilon, \rho\left(\lambda\right)Q_\Omega + \varepsilon\right\rangle \\ &= -\lambda\left\langle\varphi_\Omega, \varepsilon\right\rangle - \frac{\rho\left(\lambda\right)^2}{2}\left\langle Q_\Omega, Q_\Omega\right\rangle - \frac{1}{2}\left\langle\varepsilon, \varepsilon\right\rangle - \rho\left(\lambda\right)\left\langle\varepsilon, Q_\Omega\right\rangle. \end{aligned} \tag{2.33}$$

By (2.30) and the Cauchy-Schwarz inequality, we can obtain the result. □

The following two lemmas show that the landscape of the action functional $\mathcal{S}_\Omega(u)$ along the unstable direction $\lambda\varphi_\Omega + \rho\left(\lambda\right)Q_\Omega$ around the solitary wave Q_Ω is definite. Firstly we have

Lemma 2.9. *There exist* $0 < \tilde{\lambda}_2 \ll 1$*, such that if* $0 < |\lambda| < \tilde{\lambda}_2$*, we have*

$$\mathcal{S}_\Omega\left(Q_\Omega + \lambda\varphi_\Omega + \rho\left(\lambda\right)Q_\Omega\right) = \mathcal{S}_\Omega\left(Q_\Omega\right) + \frac{1}{6}\mathbf{d}'''\left(\Omega\right)\cdot\lambda^3 + \mathrm{o}\left(|\lambda|^3\right), \tag{2.34}$$

where $\rho(\lambda)$ *is defined by* (2.30)*.*

Proof. By the definition (2.30) of $\rho(\lambda)$, $\lambda\varphi_\Omega + \rho\left(\lambda\right)Q_\Omega$ is sufficiently small in $H^1(\mathbb{R})$. Therefore, by taking the Taylor series expression of $\mathcal{S}_\Omega$ at Q_Ω, we have

$$
\begin{aligned}
(2.35)\quad & \mathcal{S}_\Omega\left(Q_\Omega+\lambda\varphi_\Omega+\rho(\lambda)Q_\Omega\right)\\
&=\mathcal{S}_\Omega\left(Q_\Omega\right)+\frac{1}{2}\mathcal{S}''_\Omega\left(Q_\Omega\right)\left(\lambda\varphi_\Omega+\rho(\lambda)Q_\Omega,\lambda\varphi_\Omega+\rho(\lambda)Q_\Omega\right)\\
&\quad+\frac{1}{6}\mathcal{S}'''_\Omega\left(Q_\Omega\right)\left(\lambda\varphi_\Omega+\rho(\lambda)Q_\Omega,\lambda\varphi_\Omega+\rho(\lambda)Q_\Omega,\lambda\varphi_\Omega+\rho(\lambda)Q_\Omega\right)\\
&\quad+\mathrm{o}\left(\|\lambda\varphi_\Omega+\rho(\lambda)Q_\Omega\|_{H^1}^3\right).
\end{aligned}
$$

Firstly, it follows from (2.6), (2.17), (2.20) and (2.30) that

$$
\begin{aligned}
&\frac{1}{2}\mathcal{S}''_\Omega\left(Q_\Omega\right)\left(\lambda\varphi_\Omega+\rho(\lambda)Q_\Omega,\lambda\varphi_\Omega+\rho(\lambda)Q_\Omega\right)\\
&\qquad=\frac{1}{2}\lambda^3\left\langle\varphi_\Omega,\varphi_\Omega\right\rangle+\mathrm{o}\left(|\lambda|^3\right).
\end{aligned}
\tag{2.36}
$$

Secondly, by (2.30) again, one can get

$$
\begin{aligned}
&\mathcal{S}'''_\Omega\left(Q_\Omega\right)\left(\lambda\varphi_\Omega+\rho(\lambda)Q_\Omega,\lambda\varphi_\Omega+\rho(\lambda)Q_\Omega,\lambda\varphi_\Omega+\rho(\lambda)Q_\Omega\right)\\
&\qquad=\lambda^3\mathcal{S}'''_\Omega\left(Q_\Omega\right)\left(\varphi_\Omega,\varphi_\Omega,\varphi_\Omega\right)+\mathrm{o}\left(|\lambda|^3\right),
\end{aligned}
\tag{2.37}
$$

which together with (2.21), (2.35), (2.36) implies the result. □

Secondly, we have

Lemma 2.10. *There exist $0<\tilde{\lambda}_3\ll 1$ and $0<\tilde{\eta}_3\ll 1$ such that if λ satisfies $0<|\lambda|<\tilde{\lambda}_3$ and $\varepsilon\in H^1(\mathbb{R})$ with $\|\varepsilon\|_{H^1}\leqslant\tilde{\eta}_3$ satisfies*

$$
|\langle\varepsilon,Q_\Omega\rangle|\leqslant C\left(\|\varepsilon\|_{H^1}^2+|\lambda|\|\varepsilon\|_{H^1}+\lambda^4\right),
\tag{2.38}
$$

where C is a constant independent of λ and ε. then we have

$$
\begin{aligned}
\mathcal{S}_\Omega\left(Q_\Omega+\lambda\varphi_\Omega+\rho(\lambda)Q_\Omega+\varepsilon\right)&=\mathcal{S}_\Omega\left(Q_\Omega\right)+\mathcal{S}''_\Omega\left(Q_\Omega\right)(\varepsilon,\varepsilon)\\
&\quad+\frac{1}{6}\mathbf{d}'''(\Omega)\cdot\lambda^3+\mathrm{o}\left(|\lambda|^3+\|\varepsilon\|_{H^1}^2\right),
\end{aligned}
\tag{2.39}
$$

where $\rho(\lambda)$ is defined by (2.30).

Proof. By the Taylor series expansion of $\mathcal{S}_\Omega$ at Q_Ω and the fact that $\mathcal{S}'_\Omega(Q_\Omega) = 0$, we have

$$
\begin{aligned}
(2.40)\qquad & \mathcal{S}_\Omega(Q_\Omega + \lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon) - \mathcal{S}_\Omega(Q_\Omega) \\
&= \frac{1}{2}\mathcal{S}''_\Omega(Q_\Omega)(\lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon, \lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon) \\
&\quad + \frac{1}{6}\mathcal{S}'''_\Omega(Q_\Omega)(\lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon, \lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon, \\
&\qquad\qquad \lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon) \\
&\quad + \mathrm{o}\left(\|\lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon\|_{H^1}^3\right).
\end{aligned}
$$

Firstly, it follows from (2.17) (2.30), (2.36) and (2.38) that,

$$
\begin{aligned}
& \frac{1}{2}\mathcal{S}''_\Omega(Q_\Omega)(\lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon, \lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon) \\
&= \frac{1}{2}\lambda^3\langle\varphi_\Omega, \varphi_\Omega\rangle + \frac{1}{2}\mathcal{S}''_\Omega(Q_\Omega)(\varepsilon, \varepsilon) \\
&\quad + \mathcal{S}''_\Omega(Q_\Omega)(\lambda\varphi_\Omega + \rho(\lambda) Q_\Omega, \varepsilon) + \mathrm{o}\left(|\lambda|^3\right) \\
&= \frac{1}{2}\lambda^3\langle\varphi_\Omega, \varphi_\Omega\rangle + \frac{1}{2}\mathcal{S}''_\Omega(Q_\Omega)(\varepsilon, \varepsilon) \\
&\quad - \lambda\langle Q_\Omega, \varepsilon\rangle + \mathrm{O}\left(\lambda^2\|\varepsilon\|_{H^1}\right) + \mathrm{o}\left(|\lambda|^3\right) \\
&= \frac{1}{2}\lambda^3\langle\varphi_\Omega, \varphi_\Omega\rangle + \frac{1}{2}\mathcal{S}''_\Omega(Q_\Omega)(\varepsilon, \varepsilon) \\
&\quad + \mathrm{O}\left(|\lambda|\|\varepsilon\|_{H^1}^2 + \lambda^2\|\varepsilon\|_{H^1} + |\lambda|^5\right) + \mathrm{o}\left(|\lambda|^3\right) \\
(2.41)\qquad &= \frac{1}{2}\lambda^3\langle\varphi_\Omega, \varphi_\Omega\rangle + \frac{1}{2}\mathcal{S}''_\Omega(Q_\Omega)(\varepsilon, \varepsilon) + \mathrm{o}\left(|\lambda|^3 + \|\varepsilon\|_{H^1}^2\right)
\end{aligned}
$$

where we used the Cauchy-Schwarz inequality in the last identity.

Secondly, by (2.37) and (2.30), we get

$$
\begin{aligned}
& \frac{1}{6}\mathcal{S}'''_\Omega(Q_\Omega)(\lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon, \lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon, \\
&\qquad \lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon) \\
&= \frac{\lambda^3}{6}\mathcal{S}'''_\Omega(Q_\Omega)(\varphi_\Omega, \varphi_\Omega, \varphi_\Omega) + \mathrm{o}\left(|\lambda|^3\right) + \mathrm{O}\left(\lambda^2\|\varepsilon\|_{H^1} + \|\varepsilon\|_{H^1}^3\right) \\
(2.42)\qquad &= \frac{\lambda^3}{6}\mathcal{S}'''_\Omega(Q_\Omega)(\varphi_\Omega, \varphi_\Omega, \varphi_\Omega) + \mathrm{o}\left(|\lambda|^3 + \|\varepsilon\|_{H^1}^2\right)
\end{aligned}
$$

Lastly, by (2.40), (2.41), (2.42) and (2.21), we can obtain the result. □

2.4. Properties of the linearized operator $\mathcal{S}''_{\Omega}(Q_{\Omega})$

As shown in Lemma 2.10, we now turn to estimate the quadratic term $\mathcal{S}''_{\Omega}(Q_{\Omega})(\varepsilon,\varepsilon)$, which in fact has some coercivity property under the condition that the remainder term ε has some geometric orthogonal structure. It is the task in this subsection and related to the spectral properties of the linearized operator $\mathcal{S}''_{\Omega}(Q_{\Omega})$.

To do so, we firstly introduce the following result.

Lemma 2.11. *Let χ be the L^2-normalized function of N defined by* (2.13), *and φ_{Ω} be defined by* (2.15). *Then we have*

$$\langle \chi, \varphi_{\Omega} \rangle \neq 0. \tag{2.43}$$

Proof. We argue by contradiction, and assume that

$$\langle \chi, \varphi_{\Omega} \rangle = 0. \tag{2.44}$$

Since φ_{Ω} is real, it is easy to see that

$$\langle \varphi_{\Omega}, \mathrm{i}\, Q_{\Omega} \rangle = 0. \tag{2.45}$$

On the one hand, by (2.44), (2.45) and Proposition 2.3, we have

$$\mathcal{S}''_{\Omega}(Q_{\Omega})(\varphi_{\Omega},\varphi_{\Omega}) \geqslant c\|\varphi_{\Omega}\|^2_{H^1} > 0. \tag{2.46}$$

On the other hand, by (2.17) and (2.20), we get

$$\mathcal{S}''_{\Omega}(Q_{\Omega})(\varphi_{\Omega},\varphi_{\Omega}) = -\langle Q_{\Omega}, \varphi_{\Omega} \rangle = 0,$$

which is in contradiction with (2.46). Therefore, (2.43) holds, and this completes the proof. □

After the above lemma, one can now show the following coercive property of $\mathcal{S}''_{\Omega}(Q_{\Omega})$ by the standard arguments in [10] [29], which is a consequence of Proposition 2.3.

Lemma 2.12. *Let $\varepsilon \in H^1(\mathbb{R}) \setminus \{0\}$. If*

$$\langle \varepsilon, iQ_\Omega \rangle = 0, \quad \langle \varepsilon, \varphi_\Omega \rangle = 0 \quad \text{and} \quad \langle \varepsilon, Q_\Omega \rangle = 0, \tag{2.47}$$

then there exists a positive constant κ_1 independent of ε, such that the following result holds,

$$\mathcal{S}''_\Omega(Q_\Omega)(\varepsilon, \varepsilon) \geqslant \kappa_1 \|\varepsilon\|^2_{H^1}. \tag{2.48}$$

Proof. It suffices to show that there exists a positive constant κ_2 independent of ε, such that the following estimate holds,

$$\mathcal{S}''_\Omega(Q_\Omega)(\varepsilon, \varepsilon) \geqslant \kappa_2 \|\varepsilon\|^2_2. \tag{2.49}$$

In fact, assume that (2.49) holds, it follows from (2.12) and $\|Q_\Omega\|_\infty < +\infty$ that

$$\|\varepsilon'\|^2_2 + \Omega\|\varepsilon\|^2_2 - \gamma|\varepsilon(0)|^2 \leqslant \mathcal{S}''_\Omega(Q_\Omega)(\varepsilon, \varepsilon) + C\|\varepsilon\|^2_2, \tag{2.50}$$

where C is a positive constant which only depends on $\|Q_\Omega\|_\infty$. Using (2.9), the Cauchy-Schwarz inequality and the fact $4\Omega > \gamma^2$, we have

$$\begin{aligned} \text{LHS of (2.50)} &= \|\varepsilon'\|^2_2 + \Omega\|\varepsilon\|^2_2 - \gamma|\varepsilon(0)|^2 \\ &\geqslant \|\varepsilon'\|^2_2 + \Omega\|\varepsilon\|^2_2 - \frac{1}{2}\|\varepsilon'\|^2_2 - \frac{\gamma^2}{2}\|\varepsilon\|^2_2 \\ &\geqslant \frac{1}{2}\|\varepsilon'\|^2_2 - \frac{\gamma^2}{2}\|\varepsilon\|^2_2. \end{aligned} \tag{2.51}$$

Therefore, inserting (2.49) and (2.51) into (2.50), one immediately get

$$\begin{aligned} \frac{1}{2}\|\varepsilon'\|^2_2 &\leqslant \frac{\gamma^2}{2}\|\varepsilon\|^2_2 + \mathcal{S}''_\Omega(Q_\Omega)(\varepsilon, \varepsilon) + C\|\varepsilon\|^2_2 \\ &\leqslant \left(1 + \frac{\gamma^2 + 2C}{2\kappa_2}\right)\mathcal{S}''_\Omega(Q_\Omega)(\varepsilon, \varepsilon) \end{aligned}$$

By taking $\kappa_1 = \frac{\kappa_2}{2\kappa_2 + \gamma^2 + 2C + 1}$, we can obtain (2.48).

Now let χ be the L^2-normalized function in N. Firstly, for any nonzero ε satisfying (2.47), one can take the following decomposition by (2.43):

$$\varepsilon = \mathsf{p}_\varepsilon + a_\varepsilon \varphi_\Omega, \quad a_\varepsilon = -\frac{\langle \varepsilon, \chi \rangle}{\langle \varphi_\Omega, \chi \rangle}.$$

On the one hand, a direct calculation implies that

$$\langle \mathtt{p}_\varepsilon, \chi \rangle = 0, \quad \text{and} \quad \langle \mathtt{p}_\varepsilon, \mathrm{i}\, Q_\Omega \rangle = 0,$$

which means that $\mathtt{p}_\varepsilon \in P$, where P is defined by Proposition 2.3, therefore we have

$$\begin{aligned} \mathcal{S}''_\Omega (Q_\Omega)(\mathtt{p}_\varepsilon, \mathtt{p}_\varepsilon) &\geqslant c\|\mathtt{p}_\varepsilon\|_2^2 \\ &= c\left\langle \varepsilon - a_\varepsilon \varphi_\Omega, \varepsilon - a_\varepsilon \varphi_\Omega \right\rangle \\ &= c\|\varepsilon\|_2^2 + c\,(a_\varepsilon)^2\, \|\varphi_\Omega\|_2^2 \\ &\geqslant c\|\varepsilon\|_2^2. \end{aligned} \tag{2.52}$$

On the other hand, by (2.17), (2.20) and (2.47), we have

$$\mathcal{S}''_\Omega (Q_\Omega)(\varepsilon, \varepsilon) = \mathcal{S}''_\Omega (Q_\Omega)(\mathtt{p}_\varepsilon + a_\varepsilon\varphi_\Omega, \mathtt{p}_\varepsilon + a_\varepsilon\varphi_\Omega) = \mathcal{S}''_\Omega (Q_\Omega)(\mathtt{p}_\varepsilon, \mathtt{p}_\varepsilon). \tag{2.53}$$

Combining (2.52) and (2.53), we can obtain

$$\mathcal{S}''_\Omega (Q_\Omega)(\varepsilon, \varepsilon) \geqslant c\|\varepsilon\|_2^2.$$

This completes the proof of (2.49) with $\kappa_2 = c$ and the proof of Lemma 2.12. □

As a consequence of Lemma 2.12, we have

Corollary 2.13. *Let $\varepsilon \in H^1(\mathbb{R}) \setminus \{0\}$ and satisfy*

$$\langle \varepsilon,\, iQ_\Omega \rangle = 0 \quad \text{and} \quad \langle \varepsilon, \varphi_\Omega \rangle = 0, \tag{2.54}$$

then there exists a positive constant κ independent of ε, such that the following estimate holds,

$$\mathcal{S}''_\Omega (Q_\Omega)(\varepsilon, \varepsilon) \geqslant \kappa\|\varepsilon\|_{H^1}^2 - \frac{1}{\kappa}\left\langle \varepsilon, Q_\Omega \right\rangle^2. \tag{2.55}$$

Proof. The proof is standard, we can refer the analogue proof as the nonlinear Schrödinger equation in [21, page 186]. □

Combining Lemma 2.8 and Corollary 2.13, we have

Lemma 2.14. *There exist* $0 < \tilde{\eta}_4 \ll 1$ *and* $0 < \tilde{\lambda}_4 \ll 1$ *such that if* $\varepsilon \in H^1(\mathbb{R})$ *with* $\|\varepsilon\|_{H^1} \leqslant \tilde{\eta}_4$, *and* λ *with* $|\lambda| \leqslant \tilde{\lambda}_4$ *satisfy*

$$\langle \varepsilon\,,\ \mathrm{i}Q_\Omega \rangle = 0 \qquad \text{and} \quad \langle \varepsilon\,,\ \varphi_\Omega \rangle = 0,$$

and

$$\mathcal{M}\left(Q_\Omega + \lambda\varphi_\Omega + \rho(\lambda)Q_\Omega + \varepsilon\right) = \mathcal{M}\left(Q_\Omega\right),$$

where $\rho(\lambda)$ *is determined by* (2.30), *then we have*

$$\mathcal{S}''_\Omega\left(Q_\Omega\right)(\varepsilon,\varepsilon) \geqslant \frac{\kappa}{2}\|\varepsilon\|^2_{H^1} + \mathrm{o}\left(\lambda^4\right).$$

Proof. First, by Lemma 2.8, we have

$$\begin{aligned}\langle \varepsilon\,,\ Q_\Omega \rangle^2 &= C^2\left(|\lambda|\|\varepsilon\|_{H^1} + \|\varepsilon\|^2_{H^1} + \lambda^4\right)^2 \\ &\leqslant 3C^2\left(\lambda^2\|\varepsilon\|^2_{H^1} + \|\varepsilon\|^4_{H^1} + \lambda^8\right).\end{aligned} \tag{2.56}$$

It implies by taking $|\lambda|$ and $\|\varepsilon\|_{H^1}$ sufficiently small that

$$\langle \varepsilon\,,\ Q_\Omega \rangle^2 = \mathrm{o}\left(\|\varepsilon\|^2_{H^1}\right) + \mathrm{o}\left(\lambda^4\right), \tag{2.57}$$

which together with (2.55) implies that

$$\begin{aligned}\mathcal{S}''_\Omega\left(Q_\Omega\right)(\varepsilon,\varepsilon) &\geqslant \kappa\|\varepsilon\|^2_{H^1} - \frac{1}{\kappa}\langle \varepsilon\,,\ Q_\Omega \rangle^2 \\ &= \kappa\|\varepsilon\|^2_{H^1} + \mathrm{o}\left(\|\varepsilon\|^2_{H^1}\right) + \mathrm{o}\left(\lambda^4\right) \\ &\geqslant \frac{\kappa}{2}\|\varepsilon\|^2_{H^1} + \mathrm{o}\left(\lambda^4\right).\end{aligned}$$

This completes the proof. ☐

3. The ε-variable equation and the dynamics of the parameters

In this section, we derive the equation obeyed by the remainder term

$$\varepsilon(t,x) = u(t,x)\,\mathrm{e}^{\mathrm{i}\,\theta(t)} - \left(Q_\Omega(x) + \lambda(t)\,\varphi_\Omega(x) + \rho(\lambda(t))\,Q_\Omega(x)\right), \tag{3.1}$$

where u is the solution of (1.1) in $H^1(\mathbb{R})$, φ_Ω and $\rho(\lambda)$ are determined by (2.15) and (2.30) respectively, λ and θ are two $\mathcal{C}^1$ functions with respect to t.

Firstly, we have

Lemma 3.1. *Let $u(t) \in C\left([0,T), H^1(\mathbb{R})\right) \cap \mathcal{C}^1\left([0,T), H^{-1}(\mathbb{R})\right)$ be the solution to* (1.1) *for some $T > 0$, and $\varepsilon(t,x)$ be defined by* (3.1)*, then we have*

$$\begin{aligned} i\varepsilon_t = & -i\lambda_t\left(\varphi_\Omega + \frac{d\rho}{d\lambda}(\lambda)\,Q_\Omega\right) - (\theta_t + \Omega)\left(Q_\Omega + \lambda\varphi_\Omega + \rho(\lambda)\,Q + \varepsilon\right) \\ & + \mathcal{L}\left(\lambda\varphi_\Omega + \rho(\lambda)\,Q_\Omega + \varepsilon\right) \\ & - \frac{1}{2}f''(Q_\Omega)\left(\lambda\varphi_\Omega + \rho(\lambda)\,Q_\Omega + \varepsilon, \lambda\varphi_\Omega + \rho(\lambda)\,Q_\Omega + \varepsilon\right) \\ & - \mathcal{R}\left(Q_\Omega, \lambda\varphi_\Omega + \rho(\lambda)\,Q_\Omega + \varepsilon\right), \end{aligned} \tag{3.2}$$

where $f'(Q_\Omega)$, $f''(Q_\Omega)$ is defined by (2.3), (2.4)*, the linearized operator $\mathcal{L}$ and the higher order remainder term $\mathcal{R}$ are defined by*

$$\mathcal{L}g = \mathcal{S}''_\Omega(Q_\Omega)\,g = -g_{xx} + \Omega g - \gamma\delta g - f'(Q_\Omega)\,g, \tag{3.3}$$

and

$$\mathcal{R}(Q_\Omega, g) = f(Q_\Omega + g) - f(Q_\Omega) - f'(Q_\Omega)\,g - \frac{1}{2}f''(Q_\Omega)(g,g). \tag{3.4}$$

Proof. First, let $v(t,x) = u(t,x)\,\mathrm{e}^{\mathrm{i}\,\theta(t)}$, then we have

$$u_t = (v_t - \mathrm{i}\,\theta_t v)\,\mathrm{e}^{-\mathrm{i}\,\theta} \quad \text{and} \quad u_{xx} = v_{xx}\mathrm{e}^{-\mathrm{i}\,\theta}, \tag{3.5}$$

which together with (1.1) implies that

$$\mathrm{i}\,v_t + \theta_t v + v_{xx} + \gamma\delta v + f(v) = 0. \tag{3.6}$$

Next, let

$$v(t,x) = Q_\Omega(x) + g(t,x). \tag{3.7}$$

By (3.6), we have

$$0 = \mathrm{i}\,g_t + \theta_t(Q_\Omega + g) + (Q_\Omega + g)_{xx} + \gamma\delta(Q_\Omega + g) + f(Q_\Omega + g).$$

Since Q_Ω is the solution of (1.4), we have

$$0 = \mathrm{i}\,g_t + (\theta_t + \Omega)(Q_\Omega + g) + g_{xx} - \Omega g + \gamma\delta g + f(Q_\Omega + g) - f(Q_\Omega). \tag{3.8}$$

By (3.3) and (3.4), we have

$$
\begin{aligned}
(3.9)\qquad 0 = \mathrm{i}\, g_t + (\theta_t + \Omega)(Q_\Omega + g) + g_{xx} - \Omega g + \gamma\delta g + f'(Q_\Omega)\, g \\
+ \frac{1}{2} f''(Q_\Omega)(g,g) + \mathcal{R}(Q_\Omega, g)\,.
\end{aligned}
$$

Finally, by taking $g(t,x) = \lambda(t)\,\varphi_\Omega(x) + \rho(\lambda(t))\, Q_\Omega(x)$ in (3.9), we can obtain (3.2), this ends the proof. □

By the orthogonal structure of the remainder term $\varepsilon(t,x)$, we can obtain the dynamical control of the parameters $\lambda(t)$ and $\theta(t)$ as follows.

Lemma 3.2. *Suppose $T > 0$. There exist $0 < \tilde{\eta}_5 \ll 1$ and $0 < \tilde{\lambda}_5 \ll 1$, such that if for all $t \in [0,T)$, $\varepsilon(t)$, $\theta(t)$, $\lambda(t)$ satisfying (3.2), and*

$$
(3.10)\qquad \langle \varepsilon(t)\,,\, iQ_\Omega\rangle = 0 \quad \textit{and} \quad \langle \varepsilon(t)\,, \varphi_\Omega\rangle = 0,
$$

and

$$
(3.11)\qquad \|\varepsilon(t)\|_{H^1} \leqslant \tilde{\eta}_5, \quad \textit{and} \quad |\lambda(t)| \leqslant \tilde{\lambda}_5,
$$

then for all $t \in [0,T)$, we have

$$
(3.12)\qquad |\lambda_t| + |\theta_t + \Omega| \leqslant C\left(|\lambda| + \|\varepsilon(t)\|_{H^1}\right),
$$

where C is a constant which only depends on Q_Ω.

Proof. Multiplying (3.2) with Q_Ω and $\mathrm{i}\,\varphi_\Omega$ respectively, we have

$$
\begin{aligned}
(3.13)\qquad & \lambda_t \left\langle \mathrm{i}\left(\varphi_\Omega + \frac{\mathrm{d}\rho}{\mathrm{d}\lambda}(\lambda)\, Q_\Omega\right), Q_\Omega\right\rangle \\
& + (\theta_t + \Omega)\left\langle (Q_\Omega + \lambda\varphi_\Omega + \rho(\lambda)\, Q + \varepsilon)\,, Q_\Omega\right\rangle \\
& \quad = \left\langle \mathcal{L}\left(\lambda\varphi_\Omega + \rho(\lambda)\, Q_\Omega + \varepsilon\right), Q_\Omega\right\rangle \\
& \qquad - \left\langle f\left(Q_\Omega + \lambda\varphi_\Omega + \rho(\lambda)\, Q_\Omega + \varepsilon\right) - f(Q_\Omega) \right. \\
& \qquad\quad \left. - f'(Q_\Omega)\left(\lambda\varphi_\Omega + \rho(\lambda)\, Q_\Omega + \varepsilon\right), Q_\Omega\right\rangle
\end{aligned}
$$

and

$$\tag{3.14} \begin{aligned} &\lambda_t \langle \mathrm{i}\left(\varphi_\Omega + \frac{\mathrm{d}\rho}{\mathrm{d}\lambda}(\lambda) Q_\Omega\right), \mathrm{i}\,\varphi_\Omega\rangle \\ &+ (\theta_t + \Omega)\langle (Q_\Omega + \lambda\varphi_\Omega + \rho(\lambda) Q + \varepsilon), \mathrm{i}\,\varphi_\Omega\rangle \\ &\quad = \langle \mathcal{L}(\lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon), \mathrm{i}\,\varphi_\Omega\rangle \\ &- \langle f(Q_\Omega + \lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon) - f(Q_\Omega) \\ &\qquad - f'(Q_\Omega)(\lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon), \mathrm{i}\,\varphi_\Omega\rangle. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{F}(Q_\Omega, \lambda, \varepsilon) = {}& f(Q_\Omega + \lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon) - f(Q_\Omega) \\ & - f'(Q_\Omega)(\lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon), \end{aligned}$$

then by (2.2) and (2.30), $\mathcal{F}(Q_\Omega, \lambda, \varepsilon)$ is a polynomial of at least one degree with respect to λ and ε. By (3.11), we have

$$\tag{3.15} \begin{aligned} |\langle \mathcal{F}(Q_\Omega, \lambda, \varepsilon), Q_\Omega\rangle| &\leqslant C(Q_\Omega) + |\langle \mathcal{F}(Q_\Omega, \lambda, \varepsilon), \mathrm{i}\,\varphi_\Omega\rangle| \\ &\leqslant C(Q_\Omega)(|\lambda| + \|\varepsilon(t)\|_{H^1}). \end{aligned}$$

In addition, by (2.30) and (3.11), we also have

$$\tag{3.16} \begin{aligned} &|\langle \mathcal{L}(\lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon), Q_\Omega\rangle| + |\langle \mathcal{L}(\lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon), \mathrm{i}\,\varphi_\Omega\rangle| \\ &\quad \leqslant C(Q_\Omega)(|\lambda| + \|\varepsilon(t)\|_{H^1}). \end{aligned}$$

Combining (3.13), (3.14), (3.15) and (3.16), we can obtain the result. □

4. Proof of Theorem 1.3

Proof. We argue by contradiction and divide the proof of main theorem into several steps.

Step 1. Preparation of the initial data. Firstly, we can choose $0 < \lambda_0 < \tilde{\lambda}_0 \ll 1$ sufficiently small such that $\mathcal{M}(u_0) = \mathcal{M}(Q_\Omega)$, where

$$\tag{4.1} u_0(x) = Q_\Omega(x) + \lambda_0\varphi_\Omega(x) + \widetilde{\rho}(\lambda_0) Q_\Omega(x),$$

and $\widetilde{\rho}(\lambda)$ is defined by (2.22). It is easy to check that

$$\tag{4.2} \|u_0 - Q_\Omega\|_{H^1} = \|\lambda_0\varphi_\Omega + \widetilde{\rho}(\lambda_0) Q_\Omega\|_{H^1} < C\lambda_0.$$

Assume that the solitary wave $Q_\Omega e^{i\Omega t}$ is orbitally stable in the energy space. By Definition 1.2, for $\eta_0 > 0$ to be determined later, there exists sufficiently small λ_0 such that the solution $u(t)$ of (1.1) with initial data $u_0 \in \mathcal{U}(Q_\Omega, C\lambda_0)$ is global, and $u(t) \in \mathcal{U}(Q_\Omega, \eta_0)$ for all $t > 0$.

Step 2. Geometric decomposition of the solution $u(t)$ and regularity of the parameters λ and θ in t. Let $\rho(\lambda)$ be defined by (2.30). By Lemma 2.7 and $u(t) \in C\big([0,\infty), H^1(\mathbb{R})\big)$, there exist two continuous functions λ and θ with respect to t such that the remainder term

$$\varepsilon(t,x) = u(t,x)\, e^{i\theta(t)} - \Big(Q_\Omega(x) + \lambda(t)\varphi_\Omega(x) + \rho(\lambda(t))Q_\Omega(x)\Big) \tag{4.3}$$

has the following orthogonal structures

$$\langle \varepsilon(t), i\, Q_\Omega\rangle = 0, \quad \langle \varepsilon(t), \varphi_\Omega\rangle = 0, \quad \forall\, t \geqslant 0,$$

which together with $u \in \mathcal{C}\big([0,\infty), H^1(\mathbb{R})\big) \cap \mathcal{C}^1\big([0,\infty), H^{-1}(\mathbb{R})\big) \cap \mathcal{U}(Q_\Omega, \eta_0)$, the difference characterization of the $\mathcal{C}^1$ function and the limiting argument implies that the parameters λ and θ are in fact $\mathcal{C}^1$ functions with respect to t. Therefore the remainder term $\varepsilon(t,x)$ satisfies the equation

$$\begin{aligned}
i\,\varepsilon_t = &- i\,\lambda_t \left(\varphi_\Omega + \frac{d\rho}{d\lambda}(\lambda)\, Q_\Omega\right) - (\theta_t + \Omega)\,(Q_\Omega + \lambda\varphi_\Omega + \rho(\lambda)\,Q + \varepsilon)\\
&+ \mathcal{L}(\lambda\varphi_\Omega + \rho(\lambda)\,Q_\Omega + \varepsilon)\\
&- \frac{1}{2} f''(Q_\Omega)\,(\lambda\varphi_\Omega + \rho(\lambda)\,Q_\Omega + \varepsilon, \lambda\varphi_\Omega + \rho(\lambda)\,Q_\Omega + \varepsilon)\\
&- \mathcal{R}(Q_\Omega, \lambda\varphi_\Omega + \rho(\lambda)\,Q_\Omega + \varepsilon),
\end{aligned} \tag{4.4}$$

where $f''(Q_\Omega)$ is defined by (2.4), $\mathcal{L}$ and $\mathcal{R}$ are defined by (3.3) and (3.4) in Lemma 3.1, and for all $t > 0$, we have

$$\langle \varepsilon(t), i\, Q_\Omega\rangle = 0, \quad \langle \varepsilon(t), \varphi_\Omega\rangle = 0, \tag{4.5}$$

and

$$\|\varepsilon(t)\|_{H^1} + |\lambda(t)| + |\theta(t)| \leqslant C\eta_0. \tag{4.6}$$

By choosing η_0 sufficiently small such that

$$0 < \max\{1, C\}\,\eta_0 < \min\left\{\,\tilde{\eta}_0, \tilde{\eta}_1, \tilde{\lambda}_1, \tilde{\eta}_3, \tilde{\lambda}_3, \tilde{\eta}_4, \tilde{\lambda}_4, \tilde{\eta}_5, \tilde{\lambda}_5\,\right\},$$

and by Lemma 3.2, we have

$$|\lambda_t| + |\theta_t + \Omega| \leqslant C \left(|\lambda| + \|\varepsilon\,(t)\|_{H^1}\right). \tag{4.7}$$

Next, by the conservation law of mass and (4.3), we have

$$\mathcal{M}\Big(Q_\Omega + \lambda\,(t)\varphi_\Omega + \rho(\lambda\,(t))Q_\Omega + \varepsilon\,(t)\Big) = \mathcal{M}\,(Q_\Omega)\,, \tag{4.8}$$

which together with Lemma 2.8 implies that

$$|\langle \varepsilon\,(t)\,, Q_\Omega\rangle| \leqslant C\Big(\|\varepsilon\,(t)\|^2_{H^1} + |\lambda\,(t)|\|\varepsilon\|_{H^1} + \lambda\,(t)^4\Big), \quad \text{for all } t > 0. \tag{4.9}$$

Step 3. Estimates of the the remainder term $\varepsilon\,(t)$ and the parameter $\lambda\,(t)$. Combining the above estimates, we have the following estimates of remainder term $\varepsilon\,(t)$ and the parameter $\lambda\,(t)$ as a consequence of Lemma 2.10.

Proposition 4.1. *Let u_0 be defined by* (4.1) *and $\varepsilon\,(t)$ be defined by* (4.3)*. Then for all $t > 0$, we have*

$$\lambda\,(t) \geqslant \frac{1}{2}\lambda_0, \tag{4.10}$$

and

$$\|\varepsilon\,(t)\|^2_{H^1} \leqslant -\frac{2}{\kappa}\mathbf{d}'''\,(\Omega)\,\lambda\,(t)^3\,, \tag{4.11}$$

where κ is the constant defined in Corollary 2.13.

Proof. The proof is similar to that of [22, Proposition 4.1]. We give the details for the reader's convenience.

Firstly, as in the proof of Lemma 2.9, we have

$$\begin{aligned}
&\mathcal{S}_\Omega\,(u_0) - \mathcal{S}_\Omega\,(Q_\Omega)\\
&\quad= \mathcal{S}_\Omega\,(Q_\Omega + \lambda_0\varphi_\Omega + \widetilde{\rho}\,(\lambda_0)\,Q_\Omega) - \mathcal{S}_\Omega\,(Q_\Omega)\\
&\quad= \frac{1}{2}\mathcal{S}''_\Omega\,(Q_\Omega)\,(\lambda_0\varphi_\Omega + \widetilde{\rho}\,(\lambda_0)\,Q_\Omega,\ \lambda_0\varphi + \widetilde{\rho}\,(\lambda_0)\,Q_\Omega)\\
&\quad+ \frac{1}{6}\mathcal{S}'''_\Omega\,(Q_\Omega)\,(\lambda_0\varphi_\Omega + \widetilde{\rho}\,(\lambda_0)\,Q_\Omega,\ \lambda_0\varphi_\Omega + \widetilde{\rho}\,(\lambda_0)\,Q_\Omega,\\
&\qquad\qquad \lambda_0\varphi_\Omega + \widetilde{\rho}\,(\lambda_0)\,Q_\Omega)\\
&\qquad + \mathrm{o}\left(\|\lambda_0\varphi_\Omega + \widetilde{\rho}\,(\lambda_0)\,Q_\Omega\|^3_{H^1}\right),
\end{aligned} \tag{4.12}$$

where we used the fact that $\mathcal{S}'_{\Omega}(Q_{\Omega}) = 0$. By (2.22) and the fact that $\langle Q_{\Omega}\,,\,\varphi\rangle = 0$, we have

$$\begin{aligned}
&\mathcal{S}''_{\Omega}(Q_{\Omega})\left(\lambda_0\varphi_{\Omega} + \widetilde{\rho}(\lambda_0)\,Q_{\Omega},\ \lambda_0\varphi_{\Omega} + \widetilde{\rho}(\lambda_0)\,Q_{\Omega}\right)\\
&\quad = (\lambda_0)^2\,\mathcal{S}''_{\Omega}(Q_{\Omega})(\varphi_{\Omega},\ \varphi_{\Omega}) + 2\lambda_0\widetilde{\rho}(\lambda_0)\,\mathcal{S}''_{\Omega}(Q_{\Omega})(\varphi_{\Omega},\ Q_{\Omega})\\
&\qquad + \widetilde{\rho}(\lambda_0)^2\,\mathcal{S}''_{\Omega}(Q_{\Omega})(Q_{\Omega},\ Q_{\Omega})\\
&\quad = -(\lambda_0)^2\,\langle Q_{\Omega}\,,\,\varphi_{\Omega}\rangle - 2\lambda_0\widetilde{\rho}(\lambda_0)\,\langle Q_{\Omega}\,,\,Q_{\Omega}\rangle\\
&\qquad + \widetilde{\rho}(\lambda_0)^2\,\mathcal{S}''_{\Omega}(Q_{\Omega})(Q_{\Omega},\ Q_{\Omega})\\
&\quad = -2\lambda_0\widetilde{\rho}(\lambda_0)\,\langle Q_{\Omega}\,,\,Q_{\Omega}\rangle\\
&\qquad + \widetilde{\rho}(\lambda_0)^2\,\mathcal{S}''_{\Omega}(Q_{\Omega})(Q_{\Omega},\ Q_{\Omega})\\
&\quad = (\lambda_0)^3\,\langle \varphi_{\Omega}\,,\,\varphi_{\Omega}\rangle + \mathrm{o}\left(|\lambda_0|^3\right),
\end{aligned}$$

which together with (4.12) and (2.21) implies that

$$\begin{aligned}
&\mathcal{S}_{\Omega}(u_0) - \mathcal{S}_{\Omega}(Q_{\Omega})\\
&\qquad = \left(\frac{1}{2}\langle \varphi_{\Omega}\,,\,\varphi_{\Omega}\rangle + \frac{1}{6}\mathcal{S}'''_{\Omega}(Q_{\Omega})(\varphi_{\Omega},\ \varphi_{\Omega},\ \varphi_{\Omega})\right)\cdot(\lambda_0)^3\\
&\qquad\quad + \mathrm{o}\left(|\lambda_0|^3\right)\\
&\qquad = \frac{1}{6}\,\mathbf{d}'''(\Omega)\cdot(\lambda_0)^3 + \mathrm{o}\left(|\lambda_0|^3\right).
\end{aligned} \tag{4.13}$$

Secondly, by Lemma 2.10 and Lemma 2.14, we know that for any $t \geqslant 0$, there exists some $\kappa > 0$ such that

$$\begin{aligned}
\mathcal{S}_{\Omega}(u(t)) - \mathcal{S}_{\Omega}(Q_{\Omega}) &= \mathcal{S}_{\Omega}\left(Q_{\Omega} + \lambda(t)\,\varphi_{\Omega} + \rho(\lambda(t))\,Q_{\Omega} + \varepsilon(t)\right) - \mathcal{S}_{\Omega}(Q_{\Omega})\\
&= \frac{1}{6}\mathbf{d}'''(\Omega)\cdot(\lambda(t))^3 + \mathcal{S}''_{\Omega}(Q_{\Omega})(\varepsilon(t),\varepsilon(t))\\
&\quad + \mathrm{o}\left(|\lambda(t)|^3\right) + \mathrm{o}\left(\|\varepsilon(t)\|^2_{H^1}\right)\\
&\geqslant \frac{1}{6}\mathbf{d}'''(\Omega)\cdot(\lambda(t))^3 + \frac{\kappa}{4}\|\varepsilon(t)\|^2_{H^1}\\
&\quad + \mathrm{o}\left(|\lambda(t)|^3\right) + \mathrm{o}\left(\|\varepsilon(t)\|^2_{H^1}\right).
\end{aligned} \tag{4.14}$$

Finally, by the mass and energy conservation laws, we have

$$\mathcal{S}_{\Omega}(u(t)) = \mathcal{S}_{\Omega}(u_0), \quad \text{for any} \quad t \geqslant 0.$$

Therefore, by (4.13), (4.14) and the fact that $\mathbf{d}'''(\Omega) < 0$, we have

$$\begin{aligned}\frac{1}{24}\,\mathbf{d}'''(\Omega)\cdot(\lambda_0)^3 &\geqslant \frac{1}{6}\,\mathbf{d}'''(\Omega)\cdot(\lambda_0)^3 + \mathrm{o}\left(|\lambda_0|^3\right)\\ &\geqslant \frac{1}{6}\,\mathbf{d}'''(\Omega)\cdot(\lambda(t))^3 + \frac{\kappa}{4}\,\|\varepsilon(t)\|_{H^1}^2\\ &\quad + \mathrm{o}\left(|\lambda(t)|^3\right) + \mathrm{o}\left(\|\varepsilon(t)\|_{H^1}^2\right)\\ &\geqslant \frac{1}{3}\,\mathbf{d}'''(\Omega)\cdot(\lambda(t))^3 + \frac{\kappa}{6}\,\|\varepsilon(t)\|_{H^1}^2,\end{aligned}$$

which implies that

$$\lambda(t) \geqslant \frac{1}{2}\lambda_0, \quad \text{and} \quad \|\varepsilon(t)\|_{H^1}^2 \leqslant -\frac{2}{\kappa}\,\mathbf{d}'''(\Omega)\cdot(\lambda(t))^3.$$

This concludes the proof of Proposition 4.1. □

By (4.9) and (4.11), we have

$$|\langle \varepsilon(t), Q_\Omega\rangle| \leqslant C\lambda(t)^{\frac{5}{2}}, \tag{4.15}$$

where C is a constant independent of $\varepsilon(t)$ and $\lambda(t)$.

Step 4. Monotonicity formula. Let us define

$$\Phi(t,x) = \varphi_\Omega(x) - \lambda(t)\,\frac{\langle\varphi_\Omega,\varphi_\Omega\rangle}{\langle Q_\Omega, Q_\Omega\rangle}Q_\Omega(x), \tag{4.16}$$

and the Virial type quantity as follows

$$\mathscr{I}(t) = \langle \mathrm{i}\,\varepsilon(t), \Phi(t)\rangle. \tag{4.17}$$

By (4.4) and (4.5), we have the following estimates

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I}(t) &= \langle \mathrm{i}\,\partial_t\varepsilon, \Phi(t)\rangle - \lambda_t \frac{\langle \varphi_\Omega, \varphi_\Omega\rangle}{\langle Q_\Omega, Q_\Omega\rangle} \langle \mathrm{i}\,\varepsilon, Q_\Omega\rangle \qquad (4.18)\\
&= \langle \mathrm{i}\,\partial_t\varepsilon, \Phi(t)\rangle\\
&= -\lambda_t \langle \mathrm{i} \left(\varphi_\Omega + \frac{\mathrm{d}\rho}{\mathrm{d}\lambda}(\lambda(t))\, Q_\Omega\right), \Phi(t)\rangle \qquad (4.19)\\
&\quad - (\theta_t + \Omega) \langle (Q_\Omega + \lambda(t)\varphi_\Omega + \rho(\lambda(t))\, Q_\Omega + \varepsilon(t)), \Phi(t)\rangle \qquad (4.20)\\
&\quad + \langle \mathcal{L}(\lambda(t)\varphi_\Omega + \rho(\lambda(t))\, Q_\Omega + \varepsilon(t)), \Phi(t)\rangle \qquad (4.21)\\
&\quad - \frac{1}{2}\langle f''(Q_\Omega)(\lambda(t)\varphi_\Omega + \rho(\lambda(t))\, Q_\Omega + \varepsilon(t)) \qquad (4.22)\\
&\quad \times (\lambda(t)\varphi_\Omega + \rho(\lambda(t))\, Q_\Omega + \varepsilon(t)), \Phi(t)\rangle\\
&\quad - \langle \mathcal{R}(Q_\Omega, \lambda(t)\varphi_\Omega + \rho(\lambda(t))\, Q_\Omega + \varepsilon(t)), \Phi(t)\rangle . \qquad (4.23)
\end{aligned}
$$

Estimate of (4.19). By (1.5) and (2.15), we have the vanishing result

$$
(4.19) = 0. \qquad (4.24)
$$

Estimate of (4.20). By (2.20) and (4.5), we have

$$
\begin{aligned}
&\langle (Q_\Omega + \lambda(t)\varphi_\Omega + \rho(\lambda(t))\, Q_\Omega + \varepsilon(t)), \Phi(t)\rangle\\
=&\langle (Q_\Omega + \lambda(t)\varphi_\Omega + \rho(\lambda(t))\, Q_\Omega + \varepsilon(t)), \varphi_\Omega\rangle\\
&- \lambda(t)\frac{\langle \varphi_\Omega, \varphi_\Omega\rangle}{\langle Q_\Omega, Q_\Omega\rangle} \langle (Q_\Omega + \lambda(t)\varphi_\Omega + \rho(\lambda(t))\, Q_\Omega + \varepsilon(t)), Q_\Omega\rangle\\
=& - \lambda(t)\rho(\lambda(t)) \langle \varphi_\Omega, \varphi_\Omega\rangle - \lambda(t)\frac{\langle \varphi_\Omega, \varphi_\Omega\rangle}{\langle Q_\Omega, Q_\Omega\rangle} \langle \varepsilon(t), Q_\Omega\rangle .
\end{aligned}
$$

By (2.30) and (4.15), we obtain

$$
\langle (Q_\Omega + \lambda(t)\,\varphi_\Omega + \rho(\lambda(t))\, Q_\Omega + \varepsilon(t)), \Phi(t)\rangle = \mathrm{O}\left(\lambda(t)^{\frac{7}{2}}\right). \qquad (4.25)
$$

Now, inserting (4.7) and (4.25) into (4.20), we can obtain

$$
(4.20) = \mathrm{o}\left(\lambda(t)^2\right). \qquad (4.26)
$$

Estimate of (4.21). Since $\mathcal{L}$ is a self-adjoint operator, we deduced by Lemma 2.4, (2.30) and (2.20) that

$$\begin{aligned}
&\langle \mathcal{L}(\lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon), \Phi(t)\rangle \\
&\quad = \langle \mathcal{L}(\lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon), \varphi_\Omega\rangle \\
&\qquad - \lambda \frac{\langle \varphi_\Omega, \varphi_\Omega\rangle}{\langle Q_\Omega, Q_\Omega\rangle} \langle \mathcal{L}(\lambda\varphi_\Omega + \rho(\lambda) Q_\Omega + \varepsilon), Q_\Omega\rangle \\
&\quad = \frac{3}{2}\lambda^2 \langle \varphi_\Omega, \varphi_\Omega\rangle - \langle \varepsilon, Q_\Omega\rangle - \lambda\rho(\lambda) \frac{\langle \varphi_\Omega, \varphi_\Omega\rangle}{\langle Q_\Omega, Q_\Omega\rangle} \langle \mathcal{L} Q_\Omega, Q_\Omega\rangle \\
&\qquad - \lambda\rho(\lambda) \frac{\langle \varphi_\Omega, \varphi_\Omega\rangle}{\langle Q_\Omega, Q_\Omega\rangle} \langle \mathcal{L} Q_\Omega, \varepsilon\rangle. \qquad (4.27)
\end{aligned}$$

It follows from (2.30), (4.11) and (4.15) that

$$(4.21) = \frac{3}{2}\lambda(t)^2 \langle \varphi_\Omega, \varphi_\Omega\rangle + \mathrm{o}\left(\lambda(t)^2\right). \qquad (4.28)$$

Estimate of (4.22). Note that

$$\begin{aligned}
&\langle f''(Q_\Omega)(\lambda(t)\varphi_\Omega + \rho(\lambda(t)) Q_\Omega + \varepsilon(t)) \\
&\qquad \times (\lambda(t)\varphi_\Omega + \rho(\lambda(t)) Q_\Omega + \varepsilon(t)), \Phi(t)\rangle \\
&\quad = \langle f''(Q_\Omega)(\lambda(t)\varphi_\Omega + \rho(\lambda(t)) Q_\Omega + \varepsilon(t)) \\
&\qquad \times (\lambda(t)\varphi_\Omega + \rho(\lambda(t)) Q_\Omega + \varepsilon(t)), \varphi_\Omega\rangle \\
&\qquad - \lambda(t) \frac{\langle \varphi_\Omega, \varphi_\Omega\rangle}{\langle Q_\Omega, Q_\Omega\rangle} \langle f''(Q_\Omega)(\lambda(t)\varphi_\Omega + \rho(\lambda(t)) Q_\Omega + \varepsilon(t)) \\
&\qquad\quad \times (\lambda(t)\varphi_\Omega + \rho(\lambda(t)) Q_\Omega + \varepsilon(t)), Q_\Omega\rangle \\
&\quad = \lambda(t)^2 \langle f''(Q_\Omega)\varphi_\Omega\varphi_\Omega, \varphi_\Omega\rangle \\
&\qquad + 2\langle f''(Q_\Omega)(\lambda(t)\varphi_\Omega)(\rho(\lambda(t)) Q_\Omega + \varepsilon(t)), \varphi_\Omega\rangle \qquad (4.29) \\
&\qquad + \langle f''(Q_\Omega)(\rho(\lambda(t)) Q_\Omega + \varepsilon(t))(\rho(\lambda(t)) Q_\Omega + \varepsilon(t)), \varphi_\Omega\rangle \qquad (4.30) \\
&\qquad - \lambda(t) \frac{\langle \varphi_\Omega, \varphi_\Omega\rangle}{\langle Q_\Omega, Q_\Omega\rangle} \langle f''(Q_\Omega)(\lambda(t)\varphi_\Omega + \rho(\lambda(t)) Q_\Omega + \varepsilon(t)) \qquad (4.31) \\
&\qquad\quad \times (\lambda(t)\varphi_\Omega + \rho(\lambda(t)) Q_\Omega + \varepsilon(t)), Q_\Omega\rangle.
\end{aligned}$$

By (2.30) and (4.11), we have

$$\begin{aligned}
(4.29) &= \mathrm{O}\left(\lambda(t)\rho(\lambda(t)) + \lambda(t)\|\varepsilon(t)\|_{H^1}\right) = \mathrm{o}\left(\lambda(t)^2\right), \\
(4.30) &= \mathrm{O}\left(\lambda(t)^4 + \lambda(t)^2\|\varepsilon(t)\|_{H^1} + \|\varepsilon(t)\|_{H^1}^2\right) = \mathrm{o}\left(\lambda(t)^2\right), \\
(4.31) &= \mathrm{O}\left(\lambda(t)^2 + \lambda(t)^4 + \|\varepsilon(t)\|_{H^1}^2\right) = \mathrm{o}\left(\lambda(t)^2\right),
\end{aligned}$$

which implies that

$$(4.22) = -\frac{1}{2}\lambda(t)^2 \langle f''(Q_\Omega)\varphi_\Omega, \varphi_\Omega, \varphi_\Omega\rangle + \mathrm{o}\left(\lambda(t)^2\right). \tag{4.32}$$

Estimate of (4.23). By (2.30), (3.4) and (4.11), we have

$$(4.23) = \mathrm{O}\left(\lambda(t)^3 + \|\varepsilon(t)\|_{H^1}^2\right) = \mathrm{o}\left(\lambda(t)^2\right). \tag{4.33}$$

Therefore, by summing up (4.24), (4.26), (4.28), (4.32) and (4.33), we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I}(t) = \frac{\lambda(t)^2}{2}\left(-\langle f''(Q_\Omega)\varphi_\Omega\varphi_\Omega, \varphi_\Omega\rangle + 3\langle\varphi_\Omega, \varphi_\Omega\rangle\right) + \mathrm{o}\left(\lambda(t)^2\right).$$

It follows from (2.7) and Lemma 2.5 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I}(t) = \frac{1}{2}\mathbf{d}'''(\Omega)\lambda(t)^2 + \mathrm{o}\left(\lambda(t)^2\right). \tag{4.34}$$

Step 5. Conclusion. On the one hand, by (4.17) and (4.16), we obtain that $\|\varepsilon(t)\|_{H^1}$ and $\|\Phi(t)\|_{H^1}$ are uniformly bounded with respect to t. Therefore, by the Cauchy-Schwarz inequality, we have

$$|\mathscr{I}(t)| \ \text{uniformly bounded with respect to}\ t. \tag{4.35}$$

On the other hand, by $\mathbf{d}'''(\Omega) < 0$, (4.10) and (4.34), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I}(t) = \frac{1}{2}\,\mathbf{d}'''(\Omega)\lambda^2(t) + \mathrm{o}\left(\lambda(t)^2\right) \leqslant \frac{1}{4}\,\mathbf{d}'''(\Omega)\lambda(t)^2 \leqslant \frac{1}{16}\,\mathbf{d}'''(\Omega)(\lambda_0)^2,$$

by integrating the above inequality over $[0,\ t)$, we can obtain that

$$\mathscr{I}(t) = \mathscr{I}(0) + \int_0^t \mathscr{I}'(s)\,\mathrm{d}s \leqslant \mathscr{I}(0) + \frac{1}{16}\,\mathbf{d}'''(\Omega)(\lambda_0)^2\,t,$$

which means that

$$\lim_{t\to+\infty}\mathscr{I}(t) = -\infty,$$

which is in contradiction with (4.35).

Above all, we complete the proof of Theorem 1.3. □

Appendix A. Proof of Lemma 2.5

Proof of Lemma 2.5. (1) and (3) and the fact that $\mathbf{d}''(\Omega) = 0$ in (2) were proved in [8]. Now we show (2). For the convenience of the readers, we will give an alternative proof of the estimate $\mathbf{d}''(\Omega) = 0$. Since $\omega > \frac{\gamma^2}{4}$, we denote

$$\omega(\lambda) = \frac{\lambda^2\gamma^2}{4}, \quad \text{for} \quad \lambda > 1,$$
$$m(\omega) = \mathcal{M}(Q_\omega),$$

where $\mathcal{M}(Q_\omega)$ is defined by (1.2), and define

$$g(\lambda) = m(\omega(\lambda)). \tag{A.1}$$

It follows from (2.14) that

$$d''(\omega) = \frac{\mathrm{d}m}{\mathrm{d}\omega}(\omega). \tag{A.2}$$

A direct computation implies that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}g(\lambda) = \frac{\mathrm{d}m}{\mathrm{d}\omega}(\omega)\frac{\mathrm{d}\omega}{\mathrm{d}\lambda}(\lambda) = \frac{\lambda\gamma^2}{2}\frac{\mathrm{d}m}{\mathrm{d}\omega}(\omega), \tag{A.3}$$

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}g(\lambda) =& \frac{\mathrm{d}m}{\mathrm{d}\omega}(\omega)\frac{\mathrm{d}^2\omega}{\mathrm{d}\lambda^2}(\lambda) + \frac{\mathrm{d}^2m}{\mathrm{d}\omega^2}(\omega)\left(\frac{\mathrm{d}\omega}{\mathrm{d}\lambda}(\lambda)\right)^2\\ =& \frac{\gamma^2}{2}\frac{\mathrm{d}m}{\mathrm{d}\omega}(\omega) + \frac{\lambda^2\gamma^4}{4}\frac{\mathrm{d}^2m}{\mathrm{d}\omega^2}(\omega).\end{aligned} \tag{A.4}$$

Combining (A.2) with (A.3), we obtain that

$$\mathbf{d}''(\omega) = 0 \text{ if and only if } \tfrac{\mathrm{d}}{\mathrm{d}\lambda}g(\lambda) = 0.$$

By (1.5) and (A.1), we have

$$g(\lambda) = \int_0^{+\infty}\left(Q_{\omega(\lambda)}(x)\right)^2\mathrm{d}x = C(p,\gamma)\,h(\lambda)\,q(\lambda),$$

where $C(p,\gamma) = \left(\frac{p+1}{8}\right)^{\frac{2}{p-1}}\frac{4}{p-1}\gamma^{\frac{4}{p-1}-1} > 0$,

$$h(\lambda) = \lambda^{\frac{4}{p-1}-1}, \quad \text{and} \quad q(\lambda) = \int_{\operatorname{arctanh}\left(\frac{1}{\lambda}\right)}^{+\infty}\operatorname{sech}^{\frac{4}{p-1}}(y)\,\mathrm{d}y.$$

A direct calculation implies that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}g(\lambda) &= C(p,\gamma)\left(\frac{\mathrm{d}h}{\mathrm{d}\lambda}(\lambda)\,q(\lambda)+\frac{\mathrm{d}q}{\mathrm{d}\lambda}(\lambda)\,h(\lambda)\right)\\ &= C(p,\gamma)\left(\frac{5-p}{p-1}\lambda^{\frac{4}{p-1}-2}q(\lambda)+\lambda^{\frac{4}{p-1}-1}\frac{\left(\lambda^2-1\right)^{\frac{2}{p-1}-1}}{\lambda^{\frac{4}{p-1}}}\right).\end{aligned}$$

Therefore, $\mathbf{d}''(\Omega)=0$ if and only if $\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}g\right)\left(\frac{2\sqrt{\Omega}}{\gamma}\right)=0$, i.e. $\frac{2\sqrt{\Omega}}{\gamma}$ satisfies

$$\begin{aligned}q\left(\frac{2\sqrt{\Omega}}{\gamma}\right) &= \frac{p-1}{p-5}\left(\frac{2\sqrt{\Omega}}{\gamma}\right)^{\frac{p-5}{p-1}}\left(\left(\frac{2\sqrt{\Omega}}{\gamma}\right)^2-1\right)^{\frac{3-p}{p-1}}\\ &= \frac{p-1}{p-5}\left(\frac{2\sqrt{\Omega}}{\gamma}\right)^{-1+2\frac{p-3}{p-1}}\left(\left(\frac{2\sqrt{\Omega}}{\gamma}\right)^2-1\right)^{-\frac{p-3}{p-1}},\end{aligned}\tag{A.5}$$

which coincides with the fact that (1.7), i. e. we have $\mathbf{d}''(\Omega)=0$.

Next, we show that $\mathbf{d}'''(\Omega)<0$. By (A.4), we have

$$\mathbf{d}'''(\Omega)<0 \text{ if and only if } \left(\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}g\right)\left(\frac{2\sqrt{\Omega}}{\gamma}\right)<0.$$

Since

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}g(\lambda)=C(p,\gamma)\left(\frac{\mathrm{d}^2h}{\mathrm{d}\lambda^2}(\lambda)\,q(\lambda)+\frac{\mathrm{d}^2q}{\mathrm{d}\lambda^2}(\lambda)\,h(\lambda)+2\frac{\mathrm{d}h}{\mathrm{d}\lambda}(\lambda)\,\frac{\mathrm{d}q}{\mathrm{d}\lambda}(\lambda)\right),$$

it suffices to show that

$$\left(\frac{\mathrm{d}^2h}{\mathrm{d}\lambda^2}q+\frac{\mathrm{d}^2q}{\mathrm{d}\lambda^2}h+2\frac{\mathrm{d}h}{\mathrm{d}\lambda}\frac{\mathrm{d}q}{\mathrm{d}\lambda}\right)\left(\frac{2\sqrt{\Omega}}{\gamma}\right)<0.$$

By the direct computations, we have

$$\text{(A.6)} \qquad \begin{aligned} &\frac{\mathrm{d}^2 h}{\mathrm{d}\lambda^2}(\lambda)\, q(\lambda) + \frac{\mathrm{d}^2 q}{\mathrm{d}\lambda^2}(\lambda)\, h(\lambda) + 2\frac{\mathrm{d}h}{\mathrm{d}\lambda}(\lambda)\,\frac{\mathrm{d}q}{\mathrm{d}\lambda}(\lambda) \\ &= \left(\frac{4}{p-1} - 1\right)\left(\frac{4}{p-1} - 2\right)\lambda^{\frac{4}{p-1}-3} q(\lambda) \\ &+ \lambda^{\frac{4}{p-1}-1}\frac{3-p}{p-1}\left(\lambda^2 - 1\right)^{\frac{2}{p-1}-2} 2\lambda^{1-\frac{4}{p-1}} \\ &- \lambda^{\frac{4}{p-1}-1}\frac{4}{p-1}\left(\lambda^2 - 1\right)^{\frac{2}{p-1}-1}\lambda^{-1-\frac{4}{p-1}} \\ &+ 2\frac{\frac{4}{p-1} - 1}{\lambda^{\frac{4}{p-1}-2}\left(\lambda^2 - 1\right)^{\frac{2}{p-1}-1}\lambda^{-\frac{4}{p-1}}}. \end{aligned}$$

By inserting (A.5) into (A.6), we obtain

$$\left(\frac{\mathrm{d}^2 h}{\mathrm{d}\lambda^2} q + \frac{\mathrm{d}^2 q}{\mathrm{d}\lambda^2} h + 2\frac{\mathrm{d}h}{\mathrm{d}\lambda}\frac{\mathrm{d}q}{\mathrm{d}\lambda}\right)\left(\frac{2\sqrt{\Omega}}{\gamma}\right) = \frac{6-2p}{p-1}\left(\frac{4\Omega}{\gamma^2} - 1\right)^{\frac{2}{p-1}-2} < 0,$$

since $p > 5$. Therefore, we obtain $\mathbf{d}'''(\Omega) < 0$.

Finally, by (2.16) and Lemma 2.4, we have

$$\begin{aligned} \mathbf{d}'''(\Omega) &= \mathcal{S}_\Omega'''(Q_\Omega)(\varphi_\Omega, \varphi_\Omega, \varphi_\Omega) + 2\mathcal{S}_\Omega''(Q_\Omega)(\varphi_\Omega, \partial_\Omega \varphi_\Omega) \\ &\quad + \langle \varphi_\Omega, \varphi_\Omega \rangle + 2\langle \varphi_\Omega, \varphi_\Omega \rangle + 2\langle Q_\Omega, \partial_\Omega \varphi_\Omega \rangle \\ &= \mathcal{S}_\Omega'''(Q_\Omega)(\varphi_\Omega, \varphi_\Omega, \varphi_\Omega) + 3\langle \varphi_\Omega, \varphi_\Omega \rangle. \end{aligned}$$

This ends the proof of Lemma 2.5. □

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