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Semiclassical resolvent bounds for weakly decaying potentials

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In this note, we prove weighted resolvent estimates for the semiclassical Schrödinger operator $-h^2\Delta + V(x) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $n \neq 2$. The potential V is real-valued, and assumed to either decay at infinity or to obey a radial α-Hölder continuity condition, $0 \leq \alpha \leq 1$, with sufficient decay of the local radial C^α norm toward infinity. Note, however, that in the Hölder case, the potential need *not* decay. If the dimension $n \geq 3$, the resolvent bound is of the form $\exp\left(Ch^{-1-\frac{1-\alpha}{3+\alpha}}[(1-\alpha)\log(h^{-1})+c]\right)$, while for $n = 1$ it is of the form $\exp(Ch^{-1})$. A new type of weight and phase function construction allows us to reduce the necessary decay even in the pure L^∞ case.

1. Introduction and statement of results

Let $\Delta := \sum_{j=1}^n \partial_j^2 \leq 0$ be the Laplacian on $\mathbb{R}^n$, $n \neq 2$. In this article, we study the semiclassical Schrödinger operator

$$P(h) := -h^2\Delta + V : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \qquad h > 0,$$

where $V \in L^\infty(\mathbb{R}^n;\mathbb{R})$. We assume either that V satisfies a radial α-Hölder continuity condition, $0 \leq \alpha \leq 1$, or that it is only L^∞ but decaying. When $n \geq 3$, we use
$(r,\theta) = (|x|, x/|x|) \in (0,\infty) \times \mathbb{S}^{n-1}$ to denote polar coordinates on $\mathbb{R}^n \setminus \{0\}$.

When V is only L^∞, we assume

$$|V| \leq c_1\langle r\rangle^{-2}m(r), \tag{1.1}$$

for some

$$c_1 > 0, \qquad 0 < m(r) \leq 1, \qquad m(r)\langle r\rangle^{-1/2} \in L^2(0,\infty), \tag{1.2}$$

and where $\langle x\rangle = \langle r\rangle := (1+r^2)^{1/2}$.

Since $V \in L^\infty(\mathbb{R}^n;\mathbb{R})$, by the Kato-Rellich Theorem, $P(h)$ is self-adjoint $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with respect to the domain $H^2(\mathbb{R}^n)$. Therefore, the resolvent $(P-z)^{-1}$ is bounded $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ for all $z \in \mathbb{C} \setminus \mathbb{R}$, and we obtain

Theorem 1. *Let $n \geq 3$, m as in (1.2), $c_1 > 0$ and $E > 0$. Then there are $C > 0$ and $h_0 \in (0,1]$ so that for all $s > 1/2$, there is $C_s > 0$ such that for all $V \in L^\infty(\mathbb{R}^n;\mathbb{R})$ satisfying (1.1),*

$$g_s^\pm(h,\varepsilon) \leq C_s \exp\left(h^{-\frac{4}{3}}(C\log h^{-1} + C_s)\right), \qquad \varepsilon > 0,\ h \in (0,h_0], \tag{1.3}$$

where

$$g_s^\pm(h,\varepsilon) := \|\langle x\rangle^{-s}(P(h) - E \pm i\varepsilon)^{-1}\langle x\rangle^{-s}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}. \tag{1.4}$$

When V has some radial α-Hölder regularity, $0 \leq \alpha \leq 1$, we need not assume that V decays towards infinity. Instead, we suppose

$$V \in L^\infty, \qquad \sup_{\theta\in\mathbb{S}^{n-1}} \limsup_{y\to 0^+} \sup_r \frac{|V(r\theta) - V((r+y)\theta)|}{|y|^\alpha}\langle r\rangle^3 m^{-2}(r) \leq c_2, \tag{1.5}$$

for some $c_2 > 0$. We also define

$$V_\infty := \limsup_{r\to\infty} \sup_{\theta\in\mathbb{S}^{n-1}} V(r\theta), \tag{1.6}$$

$$0 < \delta_V := \inf\left\{y > 0 \mid \sup_{\theta\in\mathbb{S}^{n-1}} \sup_r \frac{|V(r\theta) - V((r+y)\theta)|}{|y|^\alpha}\langle r\rangle^3 m^{-2}(r) > 2c_2\right\}, \tag{1.7}$$

and for $E > V_\infty$,

$$R_{E,V} := \sup\left\{r \mid \sup_{\theta\in\mathbb{S}^{n-1}} V(r\theta) > \frac{E + 3V_\infty}{4}\right\}. \tag{1.8}$$

Remark: Note that when $\alpha = 0$ and (1.5) holds, V is still only L^∞, but the magnitude of its fluctuations are decaying faster than those in (1.1).

In this Hölder regular case, we obtain

Theorem 2. *Let $n \geq 3$, m as in (1.2), $c_2 > 0$, $R_E > 0$, $C_V, E_\infty \in \mathbb{R}$, and $E > E_\infty$. Then there is $C > 0$ such that for all $\delta_1 > 0$, there is $h_0 \in (0,1]$*

so that for all $s > 1/2$*, there is* $C_s > 0$ *so that for* $V \in L^\infty(\mathbb{R}^n;\mathbb{R})$ *obeying* $\sup_{\mathbb{R}^n} V \le C_V$*,* $V_\infty \le E_\infty$*,* $\delta_1 \le \delta_V$*,* $R_{E,V} \le R_E$*, and* (1.5) *for some* $0 \le \alpha \le 1$*,*

$$g_s^\pm(h,\varepsilon) \le C_s \exp\left(h^{-1-\sigma_\alpha}(C\sigma_\alpha \log h^{-1} + C_s)\right), \quad \varepsilon > 0,\ h \in (0,h_0], \tag{1.9}$$

where

$$\sigma_\alpha := \frac{1-\alpha}{3+\alpha}.$$

In the one-dimensional case, (1.5) can be relaxed further to

$$\limsup_{y\to 0} \sup_x \frac{|V(x) - V(x+y)|}{m_0(|x|)} \le c_0, \tag{1.10}$$

for some

$$c_0 > 0, \qquad 0 < m_0(r) \le 1, \qquad m_0 \in L^1(0,\infty). \tag{1.11}$$

We then define

$$0 < \delta_{0,V} := \inf\{y > 0 \mid \sup_x \frac{|V(x) - V(x+y)|}{m_0(|x|)} > 2c_0\}. \tag{1.12}$$

Then we have the following one dimensional result.

Theorem 3. *Let* $n = 1$*,* m_0 *as in* (1.11)*,* $c_0 > 0$*,* $R_E > 0$*,* $C_V, E_\infty \in \mathbb{R}$ *and* $E > E_\infty$*. Then there is* $C > 0$ *such that for all* $\delta_0 > 0$*, there is* $h_0 \in (0,1]$ *so that for all* $s > 1/2$*, there is* $C_s > 0$ *so that for* $V \in L^\infty(\mathbb{R};\mathbb{R})$ *obeying* $\delta_0 \le \delta_{0,V}$*,* $\sup_{\mathbb{R}} V \le C_V$*,* $V_\infty \le E_\infty$*,* $R_{E,V} \le R_E$*, and* (1.10)*,*

$$g_s^\pm(h,\varepsilon) \le C_s \exp\left(Ch^{-1}\right), \qquad \varepsilon > 0,\ h \in (0,h_0]. \tag{1.13}$$

Bounds on $g_s^\pm$ are known to hold under various geometric, regularity, and decay assumptions. Burq [1, 2] showed $g_s^\pm \le e^{Ch^{-1}}$ for V smooth and decaying sufficiently fast near infinity, and also for more general perturbations of the Laplacian. Cardoso and Vodev [3] extended Burq's estimate to infinite volume Riemannian manifolds which may contain cusps.

In lower regularity and $n \ne 2$, Datchev [4] showed $g_s^\pm \le e^{Ch^{-1}}$, provided $V, \partial_r V \in L^\infty(\mathbb{R}^n;\mathbb{R})$ and have long-range decay. The second author [8] obtained the same bound for $n = 2$, and under the same assumptions, except with $\partial_r V$ replaced by ∇V [8]. On the other hand, Vodev [10] showed that, if

$n \geq 3$ and V's radial α-Hölder moduli are $O(h^\nu \langle r \rangle^{-\kappa})$, where $\nu > 0$, $\kappa > 1$, and $\alpha \geq 1 - 2\nu$, then $g_s^\pm \leq e^{Ch^{-\ell}}$, where

$$\ell = \max\left\{0, \frac{2(1-\nu-\alpha)}{1-\alpha}\right\} < 1.$$

If $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n;\mathbb{R})$, $n \geq 2$, it was previously shown [7, 9] that $g_s^\pm \leq e^{Ch^{-4/3}\log(h^{-1})}$. This same bound was extended to short range potentials on $\mathbb{R}^n$ [11, 13], and then to short range potentials on a large class of asymptotically Euclidean manifolds [14]. If $n = 1$, $g_s^\pm \leq e^{Ch^{-1}}$, even if $V \in L^1(\mathbb{R};\mathbb{R})$ [6].

Theorems 1 and 2 improve upon the existing literature in several ways. First, in the pure L^∞ case (1.1), Theorem 1 reduces the required decay for V from that in [11, 13]. While we are still unable to obtain estimates when V is an arbitrary short range L^∞ potential without an additional loss of powers of h in $\log(g_s^\pm(h,\varepsilon))$, the decay assumed in (1.1) appears to improve on the existing literature by one order in r. Secondly, the assumptions for Theorem 2 (1.5) allow for *non-decaying* potentials provided some control on the local oscillations of the potential V (even if V is not Hölder continuous for any positive α). Finally, as the Hölder constant of the potential varies between 0 and 1, the results interpolate between those in the L^∞ and Lipschitz cases, with the bound on $g_s^\pm(h,\varepsilon)$ agreeing with the existing estimates at both endpoints.

Next, Theorem 3 seems to be the first semiclassical resolvent estimate in one dimension that does not require V or $\partial_x V$ to belong to $L^1(\mathbb{R};\mathbb{R})$. Again, by imposing some condition on the oscillations of V, we are able to handle even non-decaying potentials.

In dimension $n \geq 2$, it is an open problem to determine the optimal h-dependence of the resolvent for $V \in L^\infty$ or V satisfying (1.5). In contrast, it is well known that the bound $e^{Ch^{-1}}$ cannot be improved in general. See, for instance, [5] and the references cited there.

To prove Theorems 1, 2 and 3, we adapt the Carleman estimates proved in [11] and [6]. In addition to the modifications necessary to take advantage of the Hölder regularity of V, the main improvement in our argument is to determine φ and w from the logarithmic derivatives of respectively φ' and w. This dramatically simplifies the computations necessary to construct the requisite phases and weights. See (2.9) and (2.10) for the main quantities one must estimate.

In the final stages of writing this note, we learned of the article [12], in which Vodev uses a somewhat different weight and phase construction to study Hölder potentials analogous to ours. However, the assumed decay in that article is stronger than what we need here. On the other hand, Vodev's article gives the local Carleman estimates necessary to handle dimension $n = 2$ as well as the case where $\mathbb{R}^n$ is replaced by the exterior of a smooth obstacle.

2. Preliminary Calculations and Lemmata

Notation: In dimension $n \geq 3$, "prime" notation denotes differentiation with respect the radial variable r, e.g., $u' := \partial_r u$. In dimension $n = 1$, u' indicates differentiation with respect to the real variable x. Throughout, we let w and φ be a weight and phase, respectively, such that $w, w', \varphi' > 0$. When $n \geq 3$, w and φ are functions of the radial variable only, while if $n = 1$ they are functions of x. We will precisely specify w and φ in the course of our proof.

As in most previous proofs of resolvent estimates for low regularity potentials, the backbone of the proof is a Carleman estimate. We start from the identity

$$r^{\frac{n-1}{2}}(-\Delta)r^{-\frac{n-1}{2}} = -\partial_r^2 + \Lambda,$$

where

$$\Lambda := \frac{1}{r^2}\left(-\Delta_{\mathbb{S}^{n-1}} + \frac{(n-1)(n-3)}{4}\right) \geq 0, \tag{2.1}$$

and $\Delta_{\mathbb{S}^{n-1}}$ denotes the negative Laplace-Beltrami operator on $\mathbb{S}^{n-1}$.

Then, we form the conjugated operator

$$\begin{aligned} P_\varphi^{\pm}(h) &:= e^{\varphi/h} r^{\frac{n-1}{2}}\left(P(h) - E \pm i\varepsilon\right) r^{-\frac{n-1}{2}} e^{-\varphi/h} \\ &= -h^2\partial_r^2 + 2h\varphi'\partial_r + h^2\Lambda + V - (\varphi')^2 + h\varphi'' - E \pm i\varepsilon. \end{aligned} \tag{2.2}$$

Now, let $V_h \in C^\infty((0,\infty)_r; L^\infty(\mathbb{S}_\theta^{n-1}))$ be a smoothed approximation to V, and define

$$R_h := V - V_h. \tag{2.3}$$

For $n \geq 3$ and $u \in e^{\varphi/h} r^{(n-1)/2} C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, we define a spherical energy functional $F[u](r)$,

$$F(r) = F[u](r) := \|hu'(r,\cdot)\|^2 - \langle (h^2\Lambda + V_h - (\varphi')^2 - E)u(r,\cdot), u(r,\cdot)\rangle, \tag{2.4}$$

where $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the norm and inner product on $L^2(\mathbb{S}^{n-1}_\theta)$, respectively. The derivative of F, in the sense of distributions on $(0,\infty)$, is

$$\begin{aligned} F' &= 2\operatorname{Re}\langle h^2 u'', u'\rangle - 2\operatorname{Re}\langle (h^2\Lambda + V_h - (\varphi')^2 - E)u, u'\rangle \\ &\quad + 2r^{-1}\langle h^2\Lambda u, u\rangle + ((\varphi')^2 - V_h)'\|u\|^2 \\ &= -2\operatorname{Re}\langle P^{\pm}_{\varphi}(h)u, u'\rangle + 2r^{-1}\langle h^2\Lambda u, u\rangle + ((\varphi')^2 - V_h)'\|u\|^2 \\ &\quad + 4h^{-1}\varphi'\|hu'\|^2 \mp 2\varepsilon \operatorname{Im}\langle u, u'\rangle + 2\operatorname{Re}\langle (R_h + h\varphi'')u, u'\rangle. \end{aligned}$$

Thus $(wF)'$, as a distribution on $(0,\infty)$, is given by

$$\begin{aligned} (wF)' &= w'F + wF' \\ &= w'\|hu'\|^2 - w'\langle (h^2\Lambda + V_h - (\varphi')^2 - E)u, u\rangle \\ &\quad - 2w\operatorname{Re}\langle P^{\pm}_{\varphi}(h)u, u'\rangle + 2wr^{-1}\langle h^2\Lambda u, u\rangle + w((\varphi')^2 - V_h)'\|u\|^2 \\ &\quad + 4h^{-1}w\varphi\|hu'\|^2 \mp 2\varepsilon w\operatorname{Im}\langle u, u'\rangle + 2w\operatorname{Re}\langle (R_h + h\varphi'')u, u'\rangle \\ &= -2\operatorname{Re} w\langle P^{\pm}_{\varphi}(h)u, u'\rangle \mp 2\varepsilon w\operatorname{Im}\langle u, u'\rangle + (2wr^{-1} - w')\langle h^2\Lambda u, u\rangle \\ &\quad + (4h^{-1}w\varphi' + w')\|hu'\|^2 + (w(E + (\varphi')^2 - V_h))'\|u\|^2 \\ &\quad + 2w\operatorname{Re}\langle (R_h + h\varphi'')u, u'\rangle. \end{aligned} \tag{2.5}$$

Using (2.1) when $n \geq 3$, we will need

$$2wr^{-1} - w' \geq 0, \tag{2.6}$$

to control the term involving Λ. It is the absence of this condition which allows for the improved estimate in dimension one. Using (2.6) together with $2ab \geq -(\gamma a^2 + \gamma^{-1}b^2)$ for all $\gamma > 0$, we find

$$\begin{aligned} w'F + wF' \geq& -\frac{3w^2}{h^2 w'}\|P^{\pm}_{\varphi}(h)u\|^2 \mp 2\varepsilon w\operatorname{Im}\langle u, u'\rangle \\ &+ \frac{1}{3}(w' + 4h^{-1}\varphi' w)\|hu'\|^2 \\ &+ (w(E + (\varphi')^2 - V_h))'\|u\|^2 \\ &- \frac{3(w(h^{-1}|R_h| + \varphi''))^2}{w' + 4h^{-1}\varphi' w}\|u\|^2. \end{aligned} \tag{2.7}$$

In dimension $n = 1$, rather than the spherical energy (2.4), we use the pointwise energy

$$F(x) = F[u](x) := |hu'(x)|^2 - (V_h(x) - (\varphi'(x))^2 - E)|u(x)|^2.$$

Exactly the same computations then lead to

$$w'F + wF' \geq -\frac{3w^2}{h^2 w'}|P_\varphi^\pm(h)u|^2 \mp 2\varepsilon w \operatorname{Im} u\overline{u'} + \frac{1}{3}(w' + 4h^{-1}\varphi' w)|hu'|^2 + (w(E + (\varphi')^2 - V_h))'|u|^2 - \frac{3(w(h^{-1}|R_h| + \varphi''))^2}{w' + 4h^{-1}\varphi' w}|u|^2.$$

Thus, the main goal of the estimates below will be to construct φ and w such that

$$(w(E + (\varphi')^2 - V_h))' - \frac{3(w(h^{-1}|R_h| + \varphi''))^2}{w' + 4h^{-1}\varphi' w} \geq \frac{E - E_\infty}{2} w'.$$

Putting

$$A(r) := (w(E + (\varphi')^2 - V_h))', \qquad B(r) := \frac{(w(h^{-1}|R_h| + \varphi''))^2}{w' + 4h^{-1}\varphi' w},$$

our goal is thus, for $K > 0$ fixed and h small enough, to find w and φ such that

$$A(r) - \frac{K}{2}B(r) \geq \frac{E - E_\infty}{2} w'(r). \tag{2.8}$$

Now, we will assume throughout that $w', \varphi' > 0$. Therefore, putting

$$\Phi := \frac{\varphi''}{\varphi'} = (\log \varphi')', \qquad \mathcal{W} := \frac{w}{w'} = \frac{1}{(\log w)'}, \tag{2.9}$$

we calculate

$$\begin{aligned}
A(r) &- \frac{K}{2}B(r) \\
&= w'(E + (\varphi')^2 - V_h) + w(2\varphi'\varphi'' - V_h') - \frac{K}{2}\frac{(w(h^{-1}|R_h| + \varphi''))^2}{w' + 4h^{-1}\varphi' w} \\
&= w'\Big[E + (\varphi')^2 - V_h + \mathcal{W}(2\varphi'\varphi'' - V_h') - \frac{K}{2}\frac{(w(h^{-1}|R_h| + \varphi''))^2}{w'^2 + 4h^{-1}\varphi' w w'}\Big] \\
&= w'\Big[E + (\varphi')^2(1 + 2\mathcal{W}\Phi) - V_h - \mathcal{W}V_h' - \frac{K}{2}\mathcal{W}^2\frac{((h^{-1}|R_h| + \varphi''))^2}{1 + 4h^{-1}\varphi'\mathcal{W}}\Big] \\
&\geq w'\Big[E + (\varphi')^2(1 + 2\mathcal{W}\Phi) - V_h - \mathcal{W}V_h' - K\mathcal{W}^2\frac{h^{-2}|R_h|^2 + (\varphi'')^2}{1 + 4h^{-1}\varphi'\mathcal{W}}\Big].
\end{aligned}$$

Finally,

$$A(r) - \frac{K}{2}B(r) \geq w'\Big[E + (\varphi')^2(1 + 2\mathcal{W}\Phi - K\mathcal{W}\Phi^2 \min(\mathcal{W}, \tfrac{h}{4\varphi'})) \\ - V_h - \mathcal{W}V_h' - K\mathcal{W}h^{-2}|R_h|^2 \min(\mathcal{W}, \tfrac{h}{4\varphi'})\Big]. \tag{2.10}$$

The key improvement in this article is that, to prove the main estimates (3.5) and (4.4), we work with $\mathcal{W}$ and Φ rather than directly with w and φ. This simplifies the calculations dramatically and points the way to a new choice of phase function allowing us to weaken the decay requirements on V. The condition (2.6) for $n \geq 3$ translates simply to $\mathcal{W} \geq r/2$. The remainder of the article focuses on constructing appropriate $\mathcal{W}$ and Φ such that (2.8) holds.

Before proceeding with the construction of $\mathcal{W}$ and Φ, we need a few elementary lemmata:

Lemma 2.1. *Let*

$$\Phi(s) = -\frac{1}{s + 1 + \Phi_1(s)},$$

with

$$0 \leq (s+1)^{-2}\Phi_1(s) \in L^1(0,\infty). \tag{2.11}$$

Then,

$$-\log(r+1) \leq \int_0^r \Phi(s)ds \leq -\log(r+1) + \|(s+1)^{-2}\Phi_1(s)\|_{L^1(0,\infty)}.$$

Proof. First, note that

$$\log(r+1) + \int_0^r \Phi(s)ds = \int_0^r \frac{1}{1+s} - \frac{1}{s+1+\Phi_1(s)}ds \\ = \int_0^r \frac{\Phi_1(s)}{(s+1)(s+1+\Phi_1(s))}ds.$$

Next, note that

$$0 \leq \int_0^r \frac{\Phi_1(s)}{(s+1)(s+1+\Phi_1(s))}ds \leq \|(s+1)^{-2}\Phi_1(s)\|_{L^1(0,\infty)},$$

which implies

$$-\log(r+1) \leq \int_0^r \Phi(s)ds \leq -\log(r+1) + \|(s+1)^{-2}\Phi_1(s)\|_{L^1(0,\infty)}. \qquad \square$$

In the proof of Theorem 2, we will need to approximate V by smooth functions V_h. In the case (1.1), we simply approximate V by 0, defining $V_h \equiv 0$. On the other hand, when we assume (1.5), we make a non-trivial approximation to V. In the spirit of [10, Section 2], let

$$\chi \in C^\infty_{\mathrm{comp}}((0,1);[0,1]), \qquad \int \chi(s)ds = 1, \tag{2.12}$$

and define

$$\begin{aligned} V(r\theta;\gamma) &:= \int_0^\infty V((r+\gamma s)\theta)\chi(s)ds \\ &= \gamma^{-1}\int_0^\infty V(s\theta)\chi(\gamma^{-1}(s-r))ds, \qquad 0<\gamma\le 1. \end{aligned}$$

Then set

$$V_h(r\theta) := V(r\theta; h^\rho),$$

for $\rho > 0$ to be chosen later, depending on α.

Lemma 2.2. *Suppose* $0 \le \alpha \le 1$, V *satisfies* (1.5), *and* δ_V *is as in* (1.7). *Then there exists* $C_\chi > 0$ *depending only on* χ *so that, for all* $h \in (0, \delta_V^{1/\rho}]$,

$$V_h(r\theta) \le \sup_{s\in[r,r+h^\rho]} V(s\theta), \tag{2.13}$$
$$|V_h'(r\theta)| \le C_\chi c_2 h^{-\rho(1-\alpha)}\langle r\rangle^{-3}m^2(r), \qquad |R_h(r\theta)| \le 2c_2 h^{\rho\alpha}\langle r\rangle^{-3}m^2(r).$$

Proof. First observe that

$$\begin{aligned} V(r\theta;\gamma) &= \int_0^\infty [V((r+\gamma s)\theta) - \inf_{t\in[r,r+\gamma]} V(t\theta)]\chi(s)ds + \inf_{t\in[r,r+\gamma]} V(t\theta) \\ &\le (\sup_{s\in[r,r+\gamma]} V(s\theta) - \inf_{t\in[r,r+\gamma]} V(t\theta))\int \chi(s)ds + \inf_{t\in[r,r+\gamma]} V(t\theta) \\ &= \sup_{s\in[r,r+\gamma]} V(s\theta) \end{aligned} \tag{2.14}$$

where in the third line we use implicitly that $\chi \ge 0$ and for $s \in \operatorname{supp}\chi$, $[V((r+\gamma s)\theta) - \inf_{t\in[r,r+\gamma]} V(t\theta)] \ge 0$.

Next, from $\int \chi' dr = 0$,

$$\begin{aligned}|V'(r\theta;\gamma)| &= \left|\gamma^{-2}\int_0^\infty V(s\theta)\chi'(\gamma^{-1}(s-r))ds - \gamma^{-1}V(r\theta)\int_0^1 \chi'(s)ds\right| \\ &= \left|\gamma^{-1}\int_0^1 [V((r+\gamma s)\theta) - V(r\theta)]\chi'(s)ds\right| \\ &\le \left|\gamma^{-1+\alpha}\int_0^1 s^\alpha \frac{(V((r+\gamma s)\theta) - V(r\theta))\chi'(s)}{\gamma^\alpha s^\alpha} ds\right|.\end{aligned}$$

In particular, by (1.5) and the definition (1.7) of δ_V, we have, for $0 < \gamma \le \delta_V$,

(2.15)
$$|V'(r\theta;\gamma)| \le 2c_2\gamma^{-1+\alpha}\langle r\rangle^{-3}m^2(r)\int_0^1 |s^\alpha\chi'(s)|ds \le C_\chi c_2\gamma^{-1+\alpha}\langle r\rangle^{-3}m^2(r).$$

Finally, using (1.5) again,

$$\begin{aligned}|V(r\theta) - V(r\theta;\gamma)| &= \left|\int_0^\infty [V(r\theta) - V((r+\gamma s)\theta)]\chi(s)ds\right| \\ &= \left|\int_0^\infty \gamma^\alpha s^\alpha \frac{V(r\theta) - V((r+\gamma s)\theta)}{\gamma^\alpha s^\alpha}\chi(s)ds\right| \\ &\le 2c_2\gamma^\alpha\langle r\rangle^{-3}m^2(r),\end{aligned} \tag{2.16}$$

for $0 < \gamma \le \delta_V$. The lemma is proved by setting $\gamma = h^\rho$, $h \in (0, \delta_V^{1/\rho}]$, in (2.14), (2.15), and (2.16). □

3. Proof of the main estimates ($n \ge 3$)

Recall the definitions of Φ and $\mathcal{W}$ from (2.9), and put

$$\begin{aligned}&\varphi(r) = h^{-\sigma}\varphi_0(r),\ \sigma \ge 0, \qquad \varphi_0(0) = 0,\ \varphi_0'(0) = \tau_0 \ge 1, \\ &w(0) = 0,\ w'(0) = 1,\end{aligned} \tag{3.1}$$

so that

$$\Phi = (\log\varphi_0')', \qquad \mathcal{W} = \frac{1}{(\log w)'}. \tag{3.2}$$

We also set

$$\sigma = \frac{1-\alpha}{3+\alpha}, \qquad \rho = \frac{2}{3+\alpha}. \tag{3.3}$$

Finally, let

$$a = a_0 h^{-M},\ a_0 \geq 1,\ M > 0. \tag{3.4}$$

Each of the parameters τ_0, a_0, and M will be fixed shortly.

The main result of this section is Proposition 3.1. In its statement and proof, we use C for a positive constant that may change from line to line, but depends only on K, C_V, c_1, c_2, E, E_∞, R_E, and m. We also reuse constants $h_0 \in (0,1]$ and $C_\eta > 0$; they depend only on the same quantities as C, except that h_0 also depends on $\delta_1 > 0$, while $C_\eta > 0$ also depends on $0 < \eta < 1$. In particular, C and h_0 are independent of α, h and η, and C_η is independent of α and h.

Proposition 3.1. *Fix $K > 0$. Let V as in Theorem 1 or 2, σ and ρ be given by (3.3), $E > E_\infty$ and $0 < \eta < 1$. Then there exist τ_0 as in (3.1), a_0 and M as in (3.4), radial functions $\mathcal{W}$ and Φ continuous except at $r = a$, and their corresponding w and φ determined by (3.1) and (3.2), and constants $C, C_\eta > 0$, $h_0 \in (0,1]$ so that*

$$A(r) - \frac{K}{2} B(r) \geq \frac{E - E_\infty}{2} w'(r), \qquad r \neq a,\ h \in (0, h_0], \tag{3.5}$$

φ_0 satisfies,

$$|\varphi_0(r)| \leq C\Big[\frac{1-\alpha}{(1-\frac{\eta}{2})(3+\alpha)} \log h^{-1} + \frac{1}{\eta}\Big], \tag{3.6}$$

and w satisfies

$$w(r) \leq C_\eta h^{-\frac{4(1-\alpha)}{(2-\eta)(3+\alpha)}}, \tag{3.7}$$

$$w'(r) \geq (r+1)^{-1-\eta}, \qquad r \neq a, \tag{3.8}$$

$$\frac{w(r)^2}{w'(r)} \leq C_\eta h^{-\frac{4(1-\alpha)}{(2-\eta)(3+\alpha)}} (1+r)^{1+\eta}, \qquad r \neq a. \tag{3.9}$$

3.1. Small r region

We start by working with $0 < r \leq a$. Let $\omega \in C^\infty_{\mathrm{comp}}((-3/4, 3/4); [0,1])$ with $\omega = 1$ near $[-1/2, 1/2]$. In this region, define $\mathcal{W}$ and Φ by

$$\mathcal{W} = \frac{r(1+\omega(r))}{2}, \qquad \Phi = -\frac{1}{r + 1 + \Phi_1(r)}, \qquad 0 < r \leq a. \tag{3.10}$$

where $\Phi_1(s)$ obeying (2.11) is to be chosen as needed. With these conditions on Φ_1, by Lemma 2.1,

$$\frac{\tau_0}{r+1} \le \varphi_0'(r) \le \frac{e^{\|\langle s\rangle^{-2}\Phi_1(s)\|_{L^1}}\tau_0}{r+1}, \qquad 0 < r \le a. \tag{3.11}$$

In this region, we work separately on the cases (1.1) and (1.5),
Case (1.1), $\alpha = 0$**:** In this case, we have $\sigma = \frac{1}{3}$, $V_h = V_h' = 0$, $R_h = V$, and $V_\infty = 0$. Therefore, using (1.1), (2.10), and (3.11),

$$\begin{aligned} &A - \frac{K}{2}B \\ &\ge w'(E + h^{-2\sigma}(\varphi_0')^2(1 + r(1+\omega)\Phi - K(8\tau_0)^{-1}h^{1+\sigma}r(r+1)(1+\omega)\Phi^2) \\ &\qquad - CK\tau_0^{-1}h^{-1+\sigma}r(r+1)\langle r\rangle^{-4}m^2) \\ &\ge w'\frac{1}{\tau_0(r+1)^2}(h^{-2\sigma}\tau_0^3(\frac{1+\Phi_1 - r\omega}{r+1+\Phi_1}) - CK\tau_0^{-1}h^{-1+\sigma}m^2) \\ &\qquad + (E - K\tau_0 e^{2\|\langle s\rangle^{-2}\Phi_1(s)\|_{L^1}}h^{1-\sigma})w', \qquad h > 0. \end{aligned} \tag{3.12}$$

So, putting

$$\Phi_1 = \max(\Phi_2, 0), \quad \text{where } \Phi_2 \text{ solves} \quad 4\frac{1+\Phi_2 - r\omega(r)}{r+1+\Phi_2} = m^2, \tag{3.13}$$

and then choosing $\tau_0 = \tau_0(C, K, m) \ge 1$ large enough, we obtain,

$$\begin{aligned} &A - \frac{K}{2}B \ge (E - K\tau_0 e^{2\|\langle s\rangle^{-2}\Phi_1(s)\|_{L^1}}h^{1-\sigma})w' \ge \frac{E}{2}w', \\ &0 < r \le a,\ h \in (0, h_0], \end{aligned} \tag{3.14}$$

for $h_0 = h_0(K, \tau_0, E, m) \in (0, 1]$ small enough. This proves the claimed inequality (3.5) for
$0 < r \le a$.
Case (1.5), $0 \le \alpha \le 1$**:** Recall that $R_{E,V}$ and δ_V are given by (1.8) and (1.7) respectively. Because $R_{E,V} \le R_E$, and $\delta_V \ge \delta_1$, the first estimate in (2.13) implies

$$\sup_{\theta\in\mathbb{S}^{n-1}} V_h(r\theta) \le \frac{E + 3V_\infty}{4} \le \frac{E + 3E_\infty}{4} =: \tilde{E}, \qquad r \ge R_E,\ h \in (0, \delta_1^{1/\rho}]. \tag{3.15}$$

Next, let $\psi \in C^\infty_{\mathrm{comp}}((-1, R_E+1); [0,1])$ with $\psi \equiv 1$ on $[0, R_E]$. Then, $\sup_{\mathbb{R}^n} V \le C_V$ and (2.13) yield

$$V_h \le C_V \psi(r) + \tilde{E}, \qquad h \in (0, \delta_1^{1/\rho}].$$

Using (2.10), (2.13), and (3.15), we have the following modified version of the estimate (3.12) for $h \in (0, \delta_1^{1/\rho}]$,

$$\begin{aligned}
&A - \frac{K}{2}B \\
&\ge w'\Big(E + h^{-2\sigma}(\varphi_0')^2(1 + r(1+\omega)\Phi - K(8\tau_0)^{-1}h^{1+\sigma} r(r+1)(1+\omega)\Phi^2) \\
&\qquad - C_V\psi - \tilde{E} - Ch^{-\rho(1-\alpha)} r\langle r\rangle^{-3} m^2 \\
&\qquad - CK\tau_0^{-1}h^{-1+2\rho\alpha+\sigma} r(r+1)\langle r\rangle^{-6} m^4\Big) \\
&\ge \frac{w'}{(r+1)^2}\Big(h^{-2\sigma}\tau_0^2(\frac{1+\Phi_1 - r\omega}{r+1+\Phi_1}) - CK\tau_0^{-1}h^{-1+2\rho\alpha+\sigma}\langle r\rangle^{-2}m^4 \\
&\qquad - Ch^{-\rho(1-\alpha)}m^2 - C_V(R_E+2)^2\psi\Big) \\
&\quad + (\tfrac{3}{4}(E - E_\infty) - K\tau_0 e^{2\|\langle s\rangle^{-2}\Phi_1(s)\|_{L^1}} h^{1-\sigma})w'.
\end{aligned}$$

To get the second inequality, we have used

$$K(8\tau_0)^{-1}h^{1-\sigma}(\varphi_0')^2 r(r+1)(1+\omega)\Phi^2 \le K\tau_0 e^{2\|\langle s\rangle^{-2}\Phi_1(s)\|_{L^1}} h^{1-\sigma},$$
$$Ch^{-\rho(1-\alpha)} r\langle r\rangle^{-3} m^2 \le Ch^{-\rho(1-\alpha)}(r+1)^{-2}m^2,$$

as well as

$$CK\tau_0^{-1}h^{-1+2\rho\alpha+\sigma} r(r+1)\langle r\rangle^{-6}m^4 \le CK\tau_0^{-1}h^{-1+2\rho\alpha+\sigma}(r+1)^{-2}\langle r\rangle^{-2}m^4.$$

By (3.3), we have $0 \le \sigma \le 1/3$. Using also (3.13), $\langle r\rangle^{-2}m^4 \le m^2$ and choosing $\tau_0 = \tau_0(C, K, C_V, R_E, m) \ge 1$ large enough, we arrive at

$$\begin{aligned}
A - \frac{K}{2}B &\ge (\tfrac{3}{4}(E - E_\infty) - K\tau_0 e^{2\|\langle s\rangle^{-2}\Phi_1(s)\|_{L^1}} h^{1-\sigma})w' \\
&\ge \frac{E - E_\infty}{2} w', \qquad 0 < r \le a,\ h \in (0, h_0]
\end{aligned} \tag{3.16}$$

for $h_0 = h_0(K, \tau_0, E, E_\infty, \delta_1, m) \in (0,1]$ small enough. Here, to see that h_0 is independent of α, we observe that $1/2 \le \rho \le 2/3$ and hence $\delta_1^{1/\rho} \ge \min\{\delta_1^2,\ \delta_1^{3/2}\}$.

3.2. Large r region

In the region $r > a$, we handle the cases (1.1) and (1.5) together, taking the worst of the estimates on R_h, V_h, and V_h'. For notational convenience, set $\delta_1 = \rho = 1$ in the case (1.1). Then if either (1.1) or (1.5) holds, for $h \in (0, \delta_1^{1/\rho}]$,

$$V_h(r\theta) \le C_V \psi(r) + \tilde{E}, \qquad |V_h'| \le Ch^{-\rho(1-\alpha)} \langle r \rangle^{-3} m^2(r),$$
$$|R_h| \le C\langle r \rangle^{-2} m(r).$$

Define $\mathcal{W}$ and Φ for $r > a$ by

$$\mathcal{W} = \frac{(r+1)^{1+\eta}}{2}, \qquad \Phi = -\frac{1+\eta}{r+1}, \qquad 0 < \eta < 1, \qquad r > a. \tag{3.17}$$

Then,

$$\varphi_0'(r) = \varphi_0'(a) e^{\int_a^r \Phi(s) ds} = \varphi_0'(a) \frac{(a+1)^{1+\eta}}{(r+1)^{1+\eta}}, \qquad r > a.$$

Therefore, from (3.11),

$$\frac{\tau_0 (a+1)^\eta}{(r+1)^{1+\eta}} \le \varphi_0'(r) \le \frac{\tau_0 e^{\|\langle s \rangle^{-2} \Phi_1(s)\|_{L^1}} (a+1)^\eta}{(r+1)^{1+\eta}}, \qquad r > a. \tag{3.18}$$

We have, again by (2.10),

$$\begin{aligned}
A - \frac{K}{2} B \ge w' \Big[& E + h^{-2\sigma} (\varphi_0')^2 [1 - (1+\eta)(r+1)^\eta \\
& - 8^{-1} K h^{1+\sigma} (r+1)^{1+\eta} \Phi^2 (\varphi_0')^{-1}] \\
& - C_V \psi(r) - \tilde{E} - Ch^{-\rho(1-\alpha)} (r+1)^{1+\eta} \langle r \rangle^{-3} m^2 \\
& - CK (r+1)^{1+\eta} h^{-1+\sigma+2\rho\alpha} \langle r \rangle^{-4} m^2 (\varphi_0')^{-1} \Big] \\
\ge -w' & \big[C(1 + \tau_0^2 + K\tau_0^{-1}) h^{-2\sigma} \langle r \rangle^{-2+2\eta} (a+1)^{-\eta} \big] \\
& + \left(\tfrac{3}{4}(E - E_\infty) - CK\tau_0 h^{1-\sigma}\right) w', \qquad h \in (0, \delta_1^{1/\rho}].
\end{aligned}$$

To get the second inequality, we have used

$$h^{-2\sigma} (\varphi_0')^2 (1+\eta)(r+1)^\eta \le C\tau_0^2 h^{-2\sigma} \langle r \rangle^{-2+2\eta} (a+1)^{-\eta},$$
$$8^{-1} K h^{1-\sigma} \varphi_0' (r+1)^{1+\eta} \Phi^2 \le CK\tau_0 h^{1-\sigma},$$
$$Ch^{-\rho(1-\alpha)} (r+1)^{1+\eta} \langle r \rangle^{-3} m^2 \le Ch^{-2\sigma} \langle r \rangle^{-2+2\eta} (a+1)^{-\eta},$$

as well as

$$CKh^{-1+\sigma+2\rho\alpha}(r+1)^{\eta}\langle r\rangle^{-4}m^2(\varphi_0')^{-1} \le CK\tau_0^{-1}h^{-2\sigma}\langle r\rangle^{-2+2\eta}(a+1)^{-\eta}.$$

Now, in (3.4), fix

$$M = \frac{2\sigma}{2-\eta} = \frac{2(1-\alpha)}{(2-\eta)(3+\alpha)}. \tag{3.19}$$

Then taking $a_0 = a_0(C, K, \tau_0, E, E_\infty) \ge 1$ large enough,

$$\begin{aligned} A - \frac{K}{2}B &\ge (\tfrac{3}{4}(E-E_\infty) + CK\tau_0 h^{1-\sigma})w' \\ &\ge \frac{E-E_\infty}{2}w', \qquad r > a \ge R_E + 1,\ h \in (0, h_0], \end{aligned} \tag{3.20}$$

for $h_0 = h_0(K, \tau_0, E, E_\infty, \delta_1, m) \in (0, 1]$ small enough. Combining (3.14), (3.16), and (3.20) establishes (3.5) in either case (1.1) or (1.5).

3.3. Determination of w and φ_0

Lemmas 3.2 and 3.3 complete the proof of Proposition 3.1.

Lemma 3.2. *With $\mathcal{W}$ determined by* (3.10) *and* (3.17), *and with initial conditions as in* (3.1), *we have*

$$w = \begin{cases} r & 0 < r \le \frac{1}{2}, \\ \frac{1}{2}e^{\int_{1/2}^{r} \frac{2}{s(1+\omega(s))}ds} & \frac{1}{2} < r \le a, \\ w(a)e^{\frac{2}{\eta}((a+1)^{-\eta}-(r+1)^{-\eta})} & r > a, \end{cases} \tag{3.21}$$

and the estimates (3.7), (3.8), *and* (3.9) *hold.*

Proof. Recalling the definition (3.2) of w in terms of $\mathcal{W}$, for $0 < \varepsilon \le r$,

$$w(r) = w(\varepsilon)e^{\int_\varepsilon^r \frac{1}{\mathcal{W}(s)}ds}. \tag{3.22}$$

Now, if $0 \le r \le 1/2$, $\mathcal{W}(r) = r$, therefore,

$$w(r) = \frac{w(\varepsilon)}{\varepsilon}r, \qquad 0 < \varepsilon \le r \le \frac{1}{2}.$$

Sending $\varepsilon \to 0^+$ and using $w'(0) = 1$, $w(0) = 0$, we have

$$w(r) = r, \qquad 0 \le r \le \tfrac{1}{2},$$

as claimed. The remaining formulae for w in (3.21) now follow easily from (3.22) with $\varepsilon = 1/2$.

To see (3.7), note that $w' = w/\mathcal{W} \geq 0$, so we need only compute $\limsup_{r\to\infty} w(r)$. For this, observe that $\omega \equiv 0$ on $r \geq 1$. Therefore, for $1 \leq r \leq a$,

$$w(r) = w(1)r^2.$$

In particular, since

$$w(1) \leq \frac{1}{2} e^{\int_{1/2}^{1} 2s^{-1} ds} = 2,$$

$w(a) = w(1)a^2 \leq 2a^2$. Thus (using $a \geq 1$),

$$\begin{aligned}\limsup_{r\to\infty} w(r) &= \limsup_{r\to\infty} w(a) e^{\frac{2}{\eta}((a+1)^{-\eta}-(r+1)^{-\eta})} \\ &\leq 2a^2 e^{\frac{2}{\eta}(a+1)^{-\eta}} \leq C_\eta a^2 \leq C_\eta h^{-\frac{4(1-\alpha)}{(2-\eta)(3+\alpha)}},\end{aligned}$$

as claimed.

For (3.8), we first note that $w'(r) = 1$ on $0 \leq r \leq 1/2$. Then, using $0 \leq \mathcal{W} \leq (r+1)^{1+\eta}/2$, we compute

$$w'(r) = \frac{w(r)}{\mathcal{W}(r)} \geq (r+1)^{-1-\eta} e^{\int_{\frac{1}{2}}^{r} \frac{1}{\mathcal{W}(s)}} ds \geq (r+1)^{-1-\eta}, \qquad r \geq \frac{1}{2}, r \neq a.$$

Finally, to see (3.9), we observe using (3.7),

$$\frac{w^2}{w'} = \mathcal{W} w \leq C_\eta h^{-\frac{4(1-\alpha)}{(2-\eta)(3+\alpha)}} (r+1)^{1+\eta}.$$

□

Lemma 3.3. *With Φ given by* (3.10) *and* (3.17)*, and with initial conditions as in* (3.1)*, we have*

$$\varphi_0'(r) = \begin{cases} \tau_0 e^{-\int_0^r \frac{1}{s+1+\Phi_1(s)} ds} & 0 < r \leq a, \\ \varphi_0'(a) \frac{(a+1)^{1+\eta}}{(r+1)^{1+\eta}} & r > a. \end{cases} \tag{3.23}$$

and the estimate (3.6) *holds.*

Proof. The formula (3.23) follows directly from (3.2), (3.10) and (3.17). Then, by (3.11) and (3.18),

$$0 \leq \varphi_0'(r) \leq \begin{cases} \frac{\tau_0 e^{\|\langle\cdot\rangle^{-2}\Phi_1(\cdot)\|_{L^1}}}{(r+1)} & 0 \leq r \leq a, \\ \tau_0 e^{\|\langle\cdot\rangle^{-2}\Phi_1(\cdot)\|_{L^1}} \frac{(a+1)^\eta}{(r+1)^{1+\eta}} & r > a. \end{cases}$$

Using that $a = a_0 h^{-M}$, with M as in (3.19), we have, for $h \in (0,1]$,

(3.24)
$$\begin{aligned} |\varphi_0(r)| &\leq \int_0^a \frac{\tau_0 e^{\|\langle\cdot\rangle^{-2}\Phi_1(\cdot)\|_{L^1}}}{s+1} ds + \int_a^\infty \tau_0 e^{\|\langle\cdot\rangle^{-2}\Phi_1(\cdot)\|_{L^1}} \frac{(a+1)^\eta}{(s+1)^{1+\eta}} ds \\ &\leq \tau_0 e^{\|\langle\cdot\rangle^{-2}\Phi_1(\cdot)\|_{L^1}} [\log(a+1) + \frac{1}{\eta}] \\ &= \tau_0 e^{\|\langle\cdot\rangle^{-2}\Phi_1(\cdot)\|_{L^1}} [\log(a_0 h^{-\frac{2(1-\alpha)}{(2-\eta)(3+\alpha)}} + 1) + \frac{1}{\eta}] \\ &\leq \tau_0 e^{\|\langle\cdot\rangle^{-2}\Phi_1(\cdot)\|_{L^1}} \Big[\frac{1-\alpha}{(1-\frac{\eta}{2})(3+\alpha)} \log h^{-1} + \log(a_0+1) + \frac{1}{\eta} \Big]. \end{aligned}$$

□

4. The one dimensional case

The key feature we exploit in the one dimensional case is the disappearance of the term involving the operator Λ (see (2.1)). This removes the requirement that $\mathcal{W} \geq r/2$, allowing *much* more flexibility in the choice of weight function (see (4.9) below).

In one dimension we are also able to simplify the approximation of the potential. For V obeying (1.10), and χ satisfying (2.12), we take

$$V_h(x) := \int_{-\infty}^{\infty} V(x+hy)\chi(y)dy.$$

We again define $R_h := V - V_h$. The following lemma, whose proof follows that of Lemma 2.2, gives bounds on V_h, V_h' and R_h in one dimension.

Lemma 4.1. *Suppose V satisfies the assumptions of Theorem 3. Then there exists $C_\chi > 0$ depending only on χ so that, for all $h \in (0, \delta_{0,V}]$,*

$$V_h(x) \leq \sup_{y\in[x,x+h]} V(y), \tag{4.1}$$

$$|V_h'(x)| \leq C_\chi c_0 h^{-1} m_0(|x|), \tag{4.2}$$

$$|R_h(x)| \leq c_0 m_0(|x|). \tag{4.3}$$

Similar to the $n \geq 3$ case, the constants $C > 0$ and $h_0 \in (0,1]$ which appear in the ensuing estimates may change from line to line, but depend only on K, C_V, c_0, E, E_∞, R_E, δ_0 and m_0. The constant $C_\eta > 0$ may also depend on $0 < \eta < 1$. In particular, C and h_0 are independent of h and η, and C_η is independent of h.

The main result of this section is

Proposition 4.2. *Fix $K > 0$ and let V satisfy the assumptions of Theorem 3. Let $E > E_\infty$ and $0 < \eta < 1$. Then there exist functions $\mathcal{W}, \Phi : \mathbb{R} \to [0, \infty)$, and corresponding functions w and φ_0 determined by and (3.2), along with C, $C_\eta > 0$ and $h_0 \in (0,1]$ such that*

$$A(x) - \frac{K}{2}B(x) \geq \frac{E - E_\infty}{2} w'(x), \qquad h \in (0, h_0], \tag{4.4}$$

and

$$|\varphi(x)| \leq C, \tag{4.5}$$

and w satisfies,

$$w(x) \leq 1, \tag{4.6}$$

$$w'(x) \geq C_\eta e^{-C/h}(|x|+1)^{-1-\eta}, \tag{4.7}$$

$$\frac{w(x)^2}{w'(x)} \leq C_\eta (|x|+1)^{1+\eta}. \tag{4.8}$$

Proof. We assume without loss of generality that $m_0(|x|) \geq (1 + |x|[\log(|x| + 1)]^2)^{-1}$. Then, put

$$\Phi = -\frac{2\,\mathrm{sgn}(x)}{|x|+1}, \qquad \mathcal{W} = \frac{\delta h}{m_0(|x|)}. \tag{4.9}$$

for $\delta > 0$ to be chosen later. We replace the initial conditions (3.1) with

$$w(0) = e^{-\frac{1}{\delta h}\int_0^\infty m_0(s)ds}, \qquad \varphi(0) = 0, \qquad \varphi'(0) = \tau_0 \geq 1,$$

where we fix τ_0 below. We find

$$\varphi' = \frac{\tau_0}{(|x|+1)^2}, \qquad w = e^{-\frac{1}{\delta h}\int_x^\infty m_0(|s|)ds}.$$

Recall from (2.10) that

$$A - \frac{K}{2}B \geq w'(E + (\varphi')^2(1 + 2\mathcal{W}\Phi - K\mathcal{W}\Phi^2 \min(\mathcal{W}, \tfrac{h}{4\varphi'})) - V_h - \mathcal{W}V_h' - K\mathcal{W}h^{-2}|R_h|^2 \min(\mathcal{W}, \tfrac{h}{4\varphi'})). \tag{4.10}$$

Let $\psi \in C^\infty_{\mathrm{comp}}(\mathbb{R}; [0, 1])$ with $\psi \equiv 1$ on $|x| \leq R_E$ and $\operatorname{supp}\psi \subseteq (-R_E - 1, R_E + 1)$. Then, by (4.1),

$$V_h \leq \frac{E + 3V_\infty}{4} \leq \frac{E + 3E_\infty}{4}, \qquad |x| \geq R_E \geq R_{E,V}.$$

Combining this with (4.2), (4.3), the choice of Φ and $\mathcal{W}$ in (4.9), and (4.10), we have

$$\begin{aligned} A - \frac{K}{2}B \geq w'(E + \tau_0^2(|x| + 1)^{-4}(1 - 4h\delta m_0^{-1}(|x| + 1)^{-1} \\ - Kh^2\delta^2 m_0^{-2}(|x| + 1)^{-2}) \\ - C_V\psi - \tfrac{E+3E_\infty}{4} - C\delta - CK\delta^2), \end{aligned}$$

for $h \in (0, \delta_0]$. First taking $\tau_0 = \sqrt{\max(C_V, 1)}(R_E + 2)^4$, and then taking $\delta > 0$ small enough (depending on C, K, E, E_∞, τ_0, and m_0), we obtain

$$A - \frac{K}{2}B \geq \frac{E - E_\infty}{2}w', \qquad h \in (0, \delta_0].$$

To obtain the estimates (4.5), (4.6), (4.7), and (4.8), observe

$$\varphi = \tau_0 \operatorname{sgn}(x)\left(1 - \frac{1}{|x| + 1}\right),$$

and

$$w' = \frac{m_0(|x|)}{\delta h}w(x),$$

and note that $m_0(|x|) \geq C_\eta(|x| + 1)^{-1-\eta}$. □

5. Carleman estimates

Our goal in this section is to prove the Carleman estimates needed to establish (1.3), (1.9) and (1.13). As above, we use $C > 0$ to denote a constant that may change from line to line, but depends only $\sup V$, c_1, c_2, E, E_∞ R_E and m ($n \geq 3$) or $\sup V$, c_0, E, E_∞, R_E, and m_0 ($n = 1$). Besides depending on the same quantities as C does, $h_0 \in (0, 1]$ depends only on δ_1 ($n \geq 3$) or

δ_0 ($n = 1$), and $C_\eta > 0$ depends only on $0 < \eta < 1$. So in particular, C, C_η, and h_0 are independent of α, h and $\varepsilon \geq 0$.

Lemma 5.1. *Let $0 < \eta < 1$ and suppose that the assumptions of one of Theorem 1, 2, or 3 hold. Then with φ and w and $h_0 \in (0, 1]$ as in the statement of Proposition 3.1 and 4.2 respectively in $n \geq 3$ and $n = 1$, we have*

$$\|\langle x\rangle^{-\frac{1+\eta}{2}} e^{\varphi/h} v\|_{L^2}^2 \leq C_\eta e^{C/h} \|\langle x\rangle^{\frac{1+\eta}{2}} e^{\varphi/h} (P(h) - E \pm i\varepsilon)v\|_{L^2}^2 + C_\eta e^{C/h} \varepsilon \|e^{\varphi/h} v\|_{L^2}^2. \tag{5.1}$$

for all $\varepsilon \geq 0$, $h \in (0, h_0]$, and $v \in C_{\mathrm{comp}}^\infty(\mathbb{R}^n)$.

Remark: Throughout the proof of Lemma 5.1, we abuse notation slightly. In dimension $n \geq 3$, we put $\|u(r)\| = \|u(r, \cdot)\|_{L^2(\mathbb{S}_\theta^{n-1})}$, while we put $\|u(x)\| = |u(x)|$ when $n = 1$. If $n \geq 3$, $\int_{r,\theta}$ denotes the integral over $(0, \infty) \times \mathbb{S}^{n-1}$ with respect to the measure $drd\theta$, while if $n = 1$, $\int_{r,\theta} u(x)$ denotes $\int_0^\infty u(r)dr - \int_0^\infty u(-r)dr = \int_{\mathbb{R}} u(x)dx$.

Proof. Since $\langle x\rangle^{-(1+\eta)/2} \leq 1$, without loss of generality, we may assume $0 \leq \varepsilon \leq 1$.

The proof begins from (2.7). Then, applying (3.5) or (4.4), it follows that for $h \in (0, h_0]$,

$$w'F + wF' \geq -\frac{3w^2}{h^2 w'} \|P_\varphi^\pm(h)u\|^2 \mp 2\varepsilon w \operatorname{Im}\langle u, u'\rangle + \frac{1}{3} w' \|hu'\|^2 + \frac{E - E_\infty}{2} w' \|u\|^2. \tag{5.2}$$

Now we integrate both sides of (5.2). For $n \geq 3$, we integrate $\int_0^\infty dr$ and use $wF, (wF)' \in L^1((0, \infty); dr)$, and $wF(0) = wF(\infty) = 0$, hence $\int_0^\infty (wF)' dr = 0$. In dimension $n = 1$, we instead integrate $\int_{\mathbb{R}} dx$ and observe that $\int_{\mathbb{R}} (wF)' dx = 0$. Using also (3.7), (3.8) and (3.9) when $n \geq 3$, or (4.6), (4.7) and (4.8) when $n = 1$, yields, for $h \in (0, h_0]$,

$$\int_{r,\theta} (r + 1)^{-1-\eta} \left(|u|^2 + |hu'|^2\right) \leq C_\eta e^{C/h} \int_{r,\theta} (r + 1)^{1+\eta} |P_\varphi^\pm(h)u|^2 + \varepsilon C_\eta e^{C/h} \int_{r,\theta} |u|^2 + |hu'|^2. \tag{5.3}$$

Moreover,

$$\operatorname{Re}\int_{r,\theta}(P_\varphi^\pm(h)u)\overline{u} = \int_{r,\theta}|hu'|^2 + \operatorname{Re}\int_{r,\theta}2h\varphi' u'\overline{u} + \int_{r,\theta}(h^2\Lambda u)u + \int_{r,\theta}h\varphi''|u|^2 + \int_{r,\theta}\left(V - E - (\varphi')^2\right)|u|^2, \tag{5.4}$$

and

$$\int_{r,\theta}h\varphi''|u|^2 = -\operatorname{Re}\int_{r,\theta}2\varphi' hu'\overline{u}. \tag{5.5}$$

These two identities, together with the facts that $\Lambda \geq 0$ and $|V - E - (\varphi')^2| \leq e^{C/h}$ for $h \in (0,1]$, imply,

$$\int_{r,\theta}|hu'|^2 \leq e^{C/h}\int_{r,\theta}|u|^2 + \frac{\gamma}{2}\int_{r,\theta}(r+1)^{-1-\eta}|u|^2 + \frac{1}{2\gamma}\int_{r,\theta}(r+1)^{1+\eta}|P_\varphi^\pm(h)(u)|^2, \qquad h \in (0,1],\ \gamma > 0. \tag{5.6}$$

To finish, we substitute (5.6) into the right side of (5.3), recall $0 \leq \varepsilon \leq 1$, and then choose $\gamma > 0$ small enough (depending on h but independent of ε), to get

$$\int_{r,\theta}(r+1)^{-1-\eta}(|u|^2 + |hu'|^2) \leq C_\eta e^{C/h}\int_{r,\theta}(r+1)^{1+\eta}|P_\varphi^\pm(h)u|^2 + \varepsilon C_\eta e^{C/h}\int_{r,\theta}|u|^2, \qquad h \in (0,h_0]. \tag{5.7}$$

The estimate (5.1) is now an easy consequence of (5.7). □

6. Resolvent estimates

In this section, we deduce the resolvent estimates in Theorems 1, 2 and 3 from the Carleman estimate (5.1). This same argument has been presented before, see, e.g., [4, 8, 9, 11, 13]. But we include it here for the reader's convenience and for the sake of completeness.

The constants C, h_0, and C_η continue to have the same dependencies as in Section 5.

Proof of Theorems 1, 2 and 3. Since increasing s in (1.4) decreases the resolvent norm, to prove (1.3), (1.9) and (1.13), we may assume without loss of generality that $0 < 2s - 1 < 1$.

Fix $\eta = 2s - 1$. When $n \geq 3$, let $\sigma = \sigma_\alpha$ be as in (3.3). Let φ, w, and $h_0 \in (0,1]$ be as in Proposition 3.1 ($n \geq 3$) or as in Proposition 4.2 ($n = 1$). Then, Lemma 5.1 holds. Put $C_\varphi = C_\varphi(h) \coloneqq 2\max\varphi$. By (5.1), for some $C, C_s = C_\eta > 0$,

$$e^{-C_\varphi/h}\|\langle x\rangle^{-s}v\|^2_{L^2} \leq C_s e^{C/h}\|\langle x\rangle^{s}(P(h) - E \pm i\varepsilon)v\|^2_{L^2} + \varepsilon C_s e^{C/h}\|v\|^2_{L^2}, \tag{6.1}$$

for all $v \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, $\varepsilon \geq 0$, and $h \in (0, h_0]$. Moreover, for any $\gamma > 0$,

$$\begin{aligned} 2\varepsilon\|v\|^2_{L^2} &= -2\,\mathrm{Im}\langle (P(h) - E \pm i\varepsilon)v, v\rangle_{L^2} \\ &\leq \gamma^{-1}\|\langle x\rangle^{s}(P(h) - E \pm i\varepsilon)v\|^2_{L^2} + \gamma\|\langle x\rangle^{-s}v\|^2_{L^2}. \end{aligned} \tag{6.2}$$

Setting $\gamma = C_s^{-1}e^{-(C+C_\varphi)/h}$, and using (6.2) to estimate $\varepsilon\|v\|^2_{L^2}$ from above in (6.1), we absorb the $\|\langle x\rangle^{-s}v\|_{L^2}$ term that now appears on the right of (6.1) into the left side. Multiplying through by $2e^{C_\varphi/h}$, and applying (3.6) ($n \geq 3$) we arrive at

$$\|\langle x\rangle^{-s}v\|^2_{L^2} \leq C_s e^{h^{-1-\sigma_\alpha}\left(\frac{C\sigma_\alpha}{3-2s}\log(h^{-1}) + C_s\right)}\|\langle x\rangle^{s}(P(h) - E \pm i\varepsilon)v\|^2_{L^2}, \quad \varepsilon \geq 0,\ h \in (0, h_0]. \tag{6.3}$$

In the case $n = 1$, we apply instead (4.5) to obtain

$$\|\langle x\rangle^{-s}v\|^2_{L^2} \leq C_s e^{Ch^{-1}}\|\langle x\rangle^{s}(P(h) - E \pm i\varepsilon)v\|^2_{L^2}, \quad \varepsilon \geq 0,\ h \in (0, h_0]. \tag{6.4}$$

The final task is to use (6.3) and (6.4) to obtain the corresponding resolvent estimates to show

$$\begin{aligned} &\|\langle x\rangle^{-s}(P(h) - E \pm i\varepsilon)^{-1}\langle x\rangle^{-s}f\|^2_{L^2} \leq C_s e^{h^{-1-\sigma_\alpha}\left(\frac{C\sigma_\alpha}{3-2s}\log(h^{-1}) + C_s\right)}\|f\|^2_{L^2}, \\ &\varepsilon > 0,\ h \in (0, h_0],\ f \in L^2, \qquad (n \geq 3) \\[1em] &\|\langle x\rangle^{-s}(P(h) - E \pm i\varepsilon)^{-1}\langle x\rangle^{-s}f\|^2_{L^2} \leq C_s e^{Ch^{-1}}\|f\|^2_{L^2}, \\ &\varepsilon > 0,\ h \in (0, h_0],\ f \in L^2, \qquad (n = 1) \end{aligned} \tag{6.5}$$

from which Theorems 1, 2 and 3 follow. To establish (6.5), we prove a simple Sobolev space estimate and then apply a density argument that relies on (6.3).

The operator

$$[P(h), \langle x \rangle^s]\langle x \rangle^{-s} = \left(-h^2\Delta\langle x \rangle^s - 2h^2(\nabla\langle x \rangle^s) \cdot \nabla\right) \langle x \rangle^{-s}$$

is bounded $H^2 \to L^2$. So, for $v \in H^2$ such that $\langle x \rangle^s v \in H^2$,

$$\begin{aligned} \|\langle x \rangle^s (P(h) - E \pm i\varepsilon)v\|_{L^2} \leq \|(P(h) - E \pm i\varepsilon)\langle x \rangle^s v\|_{L^2} \\ + \|[P(h), \langle x \rangle^s]\langle x \rangle^{-s}\langle x \rangle^s v\|_{L^2} \\ \leq C_{\varepsilon,h}\|\langle x \rangle^s v\|_{H^2}, \end{aligned} \tag{6.6}$$

for some constant $C_{\varepsilon,h} > 0$ depending on ε and h.

Given $f \in L^2$, the function $\langle x \rangle^s (P(h) - E \pm i\varepsilon)^{-1}\langle x \rangle^{-s} f \in H^2$ because

$$\begin{aligned} &\langle x \rangle^s (P(h) - E \pm i\varepsilon)^{-1}\langle x \rangle^{-s} f \\ &\quad = (P(h) - E \pm i\varepsilon)^{-1} f + [\langle x \rangle^s, (P(h) - E \pm i\varepsilon)^{-1}]\langle x \rangle^{-s} f \\ &\quad = (P(h) - E \pm i\varepsilon)^{-1} f + (P(h) - E \pm i\varepsilon)^{-1}[P(h), \langle x \rangle^s](P(h) \\ &\qquad - E \pm i\varepsilon)^{-1}\langle x \rangle^{-s} f. \end{aligned}$$

Now, choose a sequence $v_k \in C^\infty_{\text{comp}}$ such that

$$v_k \to \langle x \rangle^s (P(h) - E \pm i\varepsilon)^{-1}\langle x \rangle^{-s} f$$

in H^2. Define
$\tilde{v}_k := \langle x \rangle^{-s} v_k$. Then, as $k \to \infty$,

$$\begin{aligned} &\|\langle x \rangle^{-s}\tilde{v}_k - \langle x \rangle^{-s}(P(h) - E \pm i\varepsilon)^{-1}\langle x \rangle^{-s} f\|_{L^2} \\ &\qquad \leq \|v_k - \langle x \rangle^s (P(h) - E \pm i\varepsilon)^{-1}\langle x \rangle^{-s} f\|_{H^2} \to 0. \end{aligned}$$

Also, applying (6.6),

$$\begin{aligned} &\|\langle x \rangle^s (P(h) - E \pm i\varepsilon)\tilde{v}_k - f\|_{L^2} \\ &\qquad \leq C_{\varepsilon,h}\|v_k - \langle x \rangle^s (P(h) - E \pm i\varepsilon)^{-1}\langle x \rangle^{-s} f\|_{H^2} \to 0. \end{aligned}$$

We then achieve (6.5) by replacing v by $\tilde{v}_k$ in (6.3) and sending $k \to \infty$. □

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