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Automorphisms of products of toric varieties

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We give an explicit description of the automorphism group of a product of complete toric varieties over an arbitrary field in terms of the respective automorphism groups of its components. More precisely, we prove that, up to permutation of isomorphic components, an automorphism of a product corresponds to a product of automorphisms of its components. We also reprove, in modern language, the classic result by Demazure describing the group-scheme of automorphisms of a complete toric variety over an arbitrary field.

Introduction

Let k be a field. A *variety* is a geometrically integral scheme of finite type over k. A (split) algebraic torus T is an algebraic group isomorphic to $\mathbb{G}_{\mathrm{m}}^n$ for some integer n. A *toric variety* X is a normal variety having T as an open set such that the action of T on itself by translations extends to an action on X. We say that a toric variety X is *decomposable* if it can be written as a product $X = X_1 \times X_2$ of two toric varieties of strictly smaller dimension. Otherwise we say that X is *indecomposable.*

In this note, we consider complete (i.e. proper) toric varieties. If X is such a variety, then its sheaf of automorphisms is represented by a locally algebraic group denoted by Aut_X (cf. [9, Thm. 3.7]). Define Aut_X^0 as the neutral component of Aut_X (it is thus an algebraic group). The aim of this short note is to prove the following theorem.

Theorem 1. *Let $X_1, \dots, X_n$ be pairwise non-isomorphic indecomposable complete toric varieties. For $1 \le i \le n$, let r_i be a positive integer. Let $X = \prod_{i=1}^n X_i^{r_i}$. Then*

$$\mathrm{Aut}_X \simeq \prod_{i=1}^{n} (\mathrm{Aut}_{X_i}^{r_i} \rtimes S_{r_i}),$$

where S_{r_i} is seen as a finite constant group-scheme acting on $\mathrm{Aut}_{X_i}^{r_i}$ by permuting coordinates.

The main ingredient in the proof of Theorem 1 is Demazure's description of Aut_X for a complete toric variety X, which is given in the foundational paper [6]. Unfortunately, this description is stated for a *smooth* complete toric variety X, even though the proofs (which are spread all over the article) naturally generalize to *any* complete toric variety. Following Demazure's ideas, we give a short proof in modern language of such a description for an *arbitrary* complete toric variety over any base field k (for notations and definitions, see Section 1).

Theorem 2. *Let Σ be a complete fan and let X be the corresponding complete toric variety. Let $\mathcal{R}(\Sigma)$ be the set of Demazure roots of Σ. For $e \in \mathcal{R}(\Sigma)$, let ρ_e be the corresponding character and define the rational map*

$$\alpha_e\colon \mathbb{G}_\mathrm{a} \times \mathrm{T} \dashrightarrow \mathrm{T}, \quad \textit{given by} \quad (s,t) \mapsto t \cdot \lambda_{\rho_e}(1 + s\chi^e(t)) \,. \tag{1}$$

- (*i*) *The rational map α_e defines a faithful $\mathbb{G}_\mathrm{a}$-action on X. In particular, the morphism $\iota_e : \mathbb{G}_\mathrm{a} \to \mathrm{Aut}_X$ induced by α_e is a closed immersion.*
- (*ii*) *The neutral component Aut_X^0 is spanned by T and the image of ι_e for all $e \in \mathcal{R}(\Sigma)$. In particular, Aut_X is smooth.*
- (*iii*) *The quotient group-scheme $\mathrm{Aut}_X / \mathrm{Aut}_X^0$ is a finite constant group-scheme generated by the images of toric automorphisms.*

Alternative proofs of this description exist in the literature for the automorphism *group* (not the *scheme*) of a complete toric variety. One of these is given in [4] for complete $\mathbb{Q}$-factorial toric varieties over $\mathbb{C}$. A similar result was later obtained in [3] for projective toric varieties over any base field. There is also the recent preprint [10], which works with arbitrary complete toric varieties over an algebraically closed field of characteristic 0. The main feature of Theorem 2 is that it describes Aut_X as a scheme, proving in particular that it is *smooth.* We should remark however that we work over a base field k, while Demazure's results are valid over $\mathbb{Z}$, so there is still room for improvement.

Theorem 2 has consequences in the theory of twisted forms of toric varieties, which was developed by Duncan in [7]. Indeed, when studying automorphisms in the proper split case (§4.2 in *loc. cit.*), Duncan is led to assume that Aut_X is a smooth algebraic group. Theorem 2 proves that this assumption is unnecessary. Moreover, Theorem 2 naturally yields an analogous description of the automorphism group of *any* k-form of a toric variety: if such a k-form is defined by twisting a (split) toric variety X with a cocycle α representing a class in $H^1(\mathrm{k}, \mathrm{Aut}_X)$, then its automorphism group is simply the inner twist ${}_\alpha\,\mathrm{Aut}_X$ of the group-scheme Aut_X.

1. Preliminaries on toric varieties

Let M be a lattice, i.e., a finitely generated free abelian group and let $N = \mathrm{Hom}(M, \mathbb{Z})$ be its dual lattice. We let $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$, $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$, and $\langle \cdot, \cdot \rangle : N_\mathbb{R} \times M_\mathbb{R} \to \mathbb{R}$ be the duality pairing given by $\langle p, m \rangle = p(m)$. Let $\mathrm{T} = N \otimes_\mathbb{Z} \mathbb{G}_\mathrm{m}$ be the algebraic torus whose character lattice is M and whose 1-parameter subgroup lattice is N. As customary, for every $m \in M$ we denote by $\chi^m \colon \mathrm{T} \to \mathbb{G}_\mathrm{m}$ the corresponding character and for every $p \in N$ we denote by $\lambda_p \colon \mathbb{G}_\mathrm{m} \to \mathrm{T}$ the corresponding 1-parameter subgroup.

Recall that a toric variety X is a normal variety having T as an open set such that the action of T on T by translations extends to an action on X. Let X and X' be toric varieties with corresponding tori T and T', respectively. A toric morphism is a morphism $\varphi \colon X \to X'$ that restricts to a morphism of algebraic groups $\mathrm{T} \to \mathrm{T}'$.

A strictly convex polyhedral cone σ in $N_\mathbb{R}$ is called rational if σ is spanned by a finite set of vectors in the lattice N. All cones will be strictly convex, rational and polyhedral. We will refer to them simply as cones. A fan $\Sigma \in N_\mathbb{R}$ is a finite collection of cones such that every face of a cone $\sigma \in \Sigma$ is contained in the fan Σ; and for all cones $\sigma, \sigma' \in \Sigma$ the intersection $\sigma \cap \sigma'$ is a face in both cones σ and σ'. Let Σ and Σ' be fans in $N_\mathbb{R}$ and $N'_\mathbb{R}$, respectively. A morphism of fans is a linear map $\psi \colon N_\mathbb{R} \to N'_\mathbb{R}$ restricting to a homomorphism $N \to N'$ such that for every $\sigma \in \Sigma$, there exists $\sigma' \in \Sigma'$ with $\psi(\sigma) \subset \sigma'$.

Let Σ be a fan in $N_\mathbb{R}$. A toric variety X_Σ is defined from Σ as follows. For every $\sigma \in \Sigma$ we define the affine toric variety $X_\sigma = \mathrm{Spec}\,\mathrm{k}[\sigma^\vee \cap M]$ where

$$\sigma^\vee := \{m \in M_\mathbb{R} \mid \langle p, m \rangle \geq 0 \text{ for all } p \in \sigma\} \in M_\mathbb{R},$$

is the dual cone of $\sigma \in N_\mathbb{R}$ and $\mathrm{k}[\sigma^\vee \cap M]$ is the semigroup algebra defined via

$$\mathrm{k}[\sigma^\vee \cap M] = \bigoplus_{m \in \sigma^\vee \cap M} \mathrm{k} \cdot \chi^m \quad \text{where} \quad \chi^m \chi^{m'} = \chi^{m+m'} \text{ and } \chi^0 = 1\,.$$

Furthermore, for every $\sigma, \sigma' \in \Sigma$ the algebra inclusions

$$\mathrm{k}[\sigma^\vee \cap M] \hookrightarrow \mathrm{k}[(\sigma \cap \sigma')^\vee \cap M] \hookleftarrow \mathrm{k}[(\sigma')^\vee \cap M],$$

induce T-equivariant affine open embeddings $X_\sigma \hookleftarrow X_{\sigma\cap\sigma'} \hookrightarrow X_{\sigma'}$. The toric variety X_Σ is obtained by glueing the family of affine toric varieties $\{X_\sigma \mid \sigma \in \Sigma\}$ along these embeddings.

The construction described above can also be defined for morphisms. This yields an equivalence of categories between the category of toric varieties with toric morphisms and the category of fans with morphisms of fans. In particular, we recall that the toric variety X_Σ is complete if and only if the union of the cones of Σ equals $N_\mathbb{R}$. We refer to any textbook on toric geometry such as [5, 8, 11] for details on this equivalence.

We end this section by recalling the definition of a Demazure root, so that every object in Theorem 2 is correctly defined. Let Σ be a fan in $N_\mathbb{R}$. For every non-negative integer i, we denote by $\Sigma(i)$ the i-skeleton of Σ, i.e., the set of i-dimensional cones in Σ. As usual, we identify a ray $\rho \in \Sigma(1)$ with its primitive vector in N. A Demazure root of the fan Σ is a lattice vector $e \in M$ such that there exists $\rho_e \in \Sigma(1)$ with $\langle \rho_e, e\rangle = -1$ and $\langle \rho, e\rangle \geq 0$ for all $\rho \in \Sigma(1) \setminus \{\rho_e\}$. Note that such a ρ_e is unique. We denote the set of roots of Σ by $\mathcal{R}(\Sigma)$.

2. Proof of Theorem 2

In this section we prove Theorem 2 following [6]. We would like to thank Michel Brion for his help with some parts of this proof.

Proof of Theorem 2 (i)

Following [6, Sec. 4, §5, Thm. 3], we compute first the comorphism corresponding to (1) at the level of function fields. Note that α_e can be decomposed as the composition

$$\mathbb{G}_\mathrm{a} \times \mathrm{T} \overset{\mu_1}{\dashrightarrow} \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{a} \times \mathrm{T} \overset{\mu_2}{\dashrightarrow} \mathbb{G}_\mathrm{a} \times \mathrm{T} \overset{\mu_3}{\dashrightarrow} \mathrm{T},$$

where μ_1 sends (s,t) to $(\chi^e(t), s, t)$, μ_2 sends (r,s,t) to $(rs+1,t)$ and μ_3 sends (s,t) to $t\lambda_{\rho_e}(s)$. Each of these comorphisms is easily computable:

- μ_3^* sends a character $\chi^m \in \mathcal{O}_X(\mathrm{T}) = \mathrm{k}[M]$ to $s^{\langle \rho_e, m\rangle}\chi^m \in \mathrm{k}(s)[M]$;
- μ_2^* sends $f\chi^m \in \mathrm{k}(s)[M]$ to $f'\chi^m \in \mathrm{k}(r,s)[M]$, where $f' \in \mathrm{k}(r,s)$ is obtained from f by replacing every appearance of s by $rs+1$.

- μ_1^* sends $g\chi^m \in \mathrm{k}(r,s)[M]$ to $g'\chi^m \in \mathrm{Frac}(\mathrm{k}(s)[M])$, where g' is obtained from g by replacing every appearance of r by χ^e.

Putting all this together, we obtain

$$\alpha_e^*(\chi^m) = \chi^m(1+s\chi^e)^{\langle \rho_e, m\rangle}, \quad \text{for all } \chi^m \in \mathcal{O}_X(\mathrm{T}). \tag{2}$$

Let $\sigma \in \Sigma$. We will show that α_e defines a morphism $\mathbb{G}_\mathrm{a} \times X_\sigma \to X$. Assume first that $\rho_e \in \sigma(1)$. In this case, α_e actually defines a morphism $\mathbb{G}_\mathrm{a} \times X_\sigma \to X_\sigma$. Indeed, letting $m \in \sigma^\vee \cap M$, we have that in (2) the character χ^{m+ie} appears with $0 \le i \le \langle \rho_e, m\rangle$. Clearly $\langle \rho, m+ie\rangle \ge 0$ for $\rho \in \sigma(1)$ different from ρ_e. On the other hand, $\langle \rho_e, m+ie\rangle = \langle \rho_e, m\rangle - i \ge 0$. So $m+ie \in \sigma^\vee \cap M$ and the claim follows.

Assume now that $\rho_e \notin \sigma(1)$ and let $f = 1+s\chi^e$ and $g = \chi^e$ in $\mathrm{k}[\sigma^\vee \cap M][s]$. It is immediate that the standard open subvarieties $(\mathbb{G}_\mathrm{a} \times X_\sigma)_f$ and $(\mathbb{G}_\mathrm{a} \times X_\sigma)_g$ cover $\mathbb{G}_\mathrm{a} \times X_\sigma$, so it will suffice to prove that α_e induces morphisms $(\mathbb{G}_\mathrm{a} \times X_\sigma)_f \to X_\sigma$ and $(\mathbb{G}_\mathrm{a} \times X_\sigma)_g \to X_{\sigma'}$ for a suitable $\sigma' \in \Sigma$. This is evident for f by (2). As for g, let σ' be the cone[1] in Σ spanned by ρ_e and $\sigma \cap e^\perp$. Letting $m \in (\sigma')^\vee \cap M$ we have

$$\alpha_e^*(\chi^m) = \chi^{m+\ell e}(1+s\chi^e)^{\langle \rho_e, m\rangle}\chi^{-\ell e}, \quad \text{for any integer } \ell.$$

The last two factors in this product are already morphisms $(\mathbb{G}_\mathrm{a} \times X_\sigma)_g \to X_{\sigma'}$ since $g = \chi^e$ and $\rho_e \in \sigma'$. To conclude, it suffices to prove that there exists an integer ℓ such that

$$\langle m+\ell e, \rho\rangle = \langle m, \rho\rangle + \ell\langle e, \rho\rangle \ge 0, \quad \text{for all } \rho \in \sigma(1).$$

Now, if $\langle e, \rho\rangle = 0$, then $\langle m, \rho\rangle \ge 0$ by definition of σ' and we are done. Otherwise, $\langle e, \rho\rangle > 0$ and it is enough to take $\ell \ge -\langle m, \rho\rangle$. Since there are finitely many rays in σ, we may always choose ℓ big enough.

We have thus proved that α_e defines a morphism $\mathbb{G}_\mathrm{a} \times X_\sigma \to X$ for any cone $\sigma \in \Sigma$. And since the X_σ cover X, this proves that α_e is a morphism $\mathbb{G}_\mathrm{a} \times X \to X$.

We are left with the faithfulness of the action, which implies that the morphism $\iota_e : \mathbb{G}_\mathrm{a} \to \mathrm{Aut}_X$ defined by α_e is a closed immersion. This is stated

[1]Note that the fact that σ' is a cone in Σ is part of Demazure's definition of a root in [6, Sec. 4, §5, Def. 4]. However, Demazure proves (cf. [6, Sec. 4, §5, Rem. 3]) that in the complete case this is in fact a consequence of our current definition of a Demazure root.

outright by Demazure in [6, Sec. 4, §5] right after proving his Théorème 3, but it is slightly less evident in the setting of non-smooth varieties.

In order to check this, we need to prove that, for every k-algebra A and for $s \in \mathbb{G}_{\mathrm{a}}(A) = A$, we have the following implication (we abusively omit the notation Spec in what follows):

$$\iota_e(s) = \mathrm{id}_{X \times_{\mathrm{k}} A} \Rightarrow s = 0.$$

Let $s \in A$ be as above and fix a cone $\sigma \in \Sigma$ containing ρ_e. Note that, since $\iota_e(s) = \mathrm{id}_{X \times_{\mathrm{k}} A}$, it sends $X_\sigma \times_{\mathrm{k}} A$ to $X_\sigma \times_{\mathrm{k}} A$. We may use then (2) in order to describe $\iota_e(s)$ restricted to $X_\sigma \times_{\mathrm{k}} A$. Using this we see that $\chi^m \in A[\sigma^\vee \cap M]$ maps to $\chi^m(1 + s\chi^e)^{\langle \rho_e, m\rangle} \in A[\sigma^\vee \cap M]$. Since ρ_e is a primitive lattice vector and belongs to σ, there exists $m_0 \in \sigma^\vee \cap M$ such that $\langle \rho_e, m_0\rangle = 1$. Since by hypothesis $\iota_e(s)(\chi^{m_0}) = \chi^{m_0}$, we get then $s\chi^{m_0+e} = 0 \in A[\sigma^\vee \cap M]$, which implies $s = 0$, as desired. This concludes the proof of the faithfulness of the action defined by α_e. □

Proof of Theorem 2 (ii)

This item corresponds to the first part of [6, Sec. 4, §6, Prop. 11], which we follow closely. For $e \in \mathcal{R}(\Sigma)$ we have by part (i) a closed immersion $\iota_e : \mathbb{G}_{\mathrm{a}} \to \mathrm{Aut}_X$, while we also have the natural closed immersion $\iota_0 : \mathrm{T} \to \mathrm{Aut}_X$. This gives injective morphisms $i_e : \mathrm{Lie}(\mathbb{G}_{\mathrm{a}}) \to \mathrm{Lie}(\mathrm{Aut}_X)$ and $i_0 : \mathrm{Lie}(\mathrm{T}) \to \mathrm{Lie}(\mathrm{Aut}_X)$, where we recall that $\mathrm{Lie}(\mathrm{Aut}_X)$ consists of derivations of $\mathcal{O}_X$.

We claim[2] that the images of i_0 and i_e for $e \in \mathcal{R}(\Sigma)$ generate $\mathrm{Lie}(\mathrm{Aut}_X)$. Since every derivation can be decomposed as a finite sum of homogeneous derivations, it is enough to show that every homogeneous derivation is in the image of i_e for some $e \in \mathcal{R}(\Sigma) \cup \{0\}$. Moreover, every derivation of $\mathcal{O}_X$ induces a derivation of $\mathcal{O}_X(\mathrm{T}) = \mathrm{k}[M]$ and it is easy to check that every homogeneous derivation of $\mathrm{k}[M]$ is a multiple of

$$\partial_{p,e} \colon \mathrm{k}[M] \to \mathrm{k}[M], \quad \chi^m \mapsto \langle p, m\rangle \cdot \chi^{m+e},$$

for some $e \in M$ and some primitive $p \in N$. Assume now that $\partial_{p,e}$ is the restriction of a derivation of $\mathcal{O}_X$. Then for every $\sigma \in \Sigma$ we have $\partial_{p,e}(\mathrm{k}[\sigma^\vee \cap M]) \subset \mathrm{k}[\sigma^\vee \cap M]$, which is equivalent by the equation above to the following

[2] This claim corresponds to [6, Sec. 4, §5, Prop. 7], for which we give a simplified proof in the complete case.

statement:

(3) For every $m \in (\sigma^\vee \cap M) \setminus p^\perp$, we have $\langle \rho, m+e \rangle \geq 0$ for all $\rho \in \sigma(1)$.

If $e = 0$, then $\partial_{p,e}$ corresponds to the infinitesimal generator of the one parameter subgroup λ_p of T, hence it belongs to the image of i_0 and we are done. If $e \neq 0$, we claim that e is a Demazure root, $p \in \{\pm\rho_e\}$ and $\partial_{p,e}$ lies in the image of i_e. Indeed, the completeness of X implies that there exists $\rho_0 \in \Sigma(1)$ such that $e \notin \rho_0^\vee$. Fix such a ρ_0 and assume that p is not a multiple of ρ_0. Then there exists $m \in \rho_0^\vee \cap M$ such that $\langle \rho_0, m \rangle = 0$ and $\langle p, m \rangle \neq 0$. Hence, (3) implies that $\langle \rho_0, e \rangle \geq 0$ and so $e \in \rho_0^\vee$, which yields a contradiction. Thus, p is a multiple of ρ_0, which proves in particular that ρ_0 is unique.

Finally, let $m \in \rho_0^\vee \cap M$ such that $\langle \rho_0, m \rangle = 1$. Then (3) implies that $\langle \rho_0, m+e \rangle \geq 0$, which yields $\langle \rho_0, e \rangle \geq -1$ and thus $\langle \rho_0, e \rangle = -1$. This proves that e is a Demazure root and that $\rho_0 = \rho_e$. Differentiating equation (2) we get that the infinitesimal generator of α_e is $\partial_{\rho_e,e}$, so that $\partial_{p,e} = \pm\partial_{\rho_e,e}$ lies in the image of i_e, proving the claim.

In order to conclude, by [13, Exp. $\mathrm{VI_B}$, Prop. 7.1], we may consider the *smooth* connected subgroup $G \subset \mathrm{Aut}_X$ generated by the closed immersions ι_0 and ι_e for $e \in \mathcal{R}(\Sigma)$. The claim above implies that $\mathrm{Lie}(G) = \mathrm{Lie}(\mathrm{Aut}_X)$, which immediately tells us that Aut_X is smooth at the identity and hence everywhere. From this we get that $G = \mathrm{Aut}_X^0$. □

Proof of Theorem 2 (iii)

This corresponds to the second part of [6, Sec. 4, §6, Prop. 11], whose proof is a direct application of [6, Sec. 4, §6, Prop. 10], which Demazure proves by "reasoning as in the second corollary to [6, Sec. 2, §5, Thm. 4]". We give here a simplified proof that uses the well-known equivalence of categories between toric varieties and fans.

Note that there are finitely many toric automorphisms of X and that they are all defined over k since they come from automorphisms of the corresponding complete fan. We only have to prove then that these generate the quotient group-scheme $\mathrm{Aut}_X / \mathrm{Aut}_X^0$. And since Aut_X is smooth by part (ii), we only need to check this over an algebraically closed field k.

Let $\varphi \in \mathrm{Aut}_X(\mathrm{k})$ be a representative of an element in the quotient. Since T, seen as automorphisms of X, is a maximal torus of Aut_X and maximal tori are k-conjugate in Aut_X^0, up to composing φ with a suitable element in Aut_X^0, we may assume that it normalizes T.

This assumption implies in turn that φ stabilizes the torus T inside X since it is the unique open T-orbit in X. Then, Rosenlicht's Lemma (cf. [12, Lem. 6.5 (iii)]) tells us that, up to composing again φ with a suitable $\psi \in \mathrm{T} \subset \mathrm{Aut}^0_X$, we may assume that it restricts to a group automorphism of $\mathrm{T} \subset X$, and this is the definition of a toric morphism. This proves that toric automorphisms generate the quotient, as desired. □

3. Proof of Theorem 1

We start by considering the neutral component of the automorphism group, which amounts to proving the following result.

Proposition 3. *Let $X_1, \ldots, X_n$ be complete toric varieties and let $X = \prod_{i=1}^n X_i$. Then*

$$\mathrm{Aut}^0_X \simeq \prod_{i=1}^n \mathrm{Aut}^0_{X_i}.$$

This proposition is actually already known for *arbitrary* complete varieties as a consequence of Blanchard's Lemma (cf. [2, Cor. 7.2.3]), but we give nevertheless a full proof here below that only uses Theorem 2, providing a "full-toric" self-contained proof of Theorem 1.

Remark 4. Since Proposition 3 is valid for arbitrary complete varieties, one is naturally led to ask whether this is also the case for Theorem 1. This is far from being true. Indeed, it suffices to consider a general elliptic curve E and the product $E \times E$. Then $\mathrm{Aut}_E \simeq E \rtimes \mathbb{Z}/2\mathbb{Z}$, while $\mathrm{Aut}_{E\times E} \simeq (E \times E) \rtimes \mathrm{GL}_2(\mathbb{Z})$. In particular, the quotient $\mathrm{Aut}_{E\times E} / \mathrm{Aut}^0_{E\times E}$ is infinite, while $\mathrm{Aut}_E / \mathrm{Aut}^0_E$ is finite.

Proof of Proposition 3. It is enough to show the result for $n = 2$, i.e., for $X \times X'$ with X, X' complete toric varieties. We will show that $\mathrm{Aut}^0_{X\times X'}$ is spanned by $\mathrm{Aut}^0_X \times \{\mathrm{id}_{X'}\}$ and $\{\mathrm{id}_X\} \times \mathrm{Aut}^0_{X'}$ proving the proposition.

Let Σ and Σ' be the fans in $N_\mathbb{R}$ and $N'_\mathbb{R}$ corresponding to the toric varieties X and X', with acting tori T and T$'$, respectively. By [5, Prop. 3.1.14] the fan of $X \times X'$ is

$$\Sigma \times \Sigma' = \{\sigma \times \sigma' \mid \sigma \in \Sigma, \sigma' \in \Sigma'\}.$$

Remark that for every Demazure root $e \in \mathcal{R}(\Sigma)$ we obtain a root $(e, 0)$ of $\Sigma \times \Sigma'$. Analogously, for every Demazure root $e' \in \mathcal{R}(\Sigma')$ we obtain a root

$(0,e')$ of $\Sigma \times \Sigma'$. Furthermore, by Theorem 2 (ii) we have that $\mathrm{T} \times \{\mathrm{id}_{X'}\}$ and the images of $\iota_{(e,0)}$ span a group isomorphic to $\mathrm{Aut}^0_X \times \{\mathrm{id}_{X'}\}$ inside $\mathrm{Aut}^0_{X\times X'}$. Analogously, $\{\mathrm{id}_X\} \times \mathrm{T}'$ and the images of $\iota_{(0,e')}$ span a group isomorphic to $\{\mathrm{id}_X\} \times \mathrm{Aut}^0_{X'}$ inside $\mathrm{Aut}^0_{X\times X'}$.

Hence, to prove the proposition, it is enough by Theorem 2 (ii) to show that every Demazure root $\overline{e}$ of $\Sigma \times \Sigma'$ has the form $(e,0)$ for e a Demazure root of Σ or $(0,e')$ for e' a Demazure root of Σ'. Let $\overline{e} = (e,e')$ be a Demazure root of $\Sigma \times \Sigma'$ and let $\rho_{\overline{e}}$ be the unique ray of $\Sigma \times \Sigma'$ such that $\langle \rho_{\overline{e}}, \overline{e}\rangle = -1$. Since the 1-skeleton of $\Sigma \times \Sigma'$ is the disjoint union of $\Sigma(1) \times \{0\}$ and $\{0\} \times \Sigma'(1)$, we may assume, without loss of generality, that $\rho_{\overline{e}} = (\rho_0, 0)$ for some $\rho_0 \in \Sigma(1)$. Then for every ray $\rho' \in \Sigma'(1)$ we have $\langle (0,\rho'),(e,e')\rangle = \langle \rho', e'\rangle \geq 0$. Since X' is complete, the support of Σ' is $N'_{\mathbb{R}}$ and hence $\langle \rho', e'\rangle \geq 0$ for every $\rho' \in \Sigma'(1)$ yields $e' = 0$. Moreover, $\langle (\rho,0),(e,e')\rangle = \langle \rho, e\rangle$ and so $\langle \rho, e\rangle \geq 0$ for all $\rho \neq \rho_0$ and $\langle \rho_0, e\rangle = -1$. This shows that e is a Demazure root of Σ, proving the proposition. □

The second ingredient in the proof of Theorem 1 is the following lemma, which has the same flavour as a result by Ballard, Duncan and McFaddin on automorphisms of products of complete fans (cf. [1, Lem. 3.10]).

Lemma 5. *Let X, X'_1, X'_2 be complete toric varieties with X indecomposable. Let $\Sigma, \Sigma'_1, \Sigma'_2$ be the respective fans. Consider a toric morphism $\varphi : X \to X'_1 \times X'_2$ and denote by ψ the corresponding fan morphism. Assume that, for every $1 \leq i \leq \dim(X)$ and for every $\sigma \in \Sigma(i)$, we have $\psi(\sigma) \in (\Sigma'_1 \times \Sigma'_2)(i)$. Then $\varphi(X)$ is contained in either $X'_1 \times \{1\}$ or $\{1\} \times X'_2$.*

Proof. Let N and M be the lattices corresponding to X and let N'_i and M'_i be the lattices corresponding to X'_i for $i = 1,2$. In particular, Σ is a fan in $N_{\mathbb{R}}$ and Σ'_i is a fan in $N'_{i,\mathbb{R}}$. Note that the hypothesis on ψ, applied to a full dimensional cone in Σ, implies that $\psi : N_{\mathbb{R}} \to N'_{1,\mathbb{R}} \oplus N'_{2,\mathbb{R}}$ is injective.

Since the 1-skeleton of $\Sigma'_1 \times \Sigma'_2$ is the disjoint union of $\Sigma'_1(1) \times \{0\}$ and $\{0\} \times \Sigma'_2(1)$ and ψ preserves 1-skeletons, we may take preimages and write $\Sigma(1) = B_1 \sqcup B_2$. Define $N_{i,\mathbb{R}} := \mathrm{span}(B_i)$ for $i = 1,2$ and note that $N_{\mathbb{R}} = N_{1,\mathbb{R}} \oplus N_{2,\mathbb{R}}$ since the images of $N_{1,\mathbb{R}}$ and $N_{2,\mathbb{R}}$ in $N'_{1,\mathbb{R}} \oplus N'_{2,\mathbb{R}}$ have trivial intersection and X is complete. This decomposition of $N_{\mathbb{R}}$ gives a natural decomposition $M_{\mathbb{R}} = M_{1,\mathbb{R}} \oplus M_{2,\mathbb{R}}$, with $M_{i,\mathbb{R}}$ canonically isomorphic to the dual of $N_{i,\mathbb{R}}$.

Let σ be a full dimensional cone in Σ and write $\sigma(1) = S_1 \sqcup S_2$ by taking intersections with B_1 and B_2 respectively and note that $N_{i,\mathbb{R}} = \mathrm{span}(S_i)$. Let $\sigma_i \subset N_{i,\mathbb{R}}$ be the cone spanned by S_i. Let $u_i \in M_{i,\mathbb{R}}$ be an element in the relative interior of $\sigma_i^\vee \in M_{i,\mathbb{R}}$, so that $\langle p, u_i\rangle > 0$ for every $0 \neq p \in \sigma_i$.

Let $H_i := \{p \in N_{\mathbb{R}} \mid \langle p, u_i \rangle = 0\}$ be the supporting hyperplane in $N_{\mathbb{R}}$ of the vector u_i. We have then $\sigma_1 = H_2 \cap \sigma$ and $\sigma_2 = H_1 \cap \sigma$, hence σ_1 and σ_2 are both faces of σ. In particular, $\sigma_1, \sigma_2 \in \Sigma$.

Since ψ preserves cones by hypothesis, we see that $\psi(\sigma), \psi(\sigma_1), \psi(\sigma_2)$ are cones in $\Sigma_1' \times \Sigma_2'$ and it is evident then that $\psi(\sigma) = \psi(\sigma_1) \times \psi(\sigma_2)$ and thus $\sigma = \sigma_1 \times \sigma_2$. Since this is true for every full dimensional cone in Σ, this defines subfans $\Sigma_1, \Sigma_2 \subset \Sigma$ such that Σ can be seen as a subfan of $\Sigma_1 \times \Sigma_2$, which has the same dimension. Since X is complete, we see that $\Sigma = \Sigma_1 \times \Sigma_2$, which contradicts the indecomposability of X unless one of the Σ_i's is trivial, which is equivalent to $B_i = \emptyset$. This implies the assertion in the statement of the Lemma. □

We are now ready to prove our main theorem.

Proof of Theorem 1. Let φ be an automorphism of X. Then by Proposition 3 and Theorem 2 (iii), we know that up to composing φ with an element in

$$\prod_{i=1}^{n} (\mathrm{Aut}^0_{X_i})^{r_i} = \mathrm{Aut}^0_X,$$

we may assume that φ is a toric automorphism, i.e. it corresponds to an automorphism ψ of the fan Σ of X. Note that such an automorphism preserves $\Sigma(i)$ (in the sense of Lemma 5) for every $1 \leq i \leq \dim X$.

Denote by $X_{i,j}$ with $1 \leq j \leq r_i$ the different copies of X_i. Define then, for each $X_{i,j}$, the corresponding subvariety

$$Y_{i,j} := \{1\} \times \cdots \times X_{i,j} \times \cdots \times \{1\} \subset X$$

Since these are indecomposable toric varieties, Lemma 5 tells us that $\varphi(Y_{i,j}) \subset Y_{i',j'}$ for some i', j'. And since φ is invertible, we see that $Y_{i,j} \simeq Y_{i',j'}$, hence $X_{i,j} \simeq X_{i',j'}$ and thus $i = i'$. Then, up to composing φ with a permutation of isomorphic components (which is a toric morphism), we may assume that φ is toric and preserves each $Y_{i,j}$. This in turn implies that ψ preserves each component of Σ, which is the direct product of fans of the X_i's. And this gives $\varphi \in \prod_{i=1}^{n} \mathrm{Aut}^{r_i}_{X_i}$, which concludes the proof of the theorem. □

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