

The Shafarevich conjecture and some extension theorems for proper hyperbolic polycurves

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In this paper, we prove the Shafarevich conjecture for proper hyperbolic polycurves, which is a higher dimensional analogue of that for proper hyperbolic curves. We prove this by using the good reduction criterion for proper hyperbolic polycurves established in [19], which gives a new proof of the main theorem of [7] for canonically polarized surfaces smooth proper over curves. Our theorem is a generalization of the result of Javanpeykar [7] in the following two points: (i) We treat proper smooth models which are not necessarily canonically polarized. (ii) We treat higher dimensional varieties, that is, proper hyperbolic polycurves of any dimension. This paper also contains a generalization of the moduli theory of Kodaira fibrations due to Jost and Yau [13].

0. Introduction

The Shafarevich conjecture for proper hyperbolic curves, which was proved by Faltings, states the finiteness of isomorphism classes of proper hyperbolic curves of fixed genus over a fixed number field admitting good reduction away from a fixed finite set of finite places. In [9], Javanpeykar and Loughran conjectured that the Shafarevich conjecture holds for more general families of varieties. Moreover, they showed that if the Lang-Vojta conjecture on integral points of hyperbolic varieties holds, then the Shafarevich conjectures for hypersurfaces and complete intersections of general type ([9, Theorem 1.5]) hold. There are many references on the Shafarevich conjectures for several classes of varieties (e.g. del Pezzo surfaces ([21]), certain Fano threefolds ([10]), certain surfaces of general type ([11]), canonically polarized smooth fibered surfaces ([7]) (see also Remark 0.2), algebraic groups ([8]), K3 surfaces ([1], [22], [23]), Enriques surfaces ([24]), bielliptic surfaces ([25]), hypersurfaces in projective spaces ([15]), certain weighted hyper surfaces ([12]), and hypersurfaces in abelian varieties ([14])). However,

such finiteness for more general families of varieties are still open (see [10, Section 5] for counterexamples).

In this paper, as a generalization of the result in [7], we shall prove the Shafarevich conjecture for the class of proper hyperbolic polycurves, that is, varieties X which admit a structure of successive smooth fibrations (called a sequence of parametrizing morphisms (cf. Definition 1.1))

$$(\mathcal{S}) : X = X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = \text{Spec } K$$

whose fibers are proper hyperbolic curves. Let F be a number field, \mathfrak{p} a finite place of F , and $\mathcal{O}_{F, \mathfrak{p}}$ the valuation ring of F at \mathfrak{p} . For any proper hyperbolic polycurve X over F , we say that X has good reduction at \mathfrak{p} if X admits a smooth proper model over $\mathcal{O}_{F, \mathfrak{p}}$ (cf. Definition 1.4). Our main theorem is the following:

Theorem 0.1 (see Theorem 3.4 for a more general statement).

Let F be a number field, and S a finite set of finite places of F . Let χ be an integer, and n a positive integer. Then there exist only finitely many isomorphism classes of proper hyperbolic polycurves of dimension n with Euler-Poincaré characteristic χ over F which have good reduction outside S .

As a corollary of Theorem 0.1, we give another proof of Sawada's finiteness theorem of isomorphism classes of proper hyperbolic polycurves with prescribed fundamental groups (cf. Corollary 4.1 and Remark 4.2).

Theorem 0.1 and its proof are a generalization and a new proof of [7, Theorem 1.2], respectively. We explain the relation between Theorem 0.1 and [7, Theorem 1.2].

Remark 0.2. In [7], a canonically polarized surface over a discrete valuation field is said to have good reduction if there exists a smooth proper model whose relative dualizing sheaf is relatively ample (see [7, Introduction]), and he proved the Shafarevich conjecture for canonically polarized smooth fibered surfaces. Theorem 0.1 is a generalization of [7, Theorem 1.2] in the following two senses:

- Since a canonically polarized surface smooth proper over a curve of characteristic 0 is a proper hyperbolic polycurve by [7, Lemma 3.1] (, in fact, the converse is also true since a hyperbolic polycurve does not contain (-2) -curves), our result in the case of dimension 2 implies [7,

Theorem 1.2]. Therefore, our result can be seen as a higher dimensional generalization of [7, Theorem 1.2].

- Even in the case of dimension 2, our theorem looks stronger than [7, Theorem 1.2]. In our definition of the notion of good reduction, we do not assume (in contrast to [7, Introduction]) that smooth models have relatively ample dualizing sheaves. However, at the time of writing, we do not know any examples of proper hyperbolic polycurves which do not have smooth models with relatively ample dualizing sheaves.

One of the main tools which is used to generalize [7, Theorem 1.2] is the good reduction criterion for proper hyperbolic polycurves established in [19] (cf. the proof of Theorem 2.2).

In the proof of Theorem 0.1, we will use the Shafarevich conjecture for proper hyperbolic curves over finitely generated fields of characteristic 0, which was also proved by Faltings. To use Faltings’s result inductively, we study the structure of integral models of proper hyperbolic polycurves. Precisely, we shall give the following result:

Theorem 0.3 (cf. Theorem 2.2). *Let T be a connected Noetherian regular scheme, $K(T)$ the field of fractions of T , and $\mathfrak{X} \rightarrow T$ a proper smooth scheme. Write X for the scheme $\mathfrak{X} \times_T \text{Spec } K(T)$. Suppose that X is a proper hyperbolic polycurve over $K(T)$. Moreover, suppose that the residual characteristic of every point of T of codimension 1 is sufficiently large (see Theorem 2.2 for the precise bound) or equal to 0. Then, for any sequence of parametrizing morphisms (\mathcal{S}) of $X \rightarrow \text{Spec } K(T)$, there exists a unique sequence of parametrizing morphisms (\mathfrak{S}') of a proper hyperbolic polycurve $\mathfrak{X}' \rightarrow T$ (up to canonical isomorphism) such that the base change of $(\mathfrak{X}', (\mathfrak{S}'))$ to $\text{Spec } K(T)$ is isomorphic to $(X, (\mathcal{S}))$ (cf. Definition 1.1.2) and \mathfrak{X} is canonically isomorphic to \mathfrak{X}' .*

Theorem 0.3 is a sort of generalization of the result of Jost and Yau [13]. (In [13], proper hyperbolic polycurves of relative dimension 2 over complex manifolds are treated.) Theorem 0.3 is also a sort of generalization of [7, Lemma 5.5]. We prove Theorem 0.3 by applying, the good reduction criterion for proper hyperbolic polycurves, the results on Néron models of hyperbolic curves by Liu-Tong [16], and the purity of proper hyperbolic polycurves over regular schemes (cf. Theorem 1.11).

The content of each section is as follows: In Section 1, we give the precise definition of a proper hyperbolic polycurve and the proof of the purity

of proper hyperbolic polycurves over regular schemes by using the good reduction criterion for proper hyperbolic polycurves. In Section 2, we discuss structures of smooth models of a proper hyperbolic polycurve over a Dedekind scheme. In Section 3, we give the proof of the Shafarevich conjecture for proper hyperbolic polycurves by using the results of Section 2 and Faltings's result. In Section 4, we give another proof of the Sawada's finiteness theorem for proper polycurves in the case where their coefficient fields are finitely generated over \mathbb{Q} .

1. Proper hyperbolic polycurves over regular schemes

In this section, we discuss properties of proper hyperbolic polycurves over regular schemes (cf. Theorem 1.11). We start with the definition of a proper hyperbolic polycurve.

Definition 1.1. Let S be a scheme and X a scheme over S .

- 1) We shall say that X is a *proper hyperbolic curve* over S if the structure morphism $X \rightarrow S$ is proper, smooth, and of relative dimension 1 with geometrically connected fibers of genus $g \geq 2$.
- 2) We shall say that X is a *proper hyperbolic polycurve* (of relative dimension n) over S if there exists a (not necessarily unique) factorization

$$(\mathcal{S}) : X = X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0 = S$$

of the structure morphism $X \rightarrow S$ such that, for each $i \in \{1, \dots, n\}$, $X_i \rightarrow X_{i-1}$ is a proper hyperbolic curve. We refer to the above factorization of the morphism $X \rightarrow S$ as a *sequence of parametrizing morphisms*. Let g_i be the genus of the curve $X_i \rightarrow X_{i-1}$ for each $1 \leq i \leq n$. We write $g_{\mathcal{S}} \stackrel{\text{def}}{=} \max_{1 \leq i \leq n} g_i$. We also write $g_X \stackrel{\text{def}}{=} \min_{\mathcal{S}} g_{\mathcal{S}}$, where \mathcal{S} ranges over the sequences of parametrizing morphisms of $X \rightarrow S$. In the case where we consider a pair of a proper hyperbolic polycurve $X \rightarrow S$ and a sequence of parametrizing morphisms (\mathcal{S}) of $X \rightarrow S$, we write $(X, (\mathcal{S}))$. We refer to such a pair as a *proper hyperbolic polycurve with a sequence of parametrizing morphisms*. We shall say that two proper hyperbolic polycurves (over S) with a sequence of parametrizing morphisms $(X, (\mathcal{S}))$ and $(X', (\mathcal{S}'))$ are isomorphic if there exists an S -isomorphism between proper hyperbolic polycurves of relative dimension i over S defined by (\mathcal{S}) and (\mathcal{S}') for each $1 \leq i \leq n$ such that these

isomorphisms are compatible with the sequence of parametrizing morphisms (\mathcal{S}) and (\mathcal{S}') .

- 3) Let X be a proper hyperbolic polycurve of relative dimension n over S . Let

$$X = X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0 = S$$

be a sequence of parametrizing morphisms of $X \rightarrow S$. Write g_i for the genus of the proper hyperbolic curve $X_i \rightarrow X_{i-1}$ for each $1 \leq i \leq n$. We refer to the nonzero integer

$$\chi(X) = \prod_{1 \leq i \leq n} (2 - 2g_i)$$

as the *Euler-Poincaré characteristic* of the proper hyperbolic polycurve $X \rightarrow S$. Note that $\chi(X)$ does not depend on the choice of a sequence of parametrizing morphisms of the proper hyperbolic polycurve $X \rightarrow S$ by Lemma 1.2. It holds that 2^n divides $\chi(X)$. Note that we cannot determine n from $\chi(X)$.

Lemma 1.2. *Let X be a proper hyperbolic polycurve of relative dimension n over S . $\chi(X)$ does not depend on the choice of a sequence of parametrizing morphisms of $X \rightarrow S$.*

Proof. Let

$$X = X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0 = S$$

be a sequence of parametrizing morphisms of $X \rightarrow S$. Write g_i for the genus of the proper hyperbolic curve $X_i \rightarrow X_{i-1}$ for each $1 \leq i \leq n$. We may assume that S is the spectrum of an algebraically closed field k . Let $l \neq p$ be a prime number and $\chi(X, \mathbb{F}_l)$ the Euler characteristic of the trivial étale sheaf \mathbb{F}_l on X . It suffices to show that $\prod_{1 \leq i \leq n} (2 - 2g_i) = \chi(X, \mathbb{F}_l)$. By induction on n and the Leray spectral sequence for $X_n \rightarrow X_{n-1}$, one can verify this by using [6, Corollaire 2.11]. □

Remark 1.3. In the case where S has a point s whose residual characteristic is 0, Lemma 1.2 follows immediately from the fact that the Euler-Poincaré characteristic of a proper hyperbolic polycurve $X \rightarrow S$ can be determined by the étale fundamental groups of the scheme $X \times_S \bar{s}$. Here, \bar{s} is a geometric point of X over s . Indeed, let l be a prime number. Write Δ for the étale fundamental group of the scheme $X \times_S \bar{s}$, $H^i(\Delta, \mathbb{F}_l)$ for the i -th cohomology

group of the trivial Δ -module \mathbb{F}_l , and $\chi(\Delta, \mathbb{F}_l)$ for the Euler characteristic $\sum_{0 \leq i \leq \infty} (-1)^i \dim_{\mathbb{F}_l} H^i(\Delta, \mathbb{F}_l)$. Then one can verify that $\chi(X) = \chi(\Delta, \mathbb{F}_l)$.

In this paper, we shall say that a scheme T is *Dedekind* if T is a 1-dimensional connected Noetherian normal separated scheme.

Definition 1.4. Let T be a Dedekind scheme, $K(T)$ the function field of T , η the generic point of T , and $X \rightarrow \text{Spec } K(T)$ a proper smooth morphism with geometrically connected fibers.

- 1) We shall say that X has good reduction if there exists a proper smooth T -scheme \mathfrak{X} whose generic fiber \mathfrak{X}_η is isomorphic to X over $K(T)$. We refer to such \mathfrak{X} as a *smooth model* of X .
- 2) Suppose that $X \rightarrow \text{Spec } K(T)$ is a proper hyperbolic polycurve. Let

$$(\mathcal{S}) : X = X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0 = \text{Spec } K(T)$$

be a sequence of parametrizing morphisms of $X \rightarrow \text{Spec } K(T)$. We shall say that X has *good reduction with respect to* (\mathcal{S}) if there exist a proper hyperbolic polycurve $\mathfrak{X} \rightarrow T$ and a sequence of parametrizing morphisms

$$(\mathcal{S}') : \mathfrak{X} = \mathfrak{X}_n \rightarrow \mathfrak{X}_{n-1} \rightarrow \dots \rightarrow \mathfrak{X}_1 \rightarrow \mathfrak{X}_0 = T$$

of $\mathfrak{X} \rightarrow T$ such that the proper hyperbolic polycurve with a sequence of parametrizing morphisms defined by the base change of the sequence (\mathcal{S}') to $\text{Spec } K(T)$ is isomorphic to $(X, (\mathcal{S}))$.

- 3) Let \mathfrak{X} be a separated, smooth, and of finite type scheme over T whose generic fiber is isomorphic to X over $K(T)$. We shall say that \mathfrak{X} is the Néron model of X if the following property, called *Néron mapping property*, is satisfied:
for any smooth finitely presented scheme \mathfrak{Y} over T , the canonical map

$$\text{Mor}_T(\mathfrak{Y}, \mathfrak{X}) \rightarrow \text{Mor}_{K(T)}(\mathfrak{Y} \times_T \text{Spec } K(T), X)$$

is a bijection. Here, $\text{Mor}_T(\mathfrak{Y}, \mathfrak{X})$ is the set of morphisms from \mathfrak{Y} to \mathfrak{X} over T , and $\text{Mor}_{K(T)}(\mathfrak{Y} \times_T \text{Spec } K(T), X)$ is the set of morphisms from $\mathfrak{Y} \times_T \text{Spec } K(T)$ to X over $K(T)$.

Remark 1.5. If a smooth model of a proper hyperbolic curve exists, it is unique up to canonical isomorphism (cf. [2]). This also follows from Lemma 1.6 or Proposition 1.7.2.

Lemma 1.6. *Let S be an irreducible normal scheme and $K(S)$ the function field of S . Let C_1 and C_2 be proper hyperbolic curves over S , and ϕ an isomorphism $C_1 \times_S \text{Spec } K(S) \cong C_2 \times_S \text{Spec } K(S)$ over $K(S)$. Then there exists a unique isomorphism $\Phi : C_1 \cong C_2$ over S whose base change to $\text{Spec } K(S)$ coincides with ϕ .*

Proof. (See also the argument given in the discussion entitled “Curves” in [17, §0].) By [2, Theorem 1.11], the scheme $\text{Isom}_S(C_1, C_2)$ (cf. [2, Definition 1.10]) is finite over S . Write cl_ϕ for the classifying morphism

$$\text{Spec } K(S) \rightarrow \text{Isom}_{K(S)}(C_1 \times_S \text{Spec } K(S), C_2 \times_S \text{Spec } K(S))$$

over $K(S)$. Since S is normal, the scheme theoretic image of the composite morphism

$$\begin{aligned} \text{Spec } K(S) &\xrightarrow{\text{cl}_\phi} \text{Isom}_{K(S)}(C_1 \times_S \text{Spec } K(S), C_2 \times_S \text{Spec } K(S)) \\ &\rightarrow \text{Isom}_S(C_1, C_2) \end{aligned}$$

defines a section $S \rightarrow \text{Isom}_S(C_1, C_2)$. This concludes the proof. □

Proposition 1.7 (cf. [16]). *Let T , $K(T)$, and X be as in Definition 1.4.*

- 1) *Let \mathfrak{X} be a smooth model of X . Suppose that each closed fiber of the morphism $\mathfrak{X} \rightarrow T$ contains no rational curves. Then \mathfrak{X} is the Néron model of X ([16, Proposition 4.13]).*
- 2) *Suppose that X is a proper hyperbolic curve which has good reduction. Then a smooth model of X (cf. Remark 1.5) is the Néron model of X .*
- 3) *Let Y be a geometrically connected smooth proper scheme over $K(T)$. Suppose that X is a proper hyperbolic curve over $K(T)$ and that Y has good reduction. Moreover, suppose that there exists a $K(T)$ -morphism from Y to X . Then X has good reduction ([16, Corollary 4.7]).*
- 4) *If X has a proper Néron model, any smooth model of X is canonically isomorphic to the Néron model.*

Proof. We only show assertion 2 and 4. Assertion 2 follows from [16, Theorem 1.1] and the fact that a smooth model of X is the minimal regular

model of X . Assertion 4 follows from van der Waerden's purity theorem (see [4, Corollaire (21.12.16)] for a more general statement). \square

Note that Proposition 1.7.2 follows from Proposition 1.7.1. Also, one can show Proposition 1.7.2 by the Néron mapping property of the Néron model of the Jacobian variety of X (after replacing T by the strict henselization of each closed point of T).

Proposition 1.8. *Let T and $K(T)$ be as in Definition 1.4. Let*

$$\mathfrak{X} = \mathfrak{X}_n \rightarrow \mathfrak{X}_{n-1} \rightarrow \dots \rightarrow \mathfrak{X}_0 = T$$

be a proper hyperbolic polycurve. Then \mathfrak{X} is the Néron model of the scheme $\mathfrak{X} \times_T \text{Spec } K(T)$.

Proof. By Proposition 1.7.1, it suffices to show that there exist no rational curves contained in the special fiber of \mathfrak{X} . Since any morphism from a rational curve to a hyperbolic curve over a field is constant, a proper hyperbolic polycurve over a field contains no rational curves. Hence, Proposition 1.8 holds. \square

Proposition 1.9 (cf. [18] and [19, Section 7]). *Let Z be a connected Noetherian regular scheme.*

1) *Let $K(Z)$ be the function field of Z and $C_{K(Z)} \rightarrow \text{Spec } K(Z)$ a proper hyperbolic curve. The following are equivalent:*

- *There exists a proper hyperbolic curve $C_Z \rightarrow Z$ such that $C_Z \times_Z \text{Spec } K(Z)$ is isomorphic to $C_{K(Z)}$ over $K(Z)$.*
- *There exist a nonempty open subset U of Z satisfying that $Z \setminus U$ is of codimension ≥ 2 in Z and a proper hyperbolic curve $C_U \rightarrow U$ such that $C_U \times_U \text{Spec } K(Z)$ is isomorphic to $C_{K(Z)}$ over $K(Z)$.*
- *For any point $z \in Z$ of codimension 1, there exists a proper hyperbolic curve $C_{O_{Z,z}} \rightarrow \text{Spec } O_{Z,z}$ such that $C_{O_{Z,z}} \times_{O_{Z,z}} \text{Spec } K(Z)$ is isomorphic to $C_{K(Z)}$ over $K(Z)$.*

In this case, the scheme C_Z (respectively, C_U ; $C_{O_{Z,z}}$) is unique up to a canonical isomorphism over Z (respectively, U ; $O_{Z,z}$ for each point $z \in Z$ of codimension 1). Hence, the scheme $C_Z \times_Z U$ (respectively, $C_Z \times_Z \text{Spec } O_{Z,z}$) is isomorphic to C_U (respectively, $C_{O_{Z,z}}$) over U (respectively, $O_{Z,z}$ for each point $z \in Z$ of codimension 1) (cf. Remark 1.10).

2) *Let Y be a connected Noetherian regular scheme over Z , V a nonempty open subset of Y satisfying that $Y \setminus V$ is of codimension ≥ 2 in Y , and*

$C' \rightarrow Z$ a proper hyperbolic curve. Then the restriction map

$$\text{Mor}_Z(Y, C') \rightarrow \text{Mor}_Z(V, C')$$

is bijective. Here, $\text{Mor}_Z(Y, C')$ (respectively, $\text{Mor}_Z(V, C')$) is the set of morphisms from Y to C' over Z (respectively, from V to C' over Z).

Proof. Assertion 1 follows from [18] and Lemma 1.6. To show assertion 2, we may assume that $Y = Z$. Then the assertion follows from [18, Lemme 1] (or [19, Section 7]). □

Remark 1.10. The latter part of Proposition 1.9.1 does not hold in general. Let $\mathbb{P}_{\mathbb{Z}}^1$ be the projective line over $\text{Spec } \mathbb{Z}$. Write B_p for the scheme obtained by blowing up of $\mathbb{P}_{\mathbb{Z}(p)}^1$ at some closed point of $\mathbb{P}_{\mathbb{Z}(p)}^1$. Then write C_p for the scheme obtained by contraction of the strict transform of the special fiber of $\mathbb{P}_{\mathbb{Z}(p)}^1$ in B_p . Consider the family of smooth models $\{D_p \rightarrow \text{Spec } \mathbb{Z}_{(p)} \mid p \text{ is a prime number}\}$ of $\mathbb{P}_{\mathbb{Q}}^1 \rightarrow \text{Spec } \mathbb{Q}$, where D_p is $\mathbb{P}_{\mathbb{Z}(p)}^1$ or C_p . One can verify that there exists a proper smooth curve $C \rightarrow \text{Spec } \mathbb{Z}$ whose base change to $\text{Spec } \mathbb{Z}_{(p)}$ is isomorphic to C_p for all p if and only if $D_p = \mathbb{P}_{\mathbb{Z}(p)}^1$ for all but finite p .

Theorem 1.11. *Let Z be a connected Noetherian regular scheme.*

1) *Let $K(Z)$ be the function field of Z and*

$$(\mathcal{S}_{K(Z)}) : X_{n,K(Z)} \rightarrow \dots \rightarrow X_{1,K(Z)} \rightarrow \text{Spec } K(Z)$$

a sequence of parametrizing morphisms of a proper hyperbolic polycurve. The following are equivalent:

- *There exists a sequence of parametrizing morphisms of a proper hyperbolic polycurve*

$$(\mathcal{S}_Z) : X_{n,Z} \rightarrow \dots \rightarrow X_{1,Z} \rightarrow Z$$

such that the base change of $(X_{n,Z}, (\mathcal{S}_Z))$ to $\text{Spec } K(Z)$ is isomorphic to $(X_{n,K(Z)}, (\mathcal{S}_{K(Z)}))$.

- *There exist a nonempty open subset U of Z satisfying that $Z \setminus U$ is of codimension ≥ 2 in Z and a sequence of parametrizing morphisms*

of a proper hyperbolic polycurve

$$(\mathcal{S}_U) : X_{n,U} \rightarrow \dots \rightarrow X_{1,U} \rightarrow U$$

such that the base change of $(X_{n,U}, (\mathcal{S}_U))$ to $\text{Spec } K(Z)$ is isomorphic to $(X_{n,K(Z)}, (\mathcal{S}_{K(Z)}))$.

- For any point $z \in Z$ of codimension 1, there exists a sequence of parametrizing morphisms of a proper hyperbolic polycurve

$$(\mathcal{S}_{O_{z,z}}) : X_{n,O_{z,z}} \rightarrow \dots \rightarrow X_{1,O_{z,z}} \rightarrow \text{Spec } O_{z,z}$$

such that the base change of $(X_{n,O_{z,z}}, (\mathcal{S}_{O_{z,z}}))$ to $\text{Spec } K(Z)$ is isomorphic to $(X_{n,K(Z)}, (\mathcal{S}_{K(Z)}))$.

In this case, $(X_{n,Z}, (\mathcal{S}_Z))$ (respectively, $(X_{n,U}, (\mathcal{S}_U)); (X_{n,O_{z,z}}, (\mathcal{S}_{O_{z,z}}))$) is unique up to a canonical isomorphism over Z (respectively, $U; O_{z,z}$ for each point $z \in Z$ of codimension 1).

- 2) Let Y be a connected Noetherian regular scheme over Z , V a nonempty open subset of Y satisfying that $Y \setminus V$ is of codimension ≥ 2 in Y , and $X'_Z \rightarrow Z$ a proper hyperbolic polycurve. Then the restriction map

$$\text{Mor}_Z(Y, X'_Z) \rightarrow \text{Mor}_Z(V, X'_Z)$$

is bijective. Here, $\text{Mor}_Z(Y, X'_Z)$ (respectively, $\text{Mor}_Z(V, X'_Z)$) is the set of morphisms from Y to X'_Z over Z (respectively, from V to X'_Z over Z).

- 3) Let $K(Z)$ and $(\mathcal{S}_{K(Z)})$ be as in assertion 1. Suppose that the equivalent conditions of assertion 1 are satisfied. Let $X_Z \rightarrow Z$ be a proper smooth morphism such that $X_Z \times_Z \text{Spec } K(Z)$ is isomorphic to $X_{n,K(Z)}$ over $K(Z)$. Then X_Z is canonically isomorphic to $X_{n,Z}$ over Z .

Proof. To show assertion 1 and 2, we may assume that $n = 1$, in which case the assertions follow from Proposition 1.9.1 and 2. Next, we show assertion 3. By Proposition 1.7.4, $X_Z \times_Z \text{Spec } O_{z,z}$ is canonically isomorphic to $X_{n,O_{z,z}}$ over $\text{Spec } O_{z,z}$ for any point $z \in Z$ of codimension 1. Therefore, there exists an open subset U of Z such that $Z \setminus U$ is of codimension ≥ 2 in Z and $X_Z \times_Z U$ is canonically isomorphic to $X_{n,U}$ over U . By assertion 2, there exists a canonical birational morphism $\phi : X_Z \rightarrow X_{n,Z}$ over Z . ϕ is isomorphism by van der Waerden's purity theorem (cf. [4, Corollaire (21.12.16)]). \square

2. Existence of a smooth model of a proper hyperbolic polycurve with respect to a given sequence of parametrizing morphisms

In this section, we discuss structures of smooth models of a proper hyperbolic polycurve over a Dedekind scheme. For proper hyperbolic polycurves of relative dimension 2 over complex manifolds, some of the main results of this section (part of Theorem 2.2 and Corollary 2.3) are proven in [13].

Definition 2.1 (cf. [19, Theorem 1.2.3 and Theorem 1.3]). Let z be a positive integer. Define a function $f_z(m)$ for $m \geq 2$ in the following way:

- For $m = 2$, $f_z(2) = z + 1$.
- For $m = 3$, $f_z(3) = 2^{z^2}$.
- For $m \geq 3$,

$$f_z(m + 1) = (f_z(m)) \times (2^{z^2 \times f_z(m)^2})^{f_z(m)}.$$

Theorem 2.2. *Let Z be a connected Noetherian regular scheme, $K(Z)$ the field of fractions of Z , and $\mathfrak{X} \rightarrow Z$ a proper smooth scheme. Write X for the scheme $\mathfrak{X} \times_Z \text{Spec } K(Z)$. Suppose that X is a proper hyperbolic polycurve of relative dimension n over $K(Z)$.*

1) *Let*

$$(\mathcal{S}) : X = X_n \rightarrow \dots \rightarrow X_1 \rightarrow \text{Spec } K(Z)$$

be a sequence of parametrizing morphisms of a proper hyperbolic curve $X \rightarrow \text{Spec } K(Z)$. Suppose that the residual characteristic of every point of Z of codimension 1 is more than $2^{2g_S \times f_{2g_S}(n)}$ or equal to 0. Then there exists a unique sequence of parametrizing morphisms

$$(\mathfrak{S}) : \mathfrak{X}_n \rightarrow \dots \rightarrow \mathfrak{X}_1 \rightarrow Z$$

(up to canonical isomorphism) such that the base change of $(\mathfrak{X}_n, (\mathfrak{S}))$ to $\text{Spec } K(Z)$ is isomorphic to $(X_n, (\mathcal{S}))$ and \mathfrak{X} is canonically isomorphic to \mathfrak{X}_n . In particular, if the residual characteristic of every point of Z of codimension 1 is more than $2^{2g_S \times f_{2g_S}(n)}$ or equal to 0, $\mathfrak{X} \rightarrow Z$ has a structure of a proper hyperbolic polycurve. If, moreover, Z is Dedekind, \mathfrak{X} is the Néron model of X over Z .

2) *Suppose that the residual characteristic of every point of Z of codimension 1 is more than $2^{(|\chi(X)|+2) \times f_{(|\chi(X)|+2)}(n)}$ or equal to 0. Then,*

for any sequence of parametrizing morphisms (\mathcal{S}) of $X \rightarrow \text{Spec } K(Z)$, there exists a sequence of parametrizing morphisms (\mathfrak{S}') of a proper hyperbolic polycurve $\mathfrak{X}' \rightarrow Z$ such that the base change of $(\mathfrak{X}', (\mathfrak{S}'))$ to $\text{Spec } K(Z)$ is isomorphic to $(X, (\mathcal{S}))$ and \mathfrak{X} is canonically isomorphic to \mathfrak{X}' .

Proof. Theorem 2.2.2 follows from Theorem 2.2.1 and the fact that $2g_{\mathcal{S}} \leq |\chi(X)| + 2$ for any sequence of parametrizing morphisms (\mathcal{S}) of $X \rightarrow Z$. The uniqueness portion of Theorem 2.2.1 follows from Theorem 1.11.1. We show the rest of Theorem 2.2.1. Let

$$(\mathcal{S}) : X = X_n \rightarrow \dots \rightarrow X_0 = \text{Spec } K(Z)$$

be a sequence of parametrizing morphisms of X . By Theorem 1.11, we may assume that Z is the spectrum of a discrete valuation ring $O_{K(Z)}$. Then Theorem 2.2.1 follows from [19, Theorem 1.2.1, Theorem 1.2.3, and Theorem 1.3]. □

Corollary 2.3. *Let $Z, K(Z), \mathfrak{X}$, and X be as in Theorem 2.2. Suppose that Z is a Dedekind scheme and that the residual characteristic of every closed point is more than $2^{(|\chi(X)|+2) \times f_{(|\chi(X)|+2)}(n)}$ or equal to 0. Then X has good reduction with respect to any sequence of parametrizing morphisms of the proper hyperbolic polycurve $X \rightarrow \text{Spec } K(Z)$ and X has a proper Néron model.*

Proof. Corollary 2.3 follows from Theorem 2.2, Proposition 1.8, and [16, Corollary 2.5]. □

3. The Shafarevich conjecture for proper hyperbolic polycurves

In this section, we prove the Shafarevich conjecture for proper hyperbolic polycurves. Firstly, we will recall the Shafarevich conjecture for proper hyperbolic curves over finitely generated fields of characteristic of 0, which was proved by Faltings. Then we will prove the main theorem (Theorem 3.4) by using Faltings’s result and results of Section 2.

Proposition 3.1 (Faltings). *Let S be a normal connected scheme flat of finite type over $\text{Spec } \mathbb{Z}$. Let $g \geq 2$ be an integer. Then there exist only finitely many isomorphism classes of proper hyperbolic curves of genus g over $K(S)$ which have a smooth proper model over S .*

Proof. We may shrink S so that S is regular. Now Proposition 3.1 follows from [3, VI, §1, Theorem 2] and the Torelli theorem. □

Proposition 3.2. *Let S be a normal connected scheme flat of finite type over $\text{Spec } \mathbb{Z}$. Let χ be an integer, and n a positive integer. Then there exist only finitely many isomorphism classes of proper hyperbolic polycurves of dimension n with Euler-Poincaré characteristic χ over S with a sequence of parametrizing morphisms.*

Remark 3.3. As written in Definition 1.1.3, the dimension of a proper hyperbolic polycurve is bounded by the absolute value of its Euler-Poincaré characteristic. Therefore, Proposition 3.2 is still true even if we do not fix the dimension.

Proof. Let $A(n, \chi, S)$ be the set of isomorphism classes of proper hyperbolic polycurves with a sequence of parametrizing morphisms as in the statement of Proposition 3.2. We will prove Proposition 3.2 by induction on n . The case of $n = 1$ follows from Lemma 1.6 and Proposition 3.1. Let n be a positive integer greater than 1. Take a pair $(\mathfrak{X}, (\mathfrak{S})) \in A(n, \chi, S)$ with

$$(\mathfrak{S}): \mathfrak{X} = \mathfrak{X}_n \rightarrow \mathfrak{X}_{n-1} \rightarrow \cdots \rightarrow \mathfrak{X}_1 \rightarrow \mathfrak{X}_0 = S.$$

Let $(\mathfrak{X}_{n-1}, (\mathfrak{S}'))$ be the proper hyperbolic polycurve with a sequence of parametrizing morphisms cut out from $(\mathfrak{X}, \mathfrak{S})$. We have $(\mathfrak{X}_{n-1}, (\mathfrak{S}')) \in A(n - 1, \chi(\mathfrak{X}_{n-1}), S)$ and $|\chi(\mathfrak{X}_{n-1})| \leq |\chi|$. By the induction hypothesis, we may fix the isomorphism class of $(\mathfrak{X}_{n-1}, (\mathfrak{S}'))$. Since \mathfrak{X}_{n-1} is a regular connected scheme flat of finite type over $\text{Spec } \mathbb{Z}$, we have $\mathfrak{X}_n \rightarrow \mathfrak{X}_{n-1} \in A(1, \chi(\mathfrak{X}_n \rightarrow \mathfrak{X}_{n-1}), \mathfrak{X}_{n-1})$. Since $|\chi(\mathfrak{X}_n \rightarrow \mathfrak{X}_{n-1})| \leq |\chi|$, the desired finiteness follows from the case of $n = 1$. □

Theorem 3.4. *Let S be an integral scheme flat of finite type over $\text{Spec } \mathbb{Z}$. Let χ be an integer, and n a positive integer. Then there exist only finitely many isomorphism classes of proper hyperbolic polycurves of dimension n with Euler-Poincaré characteristic χ over $K(S)$ which have good reduction at any regular codimension 1 point of S .*

Remark 3.5. As in Proposition 3.2, we do not need to fix the dimension.

Proof. Note that we can replace S by another nonempty open subscheme of S . Therefore, we may assume that S is regular and that the residual characteristic of any point of S of codimension 1 is more than $2^{(|\chi|+2) \times f_{(|\chi|+2)}(n)}$

or equal to 0. For any proper hyperbolic polycurve X of dimension n with Euler-Poincaré characteristic χ over $K(S)$, one can equip X with a sequence of parametrizing morphisms (\mathcal{S}) over $K(S)$. By the assumption on the residual characteristics and Theorem 2.2, the pair $(X, (\mathcal{S}))$ extends to a proper hyperbolic polycurve with a sequence of parametrizing morphisms over S uniquely. Therefore, it suffices to show the finiteness of the isomorphism classes of proper hyperbolic polycurves with a sequence of parametrizing morphisms of dimension n with Euler-Poincaré characteristic χ over S . This follows from Proposition 3.2. \square

4. An application of the Shafarevich conjecture

In this section, we show the finiteness of isomorphism classes of proper hyperbolic polycurves over a fixed number field satisfying a condition determined by their étale fundamental groups. This finiteness was proved in [20] (cf. Remark 4.2) by examining the geometric étale fundamental groups of proper hyperbolic polycurves. We show this by using the Shafarevich conjecture of proper hyperbolic polycurves (cf. Theorem 3.4) and [19, Theorem 1.3].

Let L be a field and $X \rightarrow \text{Spec } L$ a proper hyperbolic polycurve. Take a geometric point $*$ of X and write $\pi_1(X, *) \rightarrow G_L$ for the surjective homomorphism between the étale fundamental groups induced by $X \rightarrow \text{Spec } L$. Note that G_L is isomorphic to the absolute Galois group of L defined by $*$.

Corollary 4.1. *Let K be a field finitely generated over \mathbb{Q} , G_K its absolute Galois group, and $\Pi \rightarrow G_K$ a surjective homomorphism of profinite groups. Then there are only finitely many K -isomorphism classes of proper hyperbolic polycurves whose étale fundamental groups are isomorphic to Π over G_K .*

Remark 4.2. Sawada proved Corollary 4.1 in the case where K is a generalized sub- p -adic field. Moreover, he treated general (not necessarily proper) hyperbolic polycurves. We give another proof of Corollary 4.1 because we can prove Corollary 4.1 immediately by using Theorem 3.4 and [19, Theorem 1.3] under the assumptions of Corollary 4.1.

Proof. If the étale fundamental group of a proper hyperbolic polycurve over K is isomorphic to Π over G_K , its Euler-Poincaré characteristic coincides with $\chi(\text{Ker}(\Pi \rightarrow G_K), \mathbb{F}_2)$ by Remark 1.3. Therefore, by the last sentence of Definition 1.1.3, it suffices to show that there are only finitely many K -isomorphism classes of proper hyperbolic polycurves of dimension n whose

étale fundamental groups are isomorphic to Π over G_K for every natural number n (cf. Remark 4.3). Moreover, we may assume that there exists a proper hyperbolic polycurve X_K of dimension n over K whose étale fundamental groups are isomorphic to Π over G_K . Take a regular connected flat scheme S of finite type over $\text{Spec } \mathbb{Z}$ whose function field is isomorphic to K . Since we can replace S by its open dense subscheme, we may assume that there exists a proper hyperbolic polycurve $X \rightarrow S$ whose base change to K is isomorphic to $X_K \rightarrow \text{Spec } K$ by Theorem 2.2.2. Moreover, we replace S by its sufficiently small open dense subscheme so that we can apply [19, Theorem 1.3] in this situation for $l = 2$. Hence, another proper hyperbolic polycurve X' of dimension n over K whose étale fundamental group is isomorphic to Π over G_K , has good reduction at any point of S of codimension 1. By Theorem 1.11.1, X' extends to a proper hyperbolic polycurve over S . Hence, by Theorem 3.4, Corollary 4.1 holds. \square

Remark 4.3. In fact, as in the proof of [20], the dimension of a (proper) hyperbolic polycurve X over a field L of characteristic 0 is determined by the profinite group $\text{Ker}(\pi_1(X, *) \rightarrow G_L)$.

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