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Refinements of strong multiplicity one for GL(2)

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In memory of Professor Richard Guy

For distinct unitary cuspidal automorphic representations π_1 and π_2 for GL(2) over a number field F and any $\alpha \in \mathbb{R}$, let $\mathcal{S}_\alpha$ be the set of primes v of F for which $\lambda_{\pi_1}(v) \neq e^{i\alpha}\lambda_{\pi_2}(v)$, where $\lambda_{\pi_i}(v)$ is the Fourier coefficient of π_i at v. In this article, we show that the lower Dirichlet density of $\mathcal{S}_\alpha$ is at least $\frac{1}{16}$. Moreover, if π_1 and π_2 are not twist-equivalent, we show that the lower Dirichlet densities of $\mathcal{S}_\alpha$ and $\cap_\alpha \mathcal{S}_\alpha$ are at least $\frac{2}{13}$ and $\frac{1}{11}$, respectively. Furthermore, for non-twist-equivalent π_1 and π_2, if each π_i corresponds to a non-CM newform of weight $k_i \geq 2$ and with trivial nebentypus, we obtain various upper bounds for the number of primes $p \leq x$ such that $\lambda_{\pi_1}(p)^2 = \lambda_{\pi_2}(p)^2$. These present refinements of the works of Murty-Pujahari, Murty-Rajan, Ramakrishnan, and Walji.

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1. Introduction and statement of main results

Let F be a number field and $\mathbb{A}_F$ be its adèle ring. Let π_1 and π_2 be (unitary) cuspidal automorphic representations for $\mathrm{GL}_2(\mathbb{A}_F)$. For each i, at any unramified (finite) prime v of F for π_i, we denote the trace of the Langlands conjugacy class of π_i by $\lambda_{\pi_i}(v)$. There is a question of determining whether π_1 and π_2 are globally equivalent (i.e. $\pi_1 \simeq \pi_2$) from the local information on π_1 and π_2. For instance, given the set

$$\mathcal{S}_0 = \mathcal{S}_0(\pi_1, \pi_2) = \{v \text{ unramified for both } \pi_1 \text{ and } \pi_2 \mid \lambda_{\pi_1}(v) \neq \lambda_{\pi_2}(v)\},$$

what information on $\mathcal{S}_0$ would be sufficient for one to determine the global equivalence of π_1 and π_2? In [11], Jacquet and Shalika showed that if $\mathcal{S}_0$ is finite, then $\pi_1 \simeq \pi_2$ (which is often called the strong multiplicity one theorem). This was improved by Ramakrishnan [21], who showed that if $\mathcal{S}_0$ is of density less than $\frac{1}{8}$, then π_1 and π_2 are globally equivalent. By an example given by Serre [25] (see also [28, Sec. 4.4]), such a bound is attained by a pair of dihedral automorphic representations.[1]

Naturally, one may ask if the bound can be improved if dihedral automorphic representations are excluded. In [28], Walji gave an affirmative answer by proving that if π_1 and π_2 are distinct and non-dihedral, then $\underline{\delta}(\mathcal{S}_0) \geq \frac{1}{4}$, where $\underline{\delta}(\mathcal{S}_0)$ denotes the lower Dirichlet density of $\mathcal{S}_0$.[2]

The main result of this article is the following generalisation of the work of Ramakrishnan [21] and Walji [28].

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[1] A cuspidal automorphic representation π for $\mathrm{GL}_2(\mathbb{A}_F)$ is called dihedral if it admits a non-trivial self-twist, namely, there is a non-trivial Hecke character χ of F such that $\pi \otimes \chi \simeq \pi$. Gelbart and Jacquet [7] showed that $\mathrm{Ad}(\pi)$ is cuspidal if π is non-dihedral.

[2] We recall that the lower Dirichlet density $\underline{\delta}(A)$ of a subset A of the primes v of F is defined by

$$\underline{\delta}(A) = \liminf_{s \to 1^+} \frac{\sum_{v \in A} \frac{1}{Nv^s}}{\log(\frac{1}{s-1})},$$

where Nv is the norm of v.

Theorem 1.1. *Let* π_1 *and* π_2 *be distinct cuspidal automorphic representations for* $\mathrm{GL}_2(\mathbb{A}_F)$*. Given* $\alpha \in \mathbb{R}$*, let*

$$\mathcal{S}_\alpha = \mathcal{S}_\alpha(\pi_1, \pi_2) = \{v \textit{ unramified for both } \pi_1 \textit{ and } \pi_2 \mid \lambda_{\pi_1}(v) \neq e^{i\alpha}\lambda_{\pi_2}(v)\},$$

and let $\underline{\delta}(\mathcal{S}_\alpha)$ *denote the lower Dirichlet density of* $\mathcal{S}_\alpha$*. Then*

$$\underline{\delta}(\mathcal{S}_\alpha) \geq \begin{cases} \frac{1}{6+2\cos(2\alpha)} & \textit{if } \cos(2\alpha) \geq 0 \textit{ and } \cos\alpha \geq 0; \\ \frac{1}{6+2\cos(2\alpha)-8\cos\alpha} & \textit{if } \cos(2\alpha) \geq 0 \textit{ and } \cos\alpha \leq 0; \\ \frac{1}{6} & \textit{if } \cos(2\alpha) \leq 0 \textit{ and } \cos\alpha \geq 0; \\ \frac{1}{6-8\cos\alpha} & \textit{if } \cos(2\alpha) \leq 0 \textit{ and } \cos\alpha \leq 0. \end{cases}$$

Moreover, if both π_1 *and* π_2 *are non-dihedral, then*

$$\underline{\delta}(\mathcal{S}_\alpha) \geq \begin{cases} \min\left\{\frac{1}{3+\cos(2\alpha)}, \frac{1}{3+\cos(2\alpha)-2\kappa_1\cos\alpha}\right\} & \textit{if } \cos(2\alpha) \geq 0; \\ \min\left\{\frac{1}{3}, \frac{1}{3-2\kappa_1\cos\alpha}\right\} & \textit{if } \cos(2\alpha) \leq 0, \end{cases}$$

where κ_1 *is 1 if* $\pi_1 \simeq \pi_2 \otimes \nu$ *for some cubic Hecke character* ν *and 0 otherwise.*

Remark. Let $n \geq 3$, and let π_1 and π_2 be distinct cuspidal automorphic representations for $\mathrm{GL}_n(\mathbb{A}_F)$, satisfying the Ramanujan-Petersson conjecture. It can be shown that $\underline{\delta}(\mathcal{S}_\alpha) \geq \frac{1}{2n^2}$. In addition, if both $\mathrm{Ad}(\pi_1)$ and $\mathrm{Ad}(\pi_2)$ are cuspidal, then $\underline{\delta}(\mathcal{S}_\alpha) \geq \frac{1}{8}$. (See the final remark in Section 5 for more details.) Also, as pointed out by the referee, Walji [30] obtained $\underline{\delta}(\mathcal{S}_0) \geq \frac{1}{8}$ and, assuming further that $\mathrm{Ad}(\pi_1)$ and $\mathrm{Ad}(\pi_2)$ are distinct, $\underline{\delta}(\mathcal{S}_0) \geq \frac{1}{3+2\sqrt{2}}$.

We shall note that our interest in this theorem was motivated by the following theorem of Ramakrishnan [22, Corollary] and [23, Corollary 4.1.3], which uses the information on $\mathcal{S}_0 \cap \mathcal{S}_\pi$ to determine whether two given GL(2)-forms are twist-equivalent.

Theorem 1.2 (Ramakrishnan). *Let* π_1 *and* π_2 *be cuspidal automorphic representations for* $\mathrm{GL}_2(\mathbb{A}_F)$*. If* π_1 *and* π_2 *are with trivial central character, and*

$$\lambda_{\pi_1}(v)^2 = \lambda_{\pi_2}(v)^2$$

for all unramified primes (for both π_1 *and* π_2*), then* π_1 *and* π_2 *are twist-equivalent.*

Moreover, if π_1 *and* π_2 *correspond to holomorphic newforms (over* $\mathbb{Q}$*) of same weight and with same nebentypus, and*

$$\lambda_{\pi_1}(p)^2 = \lambda_{\pi_2}(p)^2$$

outside a set S *of primes* p *of density less than* $\frac{1}{18}$*, then* π_1 *and* π_2 *are twist-equivalent. Furthermore, if both* π_1 *and* π_2 *are non-dihedral, then one has the same result assuming only that the density of* S *is less than 1.*

In light of this theorem, one may also ask a question of determining whether π_1 and π_2 are twist-equivalent from the information on a single $\mathcal{S}_\alpha$. For $\alpha = 0$, Walji [28] proved the following theorem.

Theorem 1.3 (Walji). *Let* π_1 *and* π_2 *be non-twist-equivalent cuspidal automorphic representations for* $\mathrm{GL}_2(\mathbb{A}_F)$ *with unitary central characters. Then*

(i) *if both* π_1 *and* π_2 *are dihedral, then* $\underline{\delta}(\mathcal{S}_0) \geq \frac{2}{9}$*;*

(ii) *if exactly one of* π_1 *and* π_2 *is non-dihedral, then* $\underline{\delta}(\mathcal{S}_0) \geq \frac{2}{7}$*;*

(iii) *if both* π_1 *and* π_2 *are non-dihedral, then* $\underline{\delta}(\mathcal{S}_0) \geq \frac{2}{5}$*.*

The second objective of this article is to prove the following "rotation variant" of the work of Walji, Theorem 1.3.

Theorem 1.4. *Let* $\alpha \in \mathbb{R}$*, and let* π_1 *and* π_2 *be non-twist-equivalent cuspidal automorphic representations for* $\mathrm{GL}_2(\mathbb{A}_F)$ *with unitary central characters* ω_1 *and* ω_2*, respectively. Then*

(i) *if both* π_1 *and* π_2 *are dihedral, then* $\underline{\delta}(\mathcal{S}_\alpha) \geq \min\{d_\alpha, \frac{1}{4}\}$*, where*

$$d_\alpha = \begin{cases} \frac{2}{7+2\cos(2\alpha)} & \text{if } \cos(2\alpha) \geq 0 \text{ and } \cos\alpha \geq 0; \\ \frac{2}{7+2\cos(2\alpha)-4\cos\alpha} & \text{if } \cos(2\alpha) \geq 0 \text{ and } \cos\alpha \leq 0; \\ \frac{2}{7} & \text{if } \cos(2\alpha) \leq 0 \text{ and } \cos\alpha \geq 0; \\ \frac{2}{7-4\cos\alpha} & \text{if } \cos(2\alpha) \leq 0 \text{ and } \cos\alpha \leq 0; \end{cases}$$

(ii) *if exactly one of* π_1 *and* π_2 *is non-dihedral, then*

$$\underline{\delta}(\mathcal{S}_\alpha) \geq \begin{cases} \frac{2}{5+2\cos(2\alpha)} & \text{if } \cos(2\alpha) \geq 0; \\ \frac{2}{5} & \text{if } \cos(2\alpha) \leq 0; \end{cases}$$

(iii) *if both* π_1 *and* π_2 *are non-dihedral, then* $\underline{\delta}(\mathcal{S}_\alpha) \geq \frac{2}{4+\kappa_2\cos(2\alpha)}$, *where* κ_2 *is 1 if* $\omega_1 = \omega_2$ *and 0 otherwise.*

Furthermore, we have the following refinement of Theorem 1.2.

Theorem 1.5. *Let* π_1 *and* π_2 *be non-twist-equivalent cuspidal automorphic representations for* $\mathrm{GL}_2(\mathbb{A}_F)$ *with unitary central characters, and let*

$$\mathcal{S}_* = \mathcal{S}_*(\pi_1, \pi_2) = \{v \text{ unramified for both } \pi_1 \text{ and } \pi_2 \mid |\lambda_{\pi_1}(v)| \neq |\lambda_{\pi_2}(v)|\}.$$

Then

$$\underline{\delta}(\mathcal{S}_*) \geq \begin{cases} \frac{1}{8} & \text{if both } \pi_1 \text{ and } \pi_2 \text{ are dihedral;} \\ \frac{1}{9.58} & \text{if exactly one of } \pi_1 \text{ and } \pi_2 \text{ is dihedral;} \\ \frac{1}{10.76} & \text{if both } \pi_1 \text{ and } \pi_2 \text{ are non-dihedral.} \end{cases}$$

Consequently, we have the following refined version of [23, Theorem 4.1.2] asserting that if the adjoint lifts of π_1 and π_2 agree at almost all primes, then π_1 and π_2 are twist-equivalent.

Theorem 1.6. *Let* $\pi_1 = \otimes_v' \pi_{1,v}$ *and* $\pi_2 = \otimes_v' \pi_{2,v}$ *be cuspidal automorphic representations for* $\mathrm{GL}_2(\mathbb{A}_F)$ *with unitary central characters. If*

$$\mathrm{Ad}(\pi_{1,v}) \simeq \mathrm{Ad}(\pi_{2,v})$$

outside a set of lower Dirichlet density less than $\frac{1}{10.76}$, *then*

$$\pi_2 \simeq \pi_1 \otimes \chi$$

for some idèle class character.

Proof of Theorem 1.6. Toward a contradiction, suppose that π_1 and π_2 are not twist-equivalent. By Theorem 1.5, we know that $\underline{\delta}(\mathcal{S}_*) \geq \frac{1}{10.76}$. However, from our assumption of the theorem, it follows that $\lambda_{\mathrm{Ad}(\pi_1)}(v) = \lambda_{\mathrm{Ad}(\pi_2)}(v)$ outside a set of lower Dirichlet density less than $\frac{1}{10.76}$. In other words, the lower Dirichlet density of the set of primes v for which $\lambda_{\mathrm{Ad}(\pi_1)}(v) \neq \lambda_{\mathrm{Ad}(\pi_2)}(v)$ is less than $\frac{1}{10.76}$. This, together with the fact that $|\lambda_{\pi_i}(v)|^2 = \lambda_{\mathrm{Ad}(\pi_i)}(v) + 1$ (for unramified v), leads to

$$\frac{1}{10.76} \leq \underline{\delta}(\mathcal{S}_*) = \underline{\delta}(\{v \mid \lambda_{\mathrm{Ad}(\pi_1)}(v) \neq \lambda_{\mathrm{Ad}(\pi_2)}(v)\}) < \frac{1}{10.76},$$

a contradiction. □

We also have the following interesting variant.

Theorem 1.7. *Let π_1 and π_2 be non-twist-equivalent cuspidal automorphic representations for $\mathrm{GL}_2(\mathbb{A}_F)$ with unitary central characters, and let*

$$\begin{aligned}\mathcal{S}^* &= \mathcal{S}^*(\pi_1,\pi_2)\\ &= \{v \text{ unramified for both } \pi_1 \text{ and } \pi_2 \mid |\lambda_{\pi_1}(v)|^2 + |\lambda_{\pi_2}(v)|^2 \neq 2\}.\end{aligned}$$

Then

$$\underline{\delta}(\mathcal{S}^*) \geq \begin{cases} \frac{1}{18} & \text{if } \pi_1 \text{ and } \pi_2 \text{ are simultaneously dihedral or non-dihedral;}\\ \frac{1}{12} & \text{if exactly one of } \pi_1 \text{ and } \pi_2 \text{ is dihedral.}\end{cases}$$

Remark. (i) As remarked in [23], if π_1 and π_2 correspond to Hilbert newforms, one can establish a version of Theorem 1.2 by invoking the ℓ-adic representations associated to π_1 and π_2 as done in [22].

(ii) In the case that $\pi_1 \simeq \pi_2 \otimes \chi$ for some idèle class character χ, as $\pi_1 \boxtimes \bar{\pi}_1 \simeq \pi_2 \boxtimes \bar{\pi}_2$, $|\lambda_{\pi_1}(v)|^2 = |\lambda_{\pi_2}(v)|^2$ for any unramified v, and thus $\underline{\delta}(\mathcal{S}_*) = 0$. Hence, to have positive $\underline{\delta}(\mathcal{S}_*)$ in Theorem 1.5, the non-twist-equivalence condition is necessary.

Remark. Our method is an adaption of the work of Walji [28], which subtly relies on the L-functions associated to π_1 and π_2. The crucial ingredients are the automorphy of the adjoint lift from GL(2) to GL(3) (due to Gelbart and Jacquet [7]) and the functoriality of the tensor product $\mathrm{GL}(2) \times \mathrm{GL}(2) \to \mathrm{GL}(4)$ (due to Ramakrishnan [23]). To prove Theorem 1.5, we will further require the automorphy of $\mathrm{Sym}^4 \pi$ and its cuspidality criterion established by Kim and Shahidi [13, 14].

There are other variants and refinements of the above-mentioned work of Ramakrishnan [22, 23]. For instance, when π_i is a cuspidal automorphic representation corresponding to a non-CM newform in $S_{k_i}^{\mathrm{new}}(\Gamma_0(q_i))$ with trivial nebentypus for each i, by Galois-theoretic techniques, Rajan [20, Corollary 1] showed that if

$$\limsup_{x\to\infty} \frac{\#\{p \leq x \mid \lambda_{\pi_1}(p)p^{\frac{k_1-1}{2}} = \lambda_{\pi_2}(p)p^{\frac{k_2-1}{2}}\}}{\pi(x)} > 0,$$

then π_1 and π_2 are twist-equivalent. Also, in [17], Murty and Pujahari showed that if

$$\limsup_{x\to\infty} \frac{\#\{p \leq x \mid \lambda_{\pi_1}(p) = \lambda_{\pi_2}(p)\}}{\pi(x)} > 0,$$

then π_1 and π_2 are twist-equivalent. In much the same spirit, we have the following theorem.

Theorem 1.8. *Let F be a totally real number field. For each i, let π_i be a cuspidal automorphic representation corresponding to a non-CM Hilbert newform of weights $k_{i,j} \geq 2$ (at all infinite primes v_j of F) and with trivial nebentypus. If*

$$\limsup_{x\to\infty} \frac{\#\{v \mid Nv \leq x, \;\; \lambda_{\pi_1}(v)^2 = \lambda_{\pi_2}(v)^2\}}{\pi_F(x)} > 0,$$

where $\pi_F(x)$ denotes the number of primes v of F such that $Nv \leq x$, then π_1 and π_2 are twist-equivalent.

It is also worth mentioning that when π_1 and π_2 correspond to non-CM newforms or non-dihedral Maaß forms, assuming the generalised Riemann hypothesis and certain analytic properties for Rankin-Selberg L-functions of $\mathrm{Sym}^{m_1}\pi_1$ and $\mathrm{Sym}^{m_2}\pi_2$, Murty and Rajan [18] showed that if π_1 and π_2 are not twist-equivalent, then

$$\#\{p \leq x \mid \lambda_{\pi_1}(p) = \lambda_{\pi_2}(p)\} \ll \frac{x^{5/6}(\log(Nx))^{1/3}}{(\log x)^{2/3}} \tag{1.1}$$

for some suitable constant N (depending on π_1 and π_2). As a consequence, if the number of primes $p \leq x$ for which $\lambda_{\pi_1}(p) = \lambda_{\pi_2}(p)$ is $\gg x^\theta$ for some $\theta > 5/6$, then π_1 and π_2 are twist-equivalent. In light of this and Theorems 1.2 and 1.8, it shall be reasonable to expect that $\pi_1 \simeq \pi_2 \otimes \chi$ for some Dirichlet character χ whenever $\lambda_{\pi_1}(p)^2 = \lambda_{f_2}(p)^2$ for certain "thin" sets of primes p (i.e., sets of *zero* upper density among all primes). We shall show that such an expectation holds, unconditionally, in certain instances as follows.

Theorem 1.9. *For each i, let π_i be a cuspidal automorphic representation corresponding to a non-CM newform in $S_{k_i}^{\mathrm{new}}(\Gamma_0(q_i))$ with trivial nebentypus. If*

$$\limsup_{x\to\infty} \frac{\#\{p \leq x \mid \lambda_{\pi_1}(p)^2 = \lambda_{\pi_2}(p)^2\}}{\pi(x)(\log\log\log x)^{1+\epsilon}/(\log\log x)^{1/2}} > 0$$

for some $\epsilon > 0$, then $\pi_1 \simeq \pi_2 \otimes \chi$ for some Dirichlet character χ.

Indeed, Theorem 1.9 follows from the following estimate.

Theorem 1.10. *In the notation of Theorems 1.9, if there is no Dirichlet character χ such that $\pi_1 \simeq \pi_2 \otimes \chi$, then*

$$\#\{p \le x \mid \lambda_{\pi_1}(p)^2 = \lambda_{\pi_2}(p)^2\} \ll \pi(x)\frac{\log(k_1q_1k_2q_2 \log\log x)}{\sqrt{\log\log x}}.$$

Proof of Theorem 1.9. Suppose that π_1 and π_2 are not twist-equivalent (i.e., there is no Dirichlet character such that $\pi_1 \simeq \pi_2 \otimes \chi$). Theorem 1.10 implies that

$$\frac{\#\{p \le x \mid \lambda_{\pi_1}(p)^2 = \lambda_{\pi_2}(p)^2\}}{\pi(x)(\log\log\log x)^{1+\epsilon}/(\log\log x)^{1/2}} \ll \frac{1}{(\log\log\log x)^{\epsilon}}$$

for any $\epsilon > 0$. Taking $\limsup_{x\to\infty}$ on both sides, we see that the assumption leads to $0 < 0$, a contradiction. □

Furthermore, assuming the generalised Riemann hypothesis, we have the following refined version of the above-mentioned works of Murty-Rajan (1.1) and Ramakrishnan [22, 23].

Theorem 1.11. *In the notation of Theorem 1.9, assume that π_1 and π_2 are not twist-equivalent. If for $(m_1, m_2) \in \{(n, n-2), (n-2, n), (n-2, n-2) \mid n \ge 3\}$, $L(s, \mathrm{Sym}^{m_1} \pi_1 \times \mathrm{Sym}^{m_2} \pi_2)$ satisfies the generalised Riemann hypothesis, then*

$$\#\{p \le x \mid \lambda_{\pi_1}(p)^2 = \lambda_{\pi_2}(p)^2\} \ll \frac{x^{5/6}(\log(k_1q_1k_2q_2x))^{1/3}}{(\log x)^{2/3}}.$$

Remark. (i) We shall note that in order to prove Theorems 1.10 and 1.11, we invoke the recent work of Newton and Thorne [19] (who proved that all the symmetric powers of cuspidal automorphic representations corresponding to non-CM newforms are *automorphic*).
(ii) With a little more effort, it is possible to extend Theorems 1.10 and 1.11 to non-CM Hilbert newforms under the assumption of the automorphy of the pertinent symmetric powers (and the generalised Riemann hypothesis for the extension of Theorem 1.11) by adapting the methods developed in [27, 31]. However, for the sake of conceptual clarity, we shall not do this here. Nonetheless, in Section 6, we will develop our argument in the setting of Hilbert newforms so that the corresponding results can be immediately derived once an effective version of Proposition 2.2 is established.
(iii) Similar to [17, 20] and Theorem 1.2, it is possible to prove a version of Theorems 1.8, 1.10, and 1.11 by only assuming π_1 corresponds to a newform without CM. We will discuss this in more detail in Section 7.

Remark. (i) Our method makes use of the Selberg polynomials (see Section 2.4), which is more elementary than the one used by Murty and Rajan [18] (who invoked the Erdős-Turán inequality to bound the "discrepancy" of certain sequences, associated to $\lambda_{\pi_1}(p)$ and $\lambda_{\pi_2}(p)$, in terms of exponential sums). Nonetheless, similar to the argument of [18], we apply the effective versions of the joint Sato-Tate distribution established by Thorner [27] and the author [31].
(ii) Compared to the argument used in [17], ours is slightly more complicated. Nevertheless, our argument yields better estimates. For instance, under the generalised Riemann hypothesis, using the argument of [17] would result in

$$\#\{p \leq x \mid \lambda_{\pi_1}(p)^2 = \lambda_{\pi_2}(p)^2\} \ll_{\pi_1,\pi_2} x^{7/8}/(\log x)^{1/2},$$

which is not as good as the one given in Theorem 1.11.

The rest of article is structured as follows. In the next section, we will discuss Walji's argument as well as collecting the relevant facts that will be used later. We will prove Theorems 1.1 and 1.4 in Sections 3 (for pairs of dihedral representations) and 4 (for the remaining cases); we will prove Theorems 1.5 and 1.7 in Section 5. The proofs of Theorems 1.8, 1.10, and 1.11 will be given in Section 6. In Section 7, we will discuss how to extend Theorem 1.8 to the case that one of the newforms is with CM.

2. Preliminaries

2.1. *L*-functions

We shall begin by reviewing automorphic L-functions, Rankin-Selberg L-functions, and their properties.

Let F be a number field, and let π be a cuspidal automorphic representation for $\mathrm{GL}_n(\mathbb{A}_F)$ with unitary central character, where, as later, $\mathbb{A}_F$ denotes the adèle ring of F. We define the (incomplete) automorphic L-function $L(s,\pi)$ attached to π by

$$L(s,\pi) = \prod_v \det(I_n - A_v(\pi)Nv^{-s})^{-1},$$

for $\mathfrak{Re}(s) > 1$, where the product is over unramified (finite) primes v for π, and $A_v(\pi)$ denotes the Langlands conjugacy class of π at v. By the work of Jacquet and Shalika [10], it is known that $L(s,\pi)$ is non-vanishing at $s = 1$

and with a possible simple pole at $s = 1$ that only appears if π is equivalent to the trivial idèle class character 1 of F.

Let π and π' be cuspidal automorphic representations, with unitary central characters, for $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{GL}_m(\mathbb{A}_F)$, respectively. We define the (incomplete) Rankin-Selberg L-function $L(s, \pi \times \pi')$ attached to π and π' by

$$L(s, \pi \times \pi') = \prod_v \det(I_{nm} - (A_v(\pi) \otimes A_v(\pi'))Nv^{-s})^{-1},$$

for $\mathfrak{Re}(s) > 1$, where the product is over unramified primes v for both π and π'. It can be shown that $L(s, \pi \times \pi')$ extends holomorphically to $\mathfrak{Re}(s) = 1$ except for a possible simple pole at $s = 1 - it$ that exists only if $\pi' \simeq \bar{\pi} \otimes |\cdot|^{it}$, where $\bar{\pi}$ is the dual of π. Moreover, Shahidi [26] showed that $L(s, \pi \times \pi')$ is non-vanishing on $\mathfrak{Re}(s) \geq 1$.

2.2. GL(2)-forms

We also collect some fact regarding cuspidal automorphic representations for $\mathrm{GL}_2(\mathbb{A}_F)$ that we will make use of repeatedly throughout our discussion.

For any cuspidal automorphic representation π for $\mathrm{GL}_2(\mathbb{A}_F)$ with unitary central character ω, by the work of Gelbart-Jacquet [7], the (automorphic) tensor product $\pi \boxtimes \bar{\pi}$ exists as an automorphic representation for $\mathrm{GL}_4(\mathbb{A}_F)$ and satisfies

$$\pi \boxtimes \bar{\pi} \simeq 1 \boxplus \mathrm{Ad}(\pi),$$

where $\boxplus$ denotes the (Langlands) isobaric sum, and $\mathrm{Ad}(\pi)$ is an automorphic representation for $\mathrm{GL}_3(\mathbb{A}_F)$. If π is non-dihedral, then $\mathrm{Ad}(\pi)$ is cuspidal. Also, one knows that $\mathrm{Ad}(\pi)$ satisfies the relation

$$\mathrm{Ad}(\pi) \simeq \mathrm{Sym}^2 \pi \otimes \omega^{-1},$$

where $\mathrm{Sym}^2 \pi$ is the symmetric square of π. (We shall discuss more properties of $\mathrm{Ad}(\pi)$ for dihedral representations π in Section 3.)

Now, let π_1 and π_2 be cuspidal automorphic representations for $\mathrm{GL}_2(\mathbb{A}_F)$ with unitary central characters. Ramakrishnan [23] proved that the tensor product $\pi_1 \boxtimes \pi_2$ exists as an automorphic representation for $\mathrm{GL}_4(\mathbb{A}_F)$; when both π_1 and π_2 are dihedral, one has the following cuspidality criterion of Ramakrishnan [23]:

$\pi_1 \boxtimes \pi_2$ *is cuspidal if and only if* π_1 *and* π_2 *cannot be induced from the same quadratic extension* K *of* F.

Lastly, we recall that from the work of Gelbart and Jacquet [7], it follows that if π_1 and π_2 are twist-equivalent (i.e., $\pi_1 \simeq \pi_2 \otimes \chi$ for some idèle class character χ), then

$$1 \boxplus \mathrm{Ad}(\pi_1) \simeq \pi_1 \boxtimes \bar{\pi}_1 \simeq \pi_2 \boxtimes \bar{\pi}_2 \simeq 1 \boxplus \mathrm{Ad}(\pi_2),$$

as $\chi\bar{\chi} \simeq 1$, which means that $\mathrm{Ad}(\pi_1) \simeq \mathrm{Ad}(\pi_2)$. Moreover, by [22, Theorem 4.1.2], the conserve is also true. Thus, we have the following twist-equivalence criterion:

π_1 and π_2 are twist-equivalent if and only if $\mathrm{Ad}(\pi_1) \simeq \mathrm{Ad}(\pi_2)$.

2.3. Walji's strategy [28]

Let F be a number field, and let π_1 and π_2 be cuspidal automorphic representations for $\mathrm{GL}_2(\mathbb{A}_F)$ with (unitary) central characters ω_1 and ω_2, respectively. As before, for each i, we denote the trace of the Langlands conjugacy class of π_i at unramified v by $\lambda_{\pi_i}(v)$. For any $\alpha \in \mathbb{R}$, we define

$$\mathcal{S}_\alpha = \mathcal{S}_\alpha(\pi_1, \pi_2) = \{v \text{ unramified for both } \pi_1 \text{ and } \pi_2 \mid \lambda_{\pi_1}(v) \neq e^{i\alpha}\lambda_{\pi_2}(v)\}.$$

We let $\underline{\delta}(\mathcal{S}_\alpha)$ denote the lower Dirichlet density of $\mathcal{S}_\alpha$ and let $\chi_{\mathcal{S}_\alpha}$ denote the indicator function of $\mathcal{S}_\alpha$.

Writing $a_v = \lambda_{\pi_1}(v)$ and $b_v = \lambda_{\pi_2}(v)$, in light of the argument used by Walji [28] (who considered the case that $\alpha = 0$), to study the lower bound of $\mathcal{S}_\alpha$, we shall apply the following consequence of the Cauchy-Schwarz inequality:

$$\begin{aligned}(2.1) \qquad \sum_v \frac{|a_v - e^{i\alpha}b_v|^2}{Nv^s} &= \sum_v \frac{|a_v - e^{i\alpha}b_v|^2 \chi_{S_\alpha}(v)}{Nv^s} \\ &\leq \Big(\sum_v \frac{|a_v - e^{i\alpha}b_v|^4}{Nv^s}\Big)^{\frac{1}{2}} \Big(\sum_{v\in S_\alpha} \frac{1}{Nv^s}\Big)^{\frac{1}{2}},\end{aligned}$$

where, as later, the sums are over unramified primes v for both π_1 and π_2. Recalling that

$$\underline{\delta}(S_\alpha) = \liminf_{s\to 1^+} \frac{\sum_{v\in S_\alpha} \frac{1}{Nv^s}}{\log(\frac{1}{s-1})},$$

to obtain a lower bound for $\underline{\delta}(\mathcal{S}_\alpha)$, we shall analyse the asymptotic behaviours of sums in (2.1) as real $s \to 1^+$.

From the identities $|a_v - e^{i\alpha}b_v|^2 = a_v\bar{a}_v - e^{-i\alpha}a_v\bar{b}_v - e^{i\alpha}\bar{a}_v b_v + b_v\bar{b}_v$ and

$$\begin{aligned}|a_v - e^{i\alpha}b_v|^4 = a_v^2\bar{a}_v^2 &+ e^{-2i\alpha}a_v^2\bar{b}_v^2 + e^{2i\alpha}\bar{a}_v^2 b_v^2 + b_v^2\bar{b}_v^2 + 4a_v\bar{a}_v b_v\bar{b}_v \\ &- 2e^{-i\alpha}a_v^2\bar{a}_v\bar{b}_v - 2e^{i\alpha}a_v\bar{a}_v^2 b_v \\ &- 2e^{-i\alpha}a_v b_v\bar{b}_v^2 - 2e^{i\alpha}\bar{a}_v b_v^2\bar{b}_v,\end{aligned} \tag{2.2}$$

it is not hard to see that to bound the sums in (2.1) as real $s \to 1^+$, it suffices to study the asymptotic behaviours of the Dirichlet series

$$\mathcal{D}(s; i, j, k, l) = \sum_v \frac{a_v^i\bar{a}_v^j b_v^k\bar{b}_v^l}{Nv^s}$$

as real $s \to 1^+$. Indeed, for example,

$$\log(L(s, \pi_1 \boxtimes \bar{\pi}_1 \times \pi_2 \boxtimes \bar{\pi}_2)) = \mathcal{D}(s; 1, 1, 1, 1) + O(1),$$

where the big-O term is contributed by prime powers v^k (we note that $\lambda_{\pi_1}(v^k)$ and $\lambda_{\pi_2}(v^k)$ can be controlled by the bound established by Blomer and Brumley [6]), and thus

$$\lim_{s\to 1^+} \frac{\mathcal{D}(s; 1, 1, 1, 1)}{\log(\frac{1}{s-1})} = \lim_{s\to 1^+} \frac{\log(L(s, \pi_1 \boxtimes \bar{\pi}_1 \times \pi_2 \boxtimes \bar{\pi}_2))}{\log(\frac{1}{s-1})} = \delta_{1,1,1,1},$$

where $\delta_{1,1,1,1}$ denotes the order of the pole of $L(s, \pi_1 \boxtimes \bar{\pi}_1 \times \pi_2 \boxtimes \bar{\pi}_2)$ at $s = 1$. Hence, to prove our main result, it is sufficient to bound the orders of the poles of L-functions involved at $s = 1$. We shall use this argument throughout Sections 3, 4, and 5.

To prove Theorem 1.4, we borrow the following estimates from [28, Sec. 2 and p. 4999].

Proposition 2.1. *In the notation as above, let π_1 and π_2 be distinct. For $0 \le i, j, k, l \le 2$, let*

$$\delta_{i,j,k,l} = \lim_{s\to 1^+} \frac{\mathcal{D}(s; i, j, k, l)}{\log(\frac{1}{s-1})}.$$

If π_1 is not dihedral but π_2 is dihedral, then one has

$$\delta_{i,j,k,l} \le \begin{cases} 0 & \text{if } (i,j,k,l) \in \{(2,1,0,1), (0,1,2,1), (1,2,1,0), (1,0,1,2)\}; \\ 1 & \text{if } (i,j,k,l) = (1,1,1,1); \\ 2 & \text{if } (i,j,k,l) \in \{(2,2,0,0), (2,0,0,2), (0,2,2,0)\}; \\ 4 & \text{if } (i,j,k,l) = (0,0,2,2). \end{cases} \tag{2.3}$$

If both π_1 and π_2 are non-dihedral and non-tetrahedral,[3] *then one has*

(2.4)
$$\delta_{i,j,k,l} \leq \begin{cases} 0 & \text{if } (i,j,k,l) \in \{(2,1,0,1),(0,1,2,1),(1,2,1,0),(1,0,1,2)\}; \\ 2 & \text{if } (i,j,k,l) \in \{(1,1,1,1),(2,2,0,0),(0,0,2,2),(2,0,0,2), \\ & \qquad (0,2,2,0)\}. \end{cases}$$

If both π_1 and π_2 are non-dihedral, and at least one of π_1 and π_2 is tetrahedral, then one has

(2.5)
$$\begin{cases} \delta_{i,j,k,l} = \kappa_1 & \text{if } (i,j,k,l) \in \{(2,1,0,1),(0,1,2,1),(1,2,1,0),(1,0,1,2)\}; \\ \delta_{i,j,k,l} \leq 2 & \text{otherwise}, \end{cases}$$

where κ_1 is 1 if $\pi_1 \simeq \pi_2 \otimes \nu$ for some cubic Hecke character ν and 0 otherwise.

If π_1 and π_2 are non-dihedral and non-twist-equivalent, then one has

(2.6)
$$\begin{cases} \delta_{i,j,k,l} = 0 & \text{if } (i,j,k,l) \in \{(2,1,0,1),(0,1,2,1),(1,2,1,0),(1,0,1,2)\}; \\ \delta_{i,j,k,l} \leq 1 & \text{if } (i,j,k,l) = (1,1,1,1); \\ \delta_{i,j,k,l} \leq 2 & \text{if } (i,j,k,l) \in \{(2,2,0,0),(0,0,2,2)\}; \\ \delta_{i,j,k,l} = \kappa_2 & \text{if } (i,j,k,l) \in \{(2,0,0,2),(0,2,2,0)\}, \end{cases}$$

where κ_2 is 1 if $\omega_1 = \omega_2$ and 0 otherwise.

Remark. Let π and π' be cuspidal automorphic representations, with unitary central characters, for $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{GL}_m(\mathbb{A}_F)$, respectively. By the theory of Rankin-Selberg L-functions, we know that the poles of $L(s, \pi \times \pi')$ and $L(s, \bar{\pi} \times \bar{\pi}')$ at $s = 1$ must have the same order. Consequently, the poles of $\mathcal{D}(s; 2,0,0,2)$ and $\mathcal{D}(s; 0,2,2,0)$ at $s = 1$ are of the same order, and thus $\delta_{2,0,0,2} = \delta_{0,2,2,0}$. Similarly, $\delta_{2,1,0,1} = \delta_{1,2,1,0}$ and $\delta_{0,1,2,1} = \delta_{1,0,1,2}$. We shall use this fact throughout our discussion.

2.4. Joint Sato-Tate distribution and Selberg polynomials

To adapt the strategies developed in Murty-Pujahari [17] and Murty-Rajan [18], we shall recall the joint Sato-Tate distribution and some basic properties of Selberg polynomials.

[3] A cuspidal automorphic representation π for $\mathrm{GL}_2(\mathbb{A}_F)$ is called tetrahedral if it is non-dihedral and its symmetric square $\mathrm{Sym}^2 \pi$ admits a non-trivial self-twist by a (cubic) Hecke character.

Let F be a totally real number field. For each i, let π_i be a cuspidal automorphic representation corresponding to a non-CM Hilbert newform of weights $k_{i,j} \geq 2$ (at all infinite primes v_j of F) and with trivial nebentypus. For each i, we write

$$\lambda_{\pi_i}(v) = 2\cos\theta_{i,v}$$

for some $\theta_{i,v} \in [0, \pi]$. (Recall that by the work of Blasius [5], the Ramanujan-Petersson conjecture holds for each π_i.)

We recall that the n-th Chebyshev polynomial $U_n(x)$ (of the second type) is defined by

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

(Note that $U_0(\cos\theta) \equiv 1$ and $U_1(\cos\theta) = 2\cos\theta$.)

We will require the following version of the joint Sato-Tate distribution, which is a consequence of the work of Barnet-Lamb *et al.* [2, 3] (see [31, Theorem 1.1 and Sec. 3] for more details).

Proposition 2.2 (Barnet-Lamb *et al.*). *In the notation as above, for any $m_1, m_2 \in \mathbb{N}$, one has*

$$\sum_{Nv \leq x} U_{m_1}(\cos\theta_{1,v}) U_{m_2}(\cos\theta_{2,v}) = o(\pi_F(x)),$$

where $\pi_F(x)$ denotes the number of primes v of F such that $Nv \leq x$.

We shall further require the following effective versions of the joint Sato-Tate distribution, proved by Thorner [27, Proposition 2.2] and the author [31].

Proposition 2.3 (Thorner). *For each i, let π_i be a cuspidal automorphic representation corresponding to a non-CM newform in $S_{k_i}^{\mathrm{new}}(\Gamma_0(q_i))$ with trivial nebentypus. Assume that all the symmetric powers $\mathrm{Sym}^{m_1} \pi_1$ and $\mathrm{Sym}^{m_2} \pi_2$ are automorphic. Then there exist positive constants c_1, c_2, c_3, c_4 and c_5 such that for any*

$$1 \leq m_1, m_2 \leq M \leq c_1 \sqrt{\log\log x} / \log(k_1 q_1 k_2 q_2 \log\log x),$$

one has

$$\sum_{p\le x} U_{m_1}(\cos\theta_{1,p})U_{m_2}(\cos\theta_{2,p}) \ll \pi(x)\exp\Big(\frac{-c_2\log x}{(k_1q_1k_2q_2M)^{c_3M^2}}\Big)$$
$$+\pi(x)(m_1m_2)^2\Big(x^{\frac{-1}{c_4M^2}}+\exp\Big(\frac{-c_5\log x}{M^2\log(k_1q_1k_2q_2M)}\Big)+\exp\Big(\frac{-c_5\sqrt{\log x}}{M}\Big)\Big).$$

Proposition 2.4. *[31, p. 287] In the notation of Proposition 2.3, assume that the symmetric powers* $\mathrm{Sym}^{m_1}\pi_1$ *and* $\mathrm{Sym}^{m_2}\pi_2$ *are automorphic. If the Rankin-Selberg L-function* $L(s,\mathrm{Sym}^{m_1}\pi_1\times\mathrm{Sym}^{m_2}\pi_2)$ *satisfies the generalised Riemann hypothesis, then*

$$\sum_{p\le x} U_{m_1}(\cos\theta_{1,p})U_{m_2}(\cos\theta_{2,p}) \ll m_1m_2x^{1/2}\log((k_1q_1k_2q_2)(m_1+m_2)x).$$

Remark. We shall note that the automorphy assumption of $\mathrm{Sym}^{m_1}\pi_1$ and $\mathrm{Sym}^{m_2}\pi_2$ in Propositions 2.3 and 2.4 can be removed by invoking the recent work of Newton and Thorne [19].

To end this section, we review some basic properties of Selberg polynomials. We begin by recalling that for any integer $M\ge 1$, the Vaaler polynomial $V_M(x)$ is defined by

$$V_M(x)=\frac{1}{M+1}\sum_{k=1}^{M}\Big(\frac{k}{M+1}-\frac12\Big)\Delta_{M+1}\Big(x-\frac{k}{M+1}\Big)$$
$$+\frac{1}{2\pi(M+1)}\sin(2\pi(M+1)x)-\frac{1}{2\pi}\Delta_{M+1}(x)\sin(2\pi x),$$

where $\Delta_M(x)=\frac{1}{M}(\frac{\sin(\pi Mx)}{\sin(\pi x)})^2$ is the Fejér kernel (see, e.g., [15, Sec. 1.2, Eq. (16) and (17)]). Following [15, Sec. 1.2, Eq. (21$^+$)], for any subinterval $J=[0,\delta]\subseteq[0,1]$ and integer $M\ge 1$, we define the Selberg polynomial

$$S^+_{J,M}(x)=\delta+B_M(x-\delta)+B_M(-x), \tag{2.7}$$

where $B_M(x)$ is the Beurling polynomial as defined in [15, Sec. 1.2, Eq. (20)], namely,

$$B_M(x)=V_M(x)+\frac{1}{2(M+1)}\Delta_{M+1}(x).$$

We recall that

$$\chi_J(x)\le S^+_{J,M}(x),$$

where χ_J is the indicator function of J. (In what follows, we shall regard both $S^+_{J,M}(x)$ and $\chi_J(x)$ as functions of period 1 defined over $\mathbb{R}$.) Moreover, writing the Fourier expansion of $S^+_{J,M}(x)$ as

$$S^+_{J,M}(x) = \sum_{n=-\infty}^{\infty} \hat{S}^+_{J,M}(n) e^{2\pi i n x},$$

we know that

$$\hat{S}^+_{J,M}(n) = \begin{cases} \delta + \frac{1}{M+1} & \text{if } n = 0; \\ 0 & \text{if } |n| > M; \end{cases}$$

also, for $1 \le |n| \le M$,

$$|\hat{S}^+_{J,M}(n)| \le \frac{1}{M+1} + \min\Big\{\delta, \frac{1}{\pi|n|}\Big\}$$

(see [15, pp. 6–8]). From the definition of Fourier transform, it follows directly that

$$\hat{S}^+_{J,M}(n) + \hat{S}^+_{J,M}(-n) = 2\mathfrak{Re}(\hat{S}^+_{J,M}(n)),$$

and thus

$$\begin{aligned} S^+_{J,M}(x) + S^+_{J,M}(-x) &= 2\delta + \frac{2}{M+1} + 2\sum_{0<|n|\le M} \hat{S}^+_{J,M}(n)\cos(2\pi n x) \\ &= 2\delta + \frac{2}{M+1} + 4\sum_{n=1}^{M} \mathfrak{Re}(\hat{S}^+_{J,M}(n))\cos(2\pi n x). \end{aligned}$$

To summarise, we have the following proposition.

Proposition 2.5. *In the notation as above, for any subinterval $J = [0, \delta] \subseteq [0, 1]$ and integer $M \ge 1$, one has*

$$\begin{aligned} \mathcal{I}_\delta(\theta) &:= \frac{1}{2}\Big(\chi_J\Big(\frac{\theta}{2\pi}\Big) + \chi_J\Big(-\frac{\theta}{2\pi}\Big)\Big) \\ &\le \delta + \frac{1}{M+1} + 2\sum_{n=1}^{M} \mathfrak{Re}(\hat{S}^+_{J,M}(n))\cos(n\theta), \end{aligned}$$

where for $1 \le n \le M$,

$$|\mathfrak{Re}(\hat{S}^+_{J,M}(n))| \le \frac{1}{M+1} + \min\Big\{\delta, \frac{1}{\pi n}\Big\}.$$

3. Proofs of Theorems 1.1 and 1.4, part I: both π_1 and π_2 are dihedral

In this section, we shall prove Theorems 1.1 and 1.4 for the case that both π_1 and π_2 are dihedral. We will emphasise the case that π_1 and π_2 are twist-equivalent. (The case that $\alpha = 0$ was not treated in [28] as the bound established by Ramakrishnan [21] is already sharp.)

Recall that every dihedral cuspidal automorphic representation π for $\mathrm{GL}_2(\mathbb{A}_F)$ can be induced from a Hecke character ψ of K for some quadratic extension K of F. For such an instance, we shall write $\pi = I_K^F(\psi)$. Following Walji [28, p. 4995], we say that a dihedral cuspidal automorphic representation π for $\mathrm{GL}_2(\mathbb{A}_F)$ has property $\mathcal{P}$ if $\pi = I_K^F(\psi)$, and the Hecke character ψ/ψ^τ is invariant under τ, the non-trivial element of $\mathrm{Gal}(K/F)$. We also recall that $\pi_1 \boxtimes \pi_2$ is cuspidal if and only if π_1 and π_2 cannot be induced from the same quadratic extension of F.

In this section, we let π_1 and π_2 be dihedral cuspidal automorphic representations for $\mathrm{GL}_2(\mathbb{A}_F)$ induced from ν and μ of quadratic extensions K_1 and K_2 of F, respectively. In addition, we let χ_i be the Hecke character associated to K_i/F and let τ_i be the non-trivial element of $\mathrm{Gal}(K_i/F)$. Besides, if π_1 and π_2 can be induced from the same quadratic extension K of F, then we shall choose $K_1 = K_2 = K$ and define $\chi = \chi_1 = \chi_2$ and $\tau = \tau_1 = \tau_2$.

We shall argue according to whether π_i has property $\mathcal{P}$ for each i.

3.1. Exactly one of π_1 and π_2 has property $\mathcal{P}$

Assume that exactly one of the dihedral automorphic representations π_1 and π_2 satisfies property $\mathcal{P}$. Without loss of generality, we consider the case that $\pi_1 = I_K^F(\nu)$ has property $\mathcal{P}$ (i.e., $(\nu/\nu^{\tau_1})^{\tau_1} = \nu/\nu^{\tau_1}$, where τ_1 is the non-trivial element of $\mathrm{Gal}(K_1/F)$).

3.1.1. π_1 and π_2 cannot be induced from the same quadratic extension of F. We first consider that case that π_1 and π_2 cannot be induced from the same quadratic extension of F. (This case was not discussed in [28, p. 4998]. Nonetheless, the bound for this case is better than the one for the case that π_1 and π_2 can be induced from the same quadratic extension.)

Since π_1 has property $\mathcal{P}$, we have

$$\pi_1 \boxtimes \bar{\pi}_1 \simeq 1 \boxplus \chi_1 \boxplus \nu/\nu^{\tau_1} \boxplus (\nu/\nu^{\tau_1})\chi_1, \tag{3.1}$$

where χ_1 is the Hecke character associated to K_1/F. Also, as $\pi_1 \boxtimes \bar{\pi}_1 \simeq 1 \boxplus \mathrm{Ad}(\pi_1)$, we see that

$$\mathrm{Ad}(\pi_1) \simeq \chi_1 \boxplus \nu/\nu^{\tau_1} \boxplus (\nu/\nu^{\tau_1})\chi_1. \tag{3.2}$$

On the other hand, since π_2 does not have property $\mathcal{P}$,

$$\pi_2 \boxtimes \bar{\pi}_2 \simeq 1 \boxplus \chi_2 \boxplus I_{K_2}^F(\mu/\mu^{\tau_2}), \tag{3.3}$$

where χ_2 is the Hecke character associated to K_2/F, and $I_{K_2}^F(\mu/\mu^{\tau_2})$ is cuspidal. Thus,

$$\mathrm{Ad}(\pi_2) \simeq \chi_2 \boxplus I_{K_2}^F(\mu/\mu^{\tau_2}). \tag{3.4}$$

Since each $\mathrm{Ad}(\pi_i)$ is self-dual, we know that $L(s, \mathrm{Ad}(\pi_1) \times \mathrm{Ad}(\pi_1))$ has a pole of order three at $s = 1$ and that $L(s, \mathrm{Ad}(\pi_2) \times \mathrm{Ad}(\pi_2))$ has a pole of order two at $s = 1$. (Note that $\nu/\nu^{\tau_1} \not\simeq \chi_1$; otherwise, $\pi_1 \simeq \nu \boxplus \nu\chi_1$, which contradicts to the cuspidality of π_1.) Moreover, in this case, $\pi_1 \boxtimes \pi_2$ is cuspidal. Thus, from the identity

$$\begin{aligned} L(s, (1 \boxplus \mathrm{Ad}(\pi_1)) \times (1 \boxplus \mathrm{Ad}(\pi_2))) &= L(s, \pi_1 \boxtimes \bar{\pi}_1 \times \pi_2 \boxtimes \bar{\pi}_2) \\ &= L(s, \pi_1 \boxtimes \pi_2 \times \bar{\pi}_1 \boxtimes \bar{\pi}_2), \end{aligned}$$

we see that $L(s, \mathrm{Ad}(\pi_1) \times \mathrm{Ad}(\pi_2))$ is holomorphic at $s = 1$.

Since the Ramanujan-Petersson conjecture holds for π_1 and π_2 (as they are dihedral), we have $|\lambda_{\pi_i}(v)|^2 \le 4$. Recalling that for each i,

$$\lambda_{\mathrm{Ad}(\pi_i)}(v) = |\lambda_{\pi_i}(v)|^2 - 1, \tag{3.5}$$

we derive

$$|\lambda_{\mathrm{Ad}(\pi_1)}(v) - \lambda_{\mathrm{Ad}(\pi_2)}(v)| = ||\lambda_{\pi_1}(v)|^2 - 1 - |\lambda_{\pi_2}(v)|^2 + 1| \le 4$$

and thus

$$\begin{aligned} \sum_v \frac{|\lambda_{\mathrm{Ad}(\pi_1)}(v) - \lambda_{\mathrm{Ad}(\pi_2)}(v)|^2}{Nv^s} &= \sum_v \frac{|\lambda_{\mathrm{Ad}(\pi_1)}(v) - \lambda_{\mathrm{Ad}(\pi_2)}(v)|^2 \chi_{\mathrm{Ad}}(v)}{Nv^s} \\ &\le 16 \sum_{v \in \mathcal{S}_{\mathrm{Ad}}} \frac{1}{Nv^s}, \end{aligned} \tag{3.6}$$

where

$$(3.7)\quad \mathcal{S}_{\mathrm{Ad}} = \mathcal{S}_{\mathrm{Ad}}(\pi_1,\pi_2) = \{v \text{ unramified for both } \pi_1 \text{ and } \pi_2 \mid \lambda_{\mathrm{Ad}(\pi_1)}(v) \neq \lambda_{\mathrm{Ad}(\pi_2)}(v)\}.$$

Note that as each $\mathrm{Ad}(\pi_i)$ is self-dual, for any unramified v, $|\lambda_{\mathrm{Ad}(\pi_1)}(v) - \lambda_{\mathrm{Ad}(\pi_2)}(v)|^2$ is equal to

$$\lambda_{\mathrm{Ad}(\pi_1)\times\mathrm{Ad}(\pi_1)}(v) - 2\lambda_{\mathrm{Ad}(\pi_1)\times\mathrm{Ad}(\pi_2)}(v) + \lambda_{\mathrm{Ad}(\pi_2)\times\mathrm{Ad}(\pi_2)}(v).$$

Now, dividing both sides of (3.6) by $\log(\frac{1}{s-1})$ and making real $s \to 1^+$, we deduce that $5 = 3 + 2 \leq 16\,\underline{\delta}(\mathcal{S}_{\mathrm{Ad}})$. By (3.5), we know that if $\lambda_{\mathrm{Ad}(\pi_1)}(v) \neq \lambda_{\mathrm{Ad}(\pi_2)}(v)$, then $|\lambda_{\pi_1}(v)| \neq |\lambda_{\pi_2}(v)|$. Thus, for any α, $\mathcal{S}_{\mathrm{Ad}} \subseteq \mathcal{S}_\alpha$, and we obtain

$$\frac{5}{16} \leq \underline{\delta}(\mathcal{S}_{\mathrm{Ad}}) \leq \underline{\delta}(\mathcal{S}_\alpha).$$

3.1.2. $\boldsymbol{\pi_1}$ and $\boldsymbol{\pi_2}$ can be induced from the same quadratic extension of $\boldsymbol{F}$. Suppose that π_1 and π_2 can be induced from the same quadratic extension K of F. As argued in [28, p. 4998], for any prime ω of K, we have $\nu(\omega) = \pm\nu^\tau(\omega)$, and thus for any prime v of F, $|\lambda_{\pi_1}(v)| = |\lambda_{I_K^F(\nu)}(v)|$ equals either 0 or 2.

Write $\pi_2 = I_K^F(\mu)$ as before. As $(\mu/\mu^\tau)^\tau \neq \mu/\mu^\tau$, $\mu^2/(\mu^\tau)^2$ is non-trivial, and thus there is a set S of density 1/4 consisting of primes v of F that split in K such that $\mu^2/(\mu^\tau)^2(\omega) \neq 1$ for $\omega \mid v$. Hence, if $\omega \mid v$ for some $v \in S$, then $\mu(\omega) \neq \pm\mu^\tau(\omega)$, which implies that $|\lambda_{\pi_2}(v)| = |\lambda_{I_K^F(\mu)}(v)|$ is not equal to 0 nor 2. Therefore, for any $v \in S$, $|\lambda_{\pi_1}(v)| \neq |\lambda_{\pi_2}(v)|$ and thus $|\lambda_{\mathrm{Ad}(\pi_1)}(v)| \neq |\lambda_{\mathrm{Ad}(\pi_2)}(v)|$ for any $v \in S$. In particular, for any $v \in S$ and any $\alpha \in \mathbb{R}$,

$$\lambda_{\pi_1}(v) \neq e^{i\alpha}\lambda_{\pi_2}(v).$$

From the above discussion, we see that

$$\frac{1}{4} \leq \underline{\delta}(S) = \underline{\delta}(\mathcal{S}_{\mathrm{Ad}}) \leq \underline{\delta}(\mathcal{S}_\alpha)$$

whenever exactly one of π_1 and π_2 satisfies property $\mathcal{P}$.

Remark. If we argue as in the previous section, as $L(s, \mathrm{Ad}(\pi_1) \times \mathrm{Ad}(\pi_2))$ has a pole of order one at $s = 1$, we will have a slightly weaker lower bound:

$$\underline{\delta}(\mathcal{S}_{\mathrm{Ad}}) \geq \frac{3 - 2 + 2}{16} = \frac{3}{16}$$

for the case that π_1 and π_2 can be induced from the same quadratic extension of F.

3.2. Both π_1 and π_2 do not have property $\mathcal{P}$

3.2.1. π_1 and π_2 are twist-equivalent. Assume that both π_1 and π_2 do not have property $\mathcal{P}$ and that π_1 and π_2 are twist-equivalent. Note that

$$\pi_1 \boxtimes \bar{\pi}_1 \simeq 1 \boxplus \chi_1 \boxplus I_{K_1}^F(\nu/\nu^{\tau_1}), \tag{3.8}$$

where χ_1 is the quadratic Hecke character associated to K_1/F, and $I_{K_1}^F(\nu/\nu^{\tau_1})$ is cuspidal. From (3.8), it follows that

$$\begin{aligned}
&L(s, \pi_1 \boxtimes \bar{\pi}_1 \times \pi_1 \boxtimes \bar{\pi}_1) \\
&\qquad = L(s, (1 \boxplus \chi_1 \boxplus I_{K_1}^F(\nu/\nu^{\tau_1})) \times \overline{(1 \boxplus \chi_1 \boxplus I_{K_1}^F(\nu/\nu^{\tau_1}))})
\end{aligned}$$

has a pole of order three at $s = 1$. Also, as $\mathrm{Ad}(\pi_1) \simeq \mathrm{Ad}(\pi_2)$, $\pi_2 \boxtimes \bar{\pi}_2 \simeq \pi_1 \boxtimes \bar{\pi}_1$, and thus

$$L(s, \pi_2 \boxtimes \bar{\pi}_2 \times \pi_2 \boxtimes \bar{\pi}_2) = L(s, \pi_1 \boxtimes \bar{\pi}_1 \times \pi_2 \boxtimes \bar{\pi}_2) = L(s, \pi_1 \boxtimes \bar{\pi}_1 \times \pi_1 \boxtimes \bar{\pi}_1)$$

has a pole of order three at $s = 1$.

Now, we consider the L-function $L(s, \pi_1 \boxtimes \pi_1 \times \bar{\pi}_1 \boxtimes \bar{\pi}_2)$. Writing $\pi_2 \simeq \pi_1 \otimes \chi$ for some (non-trivial) idèle class character χ, we deduce

$$\begin{aligned}
L(s, \pi_1 \boxtimes \pi_1 \times \bar{\pi}_1 \boxtimes \bar{\pi}_2) &= L(s, \pi_1 \boxtimes \bar{\pi}_1 \times \pi_1 \boxtimes (\bar{\pi}_1 \otimes \bar{\chi})) \\
&= L(s, (1 \boxplus \chi_1 \boxplus I_{K_1}^F(\nu/\nu^{\tau_1})) \times (\bar{\chi} \boxplus \chi_1\bar{\chi} \boxplus I_{K_1}^F(\nu/\nu^{\tau_1}) \otimes \bar{\chi}))
\end{aligned}$$

has a pole of order at most three at $s = 1$. Similarly, $L(s, \pi_1 \boxtimes \pi_1 \times \bar{\pi}_2 \boxtimes \bar{\pi}_2) = L(s, \pi_1 \boxtimes \bar{\pi}_1 \times \pi_1 \boxtimes (\bar{\pi}_1 \otimes \bar{\chi}^2))$ has a pole of order at most three at $s = 1$.

As $\delta_{i,j,k,l} \leq 3$ for all cases, we derive

$$2^2 \leq \begin{cases}
(3 + 6\cos(2\alpha) + 3 + 12)\, \underline{\delta}(\mathcal{S}_\alpha) & \text{if } \cos(2\alpha) \geq 0 \text{ and } \cos\alpha \geq 0; \\
(3 + 6\cos(2\alpha) + 3 + 12 - 24\cos\alpha)\, \underline{\delta}(\mathcal{S}_\alpha) & \text{if } \cos(2\alpha) \geq 0 \text{ and } \cos\alpha \leq 0; \\
(3 + 3 + 12)\, \underline{\delta}(\mathcal{S}_\alpha) & \text{if } \cos(2\alpha) \leq 0 \text{ and } \cos\alpha \geq 0; \\
(3 + 3 + 12 - 24\cos\alpha)\, \underline{\delta}(\mathcal{S}_\alpha) & \text{if } \cos(2\alpha) \leq 0 \text{ and } \cos\alpha \leq 0.
\end{cases}$$

3.2.2. π_1 and π_2 are not twist-equivalent. Now, we consider the situation that π_1 and π_2 do not have property $\mathcal{P}$, and π_1 and π_2 are not twist-equivalent. Similar to (3.4), by (3.8), we have

$$\mathrm{Ad}(\pi_1) \simeq \chi_1 \boxplus I_{K_1}^F(\nu/\nu^{\tau_1}). \tag{3.9}$$

Recalling that $I_{K_1}^F(\nu/\nu^{\tau_1})$ and $I_{K_2}^F(\mu/\mu^{\tau_2})$ are cuspidal, we note that as $\mathrm{Ad}(\pi_1) \not\simeq \mathrm{Ad}(\pi_2)$ and $\mathrm{Ad}(\pi_2)$ is self-dual, $\mathrm{Ad}(\pi_1) \not\simeq \overline{\mathrm{Ad}(\pi_2)}$ and thus either $\bar{\chi}_2 \not\simeq \chi_1$ or $\overline{I_{K_2}^F(\mu/\mu^{\tau_2})} \not\simeq I_{K_1}^F(\nu/\nu^{\tau_1})$. Hence, by (3.4) and (3.9), we see that

$$L(s, \mathrm{Ad}(\pi_1) \times \mathrm{Ad}(\pi_2)) = L(s, (\chi_1 \boxplus I_{K_1}^F(\nu/\nu^{\tau_1})) \times (\chi_2 \boxplus I_{K_2}^F(\mu/\mu^{\tau_2})))$$

has a pole of order at most one at $s = 1$. Also, from (3.4) and (3.9), it follows immediately that for each i, $L(s, \mathrm{Ad}(\pi_i) \times \mathrm{Ad}(\pi_i)) = L(s, \mathrm{Ad}(\pi_i) \times \overline{\mathrm{Ad}(\pi_i)})$ has a pole of order two at $s = 1$. Therefore, by the estimate (3.6), we obtain

$$2 = 2 - 2 + 2 \le 16\, \underline{\delta}(\mathcal{S}_{\mathrm{Ad}}),$$

where $\mathcal{S}_{\mathrm{Ad}}$ is defined as in (3.7). Since for any α, $\mathcal{S}_{\mathrm{Ad}} \subseteq \mathcal{S}_\alpha$, we obtain

$$\frac{1}{8} \le \underline{\delta}(\mathcal{S}_{\mathrm{Ad}}) \le \underline{\delta}(\mathcal{S}_\alpha).$$

This can be further improved for $\mathcal{S}_\alpha$ by Walji's argument. It was shown in [28, pp. 4995–4996] that

$$\delta_{i,j,k,l} \le \begin{cases} 1 & \text{if } (i,j,k,l) \in \{(2,1,0,1), (0,1,2,1), (1,2,1,0), (1,0,1,2)\}; \\ 2 & \text{if } (i,j,k,l) \in \{(1,1,1,1), (2,0,0,2), (0,2,2,0)\}; \\ 3 & \text{if } (i,j,k,l) \in \{(2,2,0,0), (0,0,2,2)\}. \end{cases}$$

Hence, we have

$$2^2 \le \begin{cases} (3 + 4\cos(2\alpha) + 3 + 8)\, \underline{\delta}(\mathcal{S}_\alpha) & \text{if } \cos(2\alpha) \ge 0 \text{ and } \cos\alpha \ge 0; \\ (3 + 4\cos(2\alpha) + 3 + 8 - 8\cos\alpha)\, \underline{\delta}(\mathcal{S}_\alpha) & \text{if } \cos(2\alpha) \ge 0 \text{ and } \cos\alpha \le 0; \\ (3 + 3 + 8)\, \underline{\delta}(\mathcal{S}_\alpha) & \text{if } \cos(2\alpha) \le 0 \text{ and } \cos\alpha \ge 0; \\ (3 + 3 + 8 - 8\cos\alpha)\, \underline{\delta}(\mathcal{S}_\alpha) & \text{if } \cos(2\alpha) \le 0 \text{ and } \cos\alpha \le 0. \end{cases}$$

Thus, for any $\alpha \in \mathbb{R}$, $\underline{\delta}(\mathcal{S}_\alpha) \ge \frac{2}{13}$ (which is $> \frac{1}{8}$). (We note that although our method gives a worse bound than the one given by Walji's argument, it will be useful in Section 5.)

3.3. Both π_1 and π_2 have property $\mathcal{P}$

3.3.1. π_1 and π_2 are twist-equivalent. We move on to the case of twist-equivalent π_1 and π_2 that satisfy property $\mathcal{P}$. We recall that in Section 3.1.1, we showed that

$$1 \boxplus \mathrm{Ad}(\pi_1) \simeq \pi_1 \boxtimes \bar{\pi}_1 \simeq 1 \boxplus \chi_1 \boxplus \nu/\nu^{\tau_1} \boxtimes (\nu/\nu^{\tau_1})\chi_1$$

and that $L(s, \mathrm{Ad}(\pi_1) \times \mathrm{Ad}(\pi_1))$ has a pole of order three at $s = 1$. Thus,

$$L(s, \pi_1 \boxtimes \bar{\pi}_1 \times \pi_1 \boxtimes \bar{\pi}_1) = L(s, 1)L(s, \mathrm{Ad}(\pi_1))^2 L(s, \mathrm{Ad}(\pi_1) \times \mathrm{Ad}(\pi_1))$$

has a pole of order four at $s = 1$. Since $\pi_1 \simeq \pi_2 \otimes \chi$ for some (non-trivial) Hecke character, we know that

$$L(s, \pi_1 \boxtimes \bar{\pi}_1 \times \pi_1 \boxtimes \bar{\pi}_1) = L(s, \pi_1 \boxtimes \bar{\pi}_1 \times \pi_2 \boxtimes \bar{\pi}_2) = L(s, \pi_2 \boxtimes \bar{\pi}_2 \times \pi_2 \boxtimes \bar{\pi}_2)$$

and thus $\delta_{i,j,k,l} = 4$ if $(i, j, k, l) \in \{(1, 1, 1, 1), (2, 2, 0, 0), (0, 0, 2, 2)\}$.

Moreover, $L(s, \pi_1 \boxtimes \pi_1 \times \bar{\pi}_2 \boxtimes \bar{\pi}_2) = L(s, \pi_1 \boxtimes (\bar{\pi}_1 \otimes \bar{\chi}) \times \pi_1 \boxtimes (\bar{\pi}_1 \otimes \bar{\chi}))$ equals

$$\begin{aligned} L(s, &(\bar{\chi} \boxplus \chi_1\bar{\chi} \boxplus (\nu/\nu^{\tau_1})\bar{\chi} \boxplus (\nu/\nu^{\tau_1})\chi_1\bar{\chi}) \\ &\times (\bar{\chi} \boxplus \chi_1\bar{\chi} \boxplus (\nu/\nu^{\tau_1})\bar{\chi} \boxplus (\nu/\nu^{\tau_1})\chi_1\bar{\chi})), \end{aligned}$$

which has a pole of order at most four at $s = 1$ (note that $\bar{\chi}$, $\chi_1\bar{\chi}$, $(\nu/\nu^{\tau_1})\bar{\chi}$, and $(\nu/\nu^{\tau_1})\chi_1\bar{\chi}$ are pairwise distinct). Similarly, we know that $L(s, \pi_1 \boxtimes \pi_1 \times \bar{\pi}_1 \boxtimes \bar{\pi}_2) = L(s, \pi_1 \boxtimes \bar{\pi}_1 \times \pi_1 \boxtimes (\bar{\pi}_1 \otimes \bar{\chi}))$ is equal to

$$L(s, (1 \boxplus \chi_1 \boxplus \nu/\nu^{\tau_1} \boxplus (\nu/\nu^{\tau_1})\chi_1) \times (\bar{\chi} \boxplus \chi_1\bar{\chi} \boxplus (\nu/\nu^{\tau_1})\bar{\chi} \boxplus (\nu/\nu^{\tau_1})\chi_1\bar{\chi})),$$

which has a pole of order at most four at $s = 1$.

By the above discussion, $\delta_{i,j,k,l} \leq 4$ for all cases, and thus

$$2^2 \leq \begin{cases} (4 + 8\cos(2\alpha) + 4 + 16)\, \underline{\delta}(\mathcal{S}_\alpha) & \text{if } \cos(2\alpha) \geq 0 \text{ and } \cos\alpha \geq 0; \\ (4 + 8\cos(2\alpha) + 4 + 16 - 32\cos\alpha)\, \underline{\delta}(\mathcal{S}_\alpha) & \text{if } \cos(2\alpha) \geq 0 \text{ and } \cos\alpha \leq 0; \\ (4 + 4 + 16)\, \underline{\delta}(\mathcal{S}_\alpha) & \text{if } \cos(2\alpha) \leq 0 \text{ and } \cos\alpha \geq 0; \\ (4 + 4 + 16 - 32\cos\alpha)\, \underline{\delta}(\mathcal{S}_\alpha) & \text{if } \cos(2\alpha) \leq 0 \text{ and } \cos\alpha \leq 0. \end{cases}$$

It is evident that it gives a worse lower bound for $\underline{\delta}(\mathcal{S}_\alpha)$ than most cases given in Section 3.2 (except for Section 3.2.2, where $\frac{1}{8}$ appears).

3.3.2. π_1 and π_2 are not twist-equivalent. To end this section, we analyse the case of non-twist-equivalent π_1 and π_2 that satisfy property $\mathcal{P}$. We first recall that by [28, Lemma 6], Walji showed that both $L(s, \mathrm{Ad}(\pi_1) \times \mathrm{Ad}(\pi_1))$ and $L(s, \mathrm{Ad}(\pi_2) \times \mathrm{Ad}(\pi_2))$ have a pole of order three at $s = 1$; also, if π_1 and π_2 can be induced from K/F, $L(s, \mathrm{Ad}(\pi_1) \times \mathrm{Ad}(\pi_2))$ have a pole of order one at $s = 1$.

In addition, if π_1 and π_2 cannot be induced from the same quadratic extension, then $\pi_1 \boxtimes \pi_2$ and $\bar{\pi}_1 \boxtimes \bar{\pi}_2$ are cuspidal. So $L(s, \pi_1 \boxtimes \pi_2 \times \bar{\pi}_1 \boxtimes \bar{\pi}_2)$ has a pole of order one at $s = 1$. As argued in Section 3.1.1, we deduce that $L(s, \mathrm{Ad}(\pi_1) \times \mathrm{Ad}(\pi_2))$ is holomorphic at $s = 1$. (We note that this case was not discussed in [28, pp. 4997–4998].)

Hence, from (3.6), it follows that if π_1 and π_2 are induced from the same quadratic extension of F,

$$\underline{\delta}(\mathcal{S}_{\mathrm{Ad}}) \geq \frac{3 - 2 + 3}{16} = \frac{1}{4},$$

where $\mathcal{S}_{\mathrm{Ad}}$ is defined as in (3.7); otherwise,

$$\underline{\delta}(\mathcal{S}_{\mathrm{Ad}}) \geq \frac{3 + 3}{16} = \frac{3}{8}.$$

In particular, for any α, as $\mathcal{S}_{\mathrm{Ad}} \subseteq \mathcal{S}_\alpha$, $\underline{\delta}(\mathcal{S}_\alpha) \geq \frac{1}{4}$.

4. Proofs of Theorems 1.1 and 1.4, part II: at least one of π_1 and π_2 is non-dihedral

In this section, we shall complete the proofs of Theorems 1.1 and 1.4 by analysing the case that at least one of π_1 and π_2 is non-dihedral.

4.1. Exactly one of π_1 and π_2 is dihedral

If π_1 is not dihedral but π_2 is dihedral, then using (2.3) and the fact that $\delta_{2,0,0,2} = \delta_{0,2,2,0}$ and arguing similarly as before, we obtain

$$\begin{aligned} 2^2 &\leq (2 + 2\delta_{2,0,0,2} \cos(2\alpha) + 4 + 4)\, \underline{\delta}(\mathcal{S}_\alpha) \\ &\leq \begin{cases} (10 + 4\cos(2\alpha))\, \underline{\delta}(\mathcal{S}_\alpha) & \text{if } \cos(2\alpha) \geq 0; \\ 10\, \underline{\delta}(\mathcal{S}_\alpha) & \text{if } \cos(2\alpha) \leq 0, \end{cases} \end{aligned}$$

and thus

$$\underline{\delta}(\mathcal{S}_\alpha) \geq \begin{cases} \frac{2}{5+2\cos(2\alpha)} & \text{if } \cos(2\alpha) \geq 0; \\ \frac{2}{5} & \text{if } \cos(2\alpha) \leq 0. \end{cases}$$

(By the symmetry, the bound also holds if π_1 is dihedral and π_2 is not dihedral.)

4.2. Both π_1 and π_2 are non-dihedral

4.2.1. Both π_1 and π_2 are non-tetrahedral. If both π_1 and π_2 are non-tetrahedral, from (2.4), we see that

$$2^2 \leq (2 + 2\delta_{2,0,0,2}\cos(2\alpha) + 2 + 8)\,\underline{\delta}(\mathcal{S}_\alpha)$$

and hence

$$\underline{\delta}(\mathcal{S}_\alpha) \geq \begin{cases} \frac{1}{3+\cos(2\alpha)} & \text{if } \cos(2\alpha) \geq 0; \\ \frac{1}{3} & \text{if } \cos(2\alpha) \leq 0. \end{cases}$$

4.2.2. At least one of π_1 and π_2 is tetrahedral. If at least one of π_1 and π_2 is tetrahedral, then applying (2.5), we have

$$\begin{aligned} 2^2 &\leq (2 + 2\delta_{2,0,0,2}\cos(2\alpha) + 2 + 8 - 8\kappa_1\cos\alpha)\,\underline{\delta}(\mathcal{S}_\alpha) \\ &\leq \begin{cases} (12 + 4\cos(2\alpha) - 8\kappa_1\cos\alpha)\,\underline{\delta}(\mathcal{S}_\alpha) & \text{if } \cos(2\alpha) \geq 0; \\ (12 - 8\kappa_1\cos\alpha)\,\underline{\delta}(\mathcal{S}_\alpha) & \text{if } \cos(2\alpha) \leq 0. \end{cases} \end{aligned}$$

Comparing this with the previous case, we complete the proof for the second part of Theorem 1.1.

4.2.3. π_1 and π_2 are not twist-equivalent. If π_1 and π_2 are not twist-equivalent, then from (2.6), it follows that

$$2^2 \leq (2 + 2\kappa_2\cos(2\alpha) + 2 + 4)\,\underline{\delta}(\mathcal{S}_\alpha) \leq (8 + 2\kappa_2\cos(2\alpha))\,\underline{\delta}(\mathcal{S}_\alpha)$$

and thus

$$\underline{\delta}(\mathcal{S}_\alpha) \geq \frac{2}{4 + \kappa_2\cos(2\alpha)},$$

which completes the proof of Theorem 1.4. $\square$

5. Proofs of Theorems 1.5 and 1.7

In this section, we shall prove Theorems 1.5 and 1.7. Since the proof of Theorem 1.7 follows the same line, we shall focus on the proof of Theorem 1.5 and then discuss how to modify it to prove Theorem 1.7 at the end of this section.

Our key observation is that $\mathcal{S}_{\mathrm{Ad}}$ and

$$\mathcal{S}_* = \mathcal{S}_*(\pi_1, \pi_2) = \{v \text{ unramified for both } \pi_1 \text{ and } \pi_2 \mid |\lambda_{\pi_1}(v)| \neq |\lambda_{\pi_2}(v)|\}$$

are exactly the same (which follows from the identity that $|\lambda_{\pi_i}(v)|^2 = \lambda_{\pi_i \times \bar{\pi}_i}(v) = \lambda_{\mathrm{Ad}(\pi_i)}(v) + 1$ for unramified v). As the case that both π_1 and π_2 are dihedral is already proved in Section 3, we shall only consider the situation that at least one of π_1 and π_2 is non-dihedral.

We shall require the following proposition, which is a consequence of the work of Kim and Shahidi [14], concerning the orders of poles of certain L-functions at $s = 1$.

Proposition 5.1. *Let π, π_1, and π_2 be non-dihedral cuspidal automorphic representations for $\mathrm{GL}_2(\mathbb{A}_F)$ with unitary central characters ω, ω_1, and ω_2, respectively. Suppose that π_1 and π_2 are not twist-equivalent. Then we have*
(5.1)
$$-\operatorname{ord}_{s=1} L(s, \Pi \times \Pi \times \Pi \times \Pi) = \begin{cases} 7 & \textit{if } \pi \textit{ is tetrahedral;} \\ 4 & \textit{if } \pi \textit{ is octahedral;} \\ 3 & \textit{if } \pi \textit{ is not solvable polyhedral,} \end{cases}$$

where $\Pi = \mathrm{Ad}(\pi)$.[4] *Also, if π_1 is not solvable polyhedral, then we have*

$$-\operatorname{ord}_{s=1} L(s, \Pi_1 \times \Pi_1 \times \Pi_2 \times \Pi_2) \leq \begin{cases} 1 & \textit{if } \pi_2 \textit{ is tetrahedral;} \\ 2 & \textit{otherwise,} \end{cases} \tag{5.2}$$

where $\Pi_i = \mathrm{Ad}(\pi_i)$ for each i.

Proof. We first note that Π and Π_i are self-dual cuspidal representations for $\mathrm{GL}_3(\mathbb{A}_F)$. As a consequence of Clebsch-Gordon decomposition (see, e.g, [29,

[4] A cuspidal automorphic representation π for $\mathrm{GL}_2(\mathbb{A}_F)$ is called octahedral if it is non-dihedral and non-tetrahedral, and $\mathrm{Sym}^3 \pi$ admits a non-trivial self-twist by a Hecke character; π is called solvable polyhedral if it is either dihedral, tetrahedral, or octahedral.

Lemma 3.3]), it is known that

$$\mathrm{Ad}(\Pi) \simeq \mathrm{Ad}(\pi) \boxplus \mathrm{Sym}^4 \pi \otimes \omega^{-2},$$

where by [12], $\mathrm{Sym}^4 \pi$ is automorphic. We note that 1 will never appear in the isobaric sum of $\mathrm{Ad}(\Pi) = \mathrm{Ad}(\mathrm{Ad}(\pi))$. Indeed, by the identity

$$L(s, \Pi \times \bar{\Pi}) = L(s, 1) L(s, \mathrm{Ad}(\Pi))$$

and the fact that $\Pi = \mathrm{Ad}(\pi)$ is cuspidal, the theory of Rankin-Selberg L-functions yields that

$$1 = -\operatorname{ord}_{s=1} L(s, \Pi \times \bar{\Pi}) = 1 - \operatorname{ord}_{s=1} L(s, \mathrm{Ad}(\Pi)),$$

and so $L(s, \mathrm{Ad}(\Pi))$ is holomorphic at $s = 1$.

Moreover, we recall the cuspidality criteria established by Kim and Shahidi [14, Sec 3.2 and Theorem 3.3.7]:

(i) if π is tetrahedral, one has

$$\mathrm{Sym}^4 \pi \otimes \omega^{-2} \simeq \mu \boxplus \mu^2 \boxplus \mathrm{Sym}^2 \pi \otimes \omega^{-1} \simeq \mu \boxplus \mu^2 \boxplus \mathrm{Ad}(\pi),$$

where μ is a (cubic) Hecke character such that $\mathrm{Ad}(\pi) \otimes \mu \simeq \mathrm{Ad}(\pi)$;

(ii) if π is octahedral, one has

$$\mathrm{Sym}^4 \pi \otimes \omega^{-2} \simeq \sigma \boxplus \mathrm{Ad}(\pi) \otimes \eta$$

for some dihedral representation σ and (quadratic) Hecke character η;

(iii) if π is not solvable polyhedral, then $\mathrm{Sym}^4 \pi$ is cuspidal (and so is $\mathrm{Sym}^4 \pi \otimes \omega^{-2}$).

To prove (5.1), we use the following decomposition of L-functions:

$$L(s, \Pi \times \Pi \times \Pi \times \Pi) = L(s, 1) L(s, \mathrm{Ad}(\Pi))^2 L(s, \mathrm{Ad}(\Pi) \times \mathrm{Ad}(\Pi)),$$

where $L(s, \mathrm{Ad}(\Pi))$ has no pole at $s = 1$, and

$$\begin{aligned} &L(s, \mathrm{Ad}(\Pi) \times \mathrm{Ad}(\Pi)) \\ &\qquad = L(s, (\mathrm{Ad}(\pi) \boxplus \mathrm{Sym}^4 \pi \otimes \omega^{-2}) \times (\mathrm{Ad}(\pi) \boxplus \mathrm{Sym}^4 \pi \otimes \omega^{-2})). \end{aligned}$$

Now, (5.1) follows from the above-mentioned cuspidality criteria of Kim and Shahidi and the theory of Rankin-Selberg L-functions.

Similarly, we have

$$L(s, \Pi_1 \times \Pi_1 \times \Pi_2 \times \Pi_2) \\ = L(s,1)L(s, \mathrm{Ad}(\Pi_1))L(s, \mathrm{Ad}(\Pi_2))L(s, \mathrm{Ad}(\Pi_1) \times \mathrm{Ad}(\Pi_2)),$$

where each $L(s, \mathrm{Ad}(\Pi_i))$ has no pole at $s = 1$, and

$$L(s, \mathrm{Ad}(\Pi_1) \times \mathrm{Ad}(\Pi_2)) \\ = L(s, (\mathrm{Ad}(\pi_1) \boxplus \mathrm{Sym}^4 \pi_1 \otimes \omega_1^{-2}) \times (\mathrm{Ad}(\pi_2) \boxplus \mathrm{Sym}^4 \pi_2 \otimes \omega_2^{-2})).$$

For non-dihedral π_1, we have

$$-\operatorname{ord}_{s=1} L(s, \mathrm{Ad}(\pi_1) \times \mathrm{Sym}^4 \pi_2 \otimes \omega_2^{-2}) \leq \begin{cases} 1 & \text{if } \pi_2 \text{ is octahedral;} \\ 0 & \text{otherwise} \end{cases}$$

(here we do not assume that π_1 is not solvable polyhedral). Moreover, if π_1 is not solvable polyhedral, then

$$-\operatorname{ord}_{s=1} L(s, \mathrm{Sym}^4 \pi_1 \otimes \omega_1^{-2} \times \mathrm{Sym}^4 \pi_2 \otimes \omega_2^{-2}) \\ \leq \begin{cases} 1 & \text{if } \pi_2 \text{ is not solvable polyhedral;} \\ 0 & \text{otherwise.} \end{cases}$$

Putting everything together, we obtain (5.2). $\square$

5.1. Exactly one of π_1 and π_2 is dihedral

Suppose that π_1 is non-dihedral and π_2 is dihedral. As $\mathrm{Ad}(\pi_1)$ is cuspidal and $\mathrm{Ad}(\pi_2)$ is not cuspidal, $L(s, \mathrm{Ad}(\pi_1) \times \mathrm{Ad}(\pi_1))$ has a simple pole at $s = 1$, and $L(s, \mathrm{Ad}(\pi_1) \times \mathrm{Ad}(\pi_2))$ is holomorphic at $s = 1$. Also, by (3.2) and (3.4), we know that $L(s, \mathrm{Ad}(\pi_2) \times \mathrm{Ad}(\pi_2))$ has a pole of order at least two at $s = 1$. Now, as argued in Section 3.1.1 (see, especially, (3.6)), if the Ramanujan-Petersson conjecture holds for π_1, then we have

$$\underline{\delta}(\mathcal{S}_*) = \underline{\delta}(\mathcal{S}_{\mathrm{Ad}}) \geq \frac{1+2}{16} = \frac{3}{16}.$$

As the Ramanujan-Petersson conjecture holds for solvable polyhedral representations (see, e.g., [29, Sec. 6]), it remains to discuss the case that π_1 is not solvable polyhedral. To do so, we shall apply the Cauchy-Schwarz inequality as follows. Note that as the Ramanujan-Petersson conjecture is valid for π_2 (and thus $|\lambda_{\mathrm{Ad}(\pi_2)}(v)| \le 3$), applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\lambda_{\mathrm{Ad}(\pi_1)}(v) - \lambda_{\mathrm{Ad}(\pi_2)}(v)|^2 &\le 2(|\lambda_{\mathrm{Ad}(\pi_1)}(v)|^2 + |\lambda_{\mathrm{Ad}(\pi_2)}(v)|^2)\\ &\le 2|\lambda_{\mathrm{Ad}(\pi_1)}(v)|^2 + 18.\end{aligned}$$

Hence, as $\mathcal{S}_{\mathrm{Ad}} = \mathcal{S}_*$, we have

$$\begin{aligned}(5.3)\quad &\sum_v \frac{|\lambda_{\mathrm{Ad}(\pi_1)}(v) - \lambda_{\mathrm{Ad}(\pi_2)}(v)|^2 \chi_{\mathcal{S}_{\mathrm{Ad}}}(v)}{Nv^s}\\ &\qquad \le 2\sum_v \frac{|\lambda_{\mathrm{Ad}(\pi_1)}(v)|^2 \chi_{\mathcal{S}_*}(v)}{Nv^s} + 18\sum_{v\in\mathcal{S}_*}\frac{1}{Nv^s}.\end{aligned}$$

Applying the Cauchy-Schwarz inequality again and using "positivity" as in [29, Eq. (2.2)], we can bound the first sum on the right of (5.3) as

$$\begin{aligned}\sum_v \frac{|\lambda_{\mathrm{Ad}(\pi_1)}(v)|^2 \chi_{\mathcal{S}_*}(v)}{Nv^s} &\le \Big(\sum_v \frac{|\lambda_{\mathrm{Ad}(\pi_1)}(v)|^4}{Nv^s}\Big)^{\frac{1}{2}} \Big(\sum_{v\in\mathcal{S}_*}\frac{1}{Nv^s}\Big)^{\frac{1}{2}}\\ &\le \Big(\log(L(s, \Pi_1 \times \Pi_1 \times \Pi_1 \times \Pi_1))\Big)^{\frac{1}{2}} \Big(\sum_{v\in\mathcal{S}_*}\frac{1}{Nv^s}\Big)^{\frac{1}{2}},\end{aligned}$$

where $\Pi_1 = \mathrm{Ad}(\pi_1)$. Therefore, arguing similarly as in Section 3.1.1, by (5.1), we obtain

$$3 \le 2\cdot 3^{\frac{1}{2}}\, \underline{\delta}(\mathcal{S}_*)^{\frac{1}{2}} + 18\, \underline{\delta}(\mathcal{S}_*).$$

A numerical calculation then yields

$$\underline{\delta}(\mathcal{S}_*) \ge \frac{5}{27} - \frac{\sqrt{19}}{54} \ge 0.1044... \ge \frac{1}{9.58}.$$

5.2. Both π_1 and π_2 are non-dihedral

We start with noting that if both π_1 and π_2 are non-dihedral, then $\mathrm{Ad}(\pi_1)$ and $\mathrm{Ad}(\pi_2)$ are cuspidal and thus each $L(s, \mathrm{Ad}(\pi_i) \times \mathrm{Ad}(\pi_i))$ has a simple pole at $s = 1$. Also, as π_1 and π_2 are non-twist-equivalent, $\mathrm{Ad}(\pi_1) \not\simeq \mathrm{Ad}(\pi_2)$,

$L(s, \mathrm{Ad}(\pi_1) \times \mathrm{Ad}(\pi_2))$ is holomorphic at $s = 1$. As argued previously, assuming the Ramanujan-Petersson conjecture (for both π_1 and π_2), we obtain

$$\underline{\delta}(\mathcal{S}_*) = \underline{\delta}(\mathcal{S}_{\mathrm{Ad}}) \geq \frac{2}{16} = \frac{1}{8}.$$

In the case that the Ramanujan-Petersson conjecture is unknown, we shall modify Walji's strategy, discussed in Section 2.3, to prove:

Theorem 5.2. *Let π_1 and π_2 be a non-dihedral cuspidal automorphic representation for $\mathrm{GL}_2(\mathbb{A}_F)$ with unitary central characters. Suppose, further, that π_1 and π_2 are not twist-equivalent. If π_1 is not solvable polyhedral, then*

$$\underline{\delta}(\mathcal{S}_*) \geq \begin{cases} \frac{1}{10.17} & \text{if } \pi_2 \text{ is tetrahedral;} \\ \frac{1}{10.76} & \text{if } \pi_2 \text{ is octahedral;} \\ \frac{1}{9.9} & \text{if } \pi_2 \text{ is not solvable polyhedral.} \end{cases}$$

Proof. We first remark that when at least one of π_1 and π_2 is not solvable polyhedral, the information on at least one of $L(s, \Pi_1 \times \Pi_1 \times \Pi_1 \times \Pi_2)$ and $L(s, \Pi_1 \times \Pi_2 \times \Pi_2 \times \Pi_2)$ at $s = 1$ seems unavailable at present. Therefore, instead of using an estimate similar to (2.1) directly, we shall require a modification.

From the inequality

$$\begin{aligned} &|\lambda_{\mathrm{Ad}(\pi_1)}(v) - \lambda_{\mathrm{Ad}(\pi_2)}(v)|^2 \\ &\qquad \leq |\lambda_{\mathrm{Ad}(\pi_1)}(v)|^2 + 2|\lambda_{\mathrm{Ad}(\pi_1)}(v)\lambda_{\mathrm{Ad}(\pi_2)}(v)| + |\lambda_{\mathrm{Ad}(\pi_2)}(v)|^2 \end{aligned}$$

and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} &\sum_v \frac{|\lambda_{\mathrm{Ad}(\pi_1)}(v) - \lambda_{\mathrm{Ad}(\pi_2)}(v)|^2 \chi_{\mathcal{S}_{\mathrm{Ad}}}(v)}{Nv^s} \\ &\quad \leq \Big(\Big(\sum_v \frac{|\lambda_{\Pi_1}(v)|^4}{Nv^s} \Big)^{\frac{1}{2}} + 2 \Big(\sum_v \frac{|\lambda_{\Pi_1 \times \Pi_2}(v)|^2}{Nv^s} \Big)^{\frac{1}{2}} + \Big(\sum_v \frac{|\lambda_{\Pi_2}(v)|^4}{Nv^s} \Big)^{\frac{1}{2}} \Big) \\ &\qquad \times \Big(\sum_{v \in \mathcal{S}_{\mathrm{Ad}}} \frac{1}{Nv^s} \Big)^{\frac{1}{2}}, \end{aligned}$$

where, as before, $\Pi_i = \mathrm{Ad}(\pi_i)$. From Proposition 5.1, arguing similarly as before, we derive

$$\underline{\delta}(\mathcal{S}_*) = \underline{\delta}(\mathcal{S}_{\mathrm{Ad}}) \geq \begin{cases} \frac{4}{(\sqrt{3}+2+\sqrt{7})^2} \geq \frac{1}{10.17} & \text{if } \pi_2 \text{ is tetrahedral;} \\ \frac{4}{(\sqrt{3}+2\sqrt{2}+\sqrt{4})^2} \geq \frac{1}{10.76} & \text{if } \pi_2 \text{ is octahedral;} \\ \frac{4}{(\sqrt{3}+2\sqrt{2}+\sqrt{3})^2} \geq \frac{1}{9.9} & \text{if } \pi_2 \text{ is not solvable polyhedral.} \end{cases}$$

Herein, we conclude the proof. □

5.3. Proof of Theorem 1.7

To prove Theorem 1.7, we consider the set

$$\begin{aligned} \mathcal{S}^-_{\mathrm{Ad}} &= \mathcal{S}^-_{\mathrm{Ad}}(\pi_1, \pi_2) \\ &= \{v \text{ unramified for both } \pi_1 \text{ and } \pi_2 \mid \lambda_{\mathrm{Ad}(\pi_1)}(v) \neq -\lambda_{\mathrm{Ad}(\pi_2)}(v)\}. \end{aligned}$$

and the sum

$$\sum_v \frac{|\lambda_{\mathrm{Ad}(\pi_1)}(v) + \lambda_{\mathrm{Ad}(\pi_2)}(v)|^2 \chi_{\mathcal{S}^-_{\mathrm{Ad}}}(v)}{Nv^s},$$

where $\chi_{\mathcal{S}^-_{\mathrm{Ad}}}$ is the indicator function of $\mathcal{S}^-_{\mathrm{Ad}}$. Note that the Ramanujan-Petersson conjecture gives

$$|\lambda_{\mathrm{Ad}(\pi_1)}(v) + \lambda_{\mathrm{Ad}(\pi_2)}(v)|^2 \leq (3+3)^2 \leq 36.$$

Processing a similar argument as in the previous sections (including Section 3) then results in

$$\underline{\delta}(\mathcal{S}^-_{\mathrm{Ad}}) \geq \begin{cases} \frac{1}{18} & \text{if } \pi_1 \text{ and } \pi_2 \text{ are simultaneously dihedral or non-dihedral;} \\ \frac{1}{12} & \text{if exactly one of } \pi_1 \text{ and } \pi_2 \text{ is dihedral.} \end{cases}$$

Finally, observing that

$$\lambda_{\mathrm{Ad}(\pi_1)}(v) + \lambda_{\mathrm{Ad}(\pi_2)}(v) = |\lambda_{\pi_1}(v)|^2 + |\lambda_{\pi_2}(v)|^2 - 2,$$

we completes the proof of Theorem 1.7. □

Remark. Although it seems that one would obtain a better bound for $\underline{\delta}(\mathcal{S}^*)$ in the simultaneously non-dihedral case than the simultaneously dihedral case by applying the argument used in Section 5.2, it is not always the case. Indeed, if both π_1 and π_2 are tetrahedral, $L(s, \Pi_1 \times \Pi_1 \times \Pi_2 \times \Pi_2)$ may admit a pole of order three at $s = 1$, where $\Pi_i = \mathrm{Ad}(\pi_i)$. This results in $\underline{\delta}(\mathcal{S}^*) = \underline{\delta}(\mathcal{S}^-_{\mathrm{Ad}}) \geq \frac{4}{(\sqrt{7}+2\cdot 3+\sqrt{7})^2} \approx \frac{1}{31.88}$. Nonetheless, if π_1 is not solvable polyhedral (regardless of that π_2 is non-dihedral or not), arguing as in Sections 5.1 and 5.2, one can obtain $\underline{\delta}(\mathcal{S}^*) = \underline{\delta}(\mathcal{S}^-_{\mathrm{Ad}}) \geq \frac{1}{10.76}$.

Remark. Let $n \geq 3$, and let π_1 and π_2 be distinct cuspidal automorphic representations for $\mathrm{GL}_n(\mathbb{A}_F)$, satisfying the Ramanujan-Petersson conjecture, such that $\mathrm{Ad}(\pi_1)$ and $\mathrm{Ad}(\pi_2)$ are cuspidal. Applying the Cauchy-Schwarz inequality twice, we have

$$\begin{aligned} \sum_v \frac{|\lambda_{\pi_1}(v) - e^{i\alpha}\lambda_{\pi_2}(v)|^2 \chi_{\mathcal{S}_\alpha}(v)}{Nv^s} &\leq 2\sum_{i=1}^{2}\sum_v \frac{|\lambda_{\pi_i}(v)|^2 \chi_{\mathcal{S}_\alpha}(v)}{Nv^s} \\ &\leq 2\sum_{i=1}^{2}\Big(\sum_v \frac{|\lambda_{\pi_i}(v)|^4}{Nv^s}\Big)^{\frac{1}{2}}\Big(\sum_{v\in\mathcal{S}_\alpha}\frac{1}{Nv^s}\Big)^{\frac{1}{2}}, \end{aligned} \tag{5.4}$$

where

$$\mathcal{S}_\alpha = \{v \text{ unramified for both } \pi_1 \text{ and } \pi_2 \mid \lambda_{\pi_1}(v) \neq e^{i\alpha}\lambda_{\pi_2}(v)\}.$$

Since we know

$$L(s, \pi_i \times \bar{\pi}_i \times \pi_i \times \bar{\pi}_i) = L(s,1)L(s, \mathrm{Ad}(\pi_i))^2 L(s, \mathrm{Ad}(\pi_i) \times \mathrm{Ad}(\pi_i)),$$

$L(s, \pi_i \times \bar{\pi}_i \times \pi_i \times \bar{\pi}_i)$ has a pole of order two at $s = 1$. Hence, we have

$$\underline{\delta}(\mathcal{S}_\alpha) \geq \Big(\frac{2}{4\sqrt{2}}\Big)^2 = \frac{1}{8}.$$

Furthermore, without the assumption that $\mathrm{Ad}(\pi_1)$ and $\mathrm{Ad}(\pi_2)$ are automorphic, as each $\mathrm{Ad}(\pi_i)$ satisfies the Ramanujan-Petersson conjecture, we know that $|\lambda_{\pi_i}(v)|^2 \leq n^2$. Thus, applying the first inequality in (5.4), we obtain

$$\underline{\delta}(\mathcal{S}_\alpha) \geq \frac{2}{4n^2} = \frac{1}{2n^2}.$$

6. Proofs of Theorems 1.8, 1.10, and 1.11

In this section, we will prove Theorems 1.8, 1.10, and 1.11. For the sake of convenience, we shall assume that any prime v in the consideration is unramified for both π_1 and π_2. (Note the argument presented in Section 6.1 does not make the assumption that both π_1 and π_2 are non-dihedral.)

6.1. Generality

For each i, we write

$$\lambda_{\pi_i}(v) = 2\cos\theta_{i,v}$$

for some $\theta_{i,v} \in [0,\pi]$. It is clear that

$$\lambda_{\pi_1}(v) = \lambda_{\pi_2}(v) \text{ if and only if } \theta_{1,v} = \theta_{2,v}$$

and that

$$\lambda_{\pi_1}(v) = -\lambda_{\pi_2}(v) \text{ if and only if } \theta_{1,v} = \pi - \theta_{2,v}.$$

Thus, $\#\{v \mid Nv \le x,\ \ \lambda_{\pi_1}(v)^2 = \lambda_{\pi_2}(v)^2\}$ is less than or equal to

$$\#\{v \mid Nv \le x,\ \ \theta_{1,v} = \theta_{2,v}\} + \#\{v \mid Nv \le x,\ \ \theta_{1,v} = \pi - \theta_{2,v}\}. \tag{6.1}$$

In the notation of Section 2.4, for any $M \ge 1$ and $\delta \in (0, \frac{1}{\pi(M+1)}]$, we first observe that

$$\#\{v \mid Nv \le x,\ \ \theta_{1,v} = \theta_{2,v}\} \le \sum_{Nv \le x} \Big(\mathcal{I}_\delta(\theta_{1,v} - \theta_{2,v}) + \mathcal{I}_\delta(\theta_{1,v} + \theta_{2,v})\Big). \tag{6.2}$$

From Proposition 2.5 and the identity

$$\cos(n(\theta_{1,v} - \theta_{2,v})) + \cos(n(\theta_{1,v} + \theta_{2,v})) = 2\cos(n\theta_{1,v})\cos(n\theta_{2,v}),$$

it follows that the sum on the right of (6.2) is less than or equal to

$$\Big(2\delta + \frac{2}{M+1}\Big)\pi_F(x) + 4\sum_{n=1}^{M} \mathfrak{Re}(\hat{S}^+_{J,M}(n)) \sum_{Nv \le x} \cos(n\theta_{1,v})\cos(n\theta_{2,v}), \tag{6.3}$$

where $\pi_F(x)$ denotes the number of primes v of F such that $Nv \le x$. Since $\delta \in (0, \frac{1}{\pi(M+1)}]$, $|\mathfrak{Re}(\hat{S}^+_{J,M}(n))| \le \delta + \frac{1}{M+1} \le \frac{2}{M}$, and thus

$$4\sum_{n=1}^{2} \mathfrak{Re}(\hat{S}^+_{J,M}(n)) \sum_{Nv \le x} \cos(n\theta_{1,v})\cos(n\theta_{2,v}) \ll \frac{1}{M}\pi_F(x). \tag{6.4}$$

Also, recalling that for $n \ge 2$,

$$2\cos(n\theta) = \frac{\sin((n+1)\theta)}{\sin\theta} - \frac{\sin((n-1)\theta)}{\sin\theta} = U_n(\cos\theta) - U_{n-2}(\cos\theta),$$

we see that the remaining terms[5] in the double sum of (6.3) become

$$\ll \frac{1}{M}\sum_{n=3}^{M}\Big|\sum_{Nv\le x}\Big(U_n(\cos(\theta_{1,v}))-U_{n-2}(\cos(\theta_{1,v}))\Big) \times \Big(U_n(\cos(\theta_{2,v}))-U_{n-2}(\cos(\theta_{2,v}))\Big)\Big|. \tag{6.5}$$

By (6.2), (6.3), (6.4), and (6.5), we see that $\#\{v \mid Nv\le x,\ \theta_{1,v}=\theta_{2,v}\}$ is

$$\ll \frac{\pi_F(x)}{M}+\frac{1}{M}\sum_{n=3}^{M}\Big|\sum_{Nv\le x}\prod_{i=1}^{2}\Big(U_n(\cos(\theta_{i,v}))-U_{n-2}(\cos(\theta_{i,v}))\Big)\Big|. \tag{6.6}$$

To bound $\#\{v \mid Nv\le x,\ \theta_{1,v}=\pi-\theta_{2,v}\}$, we use the estimate

$$\#\{v \mid Nv\le x,\ \theta_{1,v}=\pi-\theta_{2,v}\} \le \sum_{Nv\le x}\Big(\mathcal{I}_\delta(\theta_{1,v}-(\pi-\theta_{2,v}))+\mathcal{I}_\delta(\theta_{1,v}+(\pi-\theta_{2,v}))\Big).$$

As $\cos(n(\theta_{1,v}-(\pi-\theta_{2,v})))+\cos(n(\theta_{1,v}+(\pi-\theta_{2,v})))$ equals

$$2\cos(n\theta_{1,v})\cos(n(\pi-\theta_{2,v}))=2(-1)^n\cos(n\theta_{1,v})\cos(n\theta_{2,v}),$$

by Proposition 2.5, we see that $\#\{v \mid Nv\le x,\ \theta_{1,v}=\pi-\theta_{2,v}\}$ is less than or equal to

$$\Big(2\delta+\frac{2}{M+1}\Big)\pi_F(x)+4\sum_{n=1}^{M}\mathfrak{Re}(\hat{S}^{+}_{J,M}(n))(-1)^n\sum_{Nv\le x}\cos(n\theta_{1,v})\cos(n\theta_{2,v}).$$

By an analogous argument as above, this becomes

$$\ll \frac{\pi_F(x)}{M}+\frac{1}{M}\sum_{n=3}^{M}\Big|\sum_{Nv\le x}\prod_{i=1}^{2}\Big(U_n(\cos(\theta_{i,v}))-U_{n-2}(\cos(\theta_{i,v}))\Big)\Big|. \tag{6.7}$$

[5]We note that although [17] includes the case $n=2$ in their consideration, one has to treat it separately since [17, Proposition 2.1] (cf. Propositions 2.2) does not cover the situation that $U_{2-2}(\cos(\theta_{1,v}))U_{2-2}(\cos(\theta_{2,v}))\equiv 1$.

6.2. Proof of Theorem 1.8

Now, we are in a position to prove Theorem 1.8. Assume that both π_1 and π_2 are non-dihedral and that

$$\limsup_{x\to\infty} \frac{\#\{v \mid Nv \le x,\ \ \lambda_{\pi_1}(v)^2 = \lambda_{\pi_2}(v)^2\}}{\pi_F(x)} > 0. \tag{6.8}$$

Suppose, on the contrary, that π_1 and π_2 are not twist-equivalent. From Proposition 2.2 and estimates (6.6) and (6.7), it follows that the limit on the left of (6.8) is less than or equal to

$$\begin{aligned}&\limsup_{x\to\infty} \frac{\#\{v \mid Nv \le x,\ \ \theta_{1,v} = \theta_{2,v}\}}{\pi_F(x)}\\ &\quad + \limsup_{x\to\infty} \frac{\#\{v \mid Nv \le x,\ \ \theta_{1,v} = \pi - \theta_{2,v}\}}{\pi_F(x)} \ll \frac{1}{M}\end{aligned}$$

for any $M \ge 1$. Making $M \to \infty$ yields that

$$\limsup_{x\to\infty} \frac{\#\{v \mid Nv \le x,\ \ \lambda_{\pi_1}(v)^2 = \lambda_{\pi_2}(v)^2\}}{\pi_F(x)} = 0$$

and thus $0 < 0$, a contradiction. $\square$

6.3. Completing the proof of Theorems 1.10 and 1.11

Let $F = \mathbb{Q}$. For each i, let π_i be a cuspidal automorphic representation corresponding to a non-CM newform in $S_{k_i}^{\mathrm{new}}(\Gamma_0(q_i))$ with trivial nebentypus. Assume that π_1 and π_2 are not twist-equivalent. Applying Proposition 2.3 (and the recent work of Newton and Thorne [19] on the automorphy of symmetric powers of π_i), we have

$$\begin{aligned}&\frac{1}{M}\sum_{n=3}^{M}\Big|\sum_{p\le x}\prod_{i=1}^{2}\Big(U_n(\cos(\theta_{i,p})) - U_{n-2}(\cos(\theta_{i,p}))\Big)\Big|\\ &\quad\ll \pi(x)\exp\Big(\frac{-c_2\log x}{(k_1q_1k_2q_2M)^{c_3M^2}}\Big)\\ &\qquad + M^4\pi(x)\Big(\exp\Big(\frac{-\log x}{c_4M^2}\Big) + \exp\Big(\frac{-c_5\log x}{M^2\log(k_1q_1k_2q_2M)}\Big)\\ &\qquad\quad + \exp\Big(\frac{-c_5\sqrt{\log x}}{M}\Big)\Big).\end{aligned} \tag{6.9}$$

Thus, using (6.1), (6.6), (6.7), and (6.9) and choosing

$$M = \lceil c_6 \sqrt{\log\log x} / \log(k_1 q_1 k_2 q_2 \log\log x) \rceil \geq 3,$$

for some sufficiently small $c_6 > 0$, we arrive at

$$\#\{p \leq x \mid \lambda_{\pi_1}(p)^2 = \lambda_{\pi_2}(p)^2\} \ll \pi(x) \frac{\log(k_1 q_1 k_2 q_2 \log\log x)}{\sqrt{\log\log x}}.$$

Moreover, assuming the generalised Riemann hypothesis, by Proposition 2.4 and estimates (6.1), (6.6), and (6.7), we have

$$\#\{p \leq x \mid \lambda_{\pi_1}(p)^2 = \lambda_{\pi_2}(p)^2\} \ll \frac{1}{M} \frac{x}{\log x} + M^2 x^{1/2} \log((k_1 q_1 k_2 q_2 M)x).$$

Choosing

$$M = \lceil x^{1/6} / (\log x)^{1/3} (\log((k_1 q_1 k_2 q_2)x))^{1/3} \rceil,$$

we derive

$$\#\{p \leq x \mid \lambda_{\pi_1}(p)^2 = \lambda_{\pi_2}(p)^2\} \ll \frac{x^{5/6} (\log(k_1 q_1 k_2 q_2 x))^{1/3}}{(\log x)^{2/3}}.$$

□

7. Remarks on non-CM condition in Theorems 1.8, 1.9, 1.10, and 1.11

We shall note that in Theorems 1.8, 1.9, 1.10, and 1.11, we assume π_1 and π_2 correspond to non-CM newforms f_1 and f_2, respectively, just for the sake of simplicity of discussion. As done in [17, 20, 22], it is possible to drop the assumption that f_2 is without CM. (We note that as remarked in [17] if both f_1 and f_2 are with CM, then the theorems are not always true.)

To extend Theorem 1.8, we require the following estimate.

Proposition 7.1. *Let F be a totally real number field. For each i, let π_i be a cuspidal automorphic representation corresponding to a Hilbert newform f_i of weights $k_{i,j} \geq 2$ (at all infinite primes v_j of F) and with trivial nebentypus. Let $m_i \geq 1$. Assume that f_1 is a non-CM newform such that* $\mathrm{Sym}^{m_1} \pi_1$ *is automorphic, and suppose that f_2 is with CM. Then one has*

$$\sum_{Nv \leq x} U_{m_1}(\cos\theta_{1,v}) U_{m_2}(\cos\theta_{2,v}) = o(\pi_F(x)).$$

We note that this proposition was stated in [17, Proposition 2.1] (for $F = \mathbb{Q}$) without assuming the automorphy of $\mathrm{Sym}^{m_1}\pi_1$. However, as may be noticed, the argument in [17, Sec. 4] only works for the case that both f_1 and f_2 are without CM. Indeed, as argued in [17, Sec. 4] (see also [8, pp. 716–719, especially, Theorems 2.4 and 2.5] and [31, Proof of Theorem 1.1]), to use the "Brauer-Taylor induction" (together with the automorphy theorem from [4] and the theory of Rankin-Selberg L-functions), one would require the *cuspidality* of both $\mathrm{Sym}^{m_1}\pi_1$ and $\mathrm{Sym}^{m_2}\pi_2$ after a suitable base change, which may not hold if f_2 is with CM (see the proof below).

We also note that the proof of Hecke's theorem on the distribution of the Frobenius angles of CM modular forms does not rely on the symmetric power L-functions but the equidistribution of the values of Hecke characters (see, e.g., [1, Theorem 3.1.1 and Sec. 3.3]).

Since we did not find a reference with the precise proof for the proposition (although it seems to be known by experts, at least, implicitly), we include a proof in this section for the sake of completeness.

Proof of Proposition 7.1. As argued in [17], by the Wiener-Ikehara Tauberian theorem (see, e.g., [16]), to prove the proposition, it suffices to show that the L-function

$$L(s, \mathrm{Sym}^{m_1}\pi_1 \times \mathrm{Sym}^{m_2}\pi_2)$$

extends to a non-vanishing holomorphic function on $\mathfrak{Re}(s) \geq 1$. As remarked in [24, pp. 243 and 251], if π_2 corresponds to a CM Hilbert newform f_2 (and so π_2 is dihedral), there exists an imaginary quadratic extension K of F such that

$$\mathrm{Sym}^{m_2}\pi_2 \simeq \boxplus_j \Pi_j,$$

where Π_j is either an idèle class character of F or a two-dimensional cuspidal automorphic representation induced from a (non-trivial) character of K. Thus, we have the factorisation

$$L(s, \mathrm{Sym}^{m_1}\pi_1 \times \mathrm{Sym}^{m_2}\pi_2) = \prod_j L(s, \mathrm{Sym}^{m_1}\pi_1 \times \Pi_j).$$

By the theory of Rankin-Selberg L-functions, each $L(s, \mathrm{Sym}^{m_1}\pi_1 \times \Pi_j)$ extends holomorphically to $\mathfrak{Re}(s) \geq 1$ except for a possible simple pole at $s = 1 - it$ that exists only if $\mathrm{Sym}^{m_1}\pi_1 \simeq \bar{\Pi}_j \otimes |\cdot|^{it}$. By a dimension consideration, if $L(s, \mathrm{Sym}^{m_1}\pi_1 \times \Pi_j)$ admits a pole at $s = 1 - it$, then $m_1 = 1$ and $\dim \Pi_j = 2$ and thus $\pi_1 \simeq \bar{\Pi}_j \otimes |\cdot|^{it}$. However, it is impossible as f_1

is without CM (so π_1 is non-dihedral), but Π_j is induced from a character (so it is dihedral). Finally, it follows from the work of Shahidi [26], each $L(s, \mathrm{Sym}^{m_1} \pi_1 \times \Pi_j)$ is non-vanishing on $\mathfrak{Re}(s) \geq 1$, which concludes the proof. $\square$

Now, using Proposition 7.1 in the place of Proposition 2.2 in the proof of Theorem 1.8 given in Section 6.2, we have the following:

Theorem 7.2. *Let F be a totally real number field. For each i, let π_i be a cuspidal automorphic representation corresponding to a Hilbert newform f_i of weights $k_{i,j} \geq 2$ and with trivial nebentypus. Assume that f_1 is without CM and that all the symmetric powers $\mathrm{Sym}^{m_1} \pi_1$ are automorphic. If*

$$\limsup_{x \to \infty} \frac{\#\{v \mid Nv \leq x, \ \ \lambda_{\pi_1}(v)^2 = \lambda_{\pi_2}(v)^2\}}{\pi(x)} > 0,$$

then π_1 and π_2 are twist-equivalent.

In closing this section, we remark that it is possible to prove an effective version of Proposition 7.1 by adapting the methods used in [27, 31] and thus obtain a version of Theorems 1.10 and 1.11, without assuming that π_2 is without CM, by using the argument developed in Section 6.

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