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Quasimode and Strichartz estimates for time-dependent Schrödinger equations with singular potentials

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We generalize the Strichartz estimates for Schrödinger operators on compact manifolds of Burq, Gérard and Tzvetkov [10] by allowing critically singular potentials V. Specifically, we show that their $1/p$–loss $L^p_t L^q_x(I\times M)$-Strichartz estimates hold for e^{-itH_V} when $H_V=-\Delta_g+V(x)$ with $V\in L^{n/2}(M)$ if $n\ge 3$ or $V\in L^{1+\delta}(M)$, $\delta>0$, if $n=2$, with (p,q) being as in the Keel-Tao theorem and $I\subset\mathbb{R}$ a bounded interval. We do this by formulating and proving new "quasimode" estimates for scaled dyadic unperturbed Schrödinger operators and taking advantage of the the fact that $1/q'-1/q=2/n$ for the endpoint Strichartz estimates when $(p,q)=(2,2n/(n-2))$. We also show that the universal quasimode estimates that we obtain are saturated on *any* compact manifolds; however, we suggest that they may lend themselves to improved Strichartz estimates in certain geometries using recently developed "Kakeya-Nikodym" techniques developed to obtain improved eigenfunction estimates assuming, say, negative curvatures.

1. Introduction and main results

In [10], Burq, Gérard and Tzvetkov showed that if (M,g) is an $n\ge 2$ dimensional compact manifold then the time-dependent Schrödinger operators associated with the Laplace-Beltrami operator satisfy

$$\|e^{-it\Delta_g}\|_{H^{1/p}(M)\to L^p_tL^q_x(I\times M)}\le C_I, \tag{1.1}$$

if $I\subset\mathbb{R}$ is a compact interval and

$$n(1/2-1/q)=2/p \text{ and } 2\le p\le\infty \text{ if } n\ge 3, \text{ or } 2<p\le\infty \text{ if } n=2. \tag{1.2}$$

Here $H^\sigma(M)$ denotes the L^2-Sobolev space associated with the Laplace-Beltrami operator on M with norm

$$\|u\|_{H^\sigma(M)}=\big\|\big(\sqrt{I-\Delta_g}\big)^\sigma u\big\|_{L^2(M)}. \tag{1.3}$$

In the two-dimensional case, the bounds in (1.1) also depend on (p,q). In practice there one just takes $I=[0,1]$ since this inequality implies the bound for all compact intervals.

The main purpose of this paper is to show that we also have the bounds in (1.1) if $-\Delta_g$ is replaced by $-\Delta_g+V(x)$, with the potential V being real-valued and satisfying

$$V\in L^{n/2}(M) \text{ when } n\ge 3 \text{ and } V\in L^{1+\delta}(M) \text{ some } \delta>0 \text{ if } n=2. \tag{1.4}$$

Such a result involves critically singular potentials, since multiplication by elements of $L^{n/2}$ scale the same as Δ_g. Indeed, if we consider the Euclidean Laplacian, then $\Delta u(\lambda\,\cdot\,)=\lambda^2\big(\Delta u\big)(\lambda\,\cdot\,)$ and $\lambda^2\|V(\lambda\,\cdot\,)\|_{L^{n/2}}=\|V\|_{L^{n/2}}$, and similar formulae hold on (M,g) if we scale the metric.

We should also point out that the natural $L^1\to L^\infty$ estimates for solutions of the heat equation involving the operators

$$H_V=-\Delta_g+V \tag{1.5}$$

may break down when one merely assumes that $V\in L^{n/2}(M)$. Moreover, individual eigenfunctions need not be bounded (unlike the case where V is smooth). See, e.g., [1], [4] and [20]. On the other hand, if V is as above then H_V defines a self-adjoint operator which is bounded from below (see, e.g., [3]). Among other things, this allows us to define the time-dependent Schrödinger operators e^{-itH_V}.

Even though heat equation bounds may break down for $L^{n/2}$ potentials, we do have the analog of the Strichartz estimates (1.1) of Burq, Gérard and Tzvetkov:

Theorem 0.1. *Let the potential V be real-valued and satisfy* (1.4)*. Also, assume that the pair of exponents (p, q) is as in* (1.2)*. We then have for any compact interval $I \subset \mathbb{R}$*

$$\big\|e^{-itH_V}u\big\|_{L^p_t L^q_x(I\times M)} \lesssim \|u\|_{H^{1/p}(M)}. \tag{1.6}$$

We should point out that, Burq, Gérard and Tzvetkov [10, Theorem 6] discussed a variant of the above theorem for the Euclidean spaces with variable coefficient metrics, and their arguments can easily be adapted to the setting of compact manifolds which would show that the above results hold when $V \in L^n(M)$. Also, our main point of departure from the analysis in [10] is to use dyadic cut-offs in the time variable as opposed to the spatial variable. We need to do this since Littlewood-Paley operators associated with $-\Delta_g$ are not easily seen to be compatible with ones associated to $-\Delta_g + V$ if V is singular. We also note that the philosophy that, for solutions of dispersive equations, dyadic time-frequency cut-offs and spatial ones should be interchangeable is not new. For instance, for solutions of Schrödinger equations this is crucially used in [14] and [18] and for wave equations in [17].

Just as Burq, Gérard and Tzvetkov [10] did for the $V \equiv 0$ case, we shall prove this result by showing that if one restricts to frequencies comparable to λ, with λ large one has no-loss estimates on small intervals of size λ^{-1}. Specifically, if fix a real-valued Littlewood-Paley bump function

$$\beta \in C_0^\infty((1/2, 2)), \tag{1.7}$$

for future convenience, satisfying

$$1 = \sum_{-\infty}^{\infty} \beta(2^{-j}s) \text{ for } \ s > 0, \quad \text{and } \beta(s) = 1, \ s \in [3/4, 5/4], \tag{1.8}$$

then the main estimate in [10] is that for large λ we have

$$\|\beta(P/\lambda)e^{it\Delta_g}\|_{L^2(M)\to L^p_t L^q_x([0,\lambda^{-1}]\times M)} = O(1), \quad P = \sqrt{-\Delta_g}, \tag{1.9}$$

if (p, q) are as in (1.2). Since $e^{-it\Delta_g}$ is a unitary operator on $L^2(M)$, this of course says that one has $O(1)$ bounds on all intervals of length λ^{-1}, and so by adding up $O(\lambda)$ of these bounds they obtained the estimate

$$\|\beta(P/\lambda)e^{it\Delta_g}\|_{L^2(M)\to L^p_t L^q_x([0,1]\times M)} = O(\lambda^{1/p}), \tag{1.9'}$$

which leads to (1.6) with $V=0$ using standard Littlewood-Paley estimates associated with $-\Delta_g$.

We shall follow this strategy and ultimately prove analogous dyadic estimates for e^{-itH_V} that will allow us to obtain (1.6). We shall have to show that the Littlewood-Paley estimates for H_V are valid for the exponents q in (1.2), which we shall obtain in an appendix using a general spectral multiplier theorem of Blunck [7] and recent estimates in our collaboration with Blair and Sire [3].

In order to obtain these natural dyadic variants of (1.6) we shall rely on certain microlocalized "quasimode" estimates for the unperturbed scaled Schrödinger operators with a damping term,

$$i\lambda\partial_t + \Delta_g + i\lambda. \tag{1.10}$$

Since there is no reason to expect that the Littlewood-Paley operators associated with $-\Delta_g$ are compatible with the corresponding ones for $H_V = -\Delta_g + V(x)$ with V singular, it does not seem that we would be able to use quasimode estimates for the unperturbed operator $-\Delta_g$ to prove results for H_V if these estimates include "spatial" dyadic cutoffs $\beta(P/\lambda)$ as above. We shall mitigate this potential issue by using the Littlewood-Paley operators acting on the time variable,

$$\beta(-D_t/\lambda)h(x) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{it\tau}\beta(-\tau/\lambda)\,\hat{h}(\tau)\,d\tau,$$

with β as above.

Let us be more specific. Our main estimates will concern solutions of the scaled inhomogeneous Schrödinger equation with damping term

$$(i\lambda\partial_t + \Delta_g + i\lambda)w(t,x) = F(t,x), \quad w(0,\,\cdot\,) = 0. \tag{1.11}$$

It will be convenient to assume that the "forcing term" here satisfies

$$F(t,x) = 0, \quad t \notin [0,1]. \tag{1.12}$$

The result that we shall use to prove Theorem 0.1 then is the following.

Theorem 0.2. *Suppose that F satisfies the support assumption in* (1.12) *and that w solves* (1.11)*. Then for $\lambda \ge 1$ and exponents as in* (1.2) *we have*

$$\big\|\beta(-D_t/\lambda)w\big\|_{L^p_tL^q_x(\mathbb{R}\times M)} \lesssim \lambda^{-1+1/p}\|F\|_{L^2_{t,x}([0,1]\times M)}, \tag{1.13}$$

and also

$$\big\|\beta(-D_t/\lambda)w\big\|_{L^p_tL^q_x(\mathbb{R}\times M)} \lesssim \lambda^{-1+2/p}\|F\|_{L^{p'}_tL^{q'}_x([0,1]\times M)}. \tag{1.14}$$

Furthermore, the quasimode estimates (1.13) *are sharp on* any *manifold.*

These "microlocalized quasimode estimates" are natural analogs of the ones obtained by one of us for Laplace-Beltrami operators in [22]. Additionally, the first estimate, (1.13), essentially follows from the results of one of us and Seeger [19]. As was the case in these earlier works, and more recently in [4] and [3], it is natural to include the "damping term", $i\lambda$, to exploit the Fourier analysis that arises. In the present context, it allows use Fourier analysis in $\mathbb{R}$ to link dyadic microlocal cutoffs in the spatial variable involving $P=\sqrt{-\Delta_g}$ with the above ones involving the time variable. As we shall see, being able to prove time-microlocalized estimates for solutions of inhomogeneous equations involving the operators in (1.10) will allow us to use the Duhamel formula to prove our estimates for e^{-itH_V} in a manner that is somewhat reminiscent to arguments in a recent joint work [3] on uniform Sobolev estimates for the operators H_V. It is for this reason, and others, that it is important for us to prove natural estimates for inhomogeneous equations, as opposed to ones just involving the Cauchy problem. On the other hand, our proof of (1.13) and (1.14) will be modeled by the arguments in [10] that lead to (1.9). In § 4 we shall show that the quasimode estimates (1.13) are optimal.

This paper is organized as follows. In the next section, we shall prove Theorem 0.2. Then, in § 3 we shall show how we can use the above Theorem along with Littlewood-Paley estimates to obtain the Strichartz estimates in Theorem 0.1. We shall prove the Littlewood-Paley estimates that we require in an Appendix. In § 4 we shall show that the universal bounds (1.13) that easily imply the estimates (1.9′) are saturated on *any* manifold, even though Bourgain and Demeter [8] showed much better estimates hold on the torus when $p=q=2(n+2)/n$ with just an λ^{ε}-loss for all $\varepsilon>0$, and Burq, Gérard and Tzvetkov [10] also showed that on spheres there are improvements of (1.9′) in many cases. It seems a challenge to show that there are improvements in more general cases, such as when M has negative sectional curvature; however, the Knapp example that we shall construct in § 4 suggests that perhaps the "Kakeya-Nikodym" techniques that have been recently developed in [5], [6], [23], [24] and [26] to obtain improved eigenfunction estimates in certain geometries might lend themselves to this problem.

2. Quasimode estimates for scaled Schrödinger operators

In this section we shall prove Theorem 0.2. If β is as in (1.7)–(1.8), let us define "wider cutoffs" that we shall also use as follows

$$\widetilde{\beta}(s) = \sum_{|j|<10} \beta(2^{-j}s) \in C_0^\infty(2^{-10}, 2^{10}). \tag{2.1}$$

For future use, note that

$$\widetilde{\beta}(s) = 1 \quad \text{on} \quad (1/4, 4). \tag{2.2}$$

One of the main estimates in [10] is that one can obtain the "expected" $O(|t|^{-n/2})$ dispersive estimates for $\beta(P/\lambda)e^{it\Delta_g}$, $P = \sqrt{-\Delta_g}$, on time intervals of the form $[-\ell(\lambda), \ell(\lambda)]$ for $\lambda \gg 1$ if $\ell(\lambda) = \delta\lambda^{-1}$ for some $\delta = \delta_M > 0$. Using the Weyl formula, they also showed that these $O(|t|^{-n/2})$ $L^1 \to L^\infty$ bounds are optimal in the sense that no such uniform bounds are possible if $\lambda\ell(\lambda) \to \infty$ as $\lambda \to \infty$. Using the bounds for each fixed Littlewood-Paley bump function $\beta(2^{-j}\,\cdot)$, one can of course obtain analogous $O(|t|^{-n/2})$ dispersive estimates involving $\widetilde{\beta}$ in (2.1) on intervals $[-\delta\lambda^{-1}, \delta\lambda^{-1}]$. So, after possibly changing scales in time and correspondingly scaling the Laplace-Beltrami operator, we may always assume that we have the bounds

$$\| \, \widetilde{\beta}(P/\lambda)e^{it\Delta_g} \, \|_{L^1(M)\to L^\infty(M)} \le C|t|^{-n/2}, \quad |t| \le \lambda^{-1}, \tag{2.3}$$

by virtue of [10, Lemma 2.5]. We also, trivially for *all* times t have the bounds

$$\| \, \widetilde{\beta}(P/\lambda)e^{it\Delta_g} \, \|_{L^2(M)\to L^2(M)} \le C, \quad C = \|\widetilde{\beta}\|_{L^\infty}. \tag{2.4}$$

As was noted in [10] one can use the Keel-Tao theorem [15, Theorem 1.2] to obtain the uniform dyadic Strichartz estimates

$$\big\|\widetilde{\beta}(P/\lambda)e^{it\Delta_g} f\big\|_{L_t^p L_x^q([0,\lambda^{-1}]\times M)} \le C\|f\|_{L^2(M)}, \tag{2.5}$$

if $\quad n(1/2 - 1/q) = 2/p$ and $2 \le p < \infty$ if $n \ge 3$, or $2 < p < \infty$ if $n = 2$.

We have excluded the case of $p = \infty$ in (2.5) since then $q = 2$ and the estimate is trivial (with $[0, \lambda^{-1}]$ replaced by any interval) by the spectral theorem. Also, for future use, note that the endpoint case in dimensions $n \geq 3$ involves the exponents $p = 2$ and $q = 2n/(n-2)$, for which we have $1/q' - 1/q = 2/n$, where, as usual, q' denotes the conjugate exponent. Being able to include this estimate will allow us to handle potentials $V \in L^{n/2}$ when $n \geq 3$, while the fact that the endpoint estimate for $n = 2$ breaks down, accounts for the reason that we assume that our potentials lie in $L^{1+\delta}$, some $\delta > 0$ if $n = 2$.

We shall use an equivalent variant of this estimate and the related estimate for inhomogeneous equations that will be formulated first for the unit interval to simplify the Fourier analysis to follow. We first, trivially note that (2.5) is equivalent to the estimate

$$\begin{aligned} \big\| \widetilde{\beta}(P/\lambda) e^{it\lambda^{-1}\Delta_g} e^{-t} f \big\|_{L^p_t L^q_x([0,1]\times M)} \\ \leq C\lambda^{1/p} \|f\|_{L^2(M)}, \ (p,q) \text{ as in } (2.5), \end{aligned} \tag{2.6}$$

and since $e^{is\Delta_g}$ has $L^2(M) \to L^2(M)$ operator norm one, we also have the damped "global" estimate

$$\begin{aligned} \big\| \widetilde{\beta}(P/\lambda) e^{it\lambda^{-1}\Delta_g} e^{-t} f \big\|_{L^p_t L^q_x([0,+\infty)\times M)} \\ \leq C\lambda^{1/p} \|f\|_{L^2(M)}, \ (p,q) \text{ as in } (2.5). \end{aligned} \tag{2.7}$$

Note that for the scaled Schrödinger operator in (1.10) we have

$$\big(i\lambda\partial_t + \Delta_g + i\lambda\big)\big(e^{it\lambda^{-1}\Delta_g} e^{-t} h\big)(x) = 0. \tag{2.8}$$

To proceed, let $\mathbb{1}_+(s) = \mathbb{1}_{[0,+\infty)}(s)$ denote the Heaviside function and

$$U(t) = \mathbb{1}_+(t) \widetilde{\beta}(P/\lambda) e^{it\lambda^{-1}\Delta_g} e^{-t} \tag{2.9}$$

be the operator in (2.7). For later use, let us note that we can rewrite this operator. Indeed, if we recall that

$$(2\pi)^{-1} \int_{-\infty}^{\infty} \frac{e^{it\tau}}{i\tau + 1}\, d\tau = \mathbb{1}_+(t) e^{-t},$$

we deduce that

$$U(t)f(x) = \frac{i\lambda}{2\pi} \int_{-\infty}^{\infty} \frac{e^{it\tau}}{-\lambda\tau + \Delta_g + i\lambda}\, \widetilde{\beta}(P/\lambda) f(x)\, d\tau. \tag{2.10}$$

Also, if we regard U as an operator sending functions of x into functions of x, t, then its adjoint is the operator

$$U^*F(x) = \int_0^\infty e^{-s}\big(e^{-is\lambda^{-1}\Delta_g}\widetilde{\beta}(P/\lambda)F(s,\,\cdot\,)\big)(x)\,ds. \tag{2.11}$$

Consequently,

$$\begin{aligned}\int U(t)U^*(s)F(s,x)\,ds \\ = \mathbb{1}_+(t)\int_0^\infty \Big(e^{i(t-s)\lambda^{-1}\Delta_g}e^{-(t-s)}\widetilde{\beta}^2(P/\lambda)e^{-2s}F(s,\,\cdot\,)\Big)(x)\,ds.\end{aligned} \tag{2.12}$$

Note also that if, say,

$$F(t,x) = 0, \quad t \notin [0,1], \tag{2.13}$$

then the solution to the scaled inhomogeneous Schrödinger equation with damping term

$$(i\lambda\partial_t + \Delta_g + i\lambda)w(t,x) = F(t,x), \quad w(0,\,\cdot\,) = 0 \tag{2.14}$$

is given by

$$\begin{aligned}w(t,x) &= (i\lambda)^{-1}\int_0^t \big(\big(e^{i(t-s)\lambda^{-1}\Delta_g}e^{-(t-s)}F(s,x)\big)\,ds \\ &= (2\pi)^{-1}\int_0^1\int_{-\infty}^\infty \frac{e^{i(t-s)\tau}}{-\lambda\tau + \Delta_g + i\lambda}F(s,\,\cdot\,)(x)\,d\tau ds.\end{aligned} \tag{2.15}$$

Thus, since $w(t,\,\cdot\,) = 0$ for $t < 0$, it follows from (2.9), (2.10) and (2.15) that

$$\begin{aligned}\widetilde{\beta}^2(P/\lambda)w(t,x) &= (i\lambda)^{-1}\int_0^t U(t)U^*(s)\big(e^{2s}\widetilde{\beta}^2(P/\lambda)F(s,\,\cdot\,)\big)(x)\,ds \\ &= (2\pi)^{-1}\int_0^1\int_{-\infty}^\infty \frac{e^{i(t-s)\tau}}{-\lambda\tau + \Delta_g + i\lambda}\widetilde{\beta}^2(P/\lambda)F(s,\,\cdot\,)(x)\,d\tau ds.\end{aligned} \tag{2.16}$$

Using these formulas, we claim that we can use the arguments of Burq, Gérard and Tzvetkov [10] along with the Keel-Tao [15] theorem to deduce the following.

Proposition 0.3. *Suppose that F satisfies the support assumption in* (2.13) *and that w solves* (2.14). *Then for exponents as in* (2.5) *and $\lambda \geq 1$ we have*

$$\big\|\widetilde{\beta}^2(P/\lambda)w\big\|_{L^p_t L^q_x(\mathbb{R}\times M)} \lesssim \lambda^{-1+1/p}\|F\|_{L^2_{t,x}([0,1]\times M)}, \tag{2.17}$$

and also

$$\big\|\widetilde{\beta}^2(P/\lambda)w\big\|_{L^p_t L^q_x(\mathbb{R}\times M)} \lesssim \lambda^{-1+2/p}\|F\|_{L^{p'}_t L^{q'}_x([0,1]\times M)}. \tag{2.18}$$

We remark that, like (1.13), the bounds in (2.17) are also optimal.

Proof. To use the dispersive estimates (2.3) of Burq, Gérard and Tzvetkov, let

$$V(t')f(x) = \mathbb{1}_{[0,\lambda^{-1}]}(t')U(\lambda t')f(x) = \mathbb{1}_{[0,\lambda^{-1}]}(t')e^{-\lambda t'}\widetilde{\beta}(P/\lambda)e^{it'\Delta_g}f(x). \tag{2.19}$$

We then clearly have

$$\|V(t')\|_{L^2(M)\to L^2(M)} = 0(1),$$

and (2.3) says that

$$\|V(t')(V(s'))^*\|_{L^1(M)\to L^\infty(M)} \lesssim |t'-s'|^{-n/2}.$$

We can use the Keel-Tao theorem along with these two inequalities to deduce that

$$\|V(t')f\|_{L^p_{t'}L^q_x([0,\lambda^{-1}]\times M)} \lesssim \|f\|_{L^2(M)},$$

as well as

$$\Big\|\int_0^{t'} V(t')V^*(s')G(s',\,\cdot\,)\,ds'\Big\|_{L^p_{t'}L^q_x([0,\lambda^{-1}]\times M)} \lesssim \|G\|_{L^{p'}_t L^{q'}_x([0,\lambda^{-1}]\times M)},$$

and

$$\Big\|\int_0^{\lambda^{-1}} V^*(s')G(s',\,\cdot\,)\,ds'\Big\|_{L^2(M)} \lesssim \|G\|_{L^{p'}_t L^{q'}_x([0,\lambda^{-1}]\times M)}.$$

Using (2.19) we deduce that these inequalities are equivalent to

$$\|U(t)f\|_{L^p_t L^q_x([0,1]\times M)} \lesssim \lambda^{1/p}\|f\|_{L^2(M)}, \tag{2.20}$$

as well as

$$\tag{2.21} \Big\|\int_0^t U(t)U^*(s)H(s,\,\cdot\,)\,ds\Big\|_{L^p_tL^q_x([0,1]\times M)} \lesssim \lambda^{2/p}\,\|H\|_{L^{p'}_tL^{q'}_x([0,1]\times M)}, \quad \text{if} \;\; H(s,\,\cdot\,)=0,\;\; s\notin[0,1],$$

and

$$\tag{2.22} \Big\|\int_0^1 U^*(s)H(s,\,\cdot\,)\,ds\Big\|_{L^2(M)} \lesssim \lambda^{1/p}\|H\|_{L^{p'}_tL^{q'}_x([0,1]\times M)},$$

respectively.

Using (2.21) with $H=e^{2s}F$ along with (2.16) we obtain the analog of (2.18) where the norms are taken over $[0,1]\times M$ since $\|H\|_{L^p_tL^q_x}\approx\|F\|_{L^p_tL^q_x}$ due to (2.13). Since, as we noted before w and hence $\widetilde{\beta}^2(P/\lambda)w$ vanishes for $t<0$, to prove the remaining part of (2.21) we need that we also have

$$\tag{2.23} \|\widetilde{\beta}^2(P/\lambda)w\|_{L^p_tL^q_x([1,\infty)\times M)} \lesssim \lambda^{-1+2/p}\|F\|_{L^{p'}_tL^{q'}_x([0,1]\times M)}.$$

Since for $t>1$

$$\int_0^t U(t)U^*(s)H(s,\,\cdot\,)\,ds = U(t)\Big(\int_0^1 U^*(s)H(s,\,\cdot\,)\,ds\Big), \quad H=e^{2s}F,$$

it is simple to check that by (2.22) we would have this inequality if

$$\tag{2.24} \|U(t)f\|_{L^p_tL^q_x([1,\infty)\times M)} \lesssim \lambda^{1/p}\|f\|_{L^2(M)}.$$

However, since for $j=1,2,\dots$ and $s\in[0,1]$

$$U(j+s)=e^{-j}e^{i\lambda^{-1}j\Delta_g}U(s),$$

and $\|e^{it\lambda^{-1}\Delta_g}\|_{L^2\to L^2}=1$, we deduce that

$$\|\widetilde{\beta}^2(P/\lambda)w\|_{L^p_tL^q_x([j,j+1]\times M)} \lesssim e^{-j}\lambda^{-1+2/p}\|F\|_{L^{p'}_tL^{q'}_x([0,1]\times M)}, \; j=1,2,3,\dots,$$

which of course yields (2.23), and, as a result, (2.17).

Since $\|U^*(s)\|_{L^2(M)\to L^2(M)} = O(1)$, using (2.16) along with (2.20) and (2.24) we find that if $H = e^{2s}F$

$$\begin{aligned}
&\big\|\widetilde{\beta}^2(P/\lambda)w\big\|_{L^p_tL^q_x(\mathbb{R}\times M)}\\
&\quad\le \lambda^{-1}\int_0^1 \big\|\mathbb{1}_+(t-s)U(t)U^*(s)H(s,\,\cdot\,)\big\|_{L^p_tL^q_x(\mathbb{R}\times M)}\,ds\\
&\quad\lesssim \lambda^{-1+1/p}\int_0^1 \|U^*(s)H(s,\,\cdot\,)\|_{L^2_x}\,ds\\
&\quad\lesssim \lambda^{-1+1/p}\int_0^1 \|F(s,\,\cdot\,)\|_{L^2_x}\,ds \le \lambda^{-1+1/p}\|F\|_{L^2_{t,x}([0,1]\times M)},
\end{aligned}$$

as desired, which completes the proof. □

By an argument we shall give in the next section the quasimode estimates (2.17) for the scaled Schrödinger operators in (1.10) imply the dyadic Strichartz estimates (2.3) of Burq, Gérard and Tzvetkov [10]. Unfortunately, though, as we noted before, we do not seem to be able to directly use Proposition 0.3 to obtain analogous estimates for $-H_V = \Delta_g - V$ with $V \in L^{n/2}(M)$, $n \ge 3$ or the 2-dimensional ones in Theorem 0.1, since Littlewood-Paley operators associated with H_V are not easily seen to be compatible with the corresponding ones involving $-\Delta_g$ if V is allowed to be singular.

It is for this reason that we need the bounds in Theorem 0.2 involving the Littlewood-Paley cutoff $\beta(-D_t/\lambda)$ in the time-variable. We are now in a position to prove this result. We shall use Proposition 0.3 and the following two elementary lemmas whose proofs we postpone for the moment.

Lemma 0.4. *Let $\alpha \in C([0,\infty))$ and $1 < p \le 2 < q < \infty$. Then*

$$\|\alpha(P)f\|_{L^q(M)} \le C_{p,q}\left(\sup_{\mu\ge 0}(1+\mu)^{n(\frac1p-\frac1q)}|\alpha(\mu)|\right)\|f\|_{L^p(M)}. \tag{2.25}$$

Lemma 0.5. *Suppose that*

$$|K_\lambda(t,t')| \le \lambda(1+\lambda|t-t'|)^{-2}. \tag{2.26}$$

Then if $1 \le p \le q \le \infty$ we have the following uniform bounds for $\lambda \ge 1$

$$\Big\|\int K_\lambda(t,t')\,G(t',\,\cdot\,)\,dt'\Big\|_{L^p_tL^q_x(\mathbb{R}\times M)} \le C\|G\|_{L^p_tL^q_x(\mathbb{R}\times M)}. \tag{2.27}$$

Also, suppose that

$$WF(t,x) = \int_{-\infty}^{\infty} \int_M K(t,x;t',y)\, F(t',y)\, d\mathrm{Vol}(y)\, ds'$$

and that for each $t, t' \in \mathbb{R}$ *the operator*

$$W_{t,t'} f(x) = \int_M K(x,t;y,t') f(y)\, d\mathrm{Vol}(y)$$

satisfies

$$\|W_{t,t'} f\|_{L^q(M)} \le \lambda (1+\lambda|t-t'|)^{-2}\, \|f\|_{L^r(M)}$$

for some $1 \le r \le q \le \infty$*. Then if* $1 \le s \le p \le \infty$ *we have for* $\lambda \ge 1$

$$\|WF\|_{L^p_t L^q_x(\mathbb{R}\times M)} \le C\lambda^{\frac{1}{s}-\frac{1}{p}} \|F\|_{L^s_t L^r_x(\mathbb{R}\times M)}. \tag{2.28}$$

Proof of Theorem 0.2. We first note that the kernel of $\beta(-D_t/\lambda)$ is $O(\lambda(1+\lambda|t-t'|)^{-2})$. Therefore, by (2.27)

$$\|\beta(-D_t/\lambda)\widetilde{\beta}^2(P/\lambda)w\|_{L^p_t L^q_x(\mathbb{R}\times M)} \lesssim \|\widetilde{\beta}^2(P/\lambda)w\|_{L^p_t L^q_x(\mathbb{R}\times M)}.$$

Therefore, if as in Proposition 0.3 and our theorem our forcing term F satisfies (2.13), it suffices to show that $\beta(-D_t/\lambda)(I-\widetilde{\beta}^2(P/\lambda))w$ enjoys the bounds in (1.13) and (1.14).

Recalling (2.15), this means that it suffices to show that

$$\Big\| \int_0^1 \int_{-\infty}^{\infty} \frac{e^{i(t-s)\tau}}{-\lambda\tau - P^2 + i\lambda} \beta(-\tau/\lambda)\left(1-\widetilde{\beta}^2(P/\lambda)\right) F(s,\,\cdot\,))\, d\tau ds \Big\|_{L^p_t L^q_x(\mathbb{R}\times M)} \lesssim \lambda^{-1+1/p} \|F\|_{L^2_{t,x}([0,1]\times M)}, \tag{2.29}$$

as well as

$$\Big\| \int_0^1 \int_{-\infty}^{\infty} \frac{e^{i(t-s)\tau}}{-\lambda\tau - P^2 + i\lambda} \beta(-\tau/\lambda)\left(1-\widetilde{\beta}^2(P/\lambda)\right) F(s,\,\cdot\,))\, d\tau ds \Big\|_{L^p_t L^q_x(\mathbb{R}\times M)} \lesssim \lambda^{-1+2/p} \|F\|_{L^{p'}_t L^{q'}_x([0,1]\times M)}. \tag{2.30}$$

To use Lemma 0.5 set

$$\alpha(t,s;\mu) = \int_{-\infty}^{\infty} \frac{e^{i(t-s)\tau}}{-\lambda\tau - \mu^2 + i\lambda} \beta(-\tau/\lambda)\left(1-\widetilde{\beta}^2(\mu/\lambda)\right) d\tau,$$

and note that, by (2.2) and the support properties of β we have for $j=0,1,2$

$$\lambda\,\big|\lambda^j\,\partial_\tau^j\big((1-\widetilde{\beta}^2(\mu/\lambda))\beta(-\tau/\lambda)(-\lambda\tau-\mu^2+i\lambda)^{-1}\big)\,\big|\lesssim\lambda(\mu^2+\lambda^2)^{-1},$$

which, by a simple integration parts argument, translates to the bound

$$|\alpha(t,s;\mu)|\lesssim\lambda(1+\lambda|t-s|)^{-2}\cdot(\mu^2+\lambda^2)^{-1}.$$

If we use Lemma 0.4 we deduce from this that the "frozen operators"

$$T_{t,s}h(x)=\int_{-\infty}^{\infty}\frac{e^{i(t-s)\tau}}{-\lambda\tau-P^2+i\lambda}\beta(-\tau/\lambda)\,\big(1-\widetilde{\beta}^2(P/\lambda)\big)h(x)\,d\tau,$$

satisfy

$$\begin{aligned}(2.31)\quad \|T_{t,s}h\|_{L^q(M)}&\lesssim\lambda(1+\lambda|t-s|)^{-2}\cdot\lambda^{-2+n(1/2-1/q)}\|h\|_{L^2(M)}\\&=\lambda(1+\lambda|t-s|)^{-2}\cdot\lambda^{-2+2/p}\|h\|_{L^2(M)},\end{aligned}$$

as well as

$$\begin{aligned}(2.32)\quad \|T_{t,s}h\|_{L^q(M)}&\lesssim\lambda(1+\lambda|t-s|)^{-2}\cdot\lambda^{-2+n(1/q'-1/q)}\|h\|_{L^{q'}(M)}\\&=\lambda(1+\lambda|t-s|)^{-2}\cdot\lambda^{-2+4/p}\|h\|_{L^{q'}(M)},\end{aligned}$$

due to the fact that our assumption on the exponents in (2.5) means that $n(1/2-1/q)=2/p$ and $n(1/q'-1/q)=2n(1/2-1/q)=4/p$.

If we combine (2.31) and (2.28), we conclude that the left side of (2.29) is dominated by

$$\lambda^{\frac12-\frac1p}\cdot\lambda^{-2+2/p}\|F\|_{L^2_{t,x}(\mathbb{R}\times M)}=\lambda^{-\frac32+\frac1p}\|F\|_{L^2_{t,x}(\mathbb{R}\times M)},$$

which is better than the bounds posited in (2.29) by a factor of $\lambda^{-1/2}$.

Similarly, if we combine (2.32) and (2.28), we find that the left side of (2.30) is dominated by

$$\lambda^{\frac{1}{p'}-\frac1p}\lambda^{-2+4/p}\|F\|_{L^{p'}_tL^{q'}_x(\mathbb{R}\times M)}=\lambda^{-1+2/p}\|F\|_{L^{p'}_tL^{q'}_x(\mathbb{R}\times M)},$$

as desired, which completes the proof. □

To conclude this section, for the sake of completeness let us now prove the lemmas, both of which are well known. The first is a slight generalization of Lemma 2.3 in [9], for instance, while the second lemma is essentially Theorem 0.3.6 in [25].

Proof of Lemma 0.4. Since

$$(1+P)^{-n(\frac12-\frac1q)}:\, L^2(M)\to L^q(M)$$

and

$$(1+P)^{-n(\frac1p-\frac12)}:\, L^p(M)\to L^2(M),$$

by orthogonality, we obtain

$$\begin{aligned}\|\alpha(P)f\|_{L^q(M)} &\lesssim \big\|\,(1+P)^{n(\frac12-\frac1p)}\alpha(P)f\,\big\|_{L^2(M)}\\ &\le \big(\sup_{\mu\ge 0}(1+\mu)^{n(\frac1p-\frac1q)}|\alpha(\mu)|\big)\cdot \|(1+P)^{-n(\frac1p-\frac12)}f\|_{L^2(M)}\\ &\lesssim \big(\sup_{\mu\ge 0}(1+\mu)^{n(\frac1p-\frac1q)}|\alpha(\mu)|\big)\cdot\|f\|_{L^p(M)},\end{aligned}$$

as desired. □

Proof of Lemma 0.5. To obtain (2.27) we note that by Minkowski's inequality and (2.26)

$$\Big\|\int K_\lambda(t,t')\,G(t',\,\cdot\,)\,dt'\,\Big\|_{L^q_x(M)} \le \int \lambda(1+\lambda|t-t'|)^{-2}\,\|G(t',\,\cdot\,)\|_{L^q_x(M)}\,dt'.$$

Taking the L^p_t-norm of both sides and using Young's inequality yields

$$\begin{aligned}&\Big\|\int K_\lambda(t,t')\,G(t',\,\cdot\,)\,dt'\Big\|_{L^p_tL^q_x(\mathbb{R}\times M)}\\ &\quad\le \Big(\int\Big|\int \lambda(1+\lambda|t-t'|)^{-2}\,\|G(t',\,\cdot\,)\|_{L^q_x(M)}\,dt'\Big|^p\,dt\Big)^{1/p}\\ &\quad\le C\Big(\int\,\|G(t,\,\cdot\,)\|^p_{L^q_x(M)}\,dt\Big)^{1/p}=\|G\|_{L^p_tL^q_x(\mathbb{R}\times M)},\end{aligned}$$

as desired.

One also obtains (2.28) from this argument after noting that, by Young's inequality, convolution with $\lambda(1+\lambda|t|)^{-2}$ has $L^s(\mathbb{R})\to L^p(\mathbb{R})$ operator norm which is $O(\lambda^{\frac1s-\frac1p})$. □

3. Strichartz estimates on compact manifolds

Let us now see how we can use the first estimate in Theorem 0.2 to prove the dyadic Strichartz estimate (2.5) of Burq, Gérard and Tzvetkov [10]. This simple argument will serve as a model for the one we shall use to prove the same sort of bounds where we replace $-\Delta_g$ with $H_V = -\Delta_g + V$, with V singular.

Let us first recall that the spectrum of $\sqrt{-\Delta_g}$ is nonnegative and discrete. If we account for multiplicity, we can arrange the eigenvalues, $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \dots$ and the associated L^2-normalized eigenfunctions

$$-\Delta_g e_j = \lambda_j^2, \quad \int_M |e_j|^2 \, dV_g = 1$$

form an orthonormal basis for $L^2(M)$. If then

$$E_j f(x) = \Big(\int_M f \, \overline{e_j} \, dV_g \Big) \, e_j(x)$$

denotes the projection onto the j-th eigenspace we have

$$e^{it\Delta_g} f = \sum_{j=0}^{\infty} e^{-it\lambda_j^2} E_j f.$$

To prove (2.5) it clearly suffices to show that for large λ we have the uniform bounds

$$\text{(3.1)} \quad \Big\| \eta(\lambda t) e^{it\Delta_g} f_\lambda \Big\|_{L^p_t L^q_x(\mathbb{R}\times M)} \le C \|f_\lambda\|_{L^2(M)}, \\ \text{if } \operatorname{spec} f_\lambda \subset [9\lambda/10, 11\lambda/10] \quad \text{and} \quad \eta \in C_0^\infty((0,1)) \quad \text{is fixed.}$$

The assumption on the spectrum of f_λ is that $E_j f_\lambda = 0$ if $\lambda_j \notin [9\lambda/10, 11\lambda/10]$, and we choose this interval since we are assuming that the Littlewood-Paley bump function arising in Theorem 0.2 satisfies

$$\text{(3.2)} \qquad \beta(s) = 1 \quad \text{on} \quad [3/4, 5/4] \quad \text{and} \quad \operatorname{supp} \beta \subset (1/2, 2).$$

To be able to use (1.13) we note that, after rescaling, (3.1) is equivalent to the statement that

$$\text{(3.1}'\text{)} \qquad \|w\|_{L^p_t L^q_x(\mathbb{R}\times M)} \le C\lambda^{1/p} \|f_\lambda\|_{L^2(M)}, \\ \text{with } w(t,x) = \eta(t) \cdot e^{it\lambda^{-1}\Delta_g} f_\lambda(x).$$

To be able to use Theorem 0.2 we shall use the following simple lemma.

Lemma 0.6. *Let w be as in (3.1′) with η and f_λ as in (3.1) and suppose that the exponents (p,q) are as in (2.5). Then for large enough λ and each $N = 1, 2, \dots$ we have the uniform bounds*

$$\| (I - \beta(-D_t/\lambda))w \|_{L^p_t L^q_x(\mathbb{R}\times M)} \le C_N \lambda^{-N} \|f_\lambda\|_{L^2(M)}. \tag{3.3}$$

Proof. We first note the Fourier transform of $t \to \eta(t)e^{-it\lambda^{-1}\lambda_j^2}$ is $\hat{\eta}(\tau + \lambda_j^2/\lambda)$ and so

$$\big(I - \beta(-D_t/\lambda)\big)w(t,x) = \sum_{\lambda_j\in[9\lambda/10, 11\lambda/10]} a(t;\lambda_j)E_j f_\lambda(x), \tag{3.4}$$

where

$$a(t;\mu) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{it\tau} \hat{\eta}(\tau + \mu^2/\lambda) \big(1 - \beta(-\tau/\lambda)\big)\, d\tau. \tag{3.5}$$

Since for q as in (2.5) we have $2 < q \le 2n/(n-2)$ for $n \ge 3$ and $2 < q < \infty$ for $q = 2$, the following Sobolev estimates are valid

$$\|u\|_{L^q(M)} \lesssim \| (I - \Delta_g)^{1/2} u \|_{L^2(M)}. \tag{3.6}$$

Therefore, by the spectral theorem,

$$\begin{aligned} &\Big\| \sum_{\lambda_j\in[9\lambda/10, 11\lambda/10]} a(t;\lambda_j)E_j f_\lambda \Big\|_{L^q(M)} \\ &\qquad \lesssim \lambda \Big\| \sum_{\lambda_j\in[9\lambda/10, 11\lambda/10]} a(t;\lambda_j)E_j f_\lambda \Big\|_{L^2(M)}. \end{aligned} \tag{3.7}$$

Next, since $2 \le p < \infty$, by Minkowski's inequality and Sobolev's theorem for $\mathbb{R}$ we therefore have

$$\begin{aligned} &\|(I - \beta(-D_t/\lambda))w\|_{L^p_t L^q_x(\mathbb{R}\times M)} \\ &\quad \lesssim \lambda \Big\| \sum_{\lambda_j\in[9\lambda/10, 11\lambda/10]} a(t;\lambda_j)E_j f_\lambda \Big\|_{L^p_t L^2_x(\mathbb{R}\times M)} \\ &\quad \le \lambda \Big\| \sum_{\lambda_j\in[9\lambda/10, 11\lambda/10]} a(t;\lambda_j)E_j f_\lambda \Big\|_{L^2_x L^p_t(\mathbb{R}\times M)} \\ &\quad \le \lambda \Big\| \sum_{\lambda_j\in[9\lambda/10, 11\lambda/10]} |D_t|^{1/2-1/p} a(t;\lambda_j)E_j f_\lambda \Big\|_{L^2_x L^2_t(\mathbb{R}\times M)}. \end{aligned}$$

Since, by orthogonality

$$\Big\| \sum_{\lambda_j\in[9\lambda/10,11\lambda/10]} |D_t|^{1/2-1/p} a(t;\lambda_j)E_jf_\lambda\Big\|^2_{L^2_xL^2_t(\mathbb{R}\times M)} = \sum_{\lambda_j\in[9\lambda/10,11\lambda/10]} \big\| \, |D_t|^{1/2-1/p}a(t;\lambda_j)E_jf_\lambda\big\|^2_{L^2_xL^2_t(\mathbb{R}\times M)},$$

we conclude that

$$\|(I-\beta(-D_t/\lambda)w\|_{L^p_tL^q_x(\mathbb{R}\times M)} \lesssim \lambda \big(\sup_{\mu\in[9\lambda/10,11\lambda/10]} \big\| \, |D_t|^{1/2-1/p}a(t;\mu)\big\|_{L^2_t(\mathbb{R})}\big) \cdot \|f_\lambda\|_{L^2(M)}. \tag{3.8}$$

Next, by Plancherel's theorem, (3.2) and (3.5),

$$\begin{aligned} \| \, |D_t|^{1/2-1/p}a(t;\mu)\|^2_{L^2_t(\mathbb{R})} &= (2\pi)^{-1}\int_{-\infty}^{\infty} |\tau|^{1-2/p}\left|\hat\eta(\tau+\mu^2/\lambda)\right|^2\left|(1-\beta(-\tau/\lambda))\right|^2 d\tau \\ &\lesssim \int_{\tau\notin[-5/\lambda/4,\,-3\lambda/4]} |\tau|^{1-2/p}\,|\hat\eta(\tau+\mu^2/\lambda)|^2\,d\tau. \end{aligned}$$

Note that $|\tau+\mu^2/\lambda| \approx (|\tau|+\lambda)$ if if $\tau\notin[-5\lambda/4,-3\lambda/4]$ and $\mu\in[9\lambda/10, 11\lambda/10]$, and since $\hat\eta\in\mathcal{S}(\mathbb{R})$ the preceding inequality leads to the trivial bounds

$$\sup_{\mu\in[9\lambda/10,11\lambda/10]} \| \, |D_t|^{1/2-1/p}a(t;\mu)\|_{L^2_t(\mathbb{R})} \lesssim \lambda^{-N}. \tag{3.9}$$

Combining this inequality with (3.8) yields (3.3). □

Using the lemma and the first estimate in Theorem 0.2 it is very easy to prove (3.1′). We first note that we may apply this Theorem, since if w is as in (3.1′),

$$(i\lambda\partial_t+\Delta_g+i\lambda)w(t,x) = \big(i\lambda\eta'(t)+i\lambda\eta(t)\big)\cdot e^{it\lambda^{-1}\Delta_g}f_\lambda(x) \quad \text{vanishes if } \ t\notin[0,1], \tag{3.10}$$

and $w(0,x)=0$. Therefore by (1.13) and (3.3) we have

$$\begin{aligned}(3.11)\quad &\|w\|_{L^p_tL^q_x(\mathbb{R}\times M)}\\ &\le \|\beta(-D_t/\lambda)w\|_{L^p_tL^q_x(\mathbb{R}\times M)}+\|(I-\beta(-D_t/\lambda))w\|_{L^p_tL^q_x(\mathbb{R}\times M)}\\ &\lesssim \lambda^{-1+1/p}\big\|\, i\lambda(\eta'(t)+\eta(t))\cdot e^{it\lambda^{-1}\Delta_g}f_\lambda\,\big\|_{L^2(\mathbb{R}\times M)}\\ &\quad+\lambda^{-N}\|f_\lambda\|_{L^2(M)}\\ &\lesssim \lambda^{1/p}\|f_\lambda\|_{L^2(M)},\end{aligned}$$

as desired.

Let us now prove dyadic high-frequency estimates for e^{-itH_V} where

$$H_V=-\Delta_g+V \tag{3.12}$$

with

$$V\in L^{n/2}(M)\ \text{if}\ \ n\ge 3,\ \text{and}\ \ V\in L^{1+\delta}(M),\ \text{some}\ \ \delta>0\ \text{if}\ \ n=2. \tag{3.13}$$

Let us focus first on the case where $n\ge 3$ and then handle $n=2$ later.

Under the assumption (3.13) H_V defines a self-adjoint operator which is bounded from below. See e.g., [3]. We wish to prove the analog of (2.5) for the operators e^{-itH_V}. If necessary, we may add a constant to V so that

$$H_V\ge 0 \tag{3.14}$$

as we shall always assume. This will not affect our estimates, since, if we, say add the constant N to V the two different Schrödinger operators will agree up to a factor $e^{\pm itN}$.

Just as with the $V=0$ case, the eigenvalues of the operator $\sqrt{H_V}$ (defined by the spectral theorem) are nonnegative, discrete and tend to infinity. We can list them counting multiplicity as $0\le\mu_1\le\mu_2\le\dots$, and there is an associated orthonormal basis of eigenfunctions $\{e_j^V\}$

$$H_Ve_j^V=\mu_j^2e_j^V\quad\text{with}\quad\int_M|e_j^V|^2=1.$$

Analogous to the $V=0$ case, let E_j^V denote the projection onto the jth eigenspace,

$$E_j^Vf=\Big(\int_M f\,\overline{e_j^V}\Big)\cdot e_j^V.$$

Then for large λ we wish to prove the analog of (2.5):

$$\big\|e^{-itH_V} f_\lambda\big\|_{L^p_t L^q_x([0,\lambda^{-1}]\times M)} \le C\|f_\lambda\|_{L^2(M)} \quad \text{if } \operatorname{spec} f_\lambda \in [9\lambda/10, 11\lambda/10], \tag{3.15}$$

with the condition meaning that $E^V_j f_\lambda = 0$ if $\lambda_j \notin [9\lambda/10, 11\lambda/10]$. We are assuming the exponents (p,q) are as in (2.5). For later use, we note that since e^{-itH_V} is a unitary operator on $L^2(M)$ this estimate yields the unit-scale bounds

$$\big\|e^{-itH_V} f_\lambda\big\|_{L^p_t L^q_x([0,1]\times M)} \le C\lambda^{1/p}\,\|f_\lambda\|_{L^2(M)} \quad \text{if } \operatorname{spec} f_\lambda \in [9\lambda/10, 11\lambda/10], \tag{3.15$'$}$$

Since, by the spectral theorem

$$\|e^{-itH_V}\|_{L^2(M)\to L^2(M)} = 1,$$

the estimate trivially holds for $p=\infty$ and $q=2$. Therefore, by interpolation, since we are currently assuming that $n\ge 3$ it suffices to prove the estimate for the other endpoint, i.e., that for f_λ as in (3.15) we have

$$\big\|e^{-itH_V} f_\lambda\big\|_{L^2_t L^{2n/(n-2)}_x([0,\lambda^{-1}]\times M)} \le C\|f_\lambda\|_{L^2(M)}. \tag{3.16}$$

By scaling, this is equivalent to the statement that, for f_λ as above, we have

$$\big\|e^{-it\lambda^{-1}H_V} f_\lambda\big\|_{L^2_t L^{2n/(n-2)}_x([0,1]\times M)} \le C\lambda^{1/2}\|f_\lambda\|_{L^2(M)}.$$

Finally, as before, this is equivalent to showing that whenever

$$\eta\in C^\infty((0,1))$$

is fixed we have

$$\big\|w\big\|_{L^2_t L^{2n/(n-2)}_x(\mathbb{R}\times M)} \le C\lambda^{1/2}\|f_\lambda\|_{L^2(M)}, \quad \text{with } w(t,x) = \eta(t)\cdot e^{-it\lambda^{-1}H_V} f_\lambda, \tag{3.17}$$

with f_λ as above.

To proceed we need the analog of Lemma 0.6.

Lemma 0.7. *Let $n \ge 2$ and let w be as in* (3.17) *with $\eta \in C_0^\infty((0,1))$ and f_λ as in* (3.15) *and suppose that the exponents (p,q) are as in* (2.5)*. Then for large enough λ and each $N = 1, 2, \dots$ we have the uniform bounds*

$$\big\| (I - \beta(-D_t/\lambda))w \big\|_{L^p_t L^q_x(\mathbb{R}\times M)} \le C_N \lambda^{-N} \|f_\lambda\|_{L^2(M)}. \tag{3.18}$$

Since, for instance, by [3] and [4] we have the analog of (3.6),

$$\|u\|_{L^q(M)} \lesssim \|(I + H_V)^{1/2} u\|_{L^2(M)},$$

for q as in (2.5) it is clear that the proof of Lemma 0.6 yields (3.18). For the two-dimensional case one uses the fact that, if, as we are assuming $V \in L^{1+\delta}(M)$, $\delta > 0$, then V is in the Kato class $\mathcal{K}(M)$.

We now are positioned to prove (3.17). To take advantage of our assumption (3.13) for a given large $\ell > 1$, as in [3], let us split

$$V = V_{\le \ell} + V_{>\ell},$$

where

$$V_{>\ell}(x) = V(x) \text{ if } |V(x)| > \ell \text{ and } 0 \text{ otherwise.}$$

Our assumption (3.13) then yields

$$\|V_{>\ell}\|_{L^{n/2}(M)} = \delta(\ell), \quad \text{with } \delta(\ell) \to 0 \text{ as } \ell \to \infty, \tag{3.19}$$

and we also trivially have

$$\|V_{\le \ell}\|_{L^\infty(M)} \le \ell. \tag{3.20}$$

To use this we note that since $-H_V = \Delta_g - V$

$$\begin{aligned}(i\lambda\partial_t + \Delta_g + i\lambda)w &= (i\lambda\partial_t - H_V + i\lambda)w + Vw \\ &= (i\lambda\partial_t - H_V + i\lambda)w + V_{\le\ell}\, w + V_{>\ell}\, w,\end{aligned}$$

and also $w(0, \cdot\,) = 0$. So we can split

$$w = \widetilde{w} + w_{\le \ell} + w_{>\ell}, \tag{3.21}$$

where

$$(i\lambda\partial_t + \Delta_g + i\lambda)\widetilde{w} = (i\partial_t - H_V + i\lambda)w = \widetilde{F}, \quad \tilde{w}(0, \cdot\,) = 0, \tag{3.22}$$

$$(3.23)\qquad (i\lambda\partial_t+\Delta_g+i\lambda)w_{\le\ell}=V_{\le\ell}\,w=F_{\le\ell},\quad w_{\le\ell}(0,\,\cdot\,)=0,$$

and

$$(3.24)\qquad (i\lambda\partial_t+\Delta_g+i\lambda)w_{>\ell}=V_{>\ell}\,w=F_{>\ell},\quad w_{>\ell}(0,\,\cdot\,)=0,$$

Note that since $w(t,x)=0$, $t\notin(0,1)$ each of the forcing terms $\widetilde{F}$, $F_{\le\ell}$ and $F_{>\ell}$ also vanishes for such t which allows us to apply the estimates in Theorem 0.2 for $\widetilde{w}$, $w_{\le\ell}$ and $w_{>\ell}$.

By (3.18) and (3.21) we have for each $N=1,2,\dots$

$$(3.25)\qquad \begin{aligned}&\|w\|_{L^2_tL^{2n/(n-2)}_x(\mathbb{R}\times M)}\\ &\le \|\beta(-D_t/\lambda)w\|_{L^2_tL^{2n/(n-2)}_x(\mathbb{R}\times M)}+C_N\lambda^{-N}\|f\|_{L^2(M)}\\ &\le \|\beta(-D_t/\lambda)\widetilde{w}\|_{L^2_tL^{2n/(n-2)}_x(\mathbb{R}\times M)}\\ &\quad+\|\beta(-D_t/\lambda)w_{\le\ell}\|_{L^2_tL^{2n/(n-2)}_x(\mathbb{R}\times M)}\\ &+\|\beta(-D_t/\lambda)w_{>\ell}\|_{L^2_tL^{2n/(n-2)}_x(\mathbb{R}\times M)}+C_N\lambda^{-N}\|f\|_{L^2(M)}.\end{aligned}$$

Based on this we would obtain (3.17) if we could show that ℓ could be fixed large enough so that we have the following three inequalities

$$(3.26)\qquad \|\beta(-D_t/\lambda)\widetilde{w}\|_{L^2_tL^{2n/(n-2)}_x(\mathbb{R}\times M)}\le C\lambda^{1/2}\|f_\lambda\|_{L^2(M)},$$

as well as

$$(3.27)\qquad \|\beta(-D_t/\lambda)w_{\le\ell}\|_{L^2_tL^{2n/(n-2)}_x(\mathbb{R}\times M)}\le C\ell\lambda^{1/2}\|f_\lambda\|_{L^2(M)},$$

and finally

$$(3.28)\qquad \|\beta(-D_t/\lambda)w_{>\ell}\|_{L^2_tL^{2n/(n-2)}_x(\mathbb{R}\times M)}\le \tfrac{1}{2}\|w\|_{L^2_tL^{2n/(n-2)}_x(\mathbb{R}\times M)}.$$

Indeed we just combine (3.25)–(3.28) and use a simple bootstrapping argument which is justified since the right side of (3.28) is finite by the aforementioned Sobolev estimates for H_V.

To prove these three estimates we shall use Theorem 0.2, as we may, since, as we mentioned before, the forcing terms in (2.21), (2.22) and (2.23) obey the support assumption in (2.13).

To prove (1.13) we note that if $\widetilde{F}$ is as an (3.22) then, since w is as in (3.17), we have

$$\begin{aligned}\widetilde{F}(t,x) &= (i\partial_t - H_V + i\lambda)\big(\eta(t)e^{-it\lambda^{-1}H_V} f_\lambda(x)\big)\\ &= i\lambda(\eta'(t)+\eta(t))e^{-it\lambda^{-1}H_V} f_\lambda(x).\end{aligned}$$

Consequently, as in the $V=0$ case considered before, we may use the L^2-estimate, (1.13), in Theorem 0.2 to deduce that

$$\begin{aligned}&\big\|\beta(-D_t/\lambda)\widetilde{w}\big\|_{L^2_tL^{2n/(n-2)}_x(\mathbb{R}\times M)}\\ &\quad\le \lambda^{-1/2}\|i\lambda(\eta'(t)+\eta(t))\cdot e^{-it\lambda^{-1}H_V} f_\lambda\|_{L^2_{t,x}(\mathbb{R}\times M)}\\ &\quad\lesssim \lambda^{1/2}\|f_\lambda\|_{L^2(M)},\end{aligned}$$

as desired. Similarly, by (3.19) and the formula for w in (3.17), we obtain

$$\begin{aligned}\big\|\beta(-D_t/\lambda)w_{\le\ell}\big\|_{L^2_tL^{2n/(n-2)}_x(\mathbb{R}\times M)} &\le C\lambda^{-1/2}\|V_{\le\ell}\, w\|_{L^2_{t,x}(\mathbb{R}\times M)}\\ &\le C\ell\lambda^{-1/2}\|\eta(t)\cdot e^{-it\lambda^{-1}H_V} f_\lambda\|_{L^2_{t,x}(\mathbb{R}\times M)}\\ &\le C'\ell\lambda^{-1/2}\|f_\lambda\|_{L^2(M)},\end{aligned}$$

which is better than the inequality posited in (3.27).

Up until now we have not used the second inequality in Theorem 0.2. We need it to obtain (3.28) which allows the bootstrapping step. Note that

$$\frac{1}{q'}-\frac{1}{q}=\frac{2}{n}, \quad \text{if } \ q=2n/(n-2),\ q'=2n/(n+2).$$

Consequently if we use (1.14), (3.24), Hölder's inequality and (3.19) than we conclude that we can fix ℓ large enough so that we have

$$\begin{aligned}\big\|\beta(-D_t/\lambda)w_{>\ell}\big\|_{L^2_tL^{2n/(n-2)}_x(\mathbb{R}\times M)} &\le C\|V_{>\ell}\, w\|_{L^2_tL^{2n/(n+2)}_x(\mathbb{R}\times M)}\\ &\le C\|V_{>\ell}\|_{L^{n/2}(M)}\cdot\|w\|_{L^2_tL^{2n/(n-2)}_x(\mathbb{R}\times M)}\\ &\le \tfrac{1}{2}\|w\|_{L^2_tL^{2n/(n-2)}_x(\mathbb{R}\times M)},\end{aligned}$$

assuming, as we may, in the last step that, if $\delta(\ell)$ is as in (3.19), $C\delta(\ell)\le\frac{1}{2}$. Since this is the last of the three inequalities we had to prove, we have established (3.17) and hence (3.16).

Next, let us point out that for functions only involving low frequencies we have these types of estimates for unit time scales for all of the exponents in (2.5) in all dimensions. In other words if $C_0<\infty$ is fixed we claim that

$$\big\|e^{-itH_V}f\big\|_{L^p_tL^q_x([0,1]\times M)}\lesssim\|f\|_{L^2(M)},\quad \text{if }\ \operatorname{spec}f\subset[0,C_0]. \tag{3.29}$$

To see this we just note that by the Sobolev estimates that were used in the proof of Lemma 0.7 we have the following uniform bounds for all times t:

$$\|e^{-itH_V}f\|_{L^q(M)}\lesssim\|\sqrt{I+H_V}\,e^{-itH_V}f\|_{L^2(M)}\lesssim(1+C_0)\|f\|_{L^2(M)},$$

by the spectral theorem for f as in (3.29).

Next, let us show that, for large enough λ, when $n=2$ we have the estimates in (3.15) for each fixed (p,q) as in (2.5). Here we can take advantage of the fact that we must have $p>2$ and so the power of λ in (1.14) is negative. Since the bounds in (1.14) blow up as the exponents in (3.15) approach the "forbidden" pair $(p,q)=(2,\infty)$ for $n=2$, one needs to choose λ larger and larger as q increases. On the other hand, by interpolation, if we can establish (1.14) for a given q_0 and large enough λ, as before, by a trivial interpolation argument, we also obtain the bounds for all $q\in(2,q_0)$. To take advantage of our assumption on the potential in (3.13), let us thus fix an exponent q sufficiently large so that

$$\frac{1}{q'}-\frac{1}{q}\geq\frac{1}{1+\delta}. \tag{3.30}$$

We then can just split w into two terms, $w=\widetilde{w}+w_V$, one being $\widetilde{w}$ exactly as before and the other now solving

$$(i\lambda\partial_t+\Delta_g+i\lambda)w_V=Vw,\quad w_V(0,\,\cdot\,)=0.$$

In other words, $w_V=w_{\leq\ell}+w_{>\ell}$.

If we repeat the arguments for the $n\geq3$ case we then deduce that we would have the estimates in (3.15) for our exponents (p,q) if

$$\big\|\beta(-D_t/\lambda)\widetilde{w}\big\|_{L^p_tL^q_x(\mathbb{R}\times M)}\leq C\lambda^{1/p}\|f_\lambda\|_{L^2(M)},$$

as well as

$$\big\|\beta(-D_t/\lambda)w_V\big\|_{L^p_tL^q_x(\mathbb{R}\times M)}\leq\tfrac12\big\|w\big\|_{L^p_tL^q_x(\mathbb{R}\times M)},$$

assuming that λ is sufficiently large depending on q.

The first inequality follows from the argument used before. One just uses (1.13).

To prove the second inequality we repeat the proof of (3.28), noting that our assumptions on q and V ensure that, by Hölder's inequality $\|V\|_{L^r(M)} \le C_M\|V\|_{L^{1+\delta}(M)} < \infty$, where $1/r = 1/q' - 1/q$, due to (3.30). As a result, if we use (1.14) and repeat the proof of (3.28) we conclude that since $w(t,x) = 0$ for $t \notin [0,1]$ the left side of the second inequality is dominated by

$$\lambda^{-1+2/p}\|Vw\|_{L_t^{p'}L_x^{q'}([0,1]\times M)} \le C_q\lambda^{-1+2/p}\|V\|_{L^r(M)}\,\|w\|_{L_t^{p'}L_x^q([0,1]\times M)}$$
$$\le C_{q,V}\lambda^{-1+2/p}\|w\|_{L_t^pL_x^q([0,1]\times M)} < \tfrac{1}{2}\|w\|_{L_t^pL_x^q},$$

for large enough λ since $-1+2/p<0$. In the second inequality we used Hölder's inequality in the t variable and the fact that $p > p'$.

Proof of Theorem 0.1

Let us conclude the section by showing that the dyadic estimates that we have obtained can be used along with Littlewood-Paley estimates associated with H_V yield Theorem 0.1. For the sake of completeness, we shall give the simple proof of the Littlewood-Paley estimates involving singular potentials in an appendix.

Let us state the estimates we require. Recall that we are assuming as in (3.14), as we may, that $H_V \ge 0$, and so we may consider the operator $P_V = \sqrt{H_V}$. If β as in (1.7) and (1.8) is our Littlewood-Paley bump function, let

$$\beta_0(s) = 1 - \sum_{j=0}^{\infty}\beta(2^{-j}s) \in C_0^\infty([0,2)).$$

We shall then use the Littlewood-Paley estimates

$$\|h\|_{L^q(M)} \lesssim \|\beta_0(H_V)h\|_{L^q(M)} + \Big\|\Big(\sum_{j=0}^{\infty}|\beta(P_V/2^j)h|^2\Big)^{1/2}\Big\|_{L^q(M)}, \tag{3.31}$$

provided that V is as in Theorem 0.1 and

$$1<q<\infty \text{ if } n=2,3,4 \quad\text{and}\quad \tfrac{2n}{n+4}<q<\tfrac{2n}{n-4} \text{ if } n\ge 5. \tag{3.32}$$

We also note that since $\frac{2n}{n-2} < \frac{2n}{n-4}$ when $n \ge 5$, the exponents here include the exponents q arising in (1.2). Also, since $p\ge 2$ and $q \ge 2$ if (p,q) are as in (1.2), we obtain from (3.31) and Minkowski's inequality that we have for such exponents

$$\|e^{-itH_V}f\|_{L^p_tL^q_x([0,1]\times M)} \lesssim \|\beta_0(H_V)e^{-itH_V})f\|_{L^p_tL^q_x([0,1]\times M)} + \Big(\sum_{j=0}^{\infty}\big\|\beta(P_V/2^j)e^{-itH_V}f\big\|^2_{L^p_tL^q_x([0,1]\times M)}\Big)^{1/2}. \tag{3.33}$$

Additionally, by (3.29) we have

$$\|\beta_0(H_V)e^{-itH_V}f\|_{L^p_tL^q_x([0,1]\times M)} \le \|\beta_0(H_V)f\|_{L^2(M)}. \tag{3.34}$$

Similarly if we use (3.29) for small $j \ge 0$ and (3.15′) for large j we obtain

$$\big\|\beta(P_V/2^j)e^{-itH_V}f\big\|_{L^p_tL^q_x([0,1]\times M)} \le C_V 2^{j/p}\|\beta(P_V/2^j)f\|_{L^2(M)}, \quad j = 0, 1, \dots, \tag{3.35}$$

for some uniform constant $C_V < \infty$.

If we recall (1.7) and (1.8) and combine (3.33), (3.34) and (3.35) and use the spectral theorem we deduce that

$$\begin{aligned}\|e^{-itH_V}f\|_{L^p_tL^q_x([0,1]\times M)} &\lesssim \|\beta_0(H_V)f\|_{L^2(M)} + \Big(\sum_{j=0}^{\infty}\|2^{j/p}\beta(P_V/2^j)f\|^2_{L^2(M)}\Big)^{1/2} \\ &\lesssim \|\,(\sqrt{I+H_V})^{1/p}f\|_{L^2(M)}.\end{aligned} \tag{3.36}$$

This does not quite give us the estimate (1.6) in Theorem 0.1, since the right hand side of this inequality involves the Sobolev space $H^{1/p}(M)$ defined as in (1.3) by the operator $\sqrt{I-\Delta_g}$ as opposed to Sobolev space defined by the operator $\sqrt{I+H_V}$ as in (3.36). This, though, is easy to rectify. By standard arguments (see e.g., the appendix in [3] for the case where $n \ge 3$ and the one in [4] for the two-dimensional case), for the potentials we are considering and for the exponents q as above we have

$$\|\sqrt{I-\Delta_g}f\|_{L^2(M)} \approx \|\sqrt{I+H_V}f\|_{L^2(M)},$$

which means that the two L^2-Sobolev spaces of order 1 are comparable. By interpolation this means that we have

$$\|(\sqrt{I-\Delta_g})^{\sigma}f\|_{L^2(M)} \approx \|(\sqrt{I+H_V})^{\sigma}f\|_{L^2(M)}, \quad 0 \le \sigma \le 1, \tag{3.37}$$

since the estimate for $\sigma = 0$ is trivial.

If we combine (3.36) and (3.37) we obtain (1.6), which completes the proof of Theorem 0.1.

4. Sharpness of the quasimode estimates

Let us now show that our scaled quasimode estimates (1.13) cannot be improved on any compact manifold (M, g). We do this by a "Knapp-type" construction that is adapted to our scaled Schrödinger operators.

First, recall that we can choose local coordinates vanishing at a given point $x_0 \in M$ so that, in these coordinates,

$$\Delta_g = \partial_1^2 + \sum_{1<j,k\le n} g^{jk}(x)\partial_j\partial_k + \sum_{k=1}^n b_j(x)\partial_k. \tag{4.1}$$

Here, $\partial_k = \partial/\partial x_k$, $k = 1, \dots, n$. Here $(g^{jk})_{1<j,k\le n}$ is a smooth real positive definite matrix, and the b_j are also smooth real-valued functions. See, e.g., [13, Appendix C.5].

Fix $a \in C_0^\infty((-1/10, 1/10))$ which equals one near the origin, and set

$$w(t,x) = e^{i\lambda(x_1-t)}a(\lambda; t, x), \tag{4.2}$$

where

$$\begin{aligned} a(\lambda; t, x) &= a(x_1 + 2(t-1/2))\, a(\lambda^{1/2}(x_1 - 2t))\, a(\lambda^{1/2}|x'|), \\ &\text{with } \; x' = (x_2, \dots, x_n). \end{aligned} \tag{4.3}$$

Due to the exponential factor, the space-time Fourier transform of w is the space-time Fourier transform of $a(\lambda; t, x)$ translated by $(-\lambda, \lambda, 0, 0, \dots, 0)$, where the first-coordinate here is dual to the time coordinate and the rest dual to the x-coordinates. Since the Fouier transform of $a(\lambda; x, t)$ is $O((1 + \lambda^{-1/2}|(\tau, \xi)|)^{-N}$ for all N, and since $\beta(s) = 1$ for $s \in [3/4, 5/4]$, it follows that

$$(I - \beta(-D_t/\lambda))w = O(\lambda^{-N}), \quad \forall\, N. \tag{4.4}$$

Also, for each fixed t near $1/2$, $x \to a(\lambda; t, x)$ is one on a set of measure $\approx \lambda^{-n/2}$. Thus, by (4.3) and (4.4), for sufficiently large λ we have

$$\begin{aligned} &\|\beta(-D_t/\lambda)w\|_{L^p_t L^q_x([0,1]\times M)} \\ &\qquad \ge \|w\|_{L^p_t L^q_x([0,1]\times M)} - O(\lambda^{-N}) \ge c\lambda^{-n/2q}, \;\; \lambda \text{ large}, \end{aligned} \tag{4.5}$$

for some $c > 0$.

Note also that

$$F(t,x) = (i\lambda\partial_t + \Delta_g + i\lambda)w(t,x) = 0, \quad t \notin (0,1), \quad \text{and} \quad w(0,x) = 0. \tag{4.6}$$

Next, let us observe that

$$\Big(\sum_{1<j,k\le n} g^{jk}(x)\partial_j\partial_k + \sum_{k=1}^{n} b_j(x)\partial_k + i\lambda\Big)w = O(\lambda). \tag{4.7}$$

Also, if we rewrite $a(\lambda;t,x)$ as

$$a(\lambda;t,x) = a(\lambda^{1/2}(x_1-2t))\cdot\tilde{a}(\lambda;t,x), \quad \text{where} \quad \tilde{a}(\lambda;t,x) = a(x_1+2(t-1/2))\,a(\lambda^{1/2}|x'|), \tag{4.8}$$

then

$$i\lambda\partial_t\tilde{a}(\lambda;x,t) = O(\lambda), \text{ and } \partial_1^j\tilde{a}(\lambda;x,t) = O(1),\ j=1,2. \tag{4.9}$$

Consequently, by Leibniz's rule we have

$$\begin{aligned}\big(i\lambda\partial_t + \partial_1^2\big)w(t,x) = a(\lambda;t,x)\cdot\big(i\lambda\partial_t + \partial_1^2\big)e^{i\lambda(x_1-t)} \\ + e^{i\lambda(x_1-t)}\cdot i\lambda\partial_t\big(a(\lambda^{1/2}(x_1-2t))\big) \\ + 2\partial_1\big(a(\lambda^{1/2}(x_1-2t))\big)\cdot\partial_1\big(e^{i\lambda(x_1-t)}\big) + O(\lambda).\end{aligned} \tag{4.10}$$

Note that the first term in the right vanishes, as does the sum of the second and third terms.

Therefore, by (4.1) and (4.7)–(4.10), we conclude that if F is as in (4.6) we have

$$F = O(\lambda),$$

and since F is supported on a set of measure $\approx \lambda^{-n/2}$, we deduce that

$$\|F\|_{L^2_{t,x}(\mathbb{R}\times M)} \le \lambda^{1-n/4}. \tag{4.11}$$

If we combine (4.11) and (4.5) we deduce that there must be a $c_0 > 0$ so that for sufficiently large λ we have

$$\frac{\|\beta(-D_t/\lambda)w\|_{L^p_t L^q_x(\mathbb{R}\times M)}}{\|F\|_{L^2_{t,x}(\mathbb{R}\times M)}} \ge c_0\,\lambda^{-1}\cdot\,\lambda^{n(\frac14-\frac1{2q})} = c_0\,\lambda^{-1+1/p}, \qquad \text{if } \; n(1/2-1/q)=2/p. \tag{4.12}$$

By (4.6) and (4.12), we deduce that our L^2-quasimode estimate (1.13) is saturated on *any compact manifold*.

Some remarks

A challenging problem is to determine when the results of Burq, Gérard and Tzvetkov [10] can be improved, even just for the $V \equiv 0$ case. As they point out, the sharpness of $O(\lambda^{1/2})$ bounds for the $L^{2n/(n-2)}(S^n)$-norms of L^2-normalized spherical harmonics of the second author [21] imply that, on the sphere, the $L^2_t L^{2n/(n-2)}_x(S^n)$ Strichartz estimates (1.6) cannot be improved when $V \equiv 0$. On the other hand, they were able to use results from [21] and the special nature of the Laplacian on the sphere to show that for many cases besides this endpoint Strichartz estimate improved bounds hold here.

More dramatically, Bourgain and Demeter [8] were able to show on the torus $\mathbb{T}^n$ for the case where $q = p = 2(n+2)/n$ and $V \equiv 0$, the analog of (1.6) is valid with any Sobolev norm $H^\varepsilon(\mathbb{T}^n)$ in the right. It does not seem clear, though, how much improvements are possible here as one approaches the endpoint case of $(p, q) = (2, 2n/(n-2))$ beyond what holds just by interpolation with (1.6) for this exponent and the dramatic improvements for $p = q = 2(n+2)/n$. It would also be interesting to determine which singular potentials could be added so that e^{-itH_V} enjoys similar bounds for the latter pair of exponents on tori. Related partial results for resolvent problems were obtained in our joint work with Blair and Sire [3].

It would also be very interesting to determine whether there is a wide class of manifolds (beyond just spheres and tori) for which some of the estimates in (1.6) could be improved even for $V \equiv 0$. This seems to be a very challenging problem. One avenue, which is suggested by the Knapp example above and recent work on eigenfunctions (e.g., [5], [6], [23], [24] and [26]) might be to try to prove "Kakeya-Nikodym" estimates that link Strichartz estimates to ones involving products of powers of $L^2(M)$ norms and powers of supremums of L^2-norms over shrinking tubes.

The construction above suggests (not unexpectedly) that the tubes should be $\lambda^{-1/2}$-neighborhoods of the projection onto (t, x) space of integral curves of the Hamilton flow of

$$p(x,t,\tau,\xi) = \tau + Q(x,\xi),$$

where $Q(x,\xi)$ is the principal symbol of the $-\Delta_g$, which in local coordinates is given by

$$Q(x,\xi) = \sum_{j,k=1}^{n} g^{jk}(x)\xi_j\xi_k.$$

Here $g^{jk}(x)$ denotes the cometric.

To be more specific, one might expect to control high-frequency solutions of the Schrödinger equations by “Kakeya-Nikodym norms” over shrinking tubes about curves which in local coordinates are of the form $\gamma(t) = (t_0 + t, x_0 + x(t))$ where $x(t)$ is a geodesic with $\dot{x}(0) = \frac{\partial Q}{\partial \xi}(x_0, \xi_0)$ and speed $2Q(x_0, t_0)$, with $(t_0, x_0) \in \mathbb{R} \times M$.

This approach proved to be successful even for “critical norms” in the related case of estimates for eigenfunctions and in the aforementioned works improved eigenfunction estimates versus the universal bounds [22] of one of us were obtained for manifolds of nonpositive curvature. It would be very interesting to prove a corresponding result for high-frequency solutions of the unperturbed Schrödinger equation.

5. Appendix: Littlewood-Paley and multiplier bounds involving $L^{n/2}$-potentials

Consider a nonnegative self-adjoint operator $H_V = -\Delta_g + V$ on a compact manifold. Consider also a Mikhlin-type multiplier $m \in C^\infty(\mathbb{R}_+)$, meaning that

$$|\partial_\tau^j m(\tau)| \le C(1+\tau)^{-j}, \tau > 0, \;\; 0 \le j \le n/2 + 1. \tag{5.1}$$

We shall also assume that we have finite propagation speed for the wave equation associated to H_V. By this we mean that if $u, v \in L^2(M)$ and $d_g(\text{supp } u, \text{supp } v) = R$ then

$$\big(u, \cos t\sqrt{H_V}\, v\big) = 0, \;\; |t| < R. \tag{5.2}$$

Additionally for a given $2 < q_0 < \infty$ we shall assume that one has the Bernstein (dyadic Sobolev) estimates

$$(5.3)\quad \|\beta(\sqrt{H_V}/\lambda)u\|_{L^{q_0}(M)} \le C\lambda^{n(\frac{1}{2}-\frac{1}{q_0})}\|u\|_{L^2(M)},\ \lambda \ge 1,$$
$$\text{and}\ \ \|\beta_0(\sqrt{H_V})u\|_{L^{q_0}(M)} \le C\|u\|_{L^2(M)},\ \text{if}\ \beta_0(s) = 1 - \sum_{k=1}^{\infty}\beta(2^{-k}s).$$

Note that we would automatically have these bounds if we had the natural heat estimates of Li and Yau [16] for small times (see [4]).

Using a result of Blunck [7, Theorem 1.1] we claim that we can obtain the following.

Theorem 0.8. *Assume that* (5.2) *and* (5.3) *are valid and that m is as in* (5.1). *Then*

$$(5.4)\qquad \big\|m(\sqrt{H_V})f\big\|_{L^q(M)} \le C_q\|f\|_{L^q(M)} \quad \forall\, q \in (q_0', q_0).$$

If the standard small time pointwise heat kernel bounds held, then the the results in Theorem 0.8 hold for all $1 < q < \infty$ by Alexopoulos [2]. However, as was shown in [4], following [1] and [20], the standard small time heat kernel estimates need not hold if $V \in L^{n/2}(M)$, since there can be unbounded eigenfunctions. On the other hand, assuming that $V \in \mathcal{K}(M)$ ensures that these estimates hold by Sturm [27]. Here, $\mathcal{K}(M)$ denotes the Kato class. Recall also that if $V \in L^{n/2+\delta}(M)$, $\delta > 0$, then $V \in \mathcal{K}(M)$, and that $L^{n/2}(M)$ and $\mathcal{K}(M)$ enjoy the same scaling properties.

By the results in [3], (5.3) holds for $V \in L^{n/2}(M)$ if $2 < q_0 < \infty$ if $n = 3, 4$, and if $q_0 = 2n/(n-4)$ if $n \ge 5$. By results in [4], if $n = 2$ one also has this bound for all $2 < q_0 < \infty$ if $V \in \mathcal{K}(M)$. One obtains these dyadic Sobolev estimates in higher dimensions $n \ge 5$ directly from Sobolev estimates proved in [3] and for $n = 2, 3, 4$ by a simple orthogonality argument and the quasimode estimates proved in [3] and [4] in the other cases. By a result of Coulhon and Sikora [12], (5.2) is valid when H_V is nonnegative, self-adjoint and $V \in L^1(M)$. Alternately, one can use arguments from [4] to show this for the potentials that we are considering.

Consequently, the estimates (5.4) are valid for the potentials we are considering provided that $1 < q < \infty$ if $n = 2, 3, 4$ and for $\frac{2n}{n+4} < q < \frac{2n}{n-4}$ if $n \ge 5$.[1] Therefore, by a standard argument involving Radamacher functions

[1]We should point out that these results also are a consequence of estimates in [11], and, in fact, a stronger theorem involving weaker regularity assumption on

(see e.g., [25, p. 21]) we obtain the Littlewood-Paley estimates (3.31) that we used at the end of § 3:

Corollary 0.9. *If* $V \in L^{n/2}(M)$ *and* $1 < q < \infty$ *for* $n = 3, 4$ *or* $2n/(n+4) < q < 2n/(n-4)$ *for* $n \geq 5$ *then whenever* (5.1) *is valid we have* $m(\sqrt{H_V}) : L^q(M) \to L^q(M)$. *If* $n = 2$, $V \in \mathcal{K}(M)$ *and* $1 < q < \infty$ *then these bounds also hold. Consequently, under these hypotheses we have the Littlewood-Paley estimates*

$$\|h\|_{L^q(M)} \leq C_{q,V} \, \|\beta_0(\sqrt{H_V})h\|_{L^q(M)} + \Big\| \Big(\sum_{k=1}^{\infty} \big|\beta(\sqrt{H_V}/2^k)h\big|^2 \Big)^{1/2} \Big\|_{L^q(M)}, \tag{5.5}$$

for q *and* V *as above.*

Proof of Theorem 0.8. By Theorem 1.1 in Blunck [7], it suffices to show that, if $q \in (2, q_0)$, we have for some $\varepsilon_q > 0$

$$\Big\| \mathbb{1}_{B(x_0,r)} \, e^{-\frac{r^2}{2}H_V} \, \mathbb{1}_{B(y_0,r)} \Big\|_{L^{q'}(M) \to L^q(M)} \lesssim r^{-n(\frac{1}{q'}-\frac{1}{q})} e^{-\varepsilon_q \frac{d_g(x_0,y_0)}{r}}. \tag{5.6}$$

Here $\mathbb{1}_{B(x_0,r)}$ is the operator which is multiplication by the indicator function of the geodesic ball $B(x_0, r)$ of radius r centered at x_0. We may assume that r is small, say, smaller than half the injectivity radius, since otherwise the estimate is trivial due to (5.3) and a simple TT^* argument.

Let us first use (5.2) to deduce that

$$\Big\| \mathbb{1}_{B(x_0,r)} \, e^{-\frac{r^2}{2}H_V} \, \mathbb{1}_{B(y_0,r)} \Big\|_{L^2(M) \to L^2(M)} \lesssim e^{-c d_g(x_0,y_0)/r}, \tag{5.7}$$

for some $c > 0$. We should note that, like (5.3), (5.7) automatically holds when one has the standard small-time heat kernel estimates.

We may assume that $d_g(x_0, y_0) \geq 10r$, since otherwise the result is trivial. In this case, choose $\rho \in C_0^\infty((-1/2, 1/2))$ with $\rho(s) = 1$ near the origin.

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the multiplier m also holds. On the other hand, since the proof of Theorem 0.8 is simple, and since it easy to use the main Theorem in [7] to see the ingredients that are needed, we have chosen to include the proof here for the sake of completeness. We also do this since checking that the hypotheses for the very general results in [11] which are needed to obtain Theorem 0.8 is a bit laborious.

Then since

$$e^{-\frac{r^2}{2}H_V} = \frac{1}{\sqrt{2\pi}} \int \frac{1}{r} e^{-\frac{1}{2}(t/r)^2} \cos t\sqrt{H_V}\, dt,$$

and by (5.2)

$$\mathbb{1}_{B(x_0,r)} \cos t\sqrt{H_V}\, \mathbb{1}_{B(y_0,r)} = 0 \quad \text{if} \ \ t < R_0 = d_g(x_0,y_0)/2,$$

we must have

$$\begin{aligned}&\mathbb{1}_{B(x_0,r)}\, e^{-\frac{r^2}{2}H_V}\, \mathbb{1}_{B(y_0,r)} \\ &\quad = \mathbb{1}_{B(x_0,r)} \Big(\frac{1}{\sqrt{2\pi}} \int (1-\rho(t/R_0))\, \frac{1}{r} e^{-\frac{1}{2}(t/r)^2} \ \cos t\sqrt{H_V}\, dt \Big)\, \mathbb{1}_{B(y_0,r)}.\end{aligned}$$

Consequently,

$$\int |(1-\rho(t/R_0))|\, \frac{1}{r} e^{-\frac{1}{2}(t/r)^2}\, dt \lesssim e^{-cd_g(x_0,y_0)/r}, \quad R_0 = d_g(x_0,y_0)/2,$$

and so, by the spectral theorem,

$$\begin{aligned}&\Big\| \mathbb{1}_{B(x_0,r)}\, e^{-\frac{r^2}{2}H_V}\, \mathbb{1}_{B(y_0,r)} \Big\|_{L^2(M)\to L^2(M)} \\ &\quad \lesssim \Big\| \frac{1}{\sqrt{2\pi}} \int (1-\rho(t/R_0))\, \frac{1}{r} e^{-\frac{1}{2}(t/r)^2} \ \cos t\sqrt{H_V}\, dt \Big\|_{L^2(M)\to L^2(M)} \\ &\quad \lesssim e^{-cd_g(x_0,y_0)/r},\end{aligned}$$

as claimed.

Next, by (5.3)

$$\begin{aligned}&\big\|e^{-\frac{r^2}{4}H_V} f\big\|_{L^{q_0}(M)} \\ &\quad \le \|\beta_0(\sqrt{H_V}) e^{-\frac{r^2}{4}H_V} f\|_{L^{q_0}(M)} + \sum_{k=1}^{\infty} \big\|\beta(\sqrt{H_V}/2^k) e^{-\frac{r^2}{4}H_V} f\big\|_{L^{q_0}(M)} \\ &\quad \lesssim \|f\|_{L^2(M)} + \sum_{k=1}^{\infty} 2^{nk(\frac{1}{2}-\frac{1}{q_0})} \big\|\beta(\sqrt{H_V}/2^k) e^{-\frac{r^2}{4}H_V} f\big\|_{L^2(M)} \\ &\quad \lesssim \Big(1 + \sum_{k=1}^{\infty} 2^{nk(\frac{1}{2}-\frac{1}{q_0})} e^{-\frac{1}{4}(r2^k)^2}\Big)\, \|f\|_{L^2(M)} \\ &\quad \lesssim r^{-n(\frac{1}{2}-\frac{1}{q_0})} \|f\|_{L^2(M)}.\end{aligned}$$

By a TT^* argument this yields

$$\|e^{-\frac{r^2}{2}H_V}\|_{L^{q_0'}(M)\to L^{q_0}(M)} \lesssim r^{-n(\frac{1}{q_0'}-\frac{1}{q_0})}. \tag{5.8}$$

By the M. Riesz interpolation theorem, (5.7) and (5.8) yield (5.6) for all $q \in (2, q_0)$, as desired. □

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