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On the finiteness of ample models

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In this paper, we generalize the finiteness of models theorem in [2] to Kawamata log terminal pairs with fixed Kodaira dimension. As a consequence, we prove that a Kawamata log terminal pair with $\mathbb{R}$-boundary has a canonical model, and it can be approximated by log pairs with $\mathbb{Q}$-boundary and the same canonical model.

1. Introduction

Throughout this paper, the ground field k is the field of complex numbers. The purpose of this paper is to prove the following theorems on the finiteness of ample models and good minimal models.

Theorem 1.1 (Finiteness of Ample Models). *Let X be a projective normal variety of dimension n. Let V be a finite dimensional affine subspace of the vector space $\mathrm{WDiv}_{\mathbb{R}}(X)$ which is defined over $\mathbb{Q}$. Fix a nonnegative integer $0 \leq k \leq n$. Suppose $L \subset \mathcal{L}(V)$ (see Definition 2.3) is a closed rational polytope, such that for any $\mathbb{Q}$-divisor $\Delta \in L$, (X, Δ) is klt and $\kappa(X, K_X + \Delta) = k$.*

Then there are finitely many rational contractions $\pi_j : X \dashrightarrow Z_j, 1 \leq i \leq l$, such that if $\pi : X \dashrightarrow Z$ is the ample model of $K_X + \Delta$ for some $\mathbb{R}$-divisor $\Delta \in L$, then there is an index $1 \leq j \leq l$ and an isomorphism $\xi : Z_j \to Z$ such that $\pi = \xi \circ \pi_j$.

Theorem 1.2 (Finiteness of Good Minimal Models). *With notation as in Theorem 1.1, if for some $\mathbb{R}$-divisor $\Delta_0 \in \mathrm{int}(L)$, $K_X + \Delta_0$ has a good minimal model. Then for any $\Delta \in L$, $K_X + \Delta$ has a good minimal model. And there are finitely many birational contractions $\phi_j : X \dashrightarrow X_j$, $1 \leq i \leq m$ such that if $\phi : X \dashrightarrow Y$ is good minimal model of $K_X + \Delta$, for some $\mathbb{R}$-divisor $\Delta \in L$, then there is an index $1 \leq j \leq l$ such that $(Y, \phi_*\Delta)$ is crepant birational with $(X_j, \phi_{j*}\Delta)$.*

There are several interesting applications of these results. The first is an approximation of an effective klt pair with $\mathbb{R}$-boundary.

Corollary 1.3. *Suppose X is a projective variety, Δ is an $\mathbb{R}$-divisor such that (X,Δ) is a klt pair and $\kappa(X, K_X+\Delta)\geq 0$. Then for any $0<\epsilon\ll 1$, we can find finitely many $\mathbb{Q}$-divisor Δ_i, $1\leq i\leq l$, such that,*

1) *Δ is a convex $\mathbb{R}$-linear combination of Δ_i.*
2) $||\Delta-\Delta_i||<\epsilon$.
3) *There is a normal variety Z, such that $Z\cong \mathrm{Proj}\ R(X, K_X+\Delta_i)$ for every i.*

The idea is to use the Kodaira-type canonical bundle formula on Iitaka fibration. Let $f:(X,\Delta)\to Z$ be an lc-trivial fibration, see definition 2.8. The Kodaira-type canonical bundle formula says that

$$D\sim_{\mathbb{Q}} K_Y+B_Y+M_Y,$$

where B_Y is the boundary part and M_Y is the moduli part. Much is known about the birational behaviour of such formulas: In particular, it is known that, after passing to a certain birational model Y' of Y, the divisor $M_{Y'}$ is nef and for any higher birational model $Y''\to Y'$ the induced moduli part $M_{Y''}$ on Y'' is the pullback of $M_{Y'}$. We call such a variety Y' the Ambro model of f.

Two of the main conjectures in higher dimensional birational geometry are:

Conjecture 1.4 (B-semiampleness Conjecture). *Let (X,Δ) be a sub pair and let $f:(X,\Delta)\to Y$ be a klt-trivial fibration to an n-dimension variety Y. If Y is an Ambro model of f, then M_Y is semiample.*

Conjecture 1.5 (Nonvanishing). *Let (X,Δ) be a klt pair of dimension n with $\mathbb{Q}$-boundary. If $K_X+\Delta$ is pseudo-effective, then there exists an effective $\mathbb{Q}$-divisor D such that $K_X+\Delta\sim_{\mathbb{Q}} D$.*

Assume that these two conjectures hold in dimension $n-1$, we prove the following statement which says that if a pseudo-effective klt pair is the limit of effective pairs with non-maximum Iitaka dimension, then it is effective.

Corollary 1.6. *Assume Conjecture 1.4 and Conjecture 1.5 hold in dimension $n-1$. Let (X,Δ) be a klt pair with $\mathbb{Q}$-boundary. Suppose $K_X+\Delta$ is pseudo-effective, and there is an effective $\mathbb{Q}$-divisor H on X, such that for* *any* $0 < \epsilon \ll 1$, *we have*

$$0 \leq \kappa(X, K_X + \Delta + \epsilon H) < n.$$

Then there exists an effective $\mathbb{Q}$*-divisor* D *such that* $K_X + \Delta \sim_{\mathbb{Q}} D$.

Another interesting application concerns the MMP with scaling.

Corollary 1.7. *Let* (X, Δ) *be a klt pair with* $\mathbb{Q}$*-boundary,* $\kappa(X, K_X + \Delta) \geq 0$ *and* H *is a pseudo-effective* $\mathbb{Q}$*-divisor, such that* $(X, \Delta + H)$ *is klt. Suppose we can run the* $K_X + \Delta$ *MMP with scaling of* H *to get a sequence* $\phi_i : X_i \dashrightarrow X_{i+1}$ *of* $K_X + \Delta$ *flips and divisorial contractions and real numbers* $1 \geq \lambda_1 \geq \lambda_2 \geq \ldots$ *such that* $K_{X_i} + \Delta_i + tH_i$ *is nef for* $t \in [\lambda_i, \lambda_{i+1}]$. *Let* $\lambda := \lim_{i \to \infty} \lambda_i$. *If* $K_X + \Delta + H$ *has a good minimal model and* $\lambda \neq \lambda_i$ *for any* $i \in \mathbb{N}$, *then* $\lambda = 0$

2. Preliminary

2.1. Notations and Conventions

We will use the notations in [2] and [8].

Let $\mathbb{Q}$ be the field of rational numbers and $\mathbb{R}$ be that of real numbers, let $\mathbb{K}$ be $\mathbb{Q}$ or $\mathbb{R}$. Let $\pi : X \to U$ be a proper morphism of normal algebraic varieties. A $\mathbb{K}$-divisor (respectively $\mathbb{K}$-Cartier divisor) D on X is a $\mathbb{K}$-linear combination of prime divisors (respectively Cartier divisors). We say two $\mathbb{K}$-divisors D and D' are $\mathbb{K}$-linearly equivalent over U, denoted $D \sim_{\mathbb{K},U} D'$, if their difference is a $\mathbb{K}$-linear combination of principal divisors and an $\mathbb{K}$-Cartier divisor pulled back from U.

If $D = \sum d_i D_i$ is an $\mathbb{R}$-divisor on a normal variety X, then the round down of D is $\lfloor D \rfloor = \sum \lfloor d_i \rfloor D_i$, where $\lfloor d \rfloor$ denotes the largest integer which is at most d, the fractional part of D is $\{D\} = D - \lfloor D \rfloor$.

The sheaf $\mathcal{O}_X(D)$ is defined by

$$\mathcal{O}_X(D)(U) = \{f \in k(X) \mid (f)|_U + D|_U \geq 0\},$$

so that $\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor)$. Similarly we define $|D| = |\lfloor D \rfloor|$.

A pair (X, Δ) consists of a normal variety X over $\mathbb{C}$ and an $\mathbb{R}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. If $g : Y \to X$ is a birational morphism and E is a divisor on Y, the discrepancy $a(E, X, \Delta)$ is $-\mathrm{coeff}_E(\Delta_Y)$ where $K_Y + \Delta_Y := g^*(K_X + \Delta)$. A pair (X, Δ) is called subklt (respectively sublc) if for every birational morphism $Y \to X$ as above, $a(E, X, \Delta) > -1$ (respectively ≥ -1) for every divisor E on Y. A pair (X, Δ) is called klt (respectively lc) if (X, Δ) is subklt (respectively sublc) and B is effective.

Definition 2.1. ([Nak04] and [Fuj17]) Let D be an $\mathbb{R}$-divisor of a projective normal variety X of dimension n. Assume that $|D| \neq \emptyset$. Then we have a rational map

$$\phi_D := \phi_{|D|} : X \dashrightarrow \mathbb{P}(\mathrm{H}^0(X, \mathcal{O}_X(D))).$$

We set W_D to be image of ϕ_D.

Let $\mathbb{N}(D) := \{m \in \mathbb{N}; |mD| \neq \emptyset\}$. The Iitaka-dimension $\kappa(D) = \kappa(X, D)$ of D is defined as follows:

$$\kappa(D) = \begin{cases} -\infty, & \text{if } \mathbb{N}(D) = \emptyset \\ \max\{\dim W_{mD} \mid m \in \mathbb{N}(D)\}, & \text{if } \mathbb{N}(D) \neq \emptyset. \end{cases} \tag{1}$$

We define the invariant Iitaka dimension of D, denoted by $\kappa_\iota(X, D)$, as follows:

(2)
$$\kappa_\iota(X, D) = \begin{cases} \kappa(X, E), & \text{if there is an } \mathbb{R}\text{-divisor } E \geq 0 \text{ such that } E \sim_{\mathbb{R}} D, \\ -\infty, & \text{otherwise.} \end{cases}$$

Let A be an ample divisor on X, set

$$\nu(X, D, A) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \limsup_{m \to \infty} m^{-k} \dim H^0(X, \lfloor mD \rfloor + A) > 0\},$$

if $H^0(X, \lfloor mD \rfloor + A) \neq 0$ for infinitely many $m \in \mathbb{N}$ or $\nu(X, D, A) = -\infty$ otherwise.

We define the numerical Iitaka dimension of D, denoted by $\nu(X, D)$, to be

$$\nu(X, D) = \max_{A \text{ ample}} \nu(X, D, A)$$

It is easy to see that $\nu(X, D) \geq \kappa(X, D)$.

The relative invariant Iitaka dimension is defined as follows.

Let $X \to U$ be a projective morphism of normal varieties and D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then the relative invariant Iitaka dimension of D, denoted by $\kappa_\iota(X/U, D)$, is defined as follows: If there is an $\mathbb{R}$-divisor $E \geq 0$ such that $D \sim_{\mathbb{R},U} E$, set $\kappa_\iota(X/U, D) = \kappa_\iota(F, D|_F)$, where F is a general fiber of the Stein factorization of $X \to U$, and otherwise we set $\kappa_\iota(X/U, D) = -\infty$. Similarly, we define the relative numerical Iitaka dimension to be $\nu(X/U, D) = \nu(F, D|_F)$. When there is an $\mathbb{R}$-divisor $E \geq 0$ such that $D \sim_{\mathbb{R},U} E$, it is easy to see that $\kappa_\iota(X/U, D)$ and $\nu(X/U, D)$ do not depend on the choice of E and F.

Definition 2.2. ([BCHM10]) Let $\pi : X \to U$ be a projective morphism of normal projective varieties and let Δ be an $\mathbb{R}$-divisor on X such that (X, Δ) is a lc pair.

We say that a birational contraction $f : X \dashrightarrow Y$ over U is a **minimal model** (respectively **good minimal model**) of $K_X + \Delta$ over U, if given a common resolution $p : W \to X$ and $q : W \to Y$, we may write

$$p^*(K_X + \Delta) = q^*(K_Y + \Delta') + E$$

where $\Delta' = f_*\Delta$, $K_Y + \Delta$ is nef (respectively semiample), $E \geq 0$ is q-exceptional and the support of E contains the strict transform of the f-exceptional divisors, Y is normal, projective and $\mathbb{Q}$-factorial.

Let D be an $\mathbb{R}$-Cartier divisor on X, we say that $g : X \dashrightarrow Z$ is the **ample model** of D over U, if g is a rational map over U, Z is normal and projective over U and there is an ample divisor H over U on Z such that if $p : W \to X$ and $q : W \to Z$ resolve g then q is a contraction morphism and we may write $p^*D \sim_{\mathbb{Q},U} q^*H + E$, where $E \geq 0$ and for every $B \in |p^*D/U|_{\mathbb{R}}$, $B \geq E$. In particular, if D is $\mathbb{Q}$-Cartier and the $\mathcal{O}_U$-algebra $R(X, D) := \oplus_{m \geq 0} \pi_* \mathcal{O}_X(mD)$ is finitely generated, then $Z \cong \mathrm{Proj}_U R(X, D)$. If (X, Δ) is lc and $K_X + \Delta$ is big, the ample model of $K_X + \Delta$ is also called the **canonical model** of $K_X + \Delta$.

Definition 2.3. ([BCHM10]) Let $\mathbb{K}$ denote either the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let $\pi : X \to U$ be a morphism of projective normal varieties, let V be a finite dimensional affine subspace of the $\mathbb{K}$-vector space $\mathrm{WDiv}_{\mathbb{K}}(X)$ of Weil divisors on X. For a $\mathbb{K}$-divisor A, define

1) $V_A = \{\Delta \mid \Delta = A + B, B \in V\}$,
2) $\mathcal{L}(V) = \{\Delta \in V \mid (X, \Delta) \text{ is a lc pair}\}$,
3) $\mathcal{L}_A(V) = \{\Delta = A + B \in V_A \mid (X, \Delta) \text{ is a lc pair and } B \geq 0\}$,
4) $\mathcal{E}(V) = \{\Delta \in \mathcal{L}(V) \mid K_X + \Delta \text{ is pseudo-effective}\}$,
5) and given a rational contraction $\phi : X \dashrightarrow Z$, define
$\mathcal{A}_{\pi,\phi}(V) = \{\Delta \in \mathcal{L}(V) \mid \phi \text{ is the ample model of } K_X + \Delta \text{ over } U\ \}$.

For an $\mathbb{R}$-divisor $D = \sum d_i D_i$ where the D_i are the irreducible components of D, define $||D_i|| := \max\{|d_i|\}$.

Proposition 2.4. *Let $X \to U$ be a projective morphism of normal varieties and D be a pseudo-effective $\mathbb{R}$-divisor over U.*

(1). *If $D_1 \geq 0, D_2 \geq 0$ are two $\mathbb{R}$-divisors, then*

$$\nu(X/U, D_1 + D_2) \geq \max\{\nu(X/U, D_1), \nu(X/U, D_2)\}.$$

(2). *If $D \geq 0$ is an $\mathbb{R}$-divisor, then $\nu(X/U, D) \geq \kappa(X/U, D)$.*

(3). *If D' is an $\mathbb{R}$-divisor with $D' - D$ being pseudo-effective, then $\nu(X, D') \geq \nu(X, D)$.*

(4). *Suppose that $D_1 \sim_{\mathbb{R},U} N_1$ and $D_2 \sim_{\mathbb{R},U} N_2$ for some $\mathbb{R}$-divisors $N_1 \geq 0$ and $N_2 \geq 0$ such that $\mathrm{Supp}(N_1) \subseteq \mathrm{Supp}(N_2)$. Then we have $\kappa_\iota(X/U, D_1) \leq \kappa_\iota(X/U, D_2)$ and $\nu(X/U, D_1) \leq \nu(X/U, D_2)$.*

Proof. (1), (2) are obvious, (3) comes from [8, Proposition 5.2.7], (4) comes from [5, Remark 2.8]. □

Lemma 2.5. *Let (X, Δ) be a klt pair with $\mathbb{R}$-boundary. Then (X, Δ) has a good minimal model if and only if $\kappa_\iota(X, K_X + \Delta) = \nu(X, K_X + \Delta)$.*

Proof. When Δ is a $\mathbb{Q}$-divisor, this is proved in [3, Theorem 4.3]. This $\mathbb{R}$-divisor version comes from [5, Lemma 2.13]. □

Lemma 2.6. *Let $f : X \to U$ be a projective morphism, (X, Δ) a klt pair with $\mathbb{Q}$-boundary and $\phi : X \dashrightarrow X_M$ and $\phi' : X \dashrightarrow X'_M$ be minimal models for $K_X + \Delta$ over U. Then*

1) *the set of ϕ-exceptional divisors coincides with the set of divisors contained in $\mathbf{B}_-(K_X + \Delta/U)$ and if ϕ is a good minimal model for $K_X + \Delta$ over U, then this set also coincides with the set of divisors contained in $\mathbf{B}(K_X + \Delta/U)$.*

2) *$X'_M \dashrightarrow X_M$ is an isomorphism in codimension 1 such that*

$$a(E; X_M, \phi_*\Delta) = a(E; X'_M, \phi'_*\Delta)$$

for any divisor E over X, and

3) *if ϕ is a good minimal model of $K_X + \Delta$ over U, then so is ϕ'.*

Proof. This is [4, Lemma 2.4]. □

Theorem 2.7. *Let $\pi : X \to U$ be a projective morphism between normal projective varieties. If (X, Δ) is a klt pair with $\mathbb{Q}$-boundary and $\kappa(X, \Delta) \geq 0$, then the $\mathcal{O}_U$-algebra*

$$R(X, K_X + \Delta) := \oplus_{m \geq 0} \pi_* \mathcal{O}_X(m(K_X + \Delta))$$

is finitely generated, therefore the ample model of $K_X + \Delta$ over U exists.

Proof. This is [2, Corollary 1.1.2]. □

Definition 2.8. An lc-trivial fibration $f : (X, \Delta) \to Y$ consists of a proper surjective morphism $f : X \to Y$ between normal varieties with connected fibers and a $\mathbb{Q}$-divisor $\Delta = \Delta_{\geq 0} - \Delta_{\leq 0}$ written as the difference of its positive and negative parts, satisfying the following properties:

1) (X, Δ) is sublc over the generic point of Y.
2) Let F denotes the generic fiber of Y, then $h^0(F, \mathcal{O}_F(\lceil \Delta_{\leq 0}|_F \rceil)) = 1$.
3) There exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that

$$K_X + \Delta \sim_{\mathbb{Q}} f^* D.$$

The following is a simplified version of the Kodaira-type canonical bundle formula given in [6].

Theorem 2.9. *Let $f : (X, \Delta) \to Z$ be an lc-trivial fibration, B a reduced divisor such that f has slc fibers in codimension 1 over $Z \setminus B$. Then we can write*

$$K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Z + J + B_\Delta)$$

where J and B_Δ have the following properties

1) *J is a $\mathbb{Q}$-linear equivalence class, called the modular part. It depends only on the generic fiber $(X_{k(Z)}, \Delta_{k(Z)})$ and it is the push-forward of a nef class by some birational morphism $Z' \to Z$.*
2) *B_Δ is a $\mathbb{Q}$-divisor, called the boundary part. It is supported on B.*
3) *Let $D \subset B$ be an irreducible divisor. Then*

$$\mathrm{coeff}_D B_\Delta = \sup_E \{1 - \frac{1 + a(E, X, \Delta)}{\mathrm{mult}_E f^* D}\}$$

where the supremum is taken over all divisors over X that dominate D.

This implies the following.

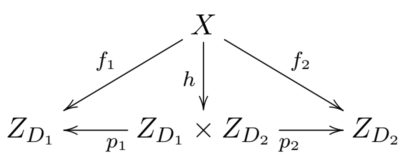
1) *If D is dominated by a divisor E such that $a(E, X, \Delta) < 0$ (respectively ≤ 0) then* $\operatorname{coeff}_D B_\Delta > 0$ *(respectively ≥ 0).*
2) *If Δ is effective then so is B_Δ.*
3) *If (X, Δ) is sublc then* $\operatorname{coeff}_D B_\Delta \leq 1$ *for every D and* $\operatorname{coeff}_D B_\Delta = 1$ *if and only if D is dominated by a divisor E such that $a(E, X, \Delta) = -1$.*

3. Proof of Main Theorem

In this section we prove the relative version of Theorem 1.1 and Theorem 1.2.

Lemma 3.1. *Let $X \to U$ be a projective morphism of normal varieties, fix an integer $0 \leq k \leq n$. Let V be a finite dimensional affine subspace of the vector space* $\mathrm{WDiv}_{\mathbb{R}}(X)$ *of Weil divisors on X which is defined over the rationals. Suppose L is a convex subset, and for every $\mathbb{Q}$-divisor $D \in L$, $\kappa(X/U, D) = k$ and the ample model of D over U exists. Suppose $f_D : X \dashrightarrow Z_D$ is the ample model of D over U. Then the generic fiber of f_D are the same for every $D \in L$, in particular, Z_D are birational equivalent for every $D \in L$.*

Proof. Choose two $\mathbb{Q}$-divisors $D_1, D_2 \in L$. After replacing X by a higher birational model and D_i by its pull back, we may assume that $X \to Z_{D_i}$ is a morphism. Then we have the following diagram



where p_1 and p_2 are two projections from the fiber product $Z_{D_1} \times Z_{D_2}$. Since Z_{D_i} is the ample model of D_i over U, there exists an ample divisor A_i on Z_{D_i}, such that

$$D_i \sim_{\mathbb{Q},U} f_i^* A_i + E_i,$$

where E_i is effective, $i = 1, 2$.

By definition of the fiber product, $A := \frac{1}{2}(p_1^* A_1 + p_2^* A_2)$ is an ample divisor on $Z_{D_1} \times Z_{D_2}$. Let Z be the normalization of image of X in $Z_{D_1} \times$

Z_{D_2}, then $A|_Z$ is ample on Z, in particular,

$$\kappa(Z/U, A|_Z) = \dim(Z) - \dim(U) \geq \dim(Z_{D_i}) - \dim(U) = \kappa(X/U, D_i) = k,$$

Because

$$\frac{1}{2}(D_1 + D_2) \sim_{\mathbb{Q},U} h^* A|_Z + \frac{1}{2}(E_1 + E_2),$$

and $\frac{1}{2}(D_1 + D_2) \in L$, we have that

$$k = \kappa(X/U, \frac{1}{2}(D_1 + D_2)) \geq \kappa(X/U, h^* A|_Z) = \kappa(Z/U, A|_Z) \geq k,$$

which means $\dim Z = \dim Z_{D_i}$. Because $f_i, i = 1, 2$ are algebraic contractions, it is easy to see that the morphisms $p_i|_Z : Z \to Z_i, i = 1, 2$ are birational. □

3.1. Finiteness of Ample Models

Let $h : X \to Y$ be an equidimensional algebraic fibration over U such that Y is smooth. Suppose (X, Δ) is klt with $\mathbb{Q}$-boundary, $\kappa(X/U, K_X + \Delta) = k \geq 0$ and h is birational equivalent to the Iitaka fibration of $K_X + \Delta$ over U, therefore $\kappa(X_\eta, (K_X + \Delta)|_\eta) = 0$, where X_η is the generic fiber of h. First we show how to get a pair (Y, C) of log general type from (X, Δ).

Since $\kappa(X/U, K_X + \Delta) = k \geq 0$, there is a $\mathbb{Q}$-effective divisor L such that

$$K_X + \Delta \sim_{\mathbb{Q},U} L.$$

We put $D := \max\{N \mid N \text{ is an effective } \mathbb{Q}\text{-divisor on } Y \text{ such that } L \geq h^* N\}$ and $F := L - h^* D$. Then we have

$$K_X + \Delta \sim_{\mathbb{Q},U} h^* D + F.$$

Because $\kappa(X_\eta, (K_X + \Delta)|_\eta) = 0$, then $h_* \mathcal{O}_X(\lfloor iF \rfloor)$ is a reflexive sheaf of rank 1 on Y. Moreover, since Y is smooth, $h_* \mathcal{O}_X(\lfloor iF \rfloor)$ is an invertible sheaf on Y. By construction, for any prime divisor P on Y, $\mathrm{Supp}(F)$ does not contain the whole fiber over the generic point of P, therefore $\mathcal{O}_Y \cong h_* \mathcal{O}_X(\lfloor iF \rfloor)$. Moreover, it is easy to see that D and F are both $\mathbb{Q}$-divisors.

Remark 3.2. Notations as above, we show how D and F vary depending on Δ. Define

$$\mathcal{D} = \{(a_1, a_2, ..., a_m) \in [0,1]^{\times m} \mid \sum_{i=1}^{m} a_i = 1\}$$

Let $h : X \to Y$ be an equidimensional algebraic fibration and Y is smooth, let X_η denote the generic fiber of h. Suppose $\{L_i, 1 \le i \le m\}$ are m linearly independent effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors, such that for every $(a_1, a_2, ..., a_m) \in \mathcal{D}$, we have $\kappa(X_\eta, \sum_{i=1}^{m} a_i L_i) = 0$. Define

$$D(a_1, ..., a_m) := \max\{N \mid N \text{ is an effective } \mathbb{Q}\text{-divisor on } Y \text{ such that } \sum_{i=1}^{m} a_i L_i \ge f^* N\}.$$

and

$$F(a_1, ..., a_m) := \sum_{i=1}^{m} a_i L_i - f^* D(a_1, ..., a_m).$$

Next we show that $D(a_1, ..., a_m)$ is a piecewise $\mathbb{Q}$-linear function on $\mathcal{D}$. Let P be a prime divisor on Y, suppose $h^* P = \sum_{j=1}^{l} b_j G_j$, where $G_j, 1 \le j \le l$ are prime divisors on X. Then the coefficient of P in $D(a_1, ..., a_m)$ is

$$\mathrm{coeff}_P D(a_1, ..., a_m) = \min\{\sum_{i=1}^{m} \frac{a_i}{b_j} \mathrm{coeff}_{G_j} L_i,\ 1 \le j \le m\}.$$

It is easy to see that $\mathrm{coeff}_P D(a_1, ..., a_m)$ is a piecewise linear function of $(a_1, ..., a_m)$. Moreover, because there are only finitely many prime divisors P such that $\mathrm{Supp}\, h^* P \subset \cup_{1 \le i \le m} \mathrm{Supp}\, L_i$, and $\frac{1}{b_j} \mathrm{coeff}_{G_j} L_i$ are rational numbers for every j, we can divide $\mathcal{D}$ into finitely many rational polytopes $\cup_k \mathcal{D}_k$ such that $D(a_1, ..., a_m)$ is a $\mathbb{Q}$-linear function in each $\mathcal{D}_k$.

Definition 3.3. Let X be a projective normal variety, V be a finite dimensional affine subspace of the vector space $\mathrm{WDiv}_{\mathbb{R}}(X)$. Let $\Omega \subset \mathcal{L}(V)$ be a subset, define

$$\mathrm{totaldiscrep}(X, \Omega) := \inf_{E,D} \{a(E, X, D) \mid E \text{ is a prime divisor over } X,\ D \in \Omega\}$$

Theorem 3.4. *Let $\pi : X \to U$ be a projective morphism of normal varieties and $\dim(X) = n$. Let V be a finite dimensional affine subspace of the vector* *space* $\mathrm{WDiv}_{\mathbb{R}}(X)$ *which is defined over the rationals. Fix an integer* $0 \leq k \leq n$. *Suppose* $L \in \mathcal{L}(V)$ *be a closed convex rational polytope, such that for any* $\Delta \in L$, (X, Δ) *is klt and* $\kappa(X/U, K_X + \Delta) = k$. *Then there exists a commutative diagram*

$$\begin{array}{ccc} X & \xleftarrow{\mu} & X' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle h} \\ U & \xleftarrow[g]{} & Y \end{array}$$

with the following properties.

1) μ *is birational morphism,* h *is an equidimensional algebraic fibration,* X' *has only* $\mathbb{Q}$*-factorial toroidal singularities and* Y *is smooth;*

2) *There exits a finite dimensional affine subspace* V' *of the vector space* $\mathrm{WDiv}_{\mathbb{R}}(X')$ *which is defined over* $\mathbb{Q}$, *a closed rational polytope* $L' \subset \mathcal{L}(V')$ *such that*

$$\mathrm{totaldiscrep}(X', L') = \mathrm{totaldiscrep}(X, L),$$

and a $\mathbb{Q}$*-linear isomorphism* $*' : L \to L'$. *For any divisor* $\Delta \in L$, $(X', \mathrm{Supp}(\Delta'))$ *is quasi-smooth (i.e.,* $(X', \mathrm{Supp}(\Delta'))$ *is toroidal), and*

$$\mu_* \mathcal{O}_{X'}(m(K_{X'} + \Delta')) \cong \mathcal{O}_X(m(K_X + \Delta)), \quad \forall m \in \mathbb{N};$$

3) h *is birational equivalent to the Iitaka fibration of* $K_X + \Delta$ *for every* $\Delta \in L$.

Proof. Fix $\Delta_1 \in L$, we may choose a birational projective morphism $\mu : X' \to X$, such that there exists a projective morphism $h : X' \to Y$ of smooth projective varieties over U and the restriction $h_\eta : X'_\eta \to Y_\eta$ over the generic point η of U is birational to the Iitaka fibration of $K_X + \Delta_1$. By Lemma 3.1, it is birational to the Iitaka fibration of $K_X + \Delta$, for every $\Delta \in L$. Let $\Delta_0 \in L$ be an inner rational point. By the weak semi-stable reduction theorem of Abramovich and Karu (cf. [1]), we can assume that $h : (X', D') \to (Y, D_Y)$ is an equidimensional toroidal morphism for some divisors D' on X' and D_Y on Y where (X', D') is quasi-smooth, Y is smooth and $\mu^{-1}(\Delta_0 \cup \mathrm{Sing}(X)) \subset D'$.

Let F be the exceptional divisor of μ, denote $a := \mathrm{totaldiscrep}(X, L)$. For a $\Delta \in L$, define Δ' by

$$\Delta' := \mu_*^{-1}\Delta + \max\{0, -a\}F$$

Since Δ_0 is an inner point of L, it is easy to see that $\mathrm{Supp}(\Delta') \in D'$, clearly (X', Δ') satisfies (1) and (2). □

The following theorem is the relative version of Theorem 1.1.

Theorem 3.5. *Let $\pi : X \to U$ be a projective morphism of normal varieties and $\dim(X) = n$. Let V be a finite dimensional affine subspace of the vector space $\mathrm{WDiv}_{\mathbb{R}}(X)$ which is defined over $\mathbb{Q}$. Fix a nonnegative integer $0 \leq k \leq n$. Suppose $L \subset \mathcal{L}(V)$ is a closed rational polytope, such that For any $\Delta \in L$, (X, Δ) is klt and $\kappa(X, K_X + \Delta) = k$.*

Then there are finitely many rational contractions $\pi_j : X \dashrightarrow Z_j, 1 \leq i \leq l$ over U, such that if $\pi : X \dashrightarrow Z$ is an ample model of $K_X + \Delta$ over U for some $\mathbb{Q}$-divisor $\Delta \in L$, then there is an index $1 \leq j \leq l$ and an isomorphism $\xi : Z_j \to Z$ such that $\pi = \xi \circ \pi_j$.

Proof. It is easy to see that we may assume that L is a convex polytope. By Theorem 3.4, there are a birational morphism $\mu : X' \to X$, an equidimensional algebraic fibration $h : X' \to Y$ over U and a $\mathbb{Q}$-linear isomorphism $*' : L \to L'$ such that $\mathrm{totaldiscrep}(X', L') = \mathrm{totaldiscrep}(X, L)$ and $\mu_* \mathcal{O}_{X'}(m(K_{X'} + \Delta')) \cong \mathcal{O}_X(m(K_X + \Delta))$, $\forall m \in \mathbb{N}$. Moreover, Z is the ample model of (X, Δ) over U if and only if it is the ample model of (X', Δ') over U. Therefore we can replace X by X' and L by L'. Then we can assume that there is an equidimensional algebraic fibration $h : X \to Y$ over U, Y is smooth and for every $\mathbb{Q}$-divisor $\Delta \in L$, $\kappa(X_\eta, (K_X + \Delta)|_\eta) = 0$, where X_η is the generic fiber of h.

By Remark 3.2, we may divide L into finitely many rational simplexes $\cup_i L_k$. For every $\Delta \in L$, there are divisors D_Δ and F_Δ satisfying the following equations

$$K_X + \Delta \sim_{\mathbb{Q},U} h^* D_\Delta + F_\Delta,$$

$$\mathcal{O}_Y \cong h_* \mathcal{O}_X(\lfloor j F_\Delta \rfloor)$$

for all $j \geq 0$. Moreover, D_Δ is a linear function of Δ in each L_k. It is easy to see that $(X, \Delta - F_\Delta) \to Y$ is an lc-trivial fibration for every $\Delta \in L$. Because this division is finite and rational, we only need to prove finiteness of the ample models for each L_k. Let $\{\Delta_i, 1 \leq i \leq m\}$ be the vertexes of L_k and let D_i, F_i be short for D_{Δ_i} and F_{Δ_i}.

By Theorem 2.9, for every $\mathbb{Q}$-divisor $\Delta_i, 1 \le i \le m$, there exist $\mathbb{Q}$-divisors B_i and J_i on Y, such that (Y, B_Y) is klt and

$$D_i \sim_{\mathbb{Q},U} K_Y + B_i + J_i.$$

Also we can find a birational morphism $f : Y' \to Y$, such that

$$K_{Y'} + B'_i + J'_i \sim_{\mathbb{Q}} f^*(K_Y + B_i + J_i).$$

where J'_i is a U-nef $\mathbb{Q}$-divisor for every $1 \le i \le m$, (Y', B'_i) is subklt and $f_*B'_i = B_i$. Because D_i is big, there are an ample $\mathbb{Q}$-divisor A' and a big $\mathbb{Q}$-divisors E_i on Y', such that $f^*D_i \sim_{\mathbb{Q}} f^*A' + E_i$. Let $\epsilon > 0$ be a small enough rational number such that $(Y', B'_i + C'_i + \epsilon f^*A'), 1 \le i \le m$ is subklt, where $C'_i \in |J'_i + \epsilon E_i|_{\mathbb{Q}}$ is a general member. Let $C_i := f_*(B'_i + C'_i)$ and $A \in |\epsilon A'|_{\mathbb{Q}}$ be a general member, then we have $(Y, A + C_i), 1 \le i \le m$ is klt and

$$(1 + \epsilon)(K_Y + B_i + J_i) \sim_{\mathbb{Q}} K_Y + A + C_i.$$

Let $W \subset \mathrm{WDiv}_{\mathbb{Q}}(Y)$ be the finite dimensional subspace spanned by the $\{C_i, 1 \le i \le m\}$. For a point $(a_1, ..., a_m) \in \mathcal{D}$, we have

$$(1 + \epsilon)(K_X + \sum_{i=1}^{m} a_i\Delta_i) \sim_{\mathbb{Q},U} h^*(K_Y + A + \sum_{i=1}^{m} a_iC_i) + R'_{(a_1,...,a_m)}$$

where $R'_{(a_1,...,a_m)}$ is effective and $\mathcal{O}_Y \cong h_*\mathcal{O}_X(\lfloor jR'_{(a_1,...,a_m)}\rfloor)$ for all $j \ge 0$. Therefore a projective variety Y_c is the ample model of $(X, \sum_{i=1}^{m} a_i\Delta_i)$ if and only if it is the canonical model of $(Y, A + \sum_{i=1}^{m} a_iC_i)$. Then we have a $\mathbb{Q}$-linear map $C_* : L_k \to \mathcal{L}_A(W)$, and for every $\mathbb{R}$-divisor $\Delta \in L_k$, (X, Δ) and (Y, C_Δ) have the same ample model over U. Therefore the claim is now immediate from [2, Corollary 1.1.5] . □

Remark 3.6. Because the map $\Delta \to D_\Delta \to C_\Delta$ is $\mathbb{Q}$-linear, by Theorem 3.5, it is easy to see that if L is a rational polytope, $\mathcal{A}_{\pi,\phi}(V) \cap L$ is also a rational polytope.

3.2. Finiteness of good minimal models

Lemma 3.7. *Let $\pi : X \to U$ be a projective morphism of normal varieties, V be a finite dimensional affine subspace of* $\mathrm{WDiv}_{\mathbb{R}}(X)$ *which is defined over the rationals. Fix a nonnegative integer $0 \le k \le n$, let $L \subset V$ be a closed* *convex rational polytope of $\mathbb{R}$-divisors, such that for any $\mathbb{Q}$-divisor $D \in L$, $\kappa(X/U, D) = k \geq 0$.*

If there is an $\mathbb{R}$-divisor $D_0 \in \operatorname{int}(L)$ satisfying $\nu(X/U, D_0) = \kappa_\iota(X/U, D_0)$. Then for any $\mathbb{R}$-divisor $D \in L$, we have $\nu(X/U, D) = \kappa_\iota(X/U, D)$.

Proof. Let $D_i, 1 \leq i \leq m$ be the vertexes of L, by assumption, D_i are $\mathbb{Q}$-divisors and $\kappa(X/U, D_i) = k$. Choose E_i such that $D_i \sim_{\mathbb{Q},U} E_i \geq 0$, then for every point $(a_1, a_2, ..., a_m) \in \mathcal{D}$, we have $\sum_{i=1}^m a_i D_i \sim_{\mathbb{R},U} \sum_{i=1}^m a_i E_i \geq 0$. By Proposition 2.4, we may replace D_i by E_i, then we may assume $D \geq 0$ for every $D \in L$ and replace $\kappa_\iota(X, D)$ by $\kappa(X, D)$.

If for some $D_0 \in \operatorname{int}(L)$, we have $\nu(X/U, D) = \kappa(X/U, D)$. Then for every $D \in L$, we have $\operatorname{Supp}(D) \subset \operatorname{Supp}(D_0)$. Thus the claim comes easily from the following inequality.

$$\kappa(X/U, D) \leq \nu(X/U, D) \leq \nu(X/U, D_0) = \kappa(X/U, D_0) = k = \kappa(X/U, D).$$

□

Theorem 3.8. *Let X be a projective variety, then there are at most countably many birational contractions $\phi_i : X \dashrightarrow X_i, i \in \mathbb{N}$, such that if X' is a projective normal $\mathbb{Q}$-factorial variety and $\phi : X \dashrightarrow X'$ is a birational contraction, then there is an isomorphism $\tau : X_i \to X'$ such that $\phi = \tau \circ \phi_i$.*

Proof. After replacing X by a resolution, we may assume that X is smooth.

If $\phi : X \dashrightarrow X'$ is a birational contraction, let $p : W \to X$ and $q : W \to X'$ resolve the indeterminacy of f. Let A be a general ample divisor on X', define $D := p_* q^* A$. By negativity lemma, $p^* D = q^* A + E$ for some p-exceptional divisor $E \geq 0$, which is also q-exceptional. Therefore

$$R(X, D) \cong R(W, p^* D) \cong R(X', A).$$

Because A is ample on X', we have that $X' = \operatorname{Proj} R(X', A) \cong \operatorname{Proj} R(X, D)$.

On the other hand, if D' is another divisor on X such that $D' \equiv D$, because X' is $\mathbb{Q}$-factorial, $q_* p^* D' \equiv q_* p^* D = A$, which means that $q_* p^* D'$ is an ample divisor on X', denote it by A'. Since $p^* D' - q^* A' \equiv p^* D - q^* A$ and they are both q-exceptional, by the negativity lemma, we have $p^* D' - q^* A' = p^* D - q^* A = E \geq 0$, hence $X' = \operatorname{Proj} R(X', A') = \operatorname{Proj} R(X, D')$. Therefore each birational contraction is determined by the numerical class of a big $\mathbb{Q}$-divisor. Since $N^1(X)$ is finite dimensional, the claim follows. □

Theorem 3.9. *Let X be a projective normal variety over a normal variety U, V be an affine finite dimensional subspace of $\mathrm{WDIV}_{\mathbb{R}}(X)$ which is defined* *over the rationals, suppose* L *is a rational polytope of* $\mathcal{L}(V)$ *such that for any* $\mathbb{Q}$*-divisor* $\Delta \in L$, (X, Δ) *has a good minimal model over* U *and has the same ample model* Z *over* U*. Then there is a birational contraction* $f : X \dashrightarrow X'$, *such that for any* $\mathbb{R}$*-divisor* $\Delta \in L$, f *is a good minimal model of* (X, Δ) *over* U*.*

Proof. Because for any $\mathbb{Q}$-divisor $\Delta \in L$, (X, Δ) have the same ample model Z over U and (X, Δ) has a good minimal model, it follows that $\kappa(X/U, K_X + \Delta)$ are the same. Then by Lemma 2.5 and Lemma 3.7, for any $\mathbb{R}$-divisor Δ, we have $\kappa_\iota(X/U, K_X + \Delta) = \nu(X/U, K_X + \Delta)$, which implies that for any $\mathbb{R}$-divisor $\Delta \in L$, $K_X + \Delta$ has a good minimal model over U.

Since a good minimal model is $\mathbb{Q}$-factorial, by Theorem 3.8, we can divide L into countably many subset $\cup_{i \geq 0} L_i$, such that for any $\mathbb{R}$-divisor $\Delta \in L_i$, f_i is a good minimal model of (X, Δ) over U. Because $f : X \dashrightarrow X'$ is a good minimal model for both (X, Δ_1) and (X, Δ_2) implies that f is also a good minimal model for $(X, \lambda\Delta_1 + (1-\lambda)\Delta_2)$, $\forall \lambda \in [0, 1]$, then each L_i is convex.

By the Pigeon-hole Principle, we may assume L_1 spans V, which means that there are finitely many $\mathbb{Q}$-divisors $\{\Delta_j \in L_1,\ 1 \leq j \leq m\}$ spanning V and f_1 is a good minimal model for (X, Δ_j), $1 \leq j \leq m$. So $K_{X_1} + f_{1*}\Delta_j, 1 \leq j \leq m$ are semiample over U. Because (X, Δ_j) and $(X_1, f_{1*}\Delta_j)$ have the same ample model Z for every $1 \leq j \leq m$, by [2, Lemma 3.6.5], $K_{X_1} + f_{1*}\Delta_j, 1 \leq j \leq m$ define a morphism $h_1 : X_1 \to Z$, and there are some ample $\mathbb{Q}$-divisors $D_i, 1 \leq j \leq m$ on Z, such that

$$K_{X_1} + f_{1*}\Delta_j \sim_{\mathbb{Q},U} h_1^* D_j, \quad 1 \leq j \leq m.$$

Because $\{\Delta_j, 1 \leq j \leq m\}$ span V, then for any $\mathbb{Q}$-divisor $\Delta \in L$, there is a $\mathbb{Q}$-divisor D on Z such that $K_{X_1} + f_{1*}\Delta \sim_{\mathbb{Q},U} h_1^* D$.

For a $\mathbb{Q}$-divisor $\Delta \in L$, let (X', Δ') be a good minimal model of (X, Δ) and $h' : X' \to Z$ be the morphism from the good minimal model to the ample model. let Δ^1 be the image of Δ on X_1. Choose a common resolution $p : W \to X_1$ and $q : W \to X'$ of X' and X_1. Because Δ' and Δ^1 are the images of Δ on X' and X_1, we have an equation

$$p^*(K_{X_1} + \Delta^1) + E = q^*(K_{X'} + \Delta') + F \tag{3}$$

where E is p-exceptional and F is q-exceptional. By assumption, there are divisors C' and C^1 on Z, such that $K_{X_1} + \Delta^1 \sim_{\mathbb{Q}} h_1^* C^1$ and $K_{X'} + \Delta' \sim_{\mathbb{Q}}$

h'^*C'. Let $f := p \circ h_1 = q \circ h'$, then (3) is equal to

$$f^*(C^1 - C') = F - E$$

Since the irreducible components of $\mathrm{Supp}(F - E)$ are either exceptional for p or exceptional for q, $\mathrm{Supp}(F - E)$ does not contain the whole fiber of any prime divisor on Z, therefore $C^1 = C'$, which means (X', Δ') is crepant birational with (X_1, Δ^1). By Lemma 2.6 (X_1, Δ^1) is a good minimal model for (X, Δ). Therefore, for every $\mathbb{Q}$-divisor $\Delta \in L$, f_1 is a good minimal model for (X, Δ). Because L is a rational polytope, any $\mathbb{R}$-divisor $\Delta \in L$ is a convex combination of $\mathbb{Q}$-divisors in L, therefore, for any $\mathbb{R}$-divisor $\Delta \in L$, f_1 is a good minimal model for (X, Δ). □

The following theorem is the relative version of theorem 1.2.

Theorem 3.10. *Let X be a projective normal variety of dimension n over a normal variety U. Let V be a finite dimensional affine subspace of the vector space $\mathrm{WDiv}_{\mathbb{R}}(X)$ which is defined over $\mathbb{Q}$. Fix a nonnegative integer $0 \leq k \leq n$. Suppose $L \subset \mathcal{L}(V)$ is a closed rational polytope, such that For any $\Delta \in L$, (X, Δ) is klt and $\kappa(X, K_X + \Delta) = k$.*

If there is an $\mathbb{R}$-divisor $\Delta_0 \in \mathrm{int}(L)$ such that $K_X + \Delta_0$ has a good minimal model over U. Then for any $\Delta \in L$, $K_X + \Delta$ has a good minimal model over U, and there are finitely many birational contractions $\phi_j : X \dashrightarrow X_j$, $1 \leq i \leq m$, such that for any $\mathbb{R}$-divisor $\Delta \in L$, if $\phi : X \dashrightarrow Y$ is a good minimal model of $K_X + \Delta$ over U, then there is an index $1 \leq j \leq l$ such that $(Y, \phi_*\Delta)$ is crepant birational with $(X_j, \phi_{j*}\Delta)$.

Proof. If for some $\mathbb{R}$-divisor $\Delta_0 \in \mathrm{int}(L)$, $K_X + \Delta_0$ has a good minimal model over U, then by Lemma 3.7 and Lemma 2.5, for any $\mathbb{R}$-divisor $\Delta \in U$, $K_X + \Delta$ has a good minimal model over U. Then the claims comes easily from Theorem 3.5 and Theorem 3.9. □

4. Applications

4.1. Approximation of pair with $\mathbb{R}$-boundary

Proof of Corollary 1.3. If Δ is a $\mathbb{Q}$-divisor, then the result is straight forward. So we assume that Δ is not a $\mathbb{Q}$-divisor.

After taking a $\mathbb{Q}$-factorialization of X, we may assume that X is $\mathbb{Q}$-factorial. First we show that there is an effective $\mathbb{Q}$-divisor B, such that $\Delta - B \geq 0$ and $\kappa(X, K_X + B) \geq 0$.

Because $\kappa(X, K_X + \Delta) = k \geq 0$, there exists a positive integer m such that

$$h^0(X, \mathcal{O}_X(\lfloor mK_X + m\Delta \rfloor)) > 0.$$

Since X is $\mathbb{Q}$-factorial, we can choose m sufficiently divisible such that mK_X is Cartier, therefore, $h^0(X, \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor)) > 0$. So we can choose $B := \frac{1}{m}\lfloor m\Delta \rfloor$.

Now we can write Δ as an $\mathbb{R}$-linear combination of $\mathbb{Q}$-divisor

$$\Delta = B + \sum_{j=1}^{m} \alpha_j E_j.$$

Let V be the space spanned by $E_j, 1 \leq j \leq m$. It is easy to see that $\Delta \subset \mathrm{int}(\mathcal{L}_B(V))$.

Because $\kappa(X, K_X + B) \geq 0$, by Proposition 2.4, for any $\mathbb{R}$-divisor $D \in \mathrm{int}(\mathcal{L}_B(V))$, we have $\kappa_\iota(X, K_X + D) = k$. Choose a rational polytope $L \subset V$ containing Δ, by Theorem 1.1, we can assume that for every $\mathbb{Q}$-divisor $D \in L$, $K_X + D$ has the same ample model.

By basic convex geometry, we can find $D_i \in L, 1 \leq i \leq l$, such that $||\Delta - D_i|| \leq \epsilon$ and Δ is a convex linear combination of D_i. Let Z be the ample model of $K_X + D_1$, then $Z \cong \mathrm{Proj}\ R(X, D_i)$ for any $D_i, 1 \leq i \leq l$. □

4.2. Nonvanishing

Proof of Corollary 1.6. Because H is effective, $\kappa(X, K_X + \Delta + tH)$ is a non-decreasing function of t. Also by Proposition 2.4, for $0 < t_1 < t_2 < t_3$, we have

$$\kappa(X, K_X + \Delta + t_2 H) \geq \max\{\kappa(X, K_X + \Delta + t_1 H), \kappa(X, K_X + \Delta + t_3 H)\}.$$

Thus there is an integer $0 \leq k < n$, such that for $t > 0$, $\kappa(X, K_X + \Delta + tH) = k$.

If $k = 0$, then

$$\kappa(X, K_X + \Delta + \epsilon H) = 0 \quad \text{and}$$
$$h^0(X, \mathcal{O}_X(m(K_X + \Delta + \epsilon H))) \leq 1$$

for ϵ small enough. Choose ϵ_0 small enough, there exists an integer m_0 and a rational function f_0 on X, such that,

$$\mathrm{div}(f_0) + m_0(K_X + \Delta + \epsilon_0 H) \geq 0,$$

Similarly, for any rational number $0 < \epsilon < \epsilon_0$, we can find f_ϵ and m_ϵ, such that

$$\mathrm{div}(f_\epsilon) + m_\epsilon(K_X + \Delta + \epsilon H) \geq 0,$$

Since $H \geq 0$, it is easy to see that $\mathrm{div}(f_\epsilon^{m_0}) + m_0 m_\epsilon(K_X + \Delta + \epsilon_0 H) \geq 0$ and $\mathrm{div}(f_0^{m_\epsilon}) + m_0 m_\epsilon(K_X + \Delta + \epsilon_0 H) \geq 0$.

Since $h^0(X, \mathcal{O}_X(m_0 m_\epsilon(K_X + \Delta + \epsilon H))) \leq 1$, we have

$$\mathrm{div}(f_\epsilon^{m_0}) + m_0 m_\epsilon(K_X + \Delta + \epsilon_0 H) = \mathrm{div}(f_0^{m_\epsilon}) + m_0 m_\epsilon(K_X + \Delta + \epsilon_0 H).$$

This implies that $\frac{1}{m_0}\mathrm{div}(f_0) = \frac{1}{m_\epsilon}\mathrm{div}(f_\epsilon)$. Therefore taking the limit for $\epsilon \to 0$, we have $\frac{1}{m_0}\mathrm{div}(f_0) + (K_X + \Delta) \geq 0$, which implies $\kappa(X, K_X + \Delta) \geq 0$.

Next we consider when $k \geq 1$. By assumption (X, Δ) is klt, so $(X, \Delta + tH)$ is still klt for $0 < t \ll 1$. Let $(X', \Delta'), \mu : X' \to X$ and $h : X' \to Y$ be the pair and morphisms given in Theorem 3.4, such that

$$\mu_* \mathcal{O}_{X'}(m(K_{X'} + \Delta' + tH')) \cong \mathcal{O}_X(m(K_X + \Delta + tH)), \quad \forall m \in \mathbb{N},$$

therefore for $0 < t \ll 1$, $\kappa(X, K_X + \Delta + tH) \geq 0$ if and only if $\kappa(X', K_{X'} + \Delta' + tH') \geq 0$. $\kappa(X, K_X + \Delta) \geq 0$ is equal to $\kappa(X', K_{X'} + \Delta') \geq 0$. So we may replace X by X'(respectively Δ by Δ', H by H') and assume that there is an equidimensional algebraic fibration $h : X \to Y$.

Let F denotes the generic fiber of h. By properties of the Iitaka fibration, we have that for all $0 < t \ll 1$,

$$\kappa(F, K_F + \Delta_F + tH_F) = 0.$$

This means on the generic fiber F, $K_F + \Delta_F$ is the limit of effective divisors, therefore it is pseudo-effective. Since we assume Conjecture 1.5 holds in dimension $\leq n - 1$, then $\kappa(F, K_F + \Delta_F) \geq 0$ and it is easy to see that

$$\kappa(F, K_F + \Delta_F) = 0.$$

Let m be sufficiently divisible, such that $m(K_X + \Delta)$ is Cartier, since h is equidimensional and $\kappa(F, K_F + \Delta_F) = 0$, $h_* \mathcal{O}_X(m(K_X + \Delta))$ is a reflexive sheaf of rank 1. Also because Y is smooth, $h_* \mathcal{O}_X(m(K_X + \Delta))$ is an invertible sheaf. So there is a Cartier divisor D' on Y, such that

$$h_* \mathcal{O}_X(m(K_X + \Delta)) = \mathcal{O}_Y(D'). \tag{4}$$

Let A' be a sufficiently ample divisor on Y, such that $D' + A'$ is ample on Y. Therefore $h_* \mathcal{O}_X(m(K_X + \Delta) + h^* A') = \mathcal{O}_Y(D' + A')$ is ample and

$m(K_X + \Delta) + h^*A'$ is effective. Let $A := \frac{1}{m}A'$, by Remark 3.2, for $0 \le t \ll 1$, we can find D'_t and F_t such that

$$K_X + \Delta + h^*A + tH \sim_{\mathbb{Q}} h^*D'_t + F_t$$

where D'_t is a linear function of t and $\mathcal{O}_Y \cong h_*\mathcal{O}_X(\lfloor iF \rfloor)$ for all $i \ge 0$. Let $D_t := D'_t - A$, which is also a linear function of t, and we have

$$K_X + \Delta + tH \sim_{\mathbb{Q}} h^*D_t + F_t.$$

Since for $t > 0$, $\kappa(X, K_X + \Delta + tH) = \dim(Y)$, then D_t is big for every $t > 0$. Therefore D_0 is pseudo-effective. By the canonical bundle formula, perhaps replacing $h : X \to Y$ by a higher model, we can find $\mathbb{Q}$-divisors B and J on Y such that

$$D_0 \sim_{\mathbb{Q}} K_Y + B + J.$$

where (Y, B) is klt pair and J is nef.

If Conjecture 1.4 holds in dimension$\le n-1$, J is semiample, choose a general member $C \in |J + B|_{\mathbb{Q}}$, then (Y, C) is klt, and $K_Y + C$ is pseudo-effective, by Conjecture 1.5 in dimension $\le n-1$, $\kappa(X, K_Y + C) \ge 0$. Therefore $\kappa(X, K_X + \Delta) \ge 0$. □

4.3. MMP with scaling

Definition 4.1. (MMP with scaling) Let (X_1, Δ_1) and $(X_1, \Delta_1 + H_1)$ be two klt pairs such that $K_{X_1} + \Delta_1 + H_1$ is nef, $\Delta_1 \ge 0$, H_1 is $\mathbb{Q}$-Cartier and pseudo-effective. Suppose that either $K_{X_1} + \Delta_1$ is nef or there is an extremal ray R_1 such that $(K_{X_1} + \Delta_1).R_1 < 0$ and $(K_{X_1} + \Delta_1 + \lambda_1 H_1).R_1 = 0$ where

$$\lambda_1 := \inf\{t \ge 0 \mid K_{X_1} + \Delta_1 + tH_1 \text{ is nef}\}$$

Now, if $K_{X_1} + \Delta_1$ is nef or if R_1 defines a Mori fibre structure, we stop. Otherwise assume that R_1 gives a divisorial contraction or a log flip $X_1 \dashrightarrow X_2$. We can now consider $(X_2, \Delta_2 + \lambda_1 H_2)$ where $\Delta_2 + \lambda_1 H_2$ is the birational transform of $\Delta_1 + \lambda_1 H_1$ and continue. That is, suppose that either $K_{X_2} + \Delta_2$ is nef or there is an extremal ray R_2 such that $(K_{X_2} + \Delta_2).R_2 < 0$ and $(K_{X_2} + \Delta_2 + \lambda_2 H_2).R_2 = 0$ where

$$\lambda_2 := \inf\{t \ge 0 \mid K_{X_2} + \Delta_2 + tH_2 \text{ is nef}\}$$

By continuing this process, we obtain a sequence of numbers λ_i and a special kind of MMP which is called the MMP on $K_{X_1} + \Delta_1$ with scaling of H_1. Note that by definition $\lambda_i \geq \lambda_{i+1}$ for every i.

Proof of Theorem 1.7. Because $K_X + \Delta + H$ has a good minimal model, $\kappa(X, K_X + \Delta + H) \geq 0$, therefore $\kappa_\iota(X, K_X + \Delta + H) = \kappa(X, K_X + \Delta + H)$, and by Theorem 2.5,

$$\nu(X, K_X + \Delta + H) = \kappa_\iota(X, K_X + \Delta + H) = \kappa(X, K_X + \Delta + H) = k$$

for some nonnegative integer k. Since H is pseudo-effective, by Proposition 2.4.(4), $\nu(X, K_X + \Delta + tH) \leq k$, $\forall t \in [0, 1]$. By assumption, $\kappa(X, K_X + \Delta) \geq 0$, therefore we have

$$\begin{aligned} k &\geq \nu(X, K_X + \Delta + tH) \geq \kappa(X, K_X + \Delta + tH) \\ &\geq \max\{\kappa(X, K_X + \Delta),\ \kappa(X, K_X + \Delta + H)\} \geq k, \end{aligned}$$

for every $t \in (0, 1]$. Then $(X, K_X + \Delta + tH)$ has a good minimal model for every $t \in (0, 1]$.

Suppose $\lambda > 0$. By Theorem 1.1, there exists $\epsilon > 0$, such that for every $t \in [\lambda, \lambda + \epsilon]$, $K_X + \Delta + tH$ has the same ample model Z. Consider the interval $I := [\lambda, \lambda + \epsilon]$. By definition of MMP with scaling, we have infinitely many birational contractions ϕ_i, such that ϕ_i is a minimal model for $K_X + \Delta + tH, t \in [\lambda_i, \lambda_{i+1}]$, and $K_{X_i} + \phi_{i*}\Delta + t\phi_{i*}H$ is not nef when $t < \lambda_i$.

On the other hand, by Theorem 1.2, there are finitely many birational contractions $f_j : X \dashrightarrow Y_j, 1 \leq j \leq m$, and we can divide I into finitely many closed interval $I = \cup_{1 \leq j \leq m}[t_j, t_{j+1}]$, such that for $t \in (t_j, t_{j+1})$, f_j is a good minimal model for $K_X + \Delta + tH$. It is easy to see that there exist i, j, such that $t_j < \lambda_i < \lambda_{i+1} < t_{j+1}$.

Consider two rational number $r_1, r_2 \in (\lambda_i, \lambda_{i+1})$. By assumption, ϕ_i and f_j are respectively good minimal models of $K_X + \Delta + r_1 H$ and $K_X + \Delta + r_2 H$. Let $h : X_i \to Z$ and $g : Y_j \to Z$ be the morphism from good minimal models to the ample model, then there are two divisors D_1, D_2 on Z, such that $K_{X_i} + \phi_{i*}\Delta + r_k\phi_{i*}H \sim_{\mathbb{Q}} h^*D_k$ and $K_{Y_j} + f_{j*}\Delta + r_k f_{j*}H \sim_{\mathbb{Q}} g^*D_k$ for $k = 1, 2$. Therefore by linearity, for all $t \in [0, 1]$, $(X_i, \phi_{i*}\Delta + t\phi_{i*}H)$ is crepant birational with $(Y_j, f_{j*}\Delta + tf_{j*}H)$, which means ϕ_i is also a minimal model for $K_X + \Delta + tH, t \in [t_j, t_{j+1}]$, this contradicts with the definition of MMP with scaling. □

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