

The principle of least action in the space of Kähler potentials

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Given a compact Kähler manifold, the space \mathcal{H} of its (relative) Kähler potentials is an infinite dimensional Fréchet manifold, on which Mabuchi and Semmes have introduced a natural connection ∇ . We study certain Lagrangians on $T\mathcal{H}$, in particular Finsler metrics, that are parallel with respect to the connection. We show that geodesics of ∇ are paths of least action; under suitable conditions the converse also holds; and we prove a certain convexity property of the least action. This generalizes earlier results of Calabi, Chen, and Darvas.

1	Introduction	786
2	Lagrangians	790
3	Extending Lagrangians	795
4	Weak geodesics, ε-geodesics, Jacobi fields	802
5	The action	805
6	Divergence of ε-Jacobi fields	809
7	Least action in \mathcal{H} and $\mathcal{H}^{1\bar{1}}$	813
8	Least action in $B(X) \cap \text{PSH}(\omega)$	815
9	Convexity of the action	818
10	Two ways to compute action	821

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11 Uniqueness of minimizing paths	825
References	830

1. Introduction

Let (X, ω) be an n dimensional, connected, compact Kähler manifold and

$$\mathcal{H} = \mathcal{H}_\omega = \{u \in C^\infty(X) : \omega + dd^c u = \omega_u > 0\}$$

its space of relative Kähler potentials. Here $C^\infty(X)$ refers to the Fréchet space of real valued smooth functions on X , and $d^c = i(\bar{\partial} - \partial)/2$, so that $dd^c = i\partial\bar{\partial}$. The space \mathcal{H} , as an open subset of a Fréchet space, inherits a Fréchet manifold structure, whose tangent bundle has a canonical trivialization $T\mathcal{H} \approx \mathcal{H} \times C^\infty(X)$. Mabuchi and Semmes [M, S] independently and with different motivations have introduced a torsion free connection ∇ on $T\mathcal{H}$. Mabuchi, as a tool to study special Kähler metrics, defined a Riemannian metric on \mathcal{H} and obtained ∇ as the Levi–Civita connection of the metric. Somewhat later Semmes found the connection in search for a geometric interpretation of interpolation of Banach spaces and of a certain homogeneous complex Monge–Ampère equation associated with interpolation. He also determined all Riemannian metrics compatible with the connection: they are linear combinations of Mabuchi’s metric and the square of a one form.

One way to explain ∇ is through its parallel transport. We will use $\dot{\cdot}$ to denote derivative of a function of one real variable, and grad_v to refer to gradient of a function $X \rightarrow \mathbb{R}$ with respect to the Kähler metric of ω_v . Let $u : [a, b] \rightarrow \mathcal{H}$ be a smooth path. By integrating the time dependent vector field $(-1/2) \text{grad}_{u(t)} \dot{u}(t)$ on X we obtain a smooth family of diffeomorphisms $\varphi(t) : X \rightarrow X$. In fact $\varphi(t) : (X, \omega_{u(0)}) \rightarrow (X, \omega_{u(t)})$ is symplectomorphic. The parallel translate of $\xi \in T_{u(t)}\mathcal{H} \approx C^\infty(X)$ to $u(0)$ along the path u is then

$$(1.1) \quad \xi \circ \varphi(t) \in C^\infty(X) \approx T_{u(0)}\mathcal{H}.$$

Understanding the geodesics of this connection was already marked in [M, S] as an interesting and potentially important problem, and Donaldson’s subsequent work [Do] gave further impetus to study them. By now the boundary value problem for geodesics is well understood. On the one hand Darvas, Hu, Vivas, and myself [D1, DL, Hu, LV] proved that points

in \mathcal{H} cannot always be connected by a geodesic, not even if they are close to each other. On the other hand work by Berman–Demailly, Berndtsson, Błocki, Chen, Chen–Feldman–Hu, Chu–Tosatti–Weinkove, Darvas, and He [BD, Be1, B11, B12, C, CFH, CTW, D2, D3, He] gave that the geodesic equation extends to various enlargements of \mathcal{H} , and in these enlargements any pair of points, or at least nearby points, can be connected by solutions of the extended geodesic equation, weak geodesics. It follows from Chen’s work that in those enlargements to which Mabuchi’s metric extends, weak geodesics minimize length. In [D2] Darvas generalized Mabuchi’s metric to certain Orlicz–type Finsler metrics on \mathcal{H} , determined the metric completions of \mathcal{H} under these metrics, and again found that weak geodesics in these completions minimize length. In a slight overstatement the length minimizing paths are independent of which of Darvas’s metric we use to compute length. This was surprising at first sight.

But in fact in geometry one encounters other similar phenomena. In a normed vector space straight line segments minimize length no matter what norm is chosen. There is also the analogy between \mathcal{H} and the space \mathcal{Q} of positive definite quadratic forms on \mathbb{R}^k . \mathcal{Q} has a natural torsion free connection that turns it in a symmetric space $\approx \mathrm{GL}_k^+(\mathbb{R})/\mathrm{SO}_k(\mathbb{R})$; and for all parallel Finsler metrics—i.e. those that are invariant under $\mathrm{GL}_k^+(\mathbb{R})$ —the shortest paths are the same: subarcs of left translates of certain one parameter subgroups in $\mathrm{GL}_k^+(\mathbb{R})$, projected to $\mathrm{GL}_k^+(\mathbb{R})/\mathrm{SO}_k(\mathbb{R})$.

Now \mathcal{H} with Mabuchi’s connection is itself a symmetric space [Do, M, S], at least according some definitions of a symmetric space (while it is not according to some others, [L3]). Although there is no group acting transitively on* (\mathcal{H}, ∇) , the holonomy groupoid Γ of (\mathcal{H}, ∇) acts on $T\mathcal{H}$. Thus $\Gamma = \bigcup_{u,v \in \mathcal{H}} \Gamma_{uv}$, where Γ_{uv} consists of linear isomorphisms $T_u\mathcal{H} \rightarrow T_v\mathcal{H}$ that arise as parallel transport along piecewise smooth paths from u to v . Concatenation of parallel transports defines an operation $\Gamma_{uv} \times \Gamma_{vw} \rightarrow \Gamma_{uw}$ that turns Γ in a grupoid. That a Finsler metric or a function $L : T\mathcal{H} \rightarrow \mathbb{R}$ is parallel means it is invariant under Γ .

Darvas’s metrics are parallel. They are defined in terms of integrals

$$(1.2) \quad \int_X \chi(\xi) \omega_u^n, \quad u \in \mathcal{H}, \quad \xi \in T_u\mathcal{H} \approx C^\infty(X),$$

*This follows from [L2]. Even though Theorem 1.2 there is formulated for isometries of Mabuchi’s metric, the proof, verbatim, gives that if ω_u is analytic while ω_v is not, then no diffeomorphism of \mathcal{H} can preserve ∇ and map u to v .

with a fixed “Young weight” $\chi : \mathbb{R} \rightarrow [0, \infty]$, and are invariant under parallel transport simply because in the formula (1.1) for parallel transport, $\varphi(t)$ satisfies $\varphi(t)^*\omega_{u(t)} = \omega_{u(0)}$. But there are many parallel Finsler metrics on $T\mathcal{H}$ beyond those considered in [D2]. The simplest is, for given $0 < \alpha < 1$,

$$p(\xi) = \sup \left\{ \int_E |\xi| \omega_u^n / \left(\int_E \omega_u^n \right)^\alpha : E \subset X \text{ is measurable} \right\},$$

$\xi \in T_u\mathcal{H}$. This is known as weak L^q -norm or $L^{q,\infty}$ Lorentz norm, $q = 1/\alpha$.

Our thesis is that the proper generality of Darvas’s results on his metrics is parallel, or holonomy invariant, Finsler metrics and more generally, fiberwise convex functions $T\mathcal{H} \rightarrow \mathbb{R}$, “Lagrangians”. In this paper and in a sequel we will show that many of his results generalize to this framework. Most of the time we will consider Lagrangians on $T\mathcal{H}$ that extend to the space $B(X) \cap \text{PSH}(\omega)$ of bounded ω -plurisubharmonic functions. Here we denote by $B(X)$ the Banach space of bounded Borel functions $\xi : X \rightarrow \mathbb{R}$ with the norm $\|\xi\| = \sup |\xi|$; the Lagrangians of interest extend to $(B(X) \cap \text{PSH}(\omega)) \times B(X)$. (The more common space $L^\infty(X)$ is a quotient of $B(X)$, but we have little use for it in this paper.) A generalization of holonomy invariance can be defined for such functions. Our results pertain to invariant Lagrangians that are convex in the $B(X)$ variable and have a certain continuity property, that we call strong continuity (Definition 3.1).

Theorem 1.1 (=Theorem 8.1, Principle of least action). *If $v : [0, T] \rightarrow B(X) \cap \text{PSH}(\omega)$ is a weak geodesic, and C^1 as a map into the Banach space $B(X)$, then it minimizes action*

$$\int_0^T L(\dot{u}(t)) dt$$

among all piecewise C^1 paths $u : [0, T] \rightarrow B(X) \cap \text{PSH}(\omega)$ with $u(0) = v(0)$, $u(T) = v(T)$.

Weak geodesics may fail to be C^1 (Example 5.4), but from Chen’s work [C] we do know that a weak geodesic with endpoints in \mathcal{H} is C^1 . The theorem can be proved for weak geodesics rather less regular than C^1 , but even this relaxed regularity is not automatic.

The next result is about how least action varies as one moves along weak geodesics; it is a manifestation of seminegative curvature. Fix $T > 0$. If $w, w' \in B(X) \cap \text{PSH}(\omega)$, the least action $\mathcal{L}_T(w, w')$ between them is the infimum of the actions $\int_0^T L(\dot{u}(t)) dt$ over all piecewise C^1 paths $u : [0, T] \rightarrow$

$B(X) \cap \text{PSH}(\omega)$ connecting w with w' . It is not obvious, but by Lemma 9.4, $\mathcal{L}_T(w, w')$ is finite.

Theorem 1.2 (=Theorem 9.1). *If $u, v : [a, b] \rightarrow B(X) \cap \text{PSH}(\omega)$ are weak geodesics, then the function $\mathcal{L}_T(u, v) : [a, b] \rightarrow \mathbb{R}$ is convex.*

Special instances of the result were known before. When L defines Mabuchi's metric, this is a consequence of the CAT(0) property, proved by Darvas in [D3]. More generally, it was known for all of Darvas's L^p -type metrics d_p , $1 \leq p < \infty$, and weak geodesics u, v in the corresponding energy spaces $\mathcal{E}^p(\omega)$. The case $p = 1$ is due to Berman, Darvas, and Lu while, as my referee indicated, other p 's to Chen and Cheng, [BDL, Proposition 5.1], [ChC, Theorem 5.1].

We also prove a converse of sorts to Theorem 1.1: under certain conditions, *only* weak geodesics minimize action, see Theorem 11.1. When L defines Mabuchi's metric, or one of Darvas's L^p -type metrics d_p , this time $1 < p < \infty$, Darvas and Lu already proved this, even more generally for paths in $\mathcal{E}^p(\omega)$, in [D3, Theorem 1], respectively [DLu, Theorem 3.5].

The tools of this paper are Chen's work on ε -geodesics, rudiments of Guedj-Zeriahi's pluripotential theory, and our results on invariant convex functions on $C^\infty(X)$ [C, GZ1, GZ2, L4]. In the proof of Theorem 1.1, even if the details are different, overall we will be able to follow the strategy of Calabi, Chen, and Darvas [C, CC, D2, D3]. Once basic properties of our Lagrangians are established, the greater generality occasionally results in less computation in the proofs for the following reason. Say, for a holonomy invariant Finsler metric $p : T\mathcal{H} \rightarrow [0, \infty)$, there is a family $\mathcal{F} \subset T\mathcal{H} \approx \mathcal{H} \times C^\infty(X)$ such that

$$(1.3) \quad p(\xi) = \sup \left\{ \int_X f \xi \omega_u^n : f \in \mathcal{F} \cap T_u \mathcal{H} \right\}, \quad \xi \in T_u \mathcal{H}$$

(see Theorem 2.4), and the integrals in (1.3), linear in ξ , can be easier to manipulate than the nonlinear integrals in (1.2).

It appears that the greatest generality in which action can be defined by an integral is the space of bounded ω -plurisubharmonic functions. Nonetheless, action can be defined for any path in $\text{PSH}(\omega)$ as a limit of integrals. Whether this action is finite or $\pm\infty$ of course depends on the path and on the Lagrangian. We plan to address this and related questions in a sequel to this paper.

Lagrangians even beyond Finsler metrics are not new to the subject. Chen's ε -geodesics are trajectories of a Lagrangian $L : T\mathcal{H} \rightarrow \mathbb{R}$ (albeit not

holonomy invariant), with kinetic energy term the square of Mabuchi's metric and potential energy a multiple of $V(u) = -\int_X u\omega^n$. Functions on \mathcal{H} that its geometry motivates, and that are used in existence problems in Kähler geometry, are also not new. Aubin's functional $I : \mathcal{H} \rightarrow \mathbb{R}$ [Au, p.146],

$$I(u) = \int_X u(\omega^n - \omega_u^n)$$

is a constant multiple of the total geodesic curvature of the line segment $[0, 1] \ni t \mapsto tu \in \mathcal{H}$, measured in Darvas's L^1 Finsler metric. Monge–Ampère energy also arises from the geometry of \mathcal{H} . It is a convex function on \mathcal{H} , for example in the sense that its restrictions to geodesics of ∇ are convex; but its negative is also convex and, up to scaling and adding a constant, it is the only continuous function that has this property.

We hope that a geometrical approach to functions on \mathcal{H} and on related spaces, in the spirit of this paper, will be of use in analytical problems on Kähler manifolds.

Contents. Section 2 is about basic properties of holonomy invariant convex Lagrangians $T\mathcal{H} \rightarrow \mathbb{R}$. Section 3 is about a subclass of Lagrangians that have an extra continuity property, which makes it possible to extend them to a larger vector bundle. Many of the results in these sections are direct consequences of results in [L4]. Section 4 reviews the notion of weak and ε -geodesics, and ε -Jacobi fields. Section 5 introduces the action and formulates Theorems 1.1 and 1.2 in precise forms. It also gives a road map to their proofs, which occupy sections 6–9. Section 10 provides a discretized formula for action, which suggests how to generalize to less regular paths, and section 11 addresses the problem of uniqueness of paths that minimize action.

In this paper we freely use basic notions of infinite dimensional analysis and geometry. There are many sources the reader can consult on these matters, one of them [L1], written with an eye on the space \mathcal{H} of Kähler potentials.

2. Lagrangians

The central objects of this paper are continuous functions $L : T\mathcal{H} \rightarrow \mathbb{R}$ that are convex on each tangent space $T_u\mathcal{H}$ and have a certain invariance property; as well as the associated action functional on paths $u : [a, b] \rightarrow \mathcal{H}$ given by $\mathcal{L}(u) = \int_a^b L(\dot{u}(t))dt$ ($= \int_a^b L \circ \dot{u}$ for brevity). In this section and in the next we record basic facts about such functions which follow more

or less directly from [L4], that dealt with the action on $C^\infty(X)$ of Hamiltonian diffeomorphisms of (X, ω) and with invariant convex functions on $C^\infty(X)$. As explained in the Introduction, for L the invariance property in question is invariance under the holonomy groupoid Γ of (\mathcal{H}, ∇) . Thus, if $\xi_1 \in T_{u(1)}\mathcal{H}$ is the parallel translate of $\xi_0 \in T_{u(0)}\mathcal{H}$ along a piecewise smooth path $u : [0, 1] \rightarrow \mathcal{H}$, then $L(\xi_0) = L(\xi_1)$. This property in fact implies a much stronger and more primitive notion of invariance.

Definition 2.1. *Given two measure spaces (X, μ) and (Y, ν) , we say that measurable functions $\xi : X \rightarrow \mathbb{R}$ and $\eta : Y \rightarrow \mathbb{R}$ are equidistributed, or are strict rearrangements of each other, if $\mu(\xi^{-1}B) = \nu(\eta^{-1}B)$ for every Borel set $B \subset \mathbb{R}$.*

In finite measure spaces this is equivalent to requiring $\mu(\xi > t) = \nu(\eta > t)$ for all $t \in \mathbb{R}$.

Back to our n dimensional Kähler manifold (X, ω) , if $u \in \mathcal{H}$ we let μ_u denote the measure induced by ω_u^n . Given measurable $\xi, \eta : X \rightarrow \mathbb{R}$ we will write

$$(2.1) \quad (\xi, u) \sim (\eta, v) \text{ if } \xi, \eta \text{ are equidistributed as functions on } (X, \mu_u), (X, \mu_v).$$

When smooth ξ, η are viewed as tangent vectors in $T_u\mathcal{H}, T_v\mathcal{H}$, we will just write $\xi \sim \eta$.

Theorem 2.2. *A function $L : T\mathcal{H} \rightarrow \mathbb{R}$, continuous and convex on each fiber $T_u\mathcal{H}$, is invariant under the holonomy groupoid Γ if and only if it is invariant under strict rearrangements: $L(\xi) = L(\eta)$ when $\xi \sim \eta$.*

For the proof we need to understand the holonomy groups Γ_{uu} . (1.1) shows that in general, elements of Γ_{uv} , isomorphisms $T_u\mathcal{H} \rightarrow T_v\mathcal{H}$, are pull-backs by certain symplectomorphisms $\varphi : (X, \omega_v) \rightarrow (X, \omega_u)$. Let us write G for those symplectomorphisms that induce elements of Γ_{00} . Thus G is a subgroup of the Fréchet–Lie group $\text{Diff } X$ of diffeomorphisms of X .

Lemma 2.3. *The closure of G in $\text{Diff } X$ contains all Hamiltonian diffeomorphisms of (X, ω) .*

Recall that Hamiltonian diffeomorphisms are time–1 maps of time dependent Hamiltonian vector fields $\text{sgrad } \zeta_t$, i.e., vector fields that are symplectic gradients with $\zeta_t \in C^\infty(X, \omega)$ a smooth family, $t \in [0, 1]$.

Proof. Let \mathfrak{g} (the “Lie algebra” of G) consist of smooth vector fields V on X for which there is a smooth map $\varphi : [0, 1] \rightarrow G \subset \text{Diff } X$ such that $\varphi(0) = \text{id}_X$ and $\dot{\varphi}(0) = d\varphi(t)/dt|_{t=0} = V$. This is a vector subspace of the space of all vector fields: for example, if φ, ψ realize vector fields $V, W \in \mathfrak{g}$, then $\varphi(t) \circ \psi(t)$ realizes $V + W$. In [S, pp. 512-513] Semmes essentially proved that \mathfrak{g} contains all Hamiltonian vector fields $\text{sgrad } \zeta$. Essentially only, because the proof of his Lemma 4.1 is given only in Sobolev spaces, not in $C^\infty(X)$. At any rate, we will need a slightly stronger, parametrized statement, to wit: If $\zeta : [a, b] \rightarrow C^\infty(X)$ is smooth, then there is a smooth family

$$(2.2) \quad [a, b] \times [0, 1] \ni (s, t) \mapsto \varphi_t^s \in G \subset \text{Diff } X$$

such that $\varphi_0^s = \text{id}_X$ and $\partial_t \varphi_t^s|_{t=0} = \text{sgrad } \zeta(s)$ for all s .

To verify this, recall Semmes’ construction in [S, top of p. 512] that, given $\xi, \eta \in C^\infty(X)$, shows that the Poisson bracket $\{\xi, \eta\} \in C^\infty(X)$, determined by ω , has symplectic gradient in \mathfrak{g} . The same construction works with a parameter appended. Thus, if $\xi, \eta : [a, b] \rightarrow C^\infty(X)$ are smooth, there is a smooth family $\varphi_t^s \in G$ as in (2.2), $\varphi_0^s = \text{id}_X$ and $\partial_t \varphi_t^s|_{t=0} = \text{sgrad}\{\xi(s), \eta(s)\}$. But any smooth $\zeta : [a, b] \rightarrow C^\infty(X)$ such that $\int_X \zeta(s)\omega^n = 0$ can be written

$$(2.3) \quad \zeta(s) = \sum_{j=1}^m \{\xi_j(s), \eta_j(s)\}, \quad m = 4n + 1,$$

with suitable smooth $\xi_j, \eta_j : [a, b] \rightarrow C^\infty(X)$. In fact ξ_j can be chosen constant, and arbitrary as long as $\xi_j(s) \equiv \bar{\xi}_j$ embed X into \mathbb{R}^m . The statement, without s -dependence, corresponds to [S, Lemma 4.1], but was already proved in [AG]. Atkin and Grabowski’s proof is easily modified to provide (2.3). The proof of [AG, (5.2) Theorem] depends on [AG, (2.6) Proposition], the s -dependent version of which says that if $\xi_j \in C^\infty(X)$, $j = 1, \dots, m$, embed X into \mathbb{R}^m , then any smooth family ψ^s of smooth k -forms on X , $s \in [a, b]$, can be written

$$\psi^s = \sum_{i_1, i_2, \dots} f_{i_1 \dots i_k}(s) d\xi_{i_1} \wedge \dots \wedge d\xi_{i_k}$$

with $f_{i_1 \dots i_k} : [a, b] \rightarrow C^\infty(X)$ smooth. This is proved by an obvious cohomology vanishing as in [AG]. Another ingredient of the proof of [AG, (5.2) Theorem], on p. 325 there, in s -dependent version says that given a smooth family α^s of exact smooth forms on X , there is a smooth family β^s of smooth forms such that $d\beta^s = \alpha^s$. One way to prove this is by Hodge theory, which

gives that the unique solution β^s of $d\beta^s = \alpha^s$ that is othogonal to $\text{Ker } d$ depends smoothly on s . The rest of the proof in [AG] manipulates identities, and changes not if a parameter s is appended. Thus (2.3) is proved.

We can now construct $\varphi_t^s \in G$ as in (2.2). First, subtracting from ζ a smooth function $c : [a, b] \rightarrow \mathbb{R}$ we obtain $\zeta' : [a, b] \rightarrow C^\infty(X)$ with $\int_X \zeta'(s)\omega^m = 0$. We find ξ_j, η_j as in (2.3), corresponding to ζ' rather than ζ , and then smooth maps $(s, t) \mapsto \varphi_{jt}^s \in G$ such that $\varphi_{j0}^s = \text{id}_X$ and $\partial_t \varphi_{jt}^s = \{\xi_j(s), \eta_j(s)\}$ at $t = 0$. Since $\text{sgrad } \zeta = \text{sgrad } \zeta'$, the diffeomorphisms

$$\varphi_t^s = \varphi_{1t}^s \circ \varphi_{2t}^s \circ \dots \circ \varphi_{mt}^s$$

have t -derivative $\text{sgrad } \zeta(s)$ at $t = 0$.

After these preparations we are ready to prove the lemma. Suppose $\varphi^1 : (X, \omega) \rightarrow (X, \omega)$ is a Hamiltonian diffeomorphism. This means it can be included in the flow φ^s of Hamiltonian vector fields $V^s = \text{sgrad } \zeta(s)$,

$$(2.4) \quad \partial_s \varphi^s = V^s(\varphi^s), \quad 0 \leq s \leq 1, \quad \varphi^0 = \text{id}_X.$$

Here $\zeta : [0, 1] \rightarrow C^\infty(X)$ is smooth. The φ_t^s constructed above for this ζ can be used as integrators in a 1-step scheme to approximate the solution of the initial value problem (2.4). General theory gives that

$$(2.5) \quad \varphi_{1/k}^{(k-1)/k} \circ \varphi_{1/k}^{(k-2)/k} \circ \dots \circ \varphi_{1/k}^0 \rightarrow \varphi^1 \quad \text{in } C^\infty(X)$$

as $k \rightarrow \infty$.

(Details are as follows. Smoothly embed X in some \mathbb{R}^m and with $p \in \mathbb{N}$, view φ^s as an element of the Banach space $B = C^p(X) \times \dots \times C^p(X)$, m copies of $C^p(X)$. Extend $\varphi_t^s : X \rightarrow X$ to a smooth family of maps $\psi_t^s : \mathbb{R}^m \rightarrow \mathbb{R}^m$ and extend V^s to a vector field on \mathbb{R}^m by $V^s = \partial_t \psi_t^s|_{t=0}$. The error analysis of e.g. [HNW, p.160, Theorem 3.4], or more directly [An, Theorem 4.1, Corollary 4.2], gives that

$$(2.6) \quad \psi_{1/k}^{(k-1)/k} \circ \psi_{1/k}^{(k-2)/k} \circ \dots \circ \psi_{1/k}^0 \circ \varphi^0 \rightarrow \varphi^1 \quad \text{in } B$$

as $k \rightarrow \infty$. Both [HNW, An] work in finite dimensional Banach spaces, the latter in \mathbb{C}^m , but the same reasoning proves the result in any Banach space.)

Since the left hand side of (2.6) is $\varphi_{1/k}^{(k-1)/k} \circ \dots \circ \varphi_{1/k}^0 \in G$, we proved that φ^1 is indeed in the closure of G . □

Proof of Theorem 2.2. That invariance under strict rearrangements implies holonomy invariance follows since parallel transport is realized by composition with a symplectomorphism, and such compositions send functions

to their strict rearrangements. The converse implication depends on Lemma 2.3. This implies that $L(\xi) = L(\xi \circ \varphi)$ if $\xi \in T_0\mathcal{H}$ and $\varphi \in \text{Diff}(X, \omega)$ is Hamiltonian. By [L4, Theorem 1.2], $L|_{T_0\mathcal{H}}$ is therefore invariant under strict rearrangements. To complete the proof, take $\xi \in T_u\mathcal{H}, \eta \in T_v\mathcal{H}$ such that $\xi \sim \eta$. Parallel translate ξ, η to $\xi', \eta' \in T_0\mathcal{H}$ along arbitrary smooth paths. Then $\xi' \sim \xi \sim \eta \sim \eta'$, whence $L(\xi) = L(\xi') = L(\eta') = L(\eta)$. \square

In what follows, a fiberwise continuous and convex function $L : T\mathcal{H} \rightarrow \mathbb{R}$ that is invariant under strict rearrangements will be called an invariant convex Lagrangian. The chief device to analyze their finer properties is the following representation theorem. We write $B(X)$ or $B(X, \mu)$ —when a Borel measure μ on X plays a role—for the Banach space of bounded Borel functions on X , with the supremum norm $\| \cdot \|$.

Theorem 2.4. *Given an invariant convex Lagrangian $L : T\mathcal{H} \rightarrow \mathbb{R}$, there are families $\mathcal{A}_u \subset \mathbb{R} \times B(X), u \in \mathcal{H}$ such that for $\xi \in T_u\mathcal{H} \approx C^\infty(X)$*

$$(2.7) \quad L(\xi) = \sup_{(a,f) \in \mathcal{A}_u} a + \int_X f \xi \omega_u^n.$$

\mathcal{A}_u can be chosen in $\mathbb{R} \times C^\infty(X)$, and have the property that whenever $(a, f) \in \mathcal{A}_u$ and $\varphi : (X, \omega_v) \rightarrow (X, \omega_u)$ is a symplectomorphism, then $(a, f \circ \varphi) \in \mathcal{A}_v$. Alternatively, \mathcal{A}_u can be chosen to be strict rearrangement invariant: if $f \in B(X, \mu_u)$ and $g \in B(X, \mu_v)$ are equidistributed, and $(a, f) \in \mathcal{A}_u$, then $(a, g) \in \mathcal{A}_v$.

If L is also positively homogeneous, ($L(c\xi) = cL(\xi)$ whenever $c \in (0, \infty)$), then in addition \mathcal{A}_u can be chosen in $\{0\} \times C^\infty(X)$, respectively, in $\{0\} \times B(X)$.

Proof. Most of the proof was done in [L4]. Lemma 2.1 there produces $\mathcal{A}_0 \subset \mathbb{R} \times C^\infty(X)$ that satisfies (2.7) when $u = 0$. If we adjoin to \mathcal{A}_0 all pairs $(a, f \circ \varphi)$ with $(a, f) \in \mathcal{A}_0$ and $\varphi : (X, \omega) \rightarrow (X, \omega)$ a symplectomorphism, because of the invariance of L the supremum in (2.7) is not going to change (for $u = 0$). So we can assume that \mathcal{A}_0 already is invariant under symplectomorphisms. We then define \mathcal{A}_u to consist of pairs $(a, f \circ \psi)$ with $(a, f) \in \mathcal{A}_0$ and $\psi : (X, \omega_v) \rightarrow (X, \omega)$ a symplectomorphism. This will do,

since if $\xi \in T_u\mathcal{H}$, with the above ψ

$$\begin{aligned} \sup_{(a,g) \in \mathcal{A}_u} a + \int_X g\xi\omega^n &= \sup_{(a,g) \in \mathcal{A}_u} a + \int_X (g \circ \psi^{-1})(\xi \circ \psi^{-1})\omega^n \\ &= \sup_{(a,f) \in \mathcal{A}_0} a + \int_X (\xi \circ \psi^{-1})f\omega^n = L(\xi \circ \psi^{-1}) = L(\xi). \end{aligned}$$

Alternatively, we can modify the above \mathcal{A}_u to \mathcal{A}'_u consisting of all $(a, g) \in \mathbb{R} \times B(X)$ for which there is $(a, f) \in \mathcal{A}_0$ such that $(f, \mu_0) \sim (g, \mu_u)$. This will not change the supremum in (2.7), with \mathcal{A}'_u now, either. It suffices to check this for $u = 0$. By a variant of a lemma of Katok, [L4, Lemma 3.2], if $(f, \mu_0) \sim (g, \mu_0)$ then there is a sequence $\varphi_k : (X, \omega) \rightarrow (X, \omega)$ of symplectomorphisms such that $\int_X |g - f \circ \varphi_k| \omega^n \rightarrow 0$. Therefore

$$\begin{aligned} \int_X g\xi\omega^n &= \lim_{k \rightarrow \infty} \int_X (f \circ \varphi_k)\xi\omega^n = \lim_{k \rightarrow \infty} \int_X (\xi \circ \varphi_k^{-1})f\omega^n, \quad \text{and so} \\ a + \int_X g\xi\omega^n &\leq \lim_{k \rightarrow \infty} L(\xi \circ \varphi_k^{-1}) = L(\xi). \end{aligned}$$

Thus replacing \mathcal{A}_u with \mathcal{A}'_u , (2.7) will still hold, and \mathcal{A}'_u is now strict rearrangement invariant.

Finally, if L is positively homogeneous, the statement of the theorem follows in the same way from the corresponding part of [L4, Lemma 2.1]. \square

The Lagrangians in this section were required to be continuous on the fibers of $T\mathcal{H}$. But, coupled with invariance, this implies continuity on $T\mathcal{H}$:

Theorem 2.5. *An invariant convex Lagrangian $L : T\mathcal{H} \rightarrow \mathbb{R}$ is a continuous function on the Fréchet manifold $T\mathcal{H}$.*

Proof. Suppose $u, u_k \in \mathcal{H}$, $\xi \in T_u\mathcal{H}$, $\xi_k \in T_{u_k}\mathcal{H}$, and $\xi_k \rightarrow \xi$. This simply means that as elements of $C^\infty(X)$, $u_k \rightarrow u$ and $\xi_k \rightarrow \xi$. Parallel translate ξ_k to $\eta_k \in T_u\mathcal{H}$ along the straight line segment $t \mapsto tu + (1 - t)u_k$. This is done by integrating the time dependent vector field $(1/2) \operatorname{grad}_{tu+(1-t)u_k}(u_k - u)$ on X , for $0 \leq t \leq 1$. If the time-1 map is $\psi_k : X \rightarrow X$, then $\eta_k = \xi_k \circ \psi_k$. Since $\psi_k \rightarrow \operatorname{id}_X$ in the C^∞ topology, $\eta_k \rightarrow \xi$ in $C^\infty(X) \approx T_u\mathcal{H}$, as $k \rightarrow \infty$. Hence $\lim_k L(\xi_k) = \lim_k L(\eta_k) = L(\xi)$, as claimed. \square

3. Extending Lagrangians

As said in the Introduction, weak geodesics tend not to stay in the space \mathcal{H} . Therefore, even in order to formulate a principle of least action we need to

evaluate the action of a Lagrangian along paths in spaces larger than \mathcal{H} . In this section we will extend invariant convex Lagrangians $T\mathcal{H} \rightarrow \mathbb{R}$ to larger Banach bundles and describe properties of the extended Lagrangians.

We start by recalling definitions. Let Y be a complex manifold and Ω a smooth real $(1, 1)$ form on it, $d\Omega = 0$. A function $u : Y \rightarrow [-\infty, \infty)$ is Ω -plurisubharmonic if $\rho + u$ is plurisubharmonic whenever ρ is a potential of Ω over some open $V \subset Y$, i.e., $\Omega|_V = dd^c\rho$. We use the convention that $\equiv -\infty$ is not plurisubharmonic, and write $\text{PSH}(\Omega)$ for the set of Ω -plurisubharmonic functions. Back to our Kähler manifold (X, ω) , we denote by $\mathcal{E}(\omega)$ the class of $u \in \text{PSH}(\omega)$ with full Monge–Ampère mass, see [GZ1]. The class contains all bounded ω -plurisubharmonic functions. The Monge–Ampère measure on X , corresponding to ω_u^n , will again be denoted μ_u . This is a Borel measure on X , its crucial property is $\mu_u(X) = \int_X \omega^n$. We endow $\mathcal{E}(\omega)$ with the discrete topology, and let

$$(3.1) \quad T^\infty\mathcal{E}(\omega) = \mathcal{E}(\omega) \times B(X),$$

a trivial Banach bundle with fibers the bounded Borel functions on X . Corresponding to usage in the subject we will not distinguish between elements $\xi \in T_u^\infty\mathcal{E}(\omega)$ and their representation $\xi \in B(X)$ in the trivialization (3.1). The embedding $C^\infty(X) \hookrightarrow B(X)$ induces an embedding $T\mathcal{H} \hookrightarrow T^\infty\mathcal{E}(\omega)$ of vector bundles, continuous if \mathcal{H} is considered with the discrete topology. We will also deal with a bundle between the image of $T\mathcal{H}$ and $T^\infty\mathcal{E}(\omega)$,

$$T^c\mathcal{H} = \mathcal{H} \times C(X).$$

Definition 3.1. *Suppose $u \in \mathcal{E}(\omega)$ and $V \subset B(X, \mu_u)$ is a vector subspace. We say that a function $p : V \rightarrow \mathbb{R}$ is strongly continuous if $p(\xi_k)$ converges whenever $\xi_k \in V$ is a uniformly bounded sequence that converges μ_u almost everywhere.*

In this case $\lim_k p(\xi_k)$ depends only on $\xi = \lim_k \xi_k$, since another sequence $\eta_k \rightarrow \xi$ can be combined with ξ_k into one sequence.

Theorem 3.2. *Any invariant convex Lagrangian $L : T\mathcal{H} \rightarrow \mathbb{R}$ has a unique fiberwise continuous extension to $T^c\mathcal{H}$. This extension is strict rearrangement invariant and fiberwise convex. If in addition L is strongly continuous on the fibers $T_u\mathcal{H}$, then it has a unique extension to $T^\infty\mathcal{E}(\omega)$ that is strict rearrangement invariant, and strongly continuous on the fibers $T_u^\infty\mathcal{E}(\omega)$. This extension is fiberwise convex.*

For example, Darvas’s metrics in [D2], coming from finite Young weights, cf. (1.2), are strongly continuous on the fibers.—The proof of the theorem will use the notion of decreasing rearrangement of measurable functions $\eta : (Y, \nu) \rightarrow \mathbb{R}$ on a measure space. This is a decreasing, upper semicontinuous function $\eta^* : (0, \nu(Y)] \rightarrow \mathbb{R}$, equidistributed with η . Thus $\nu(s \leq \eta \leq t)$ is equal to the length of the maximal interval on which $s \leq \eta^* \leq t$. The requirement of upper semicontinuity for the decreasing function η^* translates to left continuity, which differs from the more usual right continuity requirement, but the difference is of no consequence. In our setting

$$(3.2) \quad \nu(\eta > \eta^*(s)) \leq s \leq \nu(\eta \geq \eta^*(s)),$$

and more generally,

$$(3.3) \quad \nu(\eta \geq t) < \tau \text{ implies } \eta^*(\tau) < t, \quad \nu(\eta > t) > \tau \text{ implies } \eta^*(\tau) > t.$$

This is so because, for example $\nu(\eta > t)$ equals the Lebesgue measure of the set $\{\eta^* > t\}$, and this set would be contained in $(0, \tau)$ if we had $\eta^*(\tau) \leq t$.

When $\xi \in T_u^\infty \mathcal{E}(\omega)$, we compute ξ^* with respect to the measure μ_u .

Proof of Theorem 3.2. Suppose first $L : T\mathcal{H} \rightarrow \mathbb{R}$ is just continuous. By [L4, Theorem 5.2] each $L|_{T_u\mathcal{H}}$ has a unique continuous extension $C(X) \rightarrow \mathbb{R}$. These extensions are convex and strict rearrangement invariant, and together define the extension $\bar{L} : T^c\mathcal{H} \rightarrow \mathbb{R}$ of L . To see that \bar{L} is strict rearrangement invariant, given equidistributed $\xi \in T_u^c\mathcal{H}$ and $\eta \in T_v^c\mathcal{H}$, choose $\xi_j \in C^\infty(X)$ converging uniformly to ξ . Let $\varphi : (X, \omega_v) \rightarrow (X, \omega_u)$ be a symplectomorphism (for example one induced by parallel transport along some path connecting u and v). Then $\xi \circ \varphi, \eta \in T_v^c\mathcal{H}$ are equidistributed, whence

$$\bar{L}(\eta) = \bar{L}(\xi \circ \varphi) = \lim_{j \rightarrow \infty} L(\xi_j \circ \varphi) = \lim_{j \rightarrow \infty} L(\xi_j) = \bar{L}(\xi).$$

As to the second case of the theorem, if L is strongly continuous, by [L4, Theorem 5.2] $L|_{T_0\mathcal{H}} : C^\infty(X) \rightarrow \mathbb{R}$ has a unique strongly continuous and strict rearrangement invariant extension $q : B(X) \rightarrow \mathbb{R}$; this extension is convex. Further to extend q to $\bar{L} : T^\infty\mathcal{E}(\omega) \rightarrow \mathbb{R}$, take a $\xi \in T_u^\infty\mathcal{E}(\omega) \approx B(X, \mu_u)$. There is a measurable $\theta : X \rightarrow (0, \mu_0(X)]$, that pulls back Lebesgue measure to μ_0 , see e.g. [L4, Lemma 5.5]. Now let $\eta = \xi^* \circ \theta \in T_0^\infty\mathcal{E}(\omega)$ and $\bar{L}(\xi) = q(\eta)$. This is clearly the only strict rearrangement invariant way to extend $q : T_0^\infty\mathcal{E}(\omega) \rightarrow \mathbb{R}$ to $\bar{L} : T^\infty\mathcal{E}(\omega) \rightarrow \mathbb{R}$, and it is immediate that \bar{L} thus constructed has the properties claimed in the theorem. \square

Further down we will not distinguish between an invariant convex Lagrangian $T\mathcal{H} \rightarrow \mathbb{R}$ that is (strongly) continuous on the fibers and its extension to $T^c\mathcal{H}$, respectively, $T^\infty\mathcal{E}(\omega)$ provided by Theorem 3.2; we will just refer to a (strongly continuous,) invariant, convex Lagrangian $L : T^c\mathcal{H} \rightarrow \mathbb{R}$ or $L : T^\infty\mathcal{E}(\omega) \rightarrow \mathbb{R}$.

Lemma 3.3. *A strongly continuous, invariant, convex Lagrangian $L : T^\infty\mathcal{E}(\omega) \rightarrow \mathbb{R}$ is equi-Lipschitz continuous on bounded subsets of the fibers $T_u\mathcal{E}(\omega)$ in the sense that given $R \in (0, \infty)$ there is an $A \in (0, \infty)$ such that for $u \in \mathcal{E}(\omega)$ and $\xi, \eta \in T_u^\infty\mathcal{E}(\omega)$*

$$(3.4) \quad \text{if } \|\xi\|, \|\eta\| < R \text{ then } |L(\xi) - L(\eta)| \leq A\|\xi - \eta\|.$$

Proof. According to [L4, Theorem 5.4] (3.4) holds when $u = 0$. The same A will work for any u , for with a measure preserving $\theta : (X, \mu_0) \rightarrow (0, \mu_0(X))$ as in the proof of Theorem 3.2 and $\xi, \eta \in T_u^\infty\mathcal{E}(\omega)$

$$|L(\xi) - L(\eta)| = |L(\xi^* \circ \theta) - L(\eta^* \circ \theta)| \leq A \sup |\xi^* \circ \theta - \eta^* \circ \theta| \leq A\|\xi - \eta\|.$$

□

Although we have endowed $\mathcal{E}(\omega)$ with the discrete topology, we will need a continuity property of Lagrangians $T^\infty\mathcal{E}(\omega) \rightarrow \mathbb{R}$ stronger than fiberwise. This will involve the notion of Monge–Ampère capacity cap of subsets of X [BT, K, GZ2]. Recall that a function $\xi : X \rightarrow \mathbb{R}$ is quasicontinuous if for every $\varepsilon > 0$ there is an open $G \subset X$ of capacity $\text{cap}(G) < \varepsilon$ such that $\xi|_{X \setminus G}$ is continuous; and a sequence of functions $\xi_j : X \rightarrow \mathbb{R}$ converges to $\xi : X \rightarrow \mathbb{R}$ in capacity if $\lim_{j \rightarrow \infty} \text{cap}(|\xi_j - \xi| > \delta) = 0$ for every $\delta > 0$. In particular, a uniformly convergent sequence converges in capacity.

Lemma 3.4. *Let $L : T^\infty\mathcal{E}(\omega) \rightarrow \mathbb{R}$ be strongly continuous, invariant, and convex. Suppose $u_k \in \mathcal{E}(\omega)$ either decrease, or uniformly converge, to a bounded $u \in \mathcal{E}(\omega)$ as $k \rightarrow \infty$, and uniformly bounded $\xi_k \in T_{u_k}^\infty\mathcal{E}(\omega) \approx B(X)$ converge in capacity to $\xi \in T_u^\infty\mathcal{E}(\omega) \approx B(X)$. If ξ is quasicontinuous, then $\xi_k^* \rightarrow \xi^*$ away from a countable subset of $(0, \mu_0(X))$, and $\lim_k L(\xi_k) = L(\xi)$.*

Proof. If needed, we drop finitely many u_k to arrange that the remaining u_k are uniformly bounded. Upon adding a constant to the u_k and scaling u, u_k, ω , and L , we can even arrange that $0 \leq u, u_k \leq 1$. Suppose first the u_k

decrease. Define decreasing functions f, g on $(0, \mu_0(X)]$

$$f = \liminf_{k \rightarrow \infty} \xi_k^* \leq \limsup_{k \rightarrow \infty} \xi_k^* = g.$$

Let $s \in (0, \mu_0(X))$, $S = \xi^*(s)$, and $\varepsilon > 0$. With $A_k = \{|\xi_k - \xi| \geq \varepsilon\}$, $k \in \mathbb{N}$,

$$\begin{aligned} \{\xi_k \geq S + 2\varepsilon\} &\subset \{\xi \geq S + \varepsilon\} \cup A_k \quad \text{and} \\ \{\xi > S - \varepsilon\} &\subset \{\xi_k > S - 2\varepsilon\} \cup A_k. \end{aligned}$$

For $j \in \mathbb{N}$ define continuous functions $F_j, G_j : \mathbb{R} \rightarrow [0, 1]$

$$F_j(t) = \begin{cases} 0 & \text{if } t \leq S + \varepsilon - 1/j \\ 1 & \text{if } t \geq S + \varepsilon \\ \text{linear in between,} & \end{cases} \quad G_j(t) = \begin{cases} 0 & \text{if } t \leq S - \varepsilon \\ 1 & \text{if } t \geq S - \varepsilon + 1/j \\ \text{linear in between.} & \end{cases}$$

Note that F_j decrease as $j \rightarrow \infty$ to the characteristic function of $[S + \varepsilon, \infty)$ and G_j increase to the characteristic function of $(S - \varepsilon, \infty)$. We can estimate (3.5)

$$\begin{aligned} \mu_{u_k}(\xi_k \geq S + 2\varepsilon) &\leq \mu_{u_k}(\xi \geq S + \varepsilon) + \mu_{u_k}(A_k) \leq \int_X F_j \circ \xi \, d\mu_{u_k} + \text{cap}(A_k) \\ \mu_{u_k}(\xi_k \geq S - 2\varepsilon) &\geq \mu_{u_k}(\xi > S - \varepsilon) - \mu_{u_k}(A_k) \geq \int_X G_j \circ \xi \, d\mu_{u_k} - \text{cap}(A_k). \end{aligned}$$

Since $F_j \circ \xi, G_j \circ \xi$ are quasicontinuous, by [GZ2, Theorem 4.26, Proposition 4.25]

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_X F_j \circ \xi \, d\mu_{u_k} &= \int_X F_j \circ \xi \, d\mu_u, \\ \lim_{k \rightarrow \infty} \int_X G_j \circ \xi \, d\mu_{u_k} &= \int_X G_j \circ \xi \, d\mu_u. \end{aligned}$$

Therefore, letting first $k \rightarrow \infty$ in (3.5), then $j \rightarrow \infty$, and using the monotone convergence theorem and (3.2) as well,

$$\begin{aligned} \limsup_{k \rightarrow \infty} \mu_{u_k}(\xi_k \geq S + 2\varepsilon) &\leq \mu_u(\xi \geq S + \varepsilon) \leq \mu_u(\xi > S) \leq s \\ \liminf_{k \rightarrow \infty} \mu_{u_k}(\xi_k > S - 2\varepsilon) &\geq \mu_u(\xi > S - \varepsilon) \geq \mu_u(\xi \geq S) \geq s. \end{aligned}$$

Hence, given $\sigma < s < \rho$, for sufficiently large k

$$\mu_{u_k}(\xi_k \geq S + 2\varepsilon) < \rho, \quad \sigma < \mu_{u_k}(\xi_k > S - 2\varepsilon).$$

Let $\eta = \xi_k$, $\nu = \mu_{u_k}$, and apply (3.3) first with $t = S + 2\varepsilon$, $\tau = \rho$; then with $t = S - 2\varepsilon$, $\tau = \sigma$. We conclude $\xi_k^*(\rho) < S + 2\varepsilon$ and $\xi_k^*(\sigma) > S - 2\varepsilon$. In the limit $k \rightarrow \infty$

$$g(\rho) \leq S + 2\varepsilon = \xi^*(s) + 2\varepsilon \quad \text{and} \quad f(\sigma) \geq S - 2\varepsilon = \xi^*(s) - 2\varepsilon,$$

for all $\sigma < s < \rho$. If f, g are continuous at s —which occurs apart from countably many values—, $g(s) \leq \xi^*(s) \leq f(s) \leq g(s)$ follows, i.e., $\lim_k \xi_k^*(s) = \xi^*(s)$ as claimed.

It is now easy to finish the proof. With a measure preserving $\theta : (X, \mu_0) \rightarrow (0, \mu_0(X)]$, as in the proof of Theorem 3.2, $\xi_k \in B(X, \mu_{u_k})$ and $\xi_k^* \circ \theta \in B(X, \mu_0)$ are equidistributed, and $\xi_k^* \circ \theta \rightarrow \xi^* \circ \theta$ μ_0 -almost everywhere. Hence

$$\lim_k L(\xi_k) = \lim_k L(\xi_k^* \circ \theta) = L(\xi^* \circ \theta) = L(\xi).$$

We are done if u_k are known to decrease.

Now suppose that u_k converge uniformly. It suffices to prove that a subsequence of $L(\xi_k)$ converges to $L(\xi)$, and for this reason we can assume that $\|u_k - u_{k-1}\| < 2^{-k}$ for $k = 2, 3, \dots$. Then the sequence $v_k = u_k + 2^{-k}$ decreases to u , and $\mu_{u_k} = \mu_{v_k}$. We can view $\xi_k \in T_{u_k}^\infty \mathcal{E}(\omega) \approx B(X, \mu_{u_k})$ as elements $\xi'_k \in T_{v_k}^\infty \mathcal{E}(\omega) \approx B(X, \mu_{v_k})$, which are strict rearrangements of ξ_k . Hence $L(\xi_k) = L(\xi'_k) \rightarrow L(\xi)$ as $k \rightarrow \infty$, by the first part of the proof. \square

Later on we will need a variant of Lemma 3.4 in which the condition on ξ_k is relaxed.

Definition 3.5. *We will say that a family N of finite Borel measures on X is hereditarily tight if for every open $U \subset X$ the restrictions $\nu|U$, $\nu \in N$ are tight, i.e., given $\varepsilon > 0$, there is a compact $K \subset U$ such that $\nu(U \setminus K) < \varepsilon$ for all $\nu \in N$.*

For example, if all $\nu \in N$ are absolutely continuous with respect to a finite Borel measure μ , and the Radon–Nikodym derivatives $d\nu/d\mu$ are uniformly bounded, then N is hereditarily tight.

Lemma 3.6. *Let $L : T^\infty \mathcal{E}(\omega) \rightarrow \mathbb{R}$ be strongly continuous, invariant, and convex, and $u_k \in \mathcal{E}(\omega)$ either decrease, or uniformly converge, to a bounded $u \in \mathcal{E}(\omega)$ as $k \rightarrow \infty$. Suppose that the family μ_{u_k} , $k \in \mathbb{N}$, is hereditarily tight. If uniformly bounded $\xi_k \in T_{u_k}^\infty \mathcal{E}(\omega) \approx B(X)$ converge μ_u -almost everywhere to $\xi \in T_u^\infty \mathcal{E}(\omega) \approx B(X)$, then $\xi_k^* \rightarrow \xi^*$ away from a countable subset of $(0, \mu_0(X)]$, and $\lim_k L(\xi_k) = L(\xi)$.*

Proof. As in the previous proof, it suffices to argue when the u_k decrease. Let $\varepsilon > 0$. First we claim that there are uniformly bounded $\eta_k \in T_{u_k}^\infty \mathcal{E}(\omega)$ and continuous $\eta \in T_u^\infty \mathcal{E}(\omega)$ such that $\eta_k \rightarrow \eta$ uniformly and for $\varepsilon \leq s \leq \mu_0(X) - \varepsilon$

$$(3.6) \quad \eta^*(s + \varepsilon) \leq \xi^*(s) \leq \eta^*(s - \varepsilon), \quad \eta_k^*(s + \varepsilon) \leq \xi_k^*(s) \leq \eta_k^*(s - \varepsilon).$$

Indeed, by the theorems of Lusin and Egorov there are an open $U \subset X$ and $\eta \in C(X)$ such that $\mu_u(U) < \varepsilon/4$, $\eta = \xi$ on $X \setminus U$, and $\xi_k \rightarrow \xi$ uniformly on $X \setminus U$. With k_0 to be determined in a moment, let $\eta_k = \xi_k$ if $k < k_0$,

$$\eta_k = \begin{cases} \eta & \text{on } U \\ \xi_k & \text{on } X \setminus U \end{cases} \quad \text{if } k \geq k_0.$$

Thus $\eta_k \rightarrow \eta$ uniformly. Next, $\mu_u(\xi \neq \eta) \leq \mu_u(U) < \varepsilon/2$. To estimate $\mu_{u_k}(\xi_k \neq \eta_k)$ we pick a compact $K \subset U$ such that $\mu_{u_k}(U \setminus K) < \varepsilon/4$ for all k and a function $\zeta \in C(X)$ such that $\chi_K \leq \zeta \leq \chi_U$ (characteristic function). This implies $\int_X (\chi_U - \zeta) d\mu_{u_k} \leq \mu_{u_k}(U \setminus K) < \varepsilon/4$ for all k . Choose k_0 so that

$$\int_X \zeta d\mu_{u_k} < \varepsilon/4 \quad \text{if } k \geq k_0.$$

This is possible, because, for example by [GZ2, Theorem 3.18], as $k \rightarrow \infty$, the integral above tends to $\int_X \zeta d\mu_u \leq \mu_u(U) < \varepsilon/4$. Hence for $k \geq k_0$

$$(3.7) \quad \mu_{u_k}(\xi_k \neq \eta_k) \leq \mu_{u_k}(U) = \int_X (\chi_U - \zeta) d\mu_{u_k} + \int_X \zeta d\mu_{u_k} < \varepsilon/2,$$

and the same holds for $k < k_0$ trivially.

We view η, η_k as vectors in $T_u^\infty \mathcal{E}(\omega), T_{u_k}^\infty \mathcal{E}(\omega)$. Let $\sigma < \xi^*(s) < \rho$. By (3.2)

$$\mu_u(\xi \geq \rho) \leq \mu_u(\xi > \xi^*(s)) \leq s \leq \mu_u(\xi \geq \xi^*(s)) \leq \mu_u(\xi > \sigma).$$

As $\mu_u(\xi \neq \eta) < \varepsilon/2$,

$$s - \varepsilon < s - \varepsilon/2 \leq \mu_u(\eta > \sigma), \quad \mu_u(\eta \geq \rho) \leq s + \varepsilon/2 < s + \varepsilon,$$

and by (3.3) $\eta^*(s + \varepsilon) < \rho, \sigma < \eta^*(s - \varepsilon)$. This being true for all $\sigma < \xi^*(s) < \rho$, the first set of inequalities in (3.6) follows. The second set follows the same way, using (3.7).

We can apply Lemma 3.4 and conclude that $\eta_k^* \rightarrow \eta^*$ away from a countable set. Therefore

$$\limsup_{k \rightarrow \infty} \xi_k^*(s) \leq \lim_{k \rightarrow \infty} \eta_k^*(s - \varepsilon) = \eta^*(s - \varepsilon) \leq \xi^*(s - 2\varepsilon),$$

for all $s \in [2\varepsilon, \mu_0(X) - 2\varepsilon]$ except countably many; and similarly

$$\liminf_{k \rightarrow \infty} \xi_k^*(s) \geq \xi^*(s + 2\varepsilon).$$

If in addition ξ^* is continuous at s , letting $\varepsilon \rightarrow 0$ through a sequence these inequalities prove $\lim_{k \rightarrow \infty} \xi_k^*(s) = \xi^*(s)$, which therefore holds for all $s \in (0, \mu_0(X)]$ with countably many exceptions.

From this $\lim_k L(\xi_k) = L(\xi)$ follows the same way as in Lemma 3.4. \square

4. Weak geodesics, ε -geodesics, Jacobi fields

If $a < b$ are real numbers, we let

$$S_{ab} = \{s \in \mathbb{C} : a < \text{Res} < b\},$$

and denote by π the projection $S_{ab} \times X \rightarrow X$.

Following Berndtsson and Darvas [Be1, D4, section 3.3] we make the following definition.

Definition 4.1. *A path $u : (a, b) \rightarrow \text{PSH}(\omega)$ is a subgeodesic if the function $U : S_{ab} \times X \rightarrow [-\infty, \infty)$ given by $U(s, x) = u(\text{Res})(x)$ is $\pi^*\omega$ -plurisubharmonic.*

If $w, w' \in \text{PSH}(\omega)$, the weak geodesic determined by (or connecting) w, w' is $u : (a, b) \rightarrow \text{PSH}(\omega)$,

$$(4.1) \quad u = \sup\{v \mid v : (a, b) \rightarrow \text{PSH}(\omega) \text{ is subgeodesic, } \lim_{t \rightarrow a} v(t) \leq w, \lim_{t \rightarrow b} v(t) \leq w'\}.$$

The limits are understood pointwise on X ; they exist because $\pi^*\omega$ -plurisubharmonicity implies that for each $x \in X$ the function $v(\cdot)(x)$ is convex. It is possible that (4.1) gives $u \equiv -\infty$, not valued in $\text{PSH}(\omega)$; but otherwise the weak geodesic is indeed a path in $\text{PSH}(\omega)$ and is itself a subgeodesic [D4, section 3.1]. Darvas points out that in general the term “connecting” weak geodesic is misleading, as $\lim_a u$ may have little to do with w . But, if w, w' are bounded, Berndtsson proves by a simple argument that the weak geodesic indeed connects, $\lim_a u = w, \lim_b u = w'$, uniformly on X ,

[Be1, pp. 156-157]. If $c < d$, the weak geodesic $v : (c, d) \rightarrow \text{PSH}(\omega)$ between w and w' is u , composed with an affine reparametrization, because affine reparametrizations of subgeodesics yield subgeodesics.

In what follows we will only deal with weak geodesics u determined by bounded w, w' . Such a u is a Lipschitz map into $B(X)$, and we will refer to its continuous extension to the closed interval $[a, b]$ as a weak geodesic, too.

In (1.1) we defined Mabuchi's connection on $T\mathcal{H}$ through its parallel transport. A more direct definition takes a smooth path $u : [a, b] \rightarrow \mathcal{H}$ and a smooth vector field $\xi : [a, b] \rightarrow T\mathcal{H}$, $\xi(t) \in T_{u(t)}\mathcal{H}$, along it; the covariant derivative of ξ along u is then the vector field $\nabla_t \xi$ given by

$$(4.2) \quad \nabla_t \xi(t) = \dot{\xi}(t) - \frac{1}{2}(d_X \dot{u}(t), d_X \xi(t))_{u(t)} \in C^\infty(X) \approx T_{u(t)}\mathcal{H}.$$

Here d_X is differential on X , for fixed t , and $(\ , \)_{u(t)}$ is inner product on T^*X induced by the Kähler metric of $\omega_{u(t)}$. In (4.2) the left hand side is to be computed for ξ a section of $T\mathcal{H}$ along u ; on the right ξ stands for the representation of this section in the canonical trivialization $T\mathcal{H} \approx \mathcal{H} \times C^\infty(X)$, so for a function $[a, b] \rightarrow C^\infty(X)$; and the equality of the two sides again uses the trivialization of $T\mathcal{H}$.

Geodesics $u : [a, b] \rightarrow \mathcal{H}$ of ∇ satisfy $\nabla_t \dot{u}(t) = 0$. Chen, however, had the idea that the geometry of \mathcal{H} can be better accessed through ε -geodesics. Define a vector field F on \mathcal{H} by

$$(4.3) \quad F(v)\omega_v^n = \omega^n, \quad v \in \mathcal{H}, \quad F(v) \in C^\infty(X) \approx T_v\mathcal{H}.$$

If $\varepsilon > 0$, an ε -geodesic is a solution $u : [a, b] \rightarrow \mathcal{H}$ of

$$(4.4) \quad \nabla_t \dot{u}(t) = \varepsilon F(u(t)), \quad t \in [a, b].$$

In what follows, we will just write d for d_X . Chen proves [C, B11, B12]

Theorem 4.2. *Given $a < b$ and two potentials $w, w' \in \mathcal{H}$, (4.4) has a unique C^2 solution $u = u^\varepsilon : [a, b] \rightarrow \mathcal{H}$ satisfying $u(a) = w, u(b) = w'$. The solution u is smooth, and as an element of $C^\infty([a, b] \rightarrow \mathcal{H})$, it depends smoothly on w, w' (and a, b, ε). Finally, if w, w' are in a fixed compact subset of \mathcal{H} , the forms $dd^c u^\varepsilon(t), d\dot{u}^\varepsilon(t)$ and $\ddot{u}^\varepsilon(t)$ are uniformly bounded on X for $0 < \varepsilon < \varepsilon_0$ and $a \leq t \leq b$.*

It follows by the Arzelà–Ascoli theorem and a maximum principle for the Monge–Ampère operator that for fixed w, w' the uniform limit $u = \lim_{\varepsilon \rightarrow 0} u^\varepsilon$

exists. The limit u maps into the space

$$(4.5) \quad \mathcal{H}^{\overline{1}} = \{w \in C(X) \cap \text{PSH}(\omega) : \text{the current } dd^c w \text{ is bounded}\},$$

and the currents $dd^c u(t), d\dot{u}(t), \ddot{u}(t)$ are represented by uniformly bounded forms. Also, u is the weak geodesic in the sense of Definition 4.1 to connect w, w' .

Consider an ε -geodesic $u : [a, b] \rightarrow \mathcal{H}$.

Definition 4.3. *A vector field $\xi : [a, b] \rightarrow T\mathcal{H}$ along u is an ε -Jacobi field if there are an interval I containing $0 \in \mathbb{R}$ and a smooth family $I \ni s \mapsto u^s$, each $u^s : [a, b] \rightarrow \mathcal{H}$ an ε -geodesic such that $u^0 = u$ and $\xi = \partial_s u^s|_{s=0}$.*

Lemma 4.4. *(a) If $I \subset \mathbb{R}$ is an interval and $v : I \rightarrow \mathcal{H}$ a smooth path, then the covariant derivative of F along v satisfies*

$$(4.6) \quad \omega_{v(s)}^n \nabla_s F(v(s)) = -nd(F(v(s))d^c \dot{v}(s)) \wedge \omega_{v(s)}^{n-1}.$$

(b) If $\xi : [a, b] \rightarrow T\mathcal{H}$ is an ε -Jacobi field along an ε -geodesic $u : [a, b] \rightarrow \mathcal{H}$, then

$$(4.7) \quad \omega_{u(t)}^n \nabla_t^2 \xi(t) = \frac{1}{4} \{ \{ \dot{u}(t), \xi(t) \}, \dot{u}(t) \} \omega_{u(t)}^n - \varepsilon nd(F(u(t))d^c \xi(t)) \wedge \omega_{u(t)}^{n-1},$$

where $\{, \} = \{, \}_{u(t)}$ is Poisson bracket on $T_{u(t)}\mathcal{H} \approx C^\infty(X)$ for the symplectic form $\omega_{u(t)}$.

Calabi and Chen [CC, Section 2.3] derive an equivalent equation for ε -Jacobi fields.

Proof. (a) We will apply (4.2) with $\xi = F \circ v$. Differentiating $F(v(s))\omega_{v(s)}^n = \omega^n$ with respect to s gives

$$\omega_{v(s)}^n \partial_s F(v(s)) = -F(v(s))\partial_s(\omega + dd^c v(s))^n = -nF(v(s))dd^c \dot{v}(s) \wedge \omega_{v(s)}^{n-1}.$$

At the same time

$$(d\dot{v}(s), dF(v(s)))_{v(s)} \omega_{v(s)}^n = 2ndF(v(s)) \wedge d^c \dot{v}(s) \wedge \omega_{v(s)}^{n-1},$$

see e.g. [Bl2, p.103]. Substituting into (4.2) now gives (4.6).

(b) Let $u^s : [a, b] \rightarrow \mathcal{H}$ be a smooth family of ε -geodesics such that $\xi = \partial_s u^s|_{s=0}$, and set $U(s, t) = u^s(t)$. As Mabuchi's connection is torsion free, $\nabla_s \partial_t U = \nabla_t \partial_s U$. The curvature of ∇ , evaluated on $\partial_s U(s, t), \partial_t U(s, t) \in$

$T_{U(s,t)}\mathcal{H}$ is an endomorphism of $T_{U(s,t)}\mathcal{H}$ that acts on a vector field $\eta(s, t)$ by

$$R(\partial_s U, \partial_t U)\eta = (\nabla_s \nabla_t - \nabla_t \nabla_s)\eta = \{\{\partial_s U, \partial_t U\}, \eta\}/4,$$

see [M, Theorem 4.3]. (Mabuchi’s formula does not contain the factor $1/4$, due to different conventions.)

We apply ∇_s to the ε -geodesic equation $\nabla_t \partial_t U(s, t) = \varepsilon F(U(s, t))$, to obtain at $s = 0$

$$\begin{aligned} \varepsilon \nabla_s F(U) &= \nabla_s \nabla_t \partial_t U = R(\partial_s U, \partial_t U)\partial_t U + \nabla_t \nabla_s \partial_t U \\ &= (1/4)\{\{\partial_s U, \partial_t U\}, \partial_t U\} + \nabla_t \nabla_t \partial_s U = (1/4)\{\{\xi, \dot{u}\}, \dot{u}\} + \nabla_t^2 \xi. \end{aligned}$$

Combining (4.6) with this, (4.7) follows. □

5. The action

Consider an invariant convex Lagrangian $L : T\mathcal{H} \rightarrow \mathbb{R}$. If $u : [a, b] \rightarrow \mathcal{H}$ is a piecewise C^1 path, its action is

$$(5.1) \quad \mathcal{L}(u) = \int_a^b L(\dot{u}(t))dt.$$

Depending on the nature of L , this can represent length or energy of a path, but in general it is neither. No matter what L , the integral (5.1) is that of a piecewise continuous function by Theorem 2.5, so that it exists as a Riemann integral.

For the purposes of this paper we must consider action for paths beyond \mathcal{H} . The material developed in section 3 allows to define action for paths in the space $B(X) \cap \text{PSH}(\omega) \subset \mathcal{E}(\omega)$ of bounded ω -plurisubharmonic functions. This is a subset of the Banach space $B(X)$ and, viewing maps into it as maps into $B(X)$, we can talk about various regularity classes of such maps. If $a < b$ are real, the following is easy to check.

Lemma 5.1. *A map $u : [a, b] \rightarrow B(X)$ is continuous if and only if the functions*

$$(5.2) \quad u(\cdot)(x), \quad \text{for } x \in X$$

are equicontinuous, and it is C^k for $k = 1, 2, \dots$ if and only if, in addition, the functions in (5.2) are k times differentiable, and the k 'th derivatives are also equicontinuous.

According to Theorem 3.2, an invariant convex Lagrangian $T\mathcal{H} \rightarrow \mathbb{R}$ that is strongly continuous on the fibers determines a strongly continuous, invariant, convex Lagrangian $L : T^\infty\mathcal{E}(\omega) \rightarrow \mathbb{R}$. Suppose $u : [a, b] \rightarrow B(X) \cap \text{PSH}(\omega)$ is a C^1 path. Since ω -plurisubharmonic functions are quasicontinuous [BT, Theorem 3.5], [GZ2, Corollary 9.12], the difference quotients $(u(t) - u(s))/(t - s)$ are quasicontinuous and so are their uniform limits $\dot{u}(t)$. Hence by Lemma 3.4 $L \circ \dot{u} : [a, b] \rightarrow \mathbb{R}$ is continuous. Clearly if u is just piecewise C^1 , the integral in (5.1) still exists as the integral of a piecewise continuous function, and defines action $\mathcal{L}(u)$.

If $w, w' \in B(X) \cap \text{PSH}(\omega)$ and $T \in (0, \infty)$, we define the least action, or just action, $\mathcal{L}_T(w, w')$ between w, w' as

$$(5.3) \quad \mathcal{L}_T(w, w') = \inf_u \mathcal{L}(u),$$

the infimum taken over all piecewise C^1 paths $u : [0, T] \rightarrow B(X) \cap \text{PSH}(\omega)$ such that $u(0) = w, u(T) = w'$. Note that w, w' can be connected by a smooth path, e.g. $u(t) = (1 - t/T)w + (t/T)w'$ connects. We will see that $\mathcal{L}_T(w, w') > -\infty$ (Lemma 9.4).

Instead of $[0, T]$ if we minimize over paths $[a, a + T] \rightarrow B(X) \cap \text{PSH}(\omega)$, the infimum in (5.3) does not change. However, in general $\mathcal{L}_T(w, w')$ will depend on T ; it will not if L is positively homogeneous. In general

$$\mathcal{L}_T(w, w') + \mathcal{L}_S(w', w'') \geq \mathcal{L}_{T+S}(w, w'')$$

follows by concatenating paths. Of course, $\mathcal{L}_T(w, w') = \mathcal{L}_T(w', w)$ should be expected only if L is even, $L(-\xi) = L(\xi)$.

In the two results below, $L : T^\infty\mathcal{E}(\omega) \rightarrow \mathbb{R}$ is a strongly continuous, invariant, convex Lagrangian.

Theorem 5.2 (Principle of least action). *If a C^1 path $v : [0, T] \rightarrow B(X) \cap \text{PSH}(\omega)$ is a weak geodesic, then $\mathcal{L}(v) = \mathcal{L}_T(v(0), v(T))$.*

It is not hard to show that piecewise C^1 geodesics in $B(X) \cap \text{PSH}(\omega)$ are automatically C^1 , and by Chen's work, Theorem 4.2, weak geodesics connecting points in \mathcal{H} are also C^1 . Nonetheless, by Example 5.4 below, general weak geodesics in $B(X) \cap \text{PSH}(\omega)$ are not even left or right differentiable, and it is dubious if action along such paths can be defined by an integral.

Theorem 5.3. *If $u, v : [a, b] \rightarrow B(X) \cap \text{PSH}(\omega)$ are weak geodesics, then for $S > 0$ the function*

$$[a, b] \ni t \mapsto \mathcal{L}_S(u(t), v(t)) \in \mathbb{R}$$

is convex.

If L is absolutely homogeneous, $L(c\xi) = |c|L(\xi)$, and vanishes only on zero vectors, then action \mathcal{L}_S is distance measured in a Finsler metric and is independent of S ; the statement of Theorem 5.3 is an indication of seminegative curvature.

The proofs will take sections 6–9. First we prove approximate versions, with ε -geodesics in \mathcal{H} replacing weak geodesics. The approximate versions depend on two facts. First, that L is convex along ε -Jacobi fields; second, as a consequence, a triangle type inequality holds for triangles in \mathcal{H} with two sides ε -geodesics (Theorem 6.1, Lemma 7.2). It is a technical point but noteworthy that the approximate results contain no error term, no $O(\varepsilon)$. By letting $\varepsilon \rightarrow 0$ we obtain a principle of least action in $\mathcal{H}^{1\bar{1}}$ (Corollary 7.4). Approximating weak geodesics in $B(X) \cap \text{PSH}(\omega)$ by weak geodesics in $\mathcal{H}^{1\bar{1}}$ we obtain the same in $B(X) \cap \text{PSH}(\omega)$ in section 8. Theorem 5.3 is proved by the same approximation scheme in section 9.

To conclude this section we discuss smoothness of weak geodesics $v : [a, b] \rightarrow B(X) \cap \text{PSH}(\omega)$. For fixed $x \in X$, the function $v(\cdot)(x)$ is convex, hence has left and right derivatives at every $t \in (a, b)$, that we will denote $\partial_t^\mp v(t)(x)$. These one sided derivatives are bounded, since v is Lipschitz according to Berndtsson [Be1, section 2.2], see Lemma 9.3. Let (X, ω) be complex projective space \mathbb{P}_n with the Fubini–Study metric.

Example 5.4. There is a weak geodesic $v : [a, b] \rightarrow B(X) \cap \text{PSH}(\omega)$ that is not left or right differentiable (as a map into $B(X)$) at any $t \in [a, b]$. Furthermore, for each $t \in (a, b)$ the derivatives $\partial_t^- v(t)$ and $\partial_t^+ v(t)$, computed pointwise on X , differ $\mu_{v(t)}$ almost everywhere.

Our v will be a toric weak geodesic. Such weak geodesics can be obtained using an extension of Guan’s correspondence between toric geodesics and linear geodesics in a suitable Hilbert space $L^2(P)$, see [G]. Nevertheless, since the correspondence between properties of toric weak geodesics and linear geodesics is less straightforward than for geodesics in \mathcal{H} , we will just write down one possible example and verify its character directly.

We view \mathbb{P}_n as $\mathbb{C}^n \cup \mathbb{P}_{n-1}$, and denote by $|\cdot|$ Euclidean norm on \mathbb{C}^n . Keeping in mind that $\omega = dd^c \log(1 + |x|^2)$ on \mathbb{C}^n , we define $v : [a, b] \rightarrow C(X) \cap \text{PSH}(\omega)$ by

$$(5.4) \quad v(t)(x) = \begin{cases} 2 \max(t, \log |x|) - \log(1 + |x|^2), & \text{if } t \in [a, b], x \in \mathbb{C}^n \\ 0, & \text{if } t \in [a, b], x \in \mathbb{P}_{n-1}. \end{cases}$$

In fact $v(t)$ is smooth in a neighborhood of \mathbb{P}_{n-1} . To check that v is a weak geodesic, let $S_{ab} = \{s \in \mathbb{C} : a < \text{Res} < b\}$ as in section 4 and

$$V(s, x) = \begin{cases} 2 \max(\text{Res}, \log |x|) - \log(1 + |x|^2), & \text{if } s \in \bar{S}_{ab}, x \in \mathbb{C}^n \\ 0, & \text{if } s \in \bar{S}_{ab}, x \in \mathbb{P}_{n-1}. \end{cases}$$

Thus V is $\pi^*\omega$ -plurisubharmonic on $S_{ab} \times X$, i.e., v is a subgeodesic. Suppose $u : [a, b] \rightarrow B(X) \cap \text{PSH}(\omega)$ is the weak geodesic connecting $v(a), v(b)$, so that $v \leq u$. Let $U(s, x) = u(\text{Res})(x)$ for $s \in \bar{S}_{ab}, x \in X$. Since $u(a) = v(a)$ and $u(b) = v(b)$ are continuous, U is continuous at points in $\partial S_{ab} \times X$. If $y \in X$, consider the map

$$\phi_y : \bar{S}_{ab} \ni s \mapsto (s, e^s y) \in \bar{S}_{ab} \times X,$$

with the understanding that $e^s y = y$ if $y \in \mathbb{P}_{n-1}$. The pull back ϕ_y^*V is smooth, and $dd^c \phi_y^*V = 2dd^c(\text{Res} + \max(0, \log |y|)) - \phi_y^*\omega = -\phi_y^*\omega$. Hence the bounded function $\phi_y^*(U - V)$ is subharmonic on S_{ab} , and by the maximum principle it is ≤ 0 . As y ranges over X , the strips $\phi_y(S_{ab})$ foliate X . Therefore $U \leq V$. Along with $v \leq u$ this implies that $v = u$ is indeed a weak geodesic.

For fixed $x \in X$ one computes from (5.4) the one sided derivatives of $v(\cdot)(x)$,

$$(5.5) \quad \partial_t^- v(t)(x) = \begin{cases} 2 & \text{if } t > \log |x| \\ 0 & \text{otherwise,} \end{cases} \quad \partial_t^+ v(t)(x) = \begin{cases} 2 & \text{if } t \geq \log |x| \\ 0 & \text{otherwise.} \end{cases}$$

On the one hand, for each t these are discontinuous functions on X ; on the other, each difference quotient of v is continuous. Hence $\partial_t^\pm v(t)$ cannot be the limit, in $B(X)$, of difference quotients, and so v as a map into $B(X)$ has no one sided derivatives.

Finally, for each t (5.4) gives that the measure induced by $\omega_{v(t)}^n$ is invariant under $U(n)$ rotations of X and is supported on the sphere $|x| = e^t$. Hence it is a nonzero multiple of area measure on that sphere; by (5.5) therefore $\mu_{v(t)}$ -almost everywhere $\partial_t^- v(t) \neq \partial_t^+ v(t)$.

The example shows that action even along a weak geodesics in $B(X) \cap \text{PSH}(\omega)$ cannot be defined by the integral (5.1), since the integral depends on whether $\dot{u}(t)$ is interpreted as left or right derivative. Perhaps the correct interpretation is the average of the two; this is what Theorem 10.1 and Lemma 10.2 seem to suggest.

6. Divergence of ε -Jacobi fields

In this section we stay in \mathcal{H} , and consider invariant convex Lagrangians $L : T\mathcal{H} \rightarrow \mathbb{R}$, that are just continuous. Recall that given $\varepsilon > 0$, an ε -geodesic $u : [a, b] \rightarrow \mathcal{H}$ satisfies the equation

$$(6.1) \quad \nabla_t \dot{u}(t) = \varepsilon F(u(t)),$$

where the vector field $F : \mathcal{H} \rightarrow T\mathcal{H}$ is defined by $F(v)\omega_v^n = \omega^n$. Infinitesimal variations of ε -geodesics are ε -Jacobi fields. If $\xi : [a, b] \rightarrow T\mathcal{H}$ is an ε -Jacobi field along an ε -geodesic $u : [a, b] \rightarrow T\mathcal{H}$, by Lemma 4.4(b)

$$(6.2) \quad \omega_{u(t)}^n \nabla_t^2 \xi(t) = \frac{1}{4} \{ \{ \dot{u}(t), \xi(t) \}, \dot{u}(t) \} \omega_{u(t)}^n - \varepsilon nd(F(u(t))d^c \xi(t)) \wedge \omega_{u(t)}^{n-1}.$$

All our subsequent results rest on the following theorem.

Theorem 6.1. *If $\xi : [a, b] \rightarrow T\mathcal{H}$ is an ε -Jacobi field along an ε -geodesic $u : [a, b] \rightarrow \mathcal{H}$, then $L \circ \xi$ is a convex function on $[a, b]$.*

This will be derived from a special case.

Lemma 6.2. *Given $u_0 \in \mathcal{H}$ and $f_0 \in B(X)$, Theorem 6.1 holds for the Lagrangian*

$$(6.3) \quad L(\eta) = \sup_{(f,v) \sim (f_0, u_0)} \int_X f \eta d\mu_v, \quad \eta \in T_v \mathcal{H},$$

cf. (2.1).

To prove Lemma 6.2 we need some preparation. If Y is any set, we say that functions $g, h : Y \rightarrow \mathbb{R}$ are similarly ordered if $(g(x) - g(y))(h(x) - h(y)) \geq 0$ for all $x, y \in Y$. Equivalently, $g(x) < g(y)$ should imply $h(x) \leq h(y)$. The relation is not transitive, any function is similarly ordered as a constant.

Lemma 6.3. *Let Y be an oriented smooth manifold and a_{ij} smooth functions, V_i smooth vector fields on it, $i, j = 1, \dots, k$. Assume the matrix (a_{ij}) is symmetric and positive semidefinite everywhere. If $g \in C^\infty(Y)$ and a locally integrable $h : Y \rightarrow \mathbb{R}$ are similarly ordered, then the current $Q = \sum_{i,j} a_{ij}(V_i g)(V_j h)$ is ≥ 0 .*

Proof. It suffices to prove when h is bounded (in general we replace h by $h_R = \min(R, \max(-R, h))$ and let $R \rightarrow \infty$). Assume first that in addition there is a smooth increasing $H : \mathbb{R} \rightarrow \mathbb{R}$ such that $h = H \circ g$. Then $Q = H'(g) \sum a_{ij}(V_i g)(V_j g) \geq 0$. The same follows if H is any increasing function, by writing it as $\lim_p \lim_q H_{pq}$ (pointwise limit), with locally uniformly bounded smooth increasing H_{pq} .

Now consider general g, h . Let I denote the range of g , and for $t \in I$ define

$$m(t) = \inf\{h(x) : g(x) = t\}, \quad M(t) = \sup\{h(x) : g(x) = t\}.$$

If $g(x) = t < g(y) = \tau$, then $h(x) \leq h(y)$, which means that

$$m(t) \leq M(t) \leq m(\tau) \leq M(\tau) \quad \text{when } t < \tau.$$

In particular, m and M are increasing functions, and coincide on $\text{int } I$ wherever one of them is continuous, that is, apart from a countable set $T \subset I$. On $g^{-1}(I \setminus T)$ we have $h = m \circ g$. If t is a regular value of g , then $g^{-1}(t)$ has measure 0. Hence on the regular set of g the functions h and $m \circ g$ agree a.e., and the induced currents simply agree there. By what we already proved, $Q \geq 0$ on the set where $dg \neq 0$. We still need to understand the contribution of the critical set $C = (dg = 0)$.

Let $\chi : [0, \infty) \rightarrow [0, 1]$ be a smooth function, $\chi(t) = 0$ if $t \leq 1$, $\chi(t) = 1$ if $t \geq 2$. Endow Y with a Riemannian metric and denote by $\text{dist}(\cdot, C)$ distance to C . This is a Lipschitz function with Lipschitz constant 1. For $s > 0$, the function $\chi(s \text{ dist}(\cdot, C))$ has Lipschitz constant $O(s)$; it vanishes in the $1/s$ neighborhood of C and equals 1 outside the $2/s$ neighborhood. Let $\rho_s \in C^\infty(Y)$ have the same properties. To prove the lemma we need to show that if $\theta \geq 0$ is a compactly supported smooth volume form on Y , then

$$0 \leq \int_Y Q\theta = - \int_Y h \sum_{i,j} \mathcal{L}_j(\theta a_{ij} V_i g),$$

where \mathcal{L}_j stands for Lie derivative along V_j .

The inequality holds if θ is replaced by $\theta\rho_s$, because $Q \geq 0$ in a neighborhood of $\text{supp } \theta\rho_s$. The point is that the functions $\mathcal{L}_j(\theta\rho_s a_{ij} V_i g)$ are uniformly bounded and tend to $\mathcal{L}_j(\theta a_{ij} V_i g)$ a.e. as $s \rightarrow \infty$. The former is verified by applying Leibniz rule to the products, and checking each term. The only term that needs speaking for is $\theta a_{ij}(V_i g)(V_j \rho_s)$. But since $|V_i g| |V_j \rho_s|$ attains its maximum on $\{y \in Y : 1/s \leq \text{dist}(y, C) \leq 2/s\}$, this maximum is

$O(1/s)O(s) = O(1)$. As to convergence,

$$\lim_{s \rightarrow \infty} \mathcal{L}_j(\theta \rho_s a_{ij} V_i g) = \begin{cases} \mathcal{L}_j(\theta a_{ij} V_i g) & \text{on } Y \setminus C \\ 0 & \text{on } C. \end{cases}$$

If $\mathcal{L}_j(\theta a_{ij} V_i g)(y) \neq 0$ at some $y \in C$, then $(V_j V_i g)(y) \neq 0$; thus y is a zero and a regular point of $V_i g$. Such points form a hypersurface in Y , of zero measure. This proves the a.e. convergence statement.

We conclude by dominated convergence:

$$\begin{aligned} \int_Y Q\theta &= - \lim_{s \rightarrow \infty} \int_Y h \sum_{i,j} \mathcal{L}_j(\theta \rho_s a_{ij} V_i g) \\ &= \lim_{s \rightarrow \infty} \int_Y Q\theta \rho_s \geq 0. \end{aligned}$$

□

Proof of Lemma 6.2. The plan is to construct for every $t_0 \in (a, b)$ a family $f(t) \in B(X)$ with the property that $(f(t), u(t)) \sim (f_0, u_0)$ and $A(t) = \int_X f(t) \xi(t) d\mu_{u(t)} \leq L(\xi(t))$ satisfies

$$A(t_0) = L(\xi(t_0)), \quad \ddot{A}(t_0) \geq 0.$$

To simplify notation we can assume $t_0 = 0$. At the price of replacing f_0 by f_1 such that $(f_0, u_0) \sim (f_1, u(0))$, we can assume $u(0) = u_0$. Further to simplify we can arrange that $f = f_0$ realizes

$$(6.4) \quad \sup_{(f, u(0)) \sim (f_0, u(0))} \int_X f \xi(0) d\mu_{u(0)}, \quad \text{i.e.,} \quad L(\xi(0)) = \int_X f_0 \xi(0) d\mu_{u(0)};$$

this is possible simply because the supremum is attained, see e.g., [L4, Lemma 6.2]. The same lemma says that there is a maximizing f that is similarly ordered as $\xi(0)$, and accordingly we will work with f_0 similarly ordered as $\xi(0)$.

For a moment suppose $u : [a, b] \rightarrow \mathcal{H}$ is an arbitrary smooth path, and parallel transport $T_{u(0)} \mathcal{H} \rightarrow T_{u(t)} \mathcal{H}$ along u is given by pull back by a symplectomorphism $\varphi(t) : (X, \omega_{u(t)}) \rightarrow (X, \omega_{u(0)})$. Suppose $\eta : [a, b] \rightarrow T_{u(0)} \mathcal{H}$ is smooth; then $t \mapsto \eta(t) \circ \varphi(t)$ defines a vector field along u . Parallel transport

intertwines differentiation and covariant differentiation:

$$\nabla_t(\eta(t) \circ \varphi(t)) = \dot{\eta}(t) \circ \varphi(t) \quad \text{and} \quad \nabla_t^2(\eta(t) \circ \varphi(t)) = \ddot{\eta}(t) \circ \varphi(t).$$

When u is an ε -geodesic and ξ an ε -Jacobi field along it, as in the lemma, choose η so that $\eta(t) \circ \varphi(t) = \xi(t)$. By (6.2), at $t = 0$,

$$(6.5) \quad \begin{aligned} \ddot{\eta}(0)\omega_{u(0)}^n &= (1/4)\{\{\dot{u}(0), \eta(0)\}, \dot{u}(0)\}\omega_{u(0)}^n \\ &\quad - \varepsilon n d(F(u(0))d^c\eta(0)) \wedge \omega_{u(0)}^{n-1}. \end{aligned}$$

With $f(t) = f_0 \circ \varphi(t)$ we let

$$A(t) = \int_X f_0\eta(t)\omega_{u(0)}^n = \int_X f(t)\xi(t)\omega_{u(t)}^n \leq L(\xi(t)),$$

then $A(0) = L(\xi(0))$ by (6.4) and $\ddot{A}(t) = \int_X f_0\ddot{\eta}(t)\omega_{u(0)}^n$. In view of (6.5)

$$\begin{aligned} \ddot{A}(0) &= \frac{1}{4} \int_X f_0\{\{\dot{u}(0), \eta(0)\}, \dot{u}(0)\}\omega_{u(0)}^n - \varepsilon n \int_X f_0 d(F(u(0))d^c\eta(0)) \wedge \omega_{u(0)}^{n-1} \\ &= \frac{1}{4} \int_X \{\dot{u}(0), f_0\}\{\dot{u}(0), \eta(0)\}\omega_{u(0)}^n + \varepsilon n \int_X F(u(0))df_0 \wedge d^c\eta(0) \wedge \omega_{u(0)}^{n-1}. \end{aligned}$$

In the last line $\{\dot{u}(0), f_0\}$ and df_0 are currents. By Lemma 6.3 the first integrand in this last line is ≥ 0 , since f_0 and $\eta(0) = \xi(0)$ are similarly ordered; and so is, for the same reason, $2ndf_0 \wedge d^c\eta(0) \wedge \omega_{u(0)}^{n-1} = (df_0, d\eta(0))_{u(0)}\omega_{u(0)}^n$, cf. [Bl2, p.103].

To summarize, we have shown that for every $t_0 \in (a, b)$ there is a function $A \in C^\infty[a, b]$ such that

$$A(t) \leq L(\xi(t)), \text{ with equality when } t = t_0, \text{ and } \ddot{A}(t_0) \geq 0.$$

By a standard argument this implies that $L \circ \xi$ is convex. First one notes that if $p > 0$ and $q \in \mathbb{R}$, the function $L(\xi(t)) + pt^2 + qt$ cannot have a local maximum at any $t_0 \in (a, b)$, because with the A we have constructed $A(t) + pt^2 + qt$ has no local maximum at t_0 . It follows that on any subinterval $[\alpha, \beta] \subset [a, b]$, $L(\xi(t)) + pt^2 + qt$ attains its maximum at one of the endpoints, whence $L(\xi(t)) + pt^2$ is convex. Letting $p \rightarrow 0$ we see that $L \circ \xi$ itself is also convex. □

Proof of Theorem 6.1. Clearly, Lemma 6.2 implies that if $a \in \mathbb{R}$, $g \in B(X)$, and

$$L_{a,g}(\eta) = a + \sup_{(f,v) \sim (g,u_0)} \int_X f \eta d\mu_v, \quad \eta \in T_v \mathcal{H},$$

then $L_{a,g}$ is convex along any ε -Jacobi field. Since by Theorem 2.4 a general invariant convex Lagrangian is the supremum of a family of such $L_{a,g}$, the theorem follows. \square

7. Least action in \mathcal{H} and $\mathcal{H}^{1\bar{1}}$

In this section we will compare the actions along weak geodesics in $\mathcal{H}^{1\bar{1}}$ and along general paths in \mathcal{H} . Recall the notation $T^c \mathcal{H} = \mathcal{H} \times C(X)$.

Theorem 7.1. *Suppose a Lagrangian $L : T^c \mathcal{H} \rightarrow \mathbb{R}$ is invariant and convex. Consider a piecewise C^1 path $u : [0, T] \rightarrow \mathcal{H}$ and a weak geodesic $v : [0, T] \rightarrow \mathcal{H}^{1\bar{1}}$. If $u(0) = v(0)$ and $u(T) = v(T)$, then*

$$(7.1) \quad \frac{1}{T} \int_0^T L \circ \dot{u} \geq L(\dot{v}(0)).$$

First we prove a variant.

Lemma 7.2. *Suppose an invariant convex Lagrangian $L : T\mathcal{H} \rightarrow \mathbb{R}$ is positively homogeneous, $L(c\xi) = cL(\xi)$ if $c > 0$. Consider a triangle in \mathcal{H} formed by a piecewise C^1 path $u : [a, b] \rightarrow \mathcal{H}$ and ε -geodesics $v_a, v_b : [0, T] \rightarrow \mathcal{H}$; so that $v_a(0) = v_b(0)$ and $v_a(T) = u(a)$, $v_b(T) = u(b)$. Then*

$$(7.2) \quad \frac{1}{T} \int_a^b L \circ \dot{u} \geq L(\dot{v}_b(0) - \dot{v}_a(0)).$$

Note that positive homogeneity implies the triangle inequality $L(\xi + \eta) \leq L(\xi) + L(\eta)$ for $w \in \mathcal{H}$ and $\xi, \eta \in T_w \mathcal{H}$.

Proof. Because of the additive nature of (7.2), we can assume u is C^1 , not only piecewise, and then by simple approximation that it is even C^∞ . For each $s \in [a, b]$ let $U(s, \cdot) : [0, T] \rightarrow \mathcal{H}$ denote the ε -geodesic connecting $v_a(0) = v_b(0)$ with $u(s)$. According to Theorem 4.2, that is, by Chen's work, there is a unique such geodesic, and $U \in C^\infty([a, b] \times [0, T])$. Thus $\xi^s = \partial_s U(s, \cdot)$ is an ε -Jacobi field and $\xi^s(0) = 0$. By Theorem 6.1 $L \circ \xi^s$ is

convex on $[0, T]$. Using ∂_t (and later, dot) to denote right derivative, therefore

$$L(\xi^s(T)) \geq L(\xi^s(0)) + T\partial_t|_{t=0}L(\xi^s(t)).$$

By homogeneity, the first term on the right is 0. To compute the second, let $\eta(t) \in T_{U(0,0)}\mathcal{H}$ denote the parallel translate of $\xi^s(t) \in T_{U(s,t)}\mathcal{H}$ along $U(s, \cdot)$. Thus

$$\begin{aligned} \lim_{t \rightarrow 0} L(\xi^s(t))/t &= \lim_{t \rightarrow 0} L(\eta(t))/t = \lim_{t \rightarrow 0} L(\eta(t))/t \\ &= L(\nabla_t|_{t=0}\xi^s(t)) = L(\nabla_t|_{t=0}\partial_s U(s, t)) = L(\partial_s \partial_t|_{t=0}U(s, t)). \end{aligned}$$

The last equality is because ∇ has no torsion, and $U(\cdot, 0)$ is constant. Hence, using Jensen’s inequality as well,

$$\begin{aligned} \frac{1}{T} \int_a^b L(\partial_s u(s)) ds &= \frac{1}{T} \int_a^b L(\xi^s(T)) ds \geq \int_a^b L(\partial_s \partial_t|_{t=0}U(s, t)) ds \\ &\geq L\left(\int_a^b \partial_s \partial_t|_{t=0}U(s, t) ds\right) = L(\dot{v}_b(0) - \dot{v}_a(0)), \end{aligned}$$

as claimed. □

Proof of Theorem 7.1. For $\varepsilon > 0$ let $v^\varepsilon, w^\varepsilon : [0, T] \rightarrow \mathcal{H}$ denote the ε -geodesics connecting $u(0)$ with $u(T)$, respectively, $u(0)$ with itself. Again by Chen [C], see also Blocki [B11, B12], $v^\varepsilon \rightarrow v$ and $w^\varepsilon \rightarrow u(0)$ in such a way that $\dot{v}^\varepsilon(0) \rightarrow \dot{v}(0)$ and $\dot{w}^\varepsilon(0) \rightarrow 0$ in $T_{u(0)}^c\mathcal{H}$ as $\varepsilon \rightarrow 0$. Suppose first that L is positively homogeneous, and apply Lemma 7.2 with $[a, b] = [0, T]$, $v_a = w^\varepsilon$, $v_b = v^\varepsilon$. We obtain

$$\frac{1}{T} \int_0^T L \circ \dot{u} \geq L(\dot{v}^\varepsilon(0) - \dot{w}^\varepsilon(0)).$$

Letting $\varepsilon \rightarrow 0$,

$$(7.3) \quad \frac{1}{T} \int_0^T L \circ \dot{u} \geq L(\dot{v}(0))$$

follows. This is true even if L is not positively homogeneous but L plus a constant is. Since a general L is the supremum of Lagrangians of form positively homogeneous plus constant, see Theorem 2.4, (7.3) follows for a general L . □

Lemma 7.3. *If $L : T^\infty \mathcal{E}(\omega) \rightarrow \mathbb{R}$ is strict rearrangement invariant, and $v : [0, T] \rightarrow \mathcal{H}^{1\bar{1}}$ is a weak geodesic of class C^1 , then $L \circ \dot{v}$ is constant. Hence*

$$L(\dot{v}(0)) = \frac{1}{T} \int_0^T L \circ \dot{v}.$$

Proof. Berndtsson [Be2, Proposition 2.2] discovered that $\dot{v}(t) \in B(X, \mu_{v(t)})$ are equidistributed for all t , although he worked with integral Kähler classes $[\omega]$ and weak geodesics terminating in \mathcal{H} only. At any rate, [D2, Lemma 4.10] implies the general result. Since L is invariant, the lemma follows. \square

Together with Theorem 7.1 this almost proves the principle of least action in $\mathcal{H}^{1\bar{1}}$:

Corollary 7.4. *Suppose $L : T^\infty \mathcal{E}(\omega) \rightarrow \mathbb{R}$ is a strongly continuous, invariant, and convex Lagrangian. If $u : [0, T] \rightarrow \mathcal{H}$ is a piecewise C^1 path and $v : [0, T] \rightarrow \mathcal{H}^{1\bar{1}}$ is a weak geodesic between the same endpoints, then $\int_0^T L \circ \dot{u} \geq \int_0^T L \circ \dot{v}$.*

8. Least action in $B(X) \cap \text{PSH}(\omega)$

Here we will extend Corollary 7.4 to u, v taking values in $B(X) \cap \text{PSH}(\omega)$ (Theorem 5.2). In this section $L : T^\infty \mathcal{E}(\omega) \rightarrow \mathbb{R}$ is assumed to be strongly continuous, invariant, and convex.

Theorem 8.1. *Suppose $u, v : [0, T] \rightarrow B(X) \cap \text{PSH}(\omega)$ have the same endpoints: $u(0) = v(0)$, $u(T) = v(T)$. If u is piecewise C^1 and v is a C^1 weak geodesic, then $\int_0^T L \circ \dot{u} \geq \int_0^T L \circ \dot{v}$.*

This will be derived from Corollary 7.4 by approximation.

Lemma 8.2. *Suppose $u : [0, T] \rightarrow B(X) \cap \text{PSH}(\omega)$ is a piecewise C^1 path, and $w_j, w'_j \in \mathcal{H}$ decrease to $u(0)$, respectively, $u(T)$, as $j \rightarrow \infty$. Then there are a sequence $J \subset \mathbb{N}$ and for $j \in J$ piecewise linear $u_j : [0, T] \rightarrow \mathcal{H}$ such that $u_j(0) = w_j$, $u_j(T) = w'_j$, and $\int_0^T L \circ \dot{u}_j \rightarrow \int_0^T L \circ \dot{u}$ as $J \ni j \rightarrow \infty$.*

As said, at points where u, u_j are not differentiable, \dot{u}, \dot{u}_j mean right derivatives.

Proof. Choose $t_0 = 0 < t_1 < \dots < t_p = T$ so that u is C^1 on each $[t_{i-1}, t_i]$. Suppose first that u is even linear on $[t_{i-1}, t_i]$. In this case J will be all

of \mathbb{N} . A simple special instance of regularization, see [De, DP] and especially [BK], provides $z_{ij} \in \mathcal{H}$ such that z_{ij} decreases to $u(t_i)$ as $j \rightarrow \infty$ for $i = 0, \dots, p$. We take $z_{0j} = w_j$ and $z_{pj} = w'_j$, and arrange that the z_{ij} are uniformly bounded. Linearly interpolating on $[t_{i-1}, t_i]$ between $z_{i-1,j}$ and z_{ij} we obtain the functions u_j sought. Indeed, $u_j(t)$ decreases to $u(t)$, and

$$\dot{u}_j(t) = \frac{z_{ij} - z_{i-1,j}}{t_i - t_{i-1}} \in T_{u_j(t)}\mathcal{H}, \quad \text{when } t \in [t_{i-1}, t_i],$$

are uniformly bounded and tend to $\dot{u}(t)$ as $j \rightarrow \infty$. Since ω -plurisubharmonic functions are quasicontinuous [GZ2, Corollary 9.12], so are the difference quotients $\dot{u}_j(t)$. Also, by [GZ2, Proposition 9.11] z_{ij} , hence $\dot{u}_j(t)$ converge in capacity as $j \rightarrow \infty$; therefore $\lim_j L(\dot{u}_j(t)) = L(\dot{u}(t))$ by Lemma 3.4. Since $\dot{u}_j(t)$ are uniformly bounded, so are $L(\dot{u}_j(t))$ by equi-Lipschitz continuity, Lemma 3.3. The dominated convergence theorem gives therefore $\lim_j \int_0^T L \circ \dot{u}_j = \int_0^T L \circ \dot{u}$.

For general u , partition each $[t_{i-1}, t_i]$ into k equal parts. Construct $v_k : [0, T] \rightarrow B(X) \cap \text{PSH}(\omega)$ that agrees with u at each partition point, and is linear in between. Then $v_k \rightarrow u$ and $\dot{v}_k \rightarrow \dot{u}$ uniformly, because \dot{u} is uniformly continuous on $[t_{i-1}, t_i]$. Hence $L \circ \dot{v}_k \rightarrow L \circ \dot{u}$ by Lemma 3.4 and, again by dominated convergence, $\int_0^T L \circ \dot{v}_k \rightarrow \int_0^T L \circ \dot{u}$. By what we have already proved, for each k we can find $j = j_k > j_{k-1}$ and piecewise linear $u_j : [0, T] \rightarrow \mathcal{H}$ such that $u_j(0) = w_j$, $u_j(T) = w'_j$, and

$$\left| \int_0^T L \circ \dot{u}_j - \int_0^T L \circ \dot{v}_k \right| < \frac{1}{k}.$$

Thus $J = \{j_1, j_2, \dots\}$ will do. □

Lemma 8.3. *Let $v, v_j : [a, b] \rightarrow \text{PSH}(\omega)$ be weak geodesics. If $v_j(t)$ decreases to $v(t)$ when $t = a, b$, then $v_j(t)$ decreases to $v(t)$ for all $t \in [a, b]$.*

This is [D4, Proposition 3.15].—There is one more ingredient that goes into the proof of Theorem 8.1.

Lemma 8.4. *Consider a weak geodesic $v : [0, T] \rightarrow B(X) \cap \text{PSH}(\omega)$. If it is right differentiable at $t \in [0, T)$, then the right derivative $\dot{v}(t)$ is quasi-continuous. Moreover, $L \circ \dot{v}$ is constant on the subset $D \subset (0, T)$ where v is differentiable. Finally, if $v_j : [0, T] \rightarrow \mathcal{H}^{1\bar{1}}$ are weak geodesics that decrease to v , then $L \circ \dot{v}_j \rightarrow L \circ \dot{v}$ on D .*

Proof. As plurisubharmonic functions are quasicontinuous, the difference quotients $(v(t + s) - v(t))/s$ and their uniform limit $\dot{v}(t)$ are also quasicontinuous. Next we turn to the last statement, that we reduce to Lemma 3.4. First we fix $t \in D$ and show that $\dot{v}_j(t) \rightarrow \dot{v}(t)$ in sup norm $\| \cdot \|$. If $\varepsilon > 0$, there is an $s > 0$ such that

$$\left\| \dot{v}(t) - \frac{v(t \pm s) - v(t)}{\pm s} \right\| < \varepsilon,$$

and so there is a j_0 such that for $j > j_0$

$$\left\| \dot{v}(t) - \frac{v_j(t \pm s) - v_j(t)}{\pm s} \right\| < \varepsilon.$$

Convexity implies

$$\frac{v_j(t - s) - v_j(t)}{-s} \leq \dot{v}_j(t) \leq \frac{v_j(t + s) - v_j(t)}{s},$$

whence $\|\dot{v}_j(t) - \dot{v}(t)\| < \varepsilon$.

Given that $v_j(t)$ decreases to $v(t)$, that $\dot{v}_j(t) \rightarrow \dot{v}(t)$ in $B(X)$, and that $\dot{v}(t)$ is quasicontinuous, $t \in D$, Lemma 3.4 implies $L \circ \dot{v}_j \rightarrow L \circ \dot{v}$ on D .

To prove the second statement, construct $w_j, w'_j \in \mathcal{H}$ that decrease to $v(0), v(T)$, and let $v_j : [0, T] \rightarrow \mathcal{H}^{1\bar{1}}$ be the weak geodesic that joins them. By Lemma 8.3 v_j decreases to v and by Lemma 7.3 $L \circ \dot{v}_j$ is constant. According to what we just proved, $L \circ \dot{v}_j \rightarrow L \circ \dot{v}$ on D , and $L \circ \dot{v}$ must be constant there. \square

In particular, if $v : [a, b] \rightarrow B(X) \cap \text{PSH}(\omega)$ is a weak geodesic of class C^1 , then $L \circ \dot{v}$ is constant on (a, b) . Using this with different Lagrangians one can show that in fact $\dot{v}(t) \in B(X, \mu_{v(t)})$ are equidistributed for $a < t < b$.

Proof of Theorem 8.1. Construct $w_j, w'_j \in \mathcal{H}$ decreasing to $u(0), u(T)$, and let $u_j : [0, T] \rightarrow \mathcal{H}$, $j \in J$, be as in Lemma 8.2. Let $v_j : [0, T] \rightarrow \mathcal{H}^{1\bar{1}}$ be the weak geodesic connecting w_j and w'_j , $j \in J$. By Corollary 7.4

$$(8.1) \quad \int_0^T L \circ \dot{u}_j \geq \int_0^T L \circ \dot{v}_j.$$

The integral on the left tends to $\int_0^T L \circ \dot{u}$ as $j \rightarrow \infty$. The integrand on the right is constant for each j , and on $(0, T)$ converges uniformly to $L \circ \dot{v}$ by Lemma 8.4. Hence $\lim_{J \ni j \rightarrow \infty} \int_0^T L \circ \dot{v}_j = \int_0^T L \circ \dot{v}$ and letting $j \rightarrow \infty$ in (8.1) we obtain the theorem. \square

9. Convexity of the action

In this section the Lagrangian $L : T^\infty\mathcal{E}(\omega) \rightarrow \mathbb{R}$ is strongly continuous, invariant, and convex. We first investigate the least action, cf. (5.3), between two ε -geodesics, and then by letting $\varepsilon \rightarrow 0$ we prove Theorem 5.3, which was:

Theorem 9.1. *If $u, v : [a, b] \rightarrow B(X) \cap PSH(\omega)$ are weak geodesics, then for any $S \in (0, \infty)$ the function $\mathcal{L}_S(u, v) : [a, b] \rightarrow \mathbb{R}$ is convex.*

The ε -variant is as follows:

Lemma 9.2. *If $u, v : [a, b] \rightarrow \mathcal{H}$ are ε -geodesics, then for any $S \in (0, \infty)$ the function $\mathcal{L}_S(u, v) : [a, b] \rightarrow \mathbb{R}$ is convex.*

Proof. Let $a \leq \alpha < \beta \leq b$. Suppose $U : [0, S] \times [\alpha, \beta] \rightarrow \mathcal{H}$ is a smooth map such that $U(s, \cdot)$ is an ε -geodesic for all s , and $U(0, \cdot) = u, U(S, \cdot) = v$. Hence $\xi^s = \partial_s U(s, \cdot)$ is an ε -Jacobi field, $0 \leq s \leq S$, and by Theorem 6.1 $L \circ \xi^s$ is convex. Therefore, with $0 \leq \lambda \leq 1$ and $t_\lambda = (1 - \lambda)\alpha + \lambda\beta$

$$(9.1) \quad \mathcal{L}(U(\cdot, t_\lambda)) = \int_0^S L(\xi^s(t_\lambda)) ds \leq (1 - \lambda) \int_0^S L(\xi^s(\alpha)) ds + \lambda \int_0^S L(\xi^s(\beta)) ds.$$

Fix $\delta > 0$. Given u, v , we can choose U (uniquely) so that both $w_\alpha^\delta = U(\cdot, \alpha)$ and $w_\beta^\delta = U(\cdot, \beta)$ are δ -geodesics. From (9.1)

$$(9.2) \quad \mathcal{L}_S(u(t_\lambda), v(t_\lambda)) \leq \mathcal{L}(U(\cdot, t_\lambda)) \leq (1 - \lambda) \int_0^S L \circ \dot{w}_\alpha^\delta + \lambda \int_0^S L \circ \dot{w}_\beta^\delta.$$

Now $\lim_{\delta \rightarrow 0} w_\alpha^\delta = w_\alpha$ and $\lim_{\delta \rightarrow 0} w_\beta^\delta = w_\beta$ are the weak geodesics in $\mathcal{H}^{1\bar{1}}$ connecting $u(\alpha), v(\alpha)$, respectively, $u(\beta), v(\beta)$; and, as explained in section 4,

$$w_\alpha^\delta \rightarrow w_\alpha, \quad w_\beta^\delta \rightarrow w_\beta, \quad \dot{w}_\alpha^\delta \rightarrow \dot{w}_\alpha, \quad \dot{w}_\beta^\delta \rightarrow \dot{w}_\beta$$

uniformly as $\delta \rightarrow 0$. Thus by Lemma 3.4 and Theorem 8.1

$$\lim_{\delta \rightarrow 0} \int_0^S L \circ \dot{w}_\alpha^\delta = \int_0^S L \circ \dot{w}_\alpha = \mathcal{L}_S(u(\alpha), v(\alpha)),$$

and similarly for the other integral in (9.2). Hence letting $\delta \rightarrow 0$ in (9.2) gives

$$\mathcal{L}_S(u(t_\lambda), v(t_\lambda)) \leq (1 - \lambda)\mathcal{L}_S(u(\alpha), v(\alpha)) + \lambda\mathcal{L}_S(u(\beta), v(\beta)),$$

what was to be proved. □

Lemma 9.3. *If $v : [a, b] \rightarrow B(X) \cap PSH(\omega)$ is a weak geodesic, then*

$$\|v(s) - v(t)\| \leq \frac{\|v(b) - v(a)\|}{b - a} |s - t|, \quad s, t \in [a, b].$$

Proof. This is not new. Let $M = \|v(b) - v(a)\|$. As $v(\cdot)(x)$ is convex, $\dot{v}(a) \leq (v(b) - v(a))/(b - a) \leq M/(b - a)$. Furthermore, $u(t) = v(a) - M(t - a)/(b - a)$ is a subgeodesic, $u(a) = v(a)$, $u(b) \leq v(b)$. Hence $u(t) \leq v(t)$ for all t , and

$$\dot{v}(a) \geq \lim_{t \rightarrow a} \frac{u(t) - v(a)}{t - a} \geq -\frac{M}{b - a}.$$

Arguing similarly at b we find $\|\dot{v}(a)\|, \|\dot{v}(b)\| \leq M/(b - a)$, and the claim follows, again since $v(\cdot)(x)$ is convex for $x \in X$. □

Lemma 9.4. *If $w, w' \in B(X) \cap PSH(\omega)$ and $T > 0$, then $\mathcal{L}_T(w, w')$ is finite. If $w_j, w'_j \in C(X) \cap PSH(\omega)$ decrease, or converge uniformly, to w , resp. w' , then*

$$(9.3) \quad \mathcal{L}_T(w_j, w'_j) \rightarrow \mathcal{L}_T(w, w') \quad \text{as } j \rightarrow \infty.$$

We do not know if (9.3) holds when $w_j, w'_j \in B(X) \cap PSH(\omega)$.

Proof. We will prove (9.3) for decreasing sequences w_j, w'_j ; the case of uniformly convergent sequences can be reduced to decreasing sequences in a standard way, as in Lemma 3.4.

Invariance implies that L is constant on the zero section of $T^\infty \mathcal{E}(\omega)$. Since adding a constant to L will not affect the validity of the lemma, we will assume L vanishes on the zero section. Let us start with (9.3). It suffices to prove it along a subsequence $j = j_k$.

Assume first that $w_j, w'_j \in \mathcal{H}$. Let $u : [0, T] \rightarrow B(X) \cap PSH(\omega)$ be piecewise C^1 connecting w and w' . At the price of passing to a subsequence, by Lemma 8.2 there are $u_j : [0, T] \rightarrow \mathcal{H}$ piecewise C^1 such that $u_j(0) = w_j$,

$u_j(T) = w'_j$, and $\int_0^T L \circ \dot{u}_j \rightarrow \int_0^T L \circ \dot{u}$. Therefore

$$\mathcal{L}(u) = \lim_{j \rightarrow \infty} \mathcal{L}(u_j) \geq \limsup_{j \rightarrow \infty} \mathcal{L}_T(w_j, w'_j).$$

Passing to the infimum over paths u connecting w, w' ,

$$(9.4) \quad \mathcal{L}_T(w, w') \geq \limsup_{j \rightarrow \infty} \mathcal{L}_T(w_j, w'_j).$$

Let $v_j : [0, T] \rightarrow \mathcal{H}^{\text{I}}$ be the weak geodesics connecting w_j and w'_j .

We take a pause in the proof of (9.3) and show how (9.4) implies $\mathcal{L}_T(w, w') > -\infty$. By Lemma 9.3 $\|\dot{v}_j(0)\|$ is a bounded sequence. Since L is equi-Lipschitz on bounded subsets of the fibers (Lemma 3.3), using Lemma 7.3 as well, $\mathcal{L}_T(w_j, w'_j) = \mathcal{L}(v_j) = TL(\dot{v}_j(0))$ is a bounded sequence, and (9.4) implies $\mathcal{L}_T(w, w') > -\infty$.

We return to the proof of (9.3); we need to estimate $\mathcal{L}_T(w, w')$ from above. For fixed $\delta > 0$ there are infinitely many k with

$$(9.5) \quad \liminf_{j \rightarrow \infty} \mathcal{L}_T(w_j, w'_j) \geq \mathcal{L}_T(w_k, w'_k) - \delta = TL(\dot{v}_k(0)) - \delta.$$

If $0 < \varepsilon < T/2$ and $k \in \mathbb{N}$, define

$$v_k^\varepsilon(t) = \begin{cases} tw_k/\varepsilon + (\varepsilon - t)w/\varepsilon & \text{if } 0 \leq t < \varepsilon \\ v_k\left(t - \varepsilon \frac{T - 2t}{T - 2\varepsilon}\right) & \text{if } \varepsilon \leq t < T - \varepsilon \\ (T - t)w'_k/\varepsilon + (t + \varepsilon - T)w'/\varepsilon & \text{if } T - \varepsilon \leq t \leq T. \end{cases}$$

The piecewise C^1 paths $v_k^\varepsilon : [0, T] \rightarrow B(X) \cap \text{PSH}(\omega)$ connect w and w' , hence

$$(9.6) \quad \mathcal{L}_T(w, w') \leq \mathcal{L}(v_k^\varepsilon) = \left(\int_0^\varepsilon + \int_\varepsilon^{T-\varepsilon} + \int_{T-\varepsilon}^T \right) L \circ \dot{v}_k^\varepsilon.$$

The middle integral on the right is

$$\int_\varepsilon^{T-\varepsilon} L \circ \dot{v}_k^\varepsilon = (T - 2\varepsilon)L\left(\frac{T\dot{v}_k(0)}{T - 2\varepsilon}\right).$$

As we saw, the $\dot{v}_k(0)$ are uniformly bounded. By the equi-Lipschitz property of L an $\varepsilon \in (0, 1)$ can be chosen so that for all k

$$(9.7) \quad \int_\varepsilon^{T-\varepsilon} L \circ \dot{v}_k^\varepsilon \leq TL(\dot{v}_k(0)) + \delta.$$

When $0 \leq t \leq \varepsilon$, we have $\dot{v}_k^\varepsilon(t) = (w_k - w)/\varepsilon \in T_{v_k^\varepsilon(t)}^\infty \mathcal{E}(\omega)$. Again by the equi-Lipschitz property, if k is sufficiently large, $|L(\dot{v}_k^\varepsilon(t))| < \delta$; and similarly for $T - \varepsilon \leq t < T$. Putting this and (9.6), (9.7) together,

$$\mathcal{L}_T(w, w') \leq 3\delta + TL(\dot{v}_k(0))$$

if k is sufficiently large. Choosing k that also satisfies (9.5) therefore yields

$$\mathcal{L}_T(w, w') \leq 4\delta + \liminf_{j \rightarrow \infty} \mathcal{L}_T(w_j, w'_j).$$

This being true for all $\delta > 0$, (9.3) follows in view of (9.4).

So far we dealt with $w_j, w'_j \in \mathcal{H}$. If $w_j, w'_j \in C(X) \cap \text{PSH}(\omega)$ only, upon adding constants to them we can arrange that $w_j < w_{j-1}$ and $w'_j < w'_{j-1}$ everywhere. We will express this by saying that w_j, w'_j strictly decrease. We construct recursively $z_j > w_j, z'_j > w'_j$ in \mathcal{H} that strictly decrease to w, w' and satisfy $|\mathcal{L}_T(z_j, z'_j) - \mathcal{L}_T(w_j, w'_j)| < 1/j$ as follows. Suppose we already have z_{j-1}, z'_{j-1} . Construct sequences $y_i < z_{j-1}, y'_i < z'_{j-1}$ ($i \in \mathbb{N}$) in \mathcal{H} that decrease to w_j, w'_j . By what we have already proved, $|\mathcal{L}_T(y_i, y'_i) - \mathcal{L}_T(w_j, w'_j)| < 1/j$ for some i , and we let $z_j = y_i, z'_j = y'_i$ with that i . Thus

$$\mathcal{L}_T(w, w') = \lim_j \mathcal{L}_T(z_j, z'_j) = \lim_j \mathcal{L}_T(w_j, w'_j),$$

as claimed. □

Proof of Theorem 9.1. Assume first that u, v are weak geodesics in $\mathcal{H}^{1\bar{1}}$ with endpoints in \mathcal{H} , and connect $u(a), u(b)$, respectively, $v(a), v(b)$ by ε -geodesics $u^\varepsilon, v^\varepsilon$. By Chen's theorem $u^\varepsilon \rightarrow u$ and $v^\varepsilon \rightarrow v$ uniformly as $\varepsilon \rightarrow 0$. Hence by Lemma 9.4, $\mathcal{L}_S(u^\varepsilon, v^\varepsilon) \rightarrow \mathcal{L}_S(u, v)$, and so the latter, as the limit of convex functions (Lemma 9.2) is itself convex.

Second, consider general u, v . Choose $w_j, w'_j \in \mathcal{H}$ decreasing to $u(a), u(b)$ and $z_j, z'_j \in \mathcal{H}$ decreasing to $v(a), v(b)$. Join w_j, w'_j by weak geodesics $u_j : [a, b] \rightarrow \mathcal{H}^{1\bar{1}}$ and z_j, z'_j by weak geodesics $v_j : [a, b] \rightarrow \mathcal{H}^{1\bar{1}}$. By Lemma 8.3 u_j, v_j decrease to u, v , hence by Lemma 9.4 the convex functions $\mathcal{L}_S(u_j, v_j)$ converge to $\mathcal{L}_S(u, v)$. It follows that the latter is also convex. □

10. Two ways to compute action

One way is by the definition, $\int_a^b L \circ \dot{u}$, if $u : [a, b] \rightarrow B(X) \cap \text{PSH}(\omega)$. The other corresponds to computing length of a curve in a metric space as the least upper bound of the lengths of inscribed piecewise geodesic curves. The two agree in our setting as well.

The Lagrangian $L : T^\infty \mathcal{E}(\omega) \rightarrow \mathbb{R}$ in this section is strongly continuous, invariant, and convex.

Theorem 10.1. *For a piecewise C^1 path $u : [a, b] \rightarrow B(X) \cap PSH(\omega)$*

$$(10.1) \quad \int_a^b L \circ \dot{u} = \sup \sum_{i=1}^m \mathcal{L}_{t_i - t_{i-1}}(u(t_{i-1}), u(t_i)),$$

the sup over all partitions $a = t_0 < t_1 < \dots < t_m = b$.

We start with an asymptotic formula for $\mathcal{L}_T(w, z)$, valid as w, z approach each other in two different ways. One will be needed in the proof of Theorem 10.1; the other for material in section 11.

Lemma 10.2. *Consider a sequence of positive numbers $\tau_k \rightarrow 0$ and sequences $w_k, z_k \in B(X) \cap PSH(\omega)$ converging uniformly to $w \in B(X) \cap PSH(\omega)$, $\|z_k - w_k\| = O(\tau_k)$. Let $\xi \in B(X)$. Suppose that either*

- (i) $(z_k - w_k)/\tau_k \rightarrow \xi$ uniformly or only in capacity; or
- (ii) $(z_k - w_k)/\tau_k \rightarrow \xi$ μ_w -almost everywhere, and the family of $\mu_{w_k}, \mu_{z_k}, k \in \mathbb{N}$, is hereditarily tight (Definition 3.5).

Viewing ξ as a vector in $T_w^\infty \mathcal{E}(\omega)$, we then have

$$(10.2) \quad \mathcal{L}_{\tau_k}(w_k, z_k)/\tau_k \rightarrow L(\xi) \quad \text{as } k \rightarrow \infty.$$

Proof. It will suffice to prove (10.2) along a subsequence. Construct $w_k^j, z_k^j \in \mathcal{H}$ that decrease to w_k , respectively, z_k as $j \rightarrow \infty$. Let $v_k^j : [0, \tau_k] \rightarrow \mathcal{H}^{1,1}$ be the weak geodesic connecting w_k^j, z_k^j , and $u_k^j : [0, \tau_k] \rightarrow \mathcal{H}$ the line segment connecting the two,

$$u_k^j(t) = \frac{\tau_k - t}{\tau_k} w_k^j + \frac{t}{\tau_k} z_k^j.$$

Since $\|z_k - w_k\| = O(\tau_k)$ as $k \rightarrow \infty$, we can arrange that $(z_k^j - w_k^j)/\tau_k$ form a bounded set in $B(X)$. This implies by Lemma 9.3 that $\dot{v}_k^j(t)$ are uniformly bounded.

We have $v_k^j(t) \leq u_k^j(t)$, because evaluated at $x \in X$ the former is a convex function of t , the latter is a linear function, and the two agree at $t = 0, \tau_k$. Hence

$$\dot{v}_k^j(0) \leq \dot{u}_k^j(0), \quad \dot{u}_k^j(\tau_k) \leq \dot{v}_k^j(\tau_k),$$

and since the right, respectively, left derivatives $\dot{v}_k^j(0), \dot{v}_k^j(\tau_k)$ are equidistributed, see [D2, Lemma 4.10],

$$(10.3) \quad \dot{u}_k^j(\tau_k)^* \leq \dot{v}_k^j(0)^* \leq \dot{u}_k^j(0)^*.$$

As $j \rightarrow \infty$, the decreasing sequences w_k^j , respectively, z_k^j , of ω -plurisubharmonic functions converge in capacity [GZ2, Proposition 9.11], and so

$$\dot{u}_k^j(0) = \dot{u}_k^j(\tau_k) = \frac{z_k^j - w_k^j}{\tau_k} \rightarrow \frac{z_k - w_k}{\tau_k}, \quad \text{as } j \rightarrow \infty,$$

in capacity. The latter function is quasicontinuous because ω -plurisubharmonic functions are [GZ2, Corollary 9.12]. It can be viewed as an element of $T_{w_k}^\infty \mathcal{E}(\omega)$ or of $T_{z_k}^\infty \mathcal{E}(\omega)$. We will write $f_k, g_k : (0, \mu_0(X)] \rightarrow \mathbb{R}$ for its decreasing rearrangement as an element of one or the other. Lemma 3.4 then implies

$$\dot{u}_k^j(0)^* \rightarrow f_k, \quad \dot{u}_k^j(\tau_k)^* \rightarrow g_k, \quad \text{as } j \rightarrow \infty,$$

away from a countable subset of $(0, \mu_0(X)]$, and so by (10.3)

$$(10.4) \quad g_k \leq \liminf_{j \rightarrow \infty} \dot{v}_k^j(0)^* \leq \limsup_{j \rightarrow \infty} \dot{v}_k^j(0)^* \leq f_k$$

away from a countable set. In case (i) by Lemma 3.4, in case (ii) by Lemma 3.6 we obtain

$$\lim_{k \rightarrow \infty} f_k = \lim_{k \rightarrow \infty} g_k = \xi^*,$$

away from a countable set.

By Egorov's theorem for each $m \in \mathbb{N}$ there is an $E_m \subset (0, \mu_0(X)]$, whose complement has Lebesgue measure $< 2^{-m}$, and on which the sequences f_k, g_k , and for every k the sequences

$$\inf_{j \geq i} \dot{v}_k^j(0)^*, \quad \sup_{j \geq i} \dot{v}_k^j(0)^*, \quad i = 1, 2, \dots$$

converge uniformly. Upon passing to a subsequence we can arrange that for every m

$$\xi^* - 1/m \leq g_m \leq f_m \leq \xi^* + 1/m \quad \text{on } E_m.$$

In light of (10.4) for each m there is i_m such that whenever $j > i_m$,

$$(10.5) \quad \xi^* - 2/m < \dot{v}_m^j(0)^* < \xi^* + 2/m \quad \text{on } E_m.$$

Choose $j > i_m$ so that

$$(10.6) \quad \mathcal{L}_{\tau_m}(w_m, z_m)/\tau_m \quad \text{and} \quad \mathcal{L}_{\tau_m}(w_m^j, z_m^j)/\tau_m = L(\dot{v}_m^j(0))$$

are within $1/m$, and set $v_m = v_m^j$.

By the Borel–Cantelli lemma almost every point in $(0, \mu_0(X)]$ is contained in all but finitely many E_m . (10.5) therefore implies $\lim_m \dot{v}_m(0)^* = \xi^*$ a.e. With a measure preserving $\theta : (X, \mu_0) \rightarrow (0, \mu_0(X)]$, see [L4, Lemma 5.5], $\dot{v}_m(0)^* \circ \theta \in B(X, \mu_0)$ then tends to $\xi^* \circ \theta$ a.e., whence

$$L(\dot{v}_m(0)) = L(\dot{v}_m(0)^* \circ \theta) \rightarrow L(\xi^* \circ \theta) = L(\xi), \quad m \rightarrow \infty,$$

as L is invariant and strongly continuous. But then

$$\lim_{m \rightarrow \infty} \mathcal{L}_{\tau_m}(w_m, z_m) / \tau_m = L(\xi)$$

(cf. (10.6)), as needed. □

Proof of Theorem 10.1. It suffices to prove when u is C^1 , not only piecewise. Since

$$(10.7) \quad \int_{t_{i-1}}^{t_i} L \circ \dot{u} \geq \mathcal{L}_{t_i - t_{i-1}}(u(t_{i-1}), u(t_i)),$$

the left hand side of (10.1) is \geq than the right hand side. As to the converse, let $M = \max_{[a,b]} |L \circ \dot{u}|$.

Given $\varepsilon > 0$, choose $\delta > 0$ so that for any partition $a = t_0 < t_1 < \dots < t_m = b$ finer than δ any corresponding Riemann sum satisfies

$$(10.8) \quad \left| \int_a^b L \circ \dot{u} - \sum_{i=1}^m L(\dot{u}(s_i))(t_i - t_{i-1}) \right| < \varepsilon.$$

Here $t_{i-1} \leq s_i \leq t_i$. It follows from Lemma 10.2 that for every $s \in (a, b)$ there is a $\delta_s \in (0, \delta)$ such that if $0 < \tau < \delta_s$,

$$\left| \mathcal{L}_\tau(u(s - \tau/2), u(s + \tau/2)) - L(\dot{u}(s))\tau \right| < \varepsilon\tau / (b - a).$$

Vitali’s covering theorem implies that there are a partition $a = t_0 < t_1 < \dots < t_m = b$ finer than δ , and $I \subset \{1, 2, \dots, m\}$ with the following property. Let $s_i = (t_{i-1} + t_i)/2$. If $i \in I$ then $t_i - t_{i-1} < \delta_{s_i}$; while the total length of the intervals $[t_{i-1}, t_i]$ with $1 \leq i \leq m$ not in I is $< \varepsilon/M$. Write $t_i - t_{i-1} = \tau_i$.

By (10.7) $|\mathcal{L}_{\tau_i}(u(t_{i-1}), u(t_i))| \leq M\tau_i$. We have

$$\begin{aligned} & \int_a^b L \circ \dot{u} - \sum_{i=1}^m \mathcal{L}_{\tau_i}(u(t_{i-1}), u(t_i)) \\ &= \left(\int_a^b L \circ \dot{u} - \sum_{i=1}^m L(\dot{u}(s_i))\tau_i \right) + \sum_{i=1}^m (L(\dot{u}(s_i))\tau_i - \mathcal{L}_{\tau_i}(u(t_{i-1}), u(t_i))). \end{aligned}$$

The first term on the right is $< \varepsilon$ according to (10.8). The second is

$$\sum_{i \in I} + \sum_{i \notin I} \leq \sum_{i \in I} \varepsilon\tau_i/(b-a) + \sum_{i \notin I} 2M\tau_i < \varepsilon + 2\varepsilon.$$

All added up, $\int_a^b L \circ \dot{u} - \sum_{i=1}^m \mathcal{L}_{t_i-t_{i-1}}(u(t_{i-1}), u(t_i)) < 4\varepsilon$, and the theorem follows. □

11. Uniqueness of minimizing paths

In this section $L : T^\infty\mathcal{E}(\omega) \rightarrow \mathbb{R}$ will denote a strongly continuous, invariant, convex Lagrangian. The question we will entertain is whether weak geodesics are the unique minimizers of action $\int_a^b L \circ \dot{u}$ among paths connecting fixed $u(a), u(b)$.

In complete generality uniqueness, of course, fails. For example, if L is positively homogeneous, reparametrized weak geodesics will still minimize action. Uniqueness may also fail more drastically even with L a Finsler metric. Here is an example. Start with a Lagrangian $\Lambda : T^\infty\mathcal{E}(\omega) \rightarrow \mathbb{R}$ that vanishes on constants. For instance, denoting the average of $\xi \in T_u^\infty\mathcal{E}(\omega) \approx B(X)$ with respect to the measure μ_u by $\langle \xi \rangle_u$,

$$\Lambda(\xi) = \int_X |\xi - \langle \xi \rangle_u| d\mu_u, \quad \xi \in T_u^\infty\mathcal{E}(\omega),$$

is a possibility. The path $v(t) = t, 0 \leq t \leq T$, is a geodesic in \mathcal{H} , and in particular minimizes action of the Lagrangian $L(\xi) = \max(\langle \xi \rangle_u, \Lambda(\xi))$.

But take arbitrary nonconstant $w, z \in C^\infty(X)$ and with piecewise C^1 functions $f, g : [0, T] \rightarrow \mathbb{R}$ let

$$u(t) = t + f(t)w + g(t)z, \quad 0 \leq t \leq T.$$

We choose f, g so that (u maps into \mathcal{H} and) $\langle \dot{u}(t) \rangle_{u(t)} = 1$, i.e.,

$$(11.1) \quad \begin{aligned} 0 &= \int_X (\dot{f}(t)w + \dot{g}(t)z) (\omega + f(t)dd^c w + g(t)dd^c z)^n \\ &= \dot{f}(t)P(f(t), g(t)) + \dot{g}(t)Q(f(t), g(t)), \end{aligned}$$

where P, Q are polynomials determined by the choice of w, z . If f, g satisfy

$$f(0) = g(0) = 0, \quad \dot{f} = Q(f, g), \quad \dot{g} = -P(f, g) \text{ on } [0, T/2],$$

and $f(t) = f(T - t)$, $g(t) = g(T - t)$ for $T/2 \leq t \leq T$, then (11.1) holds. When T is small, this initial value problem is solvable, and furnishes a path u in \mathcal{H} connecting 0 and T . Now suppose that $P(0, 0) = \int_X w\omega^n = 0$ and $Q(0, 0) = \int_X z\omega^n$ is small but nonzero. Since, again for small T and $t \in [0, T]$

$$\Lambda(\dot{u}(t)) < 1 = \langle \dot{u}(t) \rangle_{u(t)}, \quad \text{and so} \quad \int_0^T L \circ \dot{u} = T = \int_0^T L \circ \dot{v},$$

it follows that u also minimizes action; and u is not v reparametrized.

Darvas's L^1 metric, $L(\xi) = \int_X |\xi| d\mu_u$ for $\xi \in T_u^\infty \mathcal{E}(\omega)$, supplies another example. [D4, Proposition 3.43] and Theorem 10.1 imply that a C^1 path $u : [a, b] \rightarrow B(X) \cap \text{PSH}(\omega)$ minimizes action whenever it is monotone in the sense that $u(t) \leq u(s)$ if $t \leq s$.

If mere convexity of L does not imply uniqueness of minimizers, strict versions of convexity do, at least in $\mathcal{H}^{1,1}$:

Theorem 11.1. *Suppose $u : [a, b] \rightarrow \mathcal{H}^{1,1}$ is piecewise C^1 as a map into $B(X)$, $v : [a, b] \rightarrow \mathcal{H}^{1,1}$ is a weak geodesic connecting $u(a)$ and $u(b)$, and $\int_a^b L \circ \dot{u} = \mathcal{L}_{b-a}(u(a), u(b))$. If L is strictly convex in the sense that for all $w \in \mathcal{H}^{1,1}$, $\xi, \eta \in T_w^\infty \mathcal{E}(\omega)$*

$$(11.2) \quad L\left(\frac{\xi + \eta}{2}\right) < \frac{L(\xi) + L(\eta)}{2} \quad \text{unless } \xi = \eta \text{ } \mu_w\text{-almost everywhere,}$$

then $u = v$. If, instead, L satisfies the weaker condition

$$(11.3) \quad L\left(\frac{\xi + \eta}{2}\right) < \frac{L(\xi) + L(\eta)}{2} \quad \text{unless } \xi = \lambda(\xi + \eta) \text{ } \mu_u\text{-almost everywhere}$$

with some $\lambda \in [0, 1]$, then $u = v \circ \varphi$, where $\varphi : [a, b] \rightarrow [a, b]$ is piecewise C^1 .

We have not written the minimization condition as $\int_a^b L \circ \dot{u} = \int_a^b L \circ \dot{v}$ because we need not assume a priori that v is C^1 . But we do not know if

the theorem holds more generally when u, v map into $B(X) \cap \text{PSH}(\omega)$.— It suffices to verify conditions (11.2), (11.3) for $w = 0$ only, since Rohlin’s theory of Lebesgue spaces [R, Section 2, Nos 3, 4, 7] and the fact that a general μ_w has no atoms yields a measure preserving bijection between (X, μ_0) and (X, μ_w) .

In the proof of the next lemma we will use Mabuchi length, the Lagrangian $L = M$,

$$(11.4) \quad M(\xi) = \left(\int_X \xi^2 d\mu_w \right)^{1/2}, \quad \xi \in T_w^\infty \mathcal{E}(\omega).$$

The corresponding action $\mathcal{M}_T(w, w')$, Mabuchi distance, is independent of T . We will denote it $d(w, w')$.

Lemma 11.2. *Consider a map $u : [a, b] \rightarrow \mathcal{H}^{1\bar{1}}$. If for some $c \in (a, b)$ the restrictions $u|_{[a, c]}$, $u|_{[c, b]}$ are weak geodesics, and the left and right derivatives $\partial^- u(c)$, $\partial^+ u(c)$, computed pointwise on X , agree $\mu_{u(c)}$ -almost everywhere, then u is a weak geodesic.*

Proof. It follows from He’s work [He] that if $v : [\alpha, \beta] \rightarrow \mathcal{H}^{1\bar{1}}$ is a weak geodesic, then the family $\mu_{v(t)}$, $t \in [\alpha, \beta]$, is hereditarily tight (Definition 3.5). Indeed, He constructs smooth $v_k : [\alpha, \beta] \rightarrow \mathcal{H}$ that, as maps into $B(X)$, converge to v uniformly and have $dd^c v_k(t)$ uniformly bounded, $k \in \mathbb{N}$, $t \in [\alpha, \beta]$, see [He, Theorem 1.3, and proof of Theorem 1.1]. Therefore $d\mu_{v_k(t)}/d\mu_0$ are uniformly bounded, say, by A . Since $\mu_{v(t)}$ is the weak limit of $\mu_{v_k(t)}$, e.g., by [GZ2, Theorem 3.18], for all $\xi \in C(X)$

$$\left| \int_X \xi d\mu_{v(t)} \right| = \lim_{k \rightarrow \infty} \left| \int_X \xi d\mu_{v_k(t)} \right| \leq A \int_X |\xi| d\mu_0.$$

Hence each $\mu_{v(t)}$ is absolutely continuous with respect to μ_0 , with Radon–Nikodym derivative $d\mu_{v(t)}/d\mu_0 \leq A$, and this implies hereditary tightness. (Note, though, that for the weak geodesic $v : [a, b] \rightarrow B(X) \cap \text{PSH}(\omega)$ of Example 5.4, the family $\mu_{v(t)}$ is not hereditarily tight.)

In particular, the family $d\mu_{u(t)}$, $t \in [a, b]$, is hereditarily tight. By Theorem 9.1 $h(t) = d(u(c - t), u(c + t))$ is a convex function of small $t \geq 0$, it vanishes at 0 and by Lemma 10.2 $h'(0) = 2M(\partial^- u(c)) = 2M(\partial^+ u(c))$, cf. (11.4). Hence $h(t) \geq 2M(\partial^- u(c))t$. At the same time, by Lemma 7.3

$$d(u(c - t), u(c)) = d(u(c), u(c + t)) = M(\partial^- u(c))t,$$

and so

$$h(t) = d(u(c - t), u(c + t)) \geq d(u(c - t), u(c)) + d(u(c), u(c + t)).$$

Thus $u|_{[c - t, c + t]}$ is a shortest path for d between $u(c \pm t)$; it is of constant speed, too. By [D3, Theorem 1, and the discussion following it] this means that $u|_{[c - t, c + t]}$ is a weak geodesic. Therefore u itself is a weak geodesic. Indeed, with the strip $S_{ab} = \{s \in \mathbb{C} : a < \text{Res} < b\}$ and projection $\pi : S_{ab} \times X \rightarrow X$, we need to check that

$$U : S_{ab} \times X \ni (s, x) \mapsto u(\text{Res})(x) \in \mathbb{R}$$

is $\pi^*\omega$ -plurisubharmonic and maximal in the sense that $(\pi^*\omega + dd^c U)^{n+1} = 0$. Both hold because they hold on $S_{ac} \times X$, $S_{cb} \times X$, and on a neighborhood of $\{c + i\mathbb{R}\} \times X$. □

The key to Theorem 11.1 is the following characterization of constellations in which a triangle inequality degenerates:

Lemma 11.3. *Suppose the Lagrangian L satisfies (11.3), and $w, w', w'' \in \mathcal{H}^{1\bar{1}}$. If with some $S, T > 0$*

$$\mathcal{L}_S(w', w) + \mathcal{L}_T(w, w'') = \mathcal{L}_{S+T}(w', w''),$$

then w is on the weak geodesic $v : [0, S + T] \rightarrow \mathcal{H}^{1\bar{1}}$ connecting w', w'' . If L is strictly convex (condition (11.2)), then $w = v(S)$.

Proof. Construct a path $u : [0, S + T] \rightarrow B(X) \cap \text{PSH}(\omega)$ whose restrictions $u|_{[0, S]}$, $u|_{[S, S + T]}$ are weak geodesics connecting w', w , respectively, w, w'' . By [He, Theorem 1.1] u in fact maps into $\mathcal{H}^{1\bar{1}}$. With small $\tau > 0$

$$\begin{aligned} \mathcal{L}_{S+T}(w', w'') &\leq \mathcal{L}_{S-\tau}(w', u(S - \tau)) + \mathcal{L}_{2\tau}(u(S - \tau), u(S + \tau)) \\ &\quad + \mathcal{L}_{T-\tau}(u(S + \tau), w'') \\ &\leq \mathcal{L}_{S-\tau}(w', u(S - \tau)) + \mathcal{L}_\tau(u(S - \tau), w) + \mathcal{L}_\tau(w, u(S + \tau)) \\ &\quad + \mathcal{L}_{T-\tau}(u(S + \tau), w'') \\ &\leq \mathcal{L}_S(w', w) + \mathcal{L}_T(w, w'') = \mathcal{L}_{S+T}(w', w''). \end{aligned}$$

Hence $\mathcal{L}_{2\tau}(u(S - \tau), u(S + \tau)) = \mathcal{L}_\tau(u(S - \tau), w) + \mathcal{L}_\tau(w, u(S + \tau))$. We divide by 2τ and compute the limits as $\tau \rightarrow 0$ using Lemma 10.2. This is

possible, since the family $\mu_{u(t)}$, $t \in [a, b]$, is hereditarily tight by the initial observation in the proof of Lemma 11.2. We obtain

$$(11.5) \quad L\left(\frac{\partial^- u(S) + \partial^+ u(S)}{2}\right) = \frac{L(\partial^- u(S)) + L(\partial^+ u(S))}{2},$$

If L is strictly convex, (11.5) implies $\partial^- u(S) = \partial^+ u(S)$ μ_w -almost everywhere. By Lemma 11.2 therefore u is a weak geodesic connecting w', w'' , and so coincides with v .

If L satisfies the weaker condition (11.3) only, we can still conclude $\partial^- u(S) = \lambda(\partial^- u(S) + \partial^+ u(S))$ μ_w -almost everywhere, $0 \leq \lambda \leq 1$. If one of $\partial^\pm u(S)$ is a.e. 0, then w is at 0 Mabuchi distance to w' or w'' , hence coincides with one of them. Otherwise $\lambda \neq 0, 1$, and u can be linearly reparametrized on $[S, S + T]$ to a path \tilde{u} that satisfies $\partial^- \tilde{u}(S) = \partial^+ \tilde{u}(S)$ μ_w -almost everywhere. Lemma 11.2 again implies that w is on the geodesic v . \square

Proof of Theorem 11.1. Let $a < s < b$, $S = s - a$, and $T = b - s$. Then

$$\begin{aligned} \mathcal{L}_{S+T}(u(a), u(b)) &\leq \mathcal{L}_S(u(a), u(s)) + \mathcal{L}_T(u(s), u(b)) \\ &\leq \int_a^s L \circ \dot{u} + \int_s^b L \circ \dot{u} = \int_a^b L \circ \dot{u} = \mathcal{L}_{S+T}(u(a), u(b)), \end{aligned}$$

and all inequalities here must be equalities. If L is strictly convex, this implies via Lemma 11.3 that $u(s) = v(s)$, and so $u = v$.

If L satisfies (11.3) only, Lemma 11.3 gives $u(s) = v(\varphi(s))$ with some function $\varphi : [a, b] \rightarrow [a, b]$. If v is constant, this again means $u = v$. Otherwise, on the one hand, Mabuchi distance along the weak geodesic v is $d(v(a), v(t)) = c(t - a)$, with $c \neq 0$. On the other, Mabuchi distance along u is a piecewise C^1 function of s

$$d(u(a), u(s)) = \int_a^s \left(\int_X \dot{u}(t)^2 d\mu_{u(t)} \right)^{1/2} dt.$$

It follows that $d(v(\varphi(a)), v(\varphi(s))) = c(\varphi(s) - \varphi(a))$ is a piecewise C^1 function of s , and so is $\varphi(s)$. \square

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