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Algebraic entropy for smooth projective varieties

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We show that the spectral radius for the action of a self map f of a smooth projective variety (over an arbitrary base field) on its ℓ-adic cohomology is achieved on the f^* stable sub-algebra generated by any ample class. This generalizes a result of Esnault-Srinivas who had obtained an analogous result for automorphisms of surfaces. Over $\mathbb{C}$ we also show that this sub-algebra is naturally an irreducible representation of a Looijenga-Lunts-Verbitsky type Lie algebra acting on the cohomology of a smooth projective variety.

1. Introduction

Let X be a compact Kähler manifold and $\omega \in H^2(X, \mathbb{R})$ be a $(1,1)$ form corresponding to the choice of a Kähler metric. Let $f\colon X \to X$ be a surjective and holomorphic self map of X. To any such pair (X, f) consisting of a compact metric space and a continuous self map one can associate a real number $d_{\text{top}}(f)$, the topological entropy of the pair (X, f) [Bow71, Din71].

Let $\lambda(f)$, $\lambda_{\text{even}}(f)$ and $\lambda_p(f)$, $0 \le p \le \dim(X)$ be the spectral radius of f^* acting on $\oplus_i H^i(X, \mathbb{Q})$, $\oplus_i H^{2i}(X, \mathbb{Q})$ and $H^{p,p}(X, \mathbb{R})$ respectively. The following fundamental theorem is due to Gromov-Yomdin.

Theorem 1.1. [Gro03, Yom87] With notations as above, $d_{\text{top}}(f) = \log \lambda(f) = \log \lambda_{\text{even}}(f) = \max\limits_{0 \le p \le \dim(X)} \log \lambda_p(f)$.

Theorem 1.1 implies that for a surjective self map f of a smooth projective variety over $\mathbb{C}$, the spectral radius of f^* on the Hodge classes equals the spectral radius on the entire cohomology (see [Ogu14] for a comprehensive summary of the Gromov-Yomdin theory and its generalizations).

When working over an arbitrary base field there is no obvious and useful notion of a topological entropy, however it still makes sense to look at the action of f^* on suitable cohomology theories. In this direction Esnault-Srinivas obtained the following result for automorphisms of smooth projective surfaces over an arbitrary base field.

Theorem 1.2. [ES13, Theorem 1.1] Let $f\colon X \to X$ be an automorphism of a smooth projective surface over an arbitrary algebraically closed field k. Let ℓ be a prime invertible in k. Let $\omega \in H^2(X, \mathbb{Q}_\ell)$ be an ample class. Then for any embedding of $\mathbb{Q}_\ell$ inside $\mathbb{C}$,

1) the spectral radius for the action of f^* on $H^*(X, \mathbb{Q}_\ell)$ coincides with the spectral radius for its action on the sub-space spanned by $f^{n*}\omega,\ n \in \mathbb{Z}$.
2) Let $V(f, \omega)$ be the largest f^*-stable sub-space of $H^2(X, \mathbb{Q}_\ell)$ in the orthogonal complement of ω (with respect to the cup-product pairing). Then f^* is of finite order on $V(f, \omega)$.

The proof of Theorem 1.2 is quite delicate, and uses (among other things) the classification of smooth projective surfaces in positive characteristics. It also relies on lifting of certain $K3$ surfaces to characteristic 0 based on [LM11], and uses Hodge theory to resolve this case. Given the motivic nature of Theorem 1.2, it is natural to ask for analogues of the Gromov-Yomdin theory over an arbitrary base field (see [ES13], Section 6.2). Indeed one has the following result.

Theorem 1.3. [Shu19, Corollary 1.2] Let $f\colon X \to X$ be any self map[1] of a proper scheme over an arbitrary field k. Let ℓ be a prime invertible in k and let $\bar{k}$ be an algebraic closure of k. Then (for any embedding of $\mathbb{Q}_\ell$ in $\mathbb{C}$) the spectral radius of $f^*_{\bar{k}}$ on the entire ℓ-adic cohomology equals the spectral radius for its action on $\oplus_i H^{2i}(X_{\bar{k}}, \mathbb{Q}_\ell)$.

The proof Theorem 1.3 uses the theory of weights [Del80] to obtain restrictions on the analytic properties of a zeta function associated to the self map f [Shu19, Definition 2.12]. The analytic properties of the zeta functions are then used to obtain restrictions on the behavior of the spectral radius with respect to the weight filtration [Shu19, Theorem 1.1], which in turn implies Theorem 1.3.

Now suppose X is a smooth projective variety, then one can ask more refined questions and in particular look for analogues of Theorem 1.2, (1) for higher dimensional varieties. In this direction we have the following result.

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[1]In what follows by self map we will mean an endomorphism of a scheme.

Theorem 1.4 ([Tru16]). Let $f\colon X\to X$ be a dominant self map of a smooth projective variety over an algebraically closed field k. Let ℓ be a prime invertible in k. Then for any embedding of $\mathbb{Q}_\ell$ inside $\mathbb{C}$, the spectral radius of f^* on the ℓ-adic cohomology of X equals the spectral radius of f^* acting on its Chow group modulo numerical equivalence.

Motivated by the methods in [Tru16] and in an forthcoming article [SV20, Appendix B] we obtain the following generalization of Theorem 1.2 to higher dimensions.

Theorem 1.5. Let $f\colon X\to X$ be a self map of a smooth projective variety over an arbitrary algebraically closed field k. Let ℓ be a prime invertible in k. Let $[\omega]\in H^2(X,\mathbb{Q}_\ell)$ be an ample class. Then the spectral radius of f^* acting on $H^*(X,\mathbb{Q}_\ell)$ (with respect to $\tau:\mathbb{Q}_\ell\hookrightarrow\mathbb{C}$) is independent of τ, and coincides with the spectral radius of f^* on the numerical Gromov algebra (see Definition 3.2).

Theorem 1.5 in particular shows that for smooth projective varieties with $\mathrm{Pic}(X)=\mathbb{Z}$, the spectral radius of a self map on the ℓ-adic cohomology coincides with its degree (see Corollary 3.8). Finally in a hope to generalize Theorem 1.2, (2), we propose an approach via a Looijenga-Lunts-Verbistky Lie algebra [LL97, Ver96]. The reader is referred to Section 4 for details.

2. Some preliminaries from intersection theory

Throughout this article we will work over an arbitrary algebraically closed field k. A variety (over k) is a finite type, separated and integral scheme over k. Let ℓ be a prime invertible in k. We fix once and for all an isomorphism of $\mathbb{Q}_\ell(1)$ with $\mathbb{Q}_\ell$. Hence we will talk of cycles classes with values in ℓ-adic cohomology without the Tate twist. We also fix an embedding

$$\tau\colon\mathbb{Q}_\ell\hookrightarrow\mathbb{C}. \tag{2.1}$$

2.1. Summary of results needed from intersection theory

Let X be a smooth, projective variety over k of dimension r. Let $Z^*(X)$ be the free abelian group generated by the set of closed subvarieties of X and graded by codimension [Ful98, Section 1.3]. Let $A^*(X)$ be the graded (by codimension) Chow ring of X [Ful98, Section 8.3]. The group underlying $A^*(X)$ is a graded quotient of $Z^*(X)$ by rational equivalence. We shall write $A(X) := \oplus_i A^i(X)$ when we want to ignore the grading and the ring structure.

The *components* of an algebraic cycle $[Z] \in Z^*(X)$ are the subvarieties of X which appear in $[Z]$ with non-zero coefficients. To any closed *subscheme* $Y \subseteq X$ we can associate an effective cycle $[Y]$ in $Z^*(X)$ whose components are precisely the irreducible components of Y [Ful98, Section 1.5].

Let $A^*_{\mathrm{num}}(X)$ (resp. $A^*_{\mathrm{num}}(X)_{\mathbb{Q}}$, resp. $A^*_{\mathrm{num}}(X)_{\mathbb{R}}$) be the graded (by codimension) ring of algebraic cycles on X modulo numerical equivalence with $\mathbb{Z}$ (resp. $\mathbb{Q}$, resp. $\mathbb{R}$) coefficients [Kle68, Section 1.1].

Let $A^*_{\mathrm{hom}}(X)_{\mathbb{Q}}$ be the graded (by codimension) ring of algebraic cycles on X modulo homological equivalence (with respect to ℓ-adic cohomology[2]), with $\mathbb{Q}$ coefficients (see [Del77, Chapitre 4] and [Mil16, Chapter 6] for a construction of cycle classes). Note that $A^*_{\mathrm{num}}(X)_{\mathbb{Q}}$ is a quotient of $A^*_{\mathrm{hom}}(X)_{\mathbb{Q}}$, which in turn is a $\mathbb{Q}$-subalgebra of $\oplus_i H^{2i}(X, \mathbb{Q}_\ell)$.

For a morphism $f\colon X \to Y$ of smooth, projective varieties over k, there is a pullback map $f^*\colon A^*(Y) \to A^*(X)$ and a pushforward map $f_*\colon A(X) \to A(Y)$ [Ful98, Proposition 8.3 (a) and Theorem 1.4]. The pullback is a morphism of graded rings and the pushforward is a morphism of abelian groups. Further they satisfy a projection formula [Ful98, Proposition 8.3 (c)]. In particular there exists a group homomorphism $\pi_{X*}\colon A(X) \to A(\mathrm{Spec}\,(k)) \simeq \mathbb{Z}\cdot[\mathrm{Spec}\,(k)]$ [Ful98, Definition 1.4]. $A^*_{\mathrm{num}}(X)$ and $A^*_{\mathrm{hom}}(X)_{\mathbb{Q}}$ also have similar functorial properties [Kle68, Section 1]. We shall denote the intersection product on rings by '$\cdot$'. For cycles $[Z]$ and $[Z']$ of complimentary co-dimension in X, by abuse of notation we shall also denote the integer $\pi_{X*}([Z]\cdot[Z'])$ by $[Z]\cdot[Z']$.

Let $[\mathbb{P}^s_k] \in A^{n-s}(\mathbb{P}^n_k)$, $0 \le s \le n$ be the class of a s-dimensional linear subspace of $\mathbb{P}^n_k$. The Chow ring $A^*(\mathbb{P}^n_k)$ is isomorphic to the graded ring $\mathbb{Z}[x]/(x^{n+1})$ under the map $[\mathbb{P}^{n-1}] \to x$ [Ful98, Proposition 8.4] and the class $[\mathbb{P}^s_k]$ generates the free abelian group $A^{n-s}(\mathbb{P}^n_k)$, $0 \le s \le n$ [Ful98, Example 1.9.3].

Definition 2.1. The *degree* of $[Z] \in A^s(\mathbb{P}^n_k)$ is the integer $[Z]\cdot[\mathbb{P}^s_k]$. For a subvariety Z of $\mathbb{P}^n_k$ by $\deg(Z)$ we mean $\deg([Z])$.

For any two smooth, projective varieties X and Y (over k), there is an exterior product map $A^*(X) \otimes_{\mathbb{Z}} A^*(Y) \to A^*(X \times_k Y)$ [Ful98, Section 1.10], which is a morphism of graded rings [Ful98, Example 8.3.7]. We shall denote the image of $[Z] \otimes [Z']$ by $[Z] \times [Z']$.

[2]Conjecturally $A^*_{\mathrm{hom}}(X)_{\mathbb{Q}}$ is independent of ℓ

In what follows, we will need a bound (see Proposition 2.9) well known to experts and proved using standard techniques. For ease of exposition we present a short proof using Chow's moving Lemma and the join construction [Ful98, Example 8.4.6].

Definition 2.2. Two subvarieties V and W in a smooth projective variety X are said to *intersect properly*, if the each component of $V \cap W$ has the right dimension (i.e. $\dim(V) + \dim(W) - \dim(X)$).

Remark 2.3. In a similar vein, cycles $[V]$ and $[W]$ in $Z^*(X)$ are said to *intersect properly* if each component of $[V]$ intersects each component of $[W]$ properly.

Suppose now $i\colon X \hookrightarrow \mathbb{P}^n_k$ is a closed embedding of a smooth, projective variety of dimension r. Fulton's definition of intersection multiplicities implies the following statement [Ful98, Section 6.2, Section 7.1].

Proposition 2.4. Let $[C] \in Z^*(\mathbb{P}^n_k)$ be a cycle on $\mathbb{P}^n_k$ which intersects $[X]$ properly. Then,

$$i^*([C]) = \textstyle\sum_j i(Z_j; [X], [C])[Z_j] \in A^*(X),$$

where Z_j's are the irreducible components of the intersection of X with the components of $[C]$, and $i(Z_j; [X], [C])$'s are the intersection multiplicities along the Z_j's [Ful98, Definition 7.1].

Remark 2.5. By abuse of notation the cycle $\sum_j i(Z_j; X, C)[Z_j] \in Z^*(X)$ will also be denoted by $[C] \cdot [X]$. Moreover if $[C]$ is an effective cycle so is $[C] \cdot [X]$ [Ful98, Proposition 7.1].

Let $V \subseteq X$ be a closed subvariety of dimension d. Let $L \subseteq \mathbb{P}^n_k$ be a linear subspace of dimension $n - r - 1$ disjoint from X. We denote by $C_L(V) \subseteq \mathbb{P}^n_k$, the cone of V over L [Rob70, Section 2] or equivalently the join of V and L [Ful98, Example 8.4.5]. It is a subvariety of dimension $n + d - r$, and of degree equal to the degree of V. Moreover V is an irreducible component of $C_L(V) \cap X$ and every component of $C_L(V) \cap X$ is of dimension equal to d [Rob70, Lemma 2].

Remark 2.6. Hence for any such L, we see that $C_L(V)$ and X intersect properly (see Defintion 2.2) and we let $[C_L(V)] \cdot [X]$ denotes the corresponding cycle on X (see Remark 2.5).

For an arbitrary cycle $[V] = \sum_i m_i[V_i] \in Z^{r-d}(X)$ we define

$$[C_L([V])] := \sum_i m_i[C_L(V_i)] \in Z^{r-d}(\mathbb{P}^n).$$

Let V and W be closed subvarieties of X. We define the *excess* of V (relative to W) to be $-\infty$ if they do not intersect. Else it is defined to be the maximum of the (non-negative) integers

$$\dim(Y) - \dim(V) - \dim(W) + \dim(X),$$

where Y runs through all the components of $V \cap W$. We denote the excess by $e_W(V)$. For a cycle $[V] := \sum_i m_i[V_i]$ in $Z^*(X)$, we define $e_W([V]) := \sum_i m_i e_W(V_i)$. We have the following result from [Rob70] (used there to prove the 'Chow moving Lemma').

Lemma 2.7. [Rob70, Main Lemma]

Let $i\colon X \hookrightarrow \mathbb{P}^n_k$ be a smooth, projective closed subvariety of dimension r. Let W be a subvariety of X. For any cycle $[V] \in Z^*(X)$, there exists a dense open subset U of $G(n, n-r-1)$, the Grassmanian of linear subspaces in $\mathbb{P}^n$ of dimension $n-r-1$, such that for any closed point $x \in U$, if L_x denotes the corresponding linear subspace then,

(1) $L_x \cap X = \emptyset$.

(2) $e_W\left([C_L\left([V]\right)] \cdot [X] - [V]\right) \leq \max\left(e_W([V]) - 1, 0\right)$.

2.2. An estimate for intersection product

As before let $i\colon X \hookrightarrow \mathbb{P}^n_k$ be a smooth, projective closed subvariety of dimension r. Let V and W be closed subvarieties of X. Let d be the dimension of V. The following lemma is now easy to deduce from Lemma 2.7.

Lemma 2.8. There exist a positive integer $k \leq r+1$ and a sequence of effective cycles $\{[V_j]\}_{0\leq j\leq k}$ and $\{[E_j]\}_{1\leq j\leq k}$ in $Z^{r-d}(X)$ such that,

(1) $[V_0] = [V]$ in $Z^{r-d}(X)$.

(2) $[V_j] = [E_{j+1}] - [V_{j+1}]$ in $Z^{r-d}(X)$ for all $0 \leq j \leq k-1$.

(3) For all $j \geq 1$, $[E_j] = i^*(\deg\left([V_{j-1}]\right)[\mathbb{P}^{n-d+r}_k])$ in $\mathrm{A}^{r-d}(X)$. Thus $[E_j]$'s are 'ambient' cycles.

(4) Every component of $[V_{k-1}]$ and $[V_k]$ intersects W properly (see Definition 2.2 and Remark 2.3).

In particular

$$[V] = \sum_{j=1}^{k}(-1)^{j+1}[E_j] + (-1)^k[V_k] \text{ in } Z^{r-d}(X).$$

Proof. Let

$$[V_0] := [V] \in Z^{r-d}(X).$$

For any integer $j \geq 1$, having defined $[V_{j-1}] \in Z^{r-d}(X)$ and proven that it is effective, we define

$$[E_j] := [C_{L_j}([V_{j-1}])] \cdot [X] \in Z^{r-d}(X), \tag{2.2}$$

where L_j is linear sub-space of $\mathbb{P}^n$ of dimension $n-r-1$ (see Remark 2.6), chosen such that

$$e(i^*[C_{L_j}([V_{j-1}])] - [V_{j-1}]) \leq \max(e([V_{j-1}]-)1, 0) \text{ (see Lemma 2.7).}$$

Here the excess is with respect to W. Since $[C_{L_j}(V_{j-1})]$ and $[X]$ intersect properly [Rob70, Lemma 2], Remark 2.5 implies that $[E_j]$ is an effective cycle. For any integer j having defined $[V_{j-1}]$ and $[E_j]$, we define,

$$[V_j] := [E_j] - [V_{j-1}] \text{ in } Z^{r-d}(X).$$

For any subvariety $V \subseteq X$, V is an irreducible component of $C_L(V) \cap X$ [Rob70, Lemma 2], thus the effectivity of $[V_j]$ for any $j \geq 1$, is a consequence of the effectivity of $[E_j]$.

Since $e([V_0]) = e([V]) \leq r$, for any $j \geq r$, the excess $e([V_j]) = 0$. Let $k-1$ be the smallest integer j with the property that $e([V_{k-1}]) = 0$. Then every component of the algebraic cycles $[V_{k-1}]$ and $[V_k]$ intersects W properly.

For any $j \geq 1$ since $C_L([V_{j-1}])$ and X intersect properly [Rob70, Lemma 2], Proposition 2.4 implies that

$$[E_j] = i^*\left([C_L([V_{j-1}])]\right) \in A^{r-d}(X). \tag{2.3}$$

For any $j \geq 1$ since $[C_L([V_{j-1}])]$ as a cycle on $\mathbb{P}^n_k$ has degree equal to the degree of $[V_{j-1}]$ [Ful98, Example 8.4.5], thus (2.3) implies that,

$$[E_j] = i^*\left(\deg\left([V_{j-1}]\right)[\mathbb{P}_k^{n-d+r}]\right) \text{ in } A^{r-d}(X).$$

$\square$

Now we derive a basic estimate which is needed later.

Proposition 2.9. Let $X \subseteq \mathbb{P}^n_k$ be a smooth, projective variety. Then for any two subvarieties V and W of complimentary dimension in X, $|[V] \cdot [W]| \leq C\mathrm{deg}(V)\mathrm{deg}(W)$, for a constant C independent of V and W .

Proof. We use Lemma 2.8 to construct a sequence of algebraic cycles $\{[V_j]\}_{0\leq j\leq k}$ and $\{[E_j]\}_{1\leq j\leq k}$ in $Z^{r-d}(X)$ where d is the co-dimension of V in X and satisfying properties (1)-(4) in Lemma 2.8.

Since

$$[V] = \textstyle\sum_{j=1}^k (-1)^{j+1}[E_j] + (-1)^k[V_k] \text{ in } Z^{r-d}(X),$$

one has that

$$|[V] \cdot [W]| \leq \sum_{j=1}^{k} |[E_j] \cdot [W]| + |[V_k] \cdot [W]|. \tag{2.4}$$

Note that $[E_j] = i^* \left(\mathrm{deg}([V_{j-1}])[\mathbb{P}^{n+d-r}_k]\right)$ (see Lemma 2.8, (3)) and hence for every $j \geq 1$,

$$[E_j] \cdot [W] = \mathrm{deg}(W)\mathrm{deg}([V_{j-1}]). \tag{2.5}$$

Since every component of $[V_{k-1}]$ intersects $[W]$ properly, $[V_k] \cdot [W]$ is bounded above by $[E_k] \cdot [W] = \mathrm{deg}(W)\mathrm{deg}([V_{k-1}])$ [Ful98, Proposition 7.1]. Combining (2.4) and (2.5) we get,

$$|[V] \cdot [W]| \leq \left(\sum_{j=1}^{k} \mathrm{deg}([V_{j-1}]) + \mathrm{deg}([V_{k-1}])\right) \mathrm{deg}(W). \tag{2.6}$$

Projection formula implies that for every $j \geq 1$,

$$\mathrm{deg}([E_j]) = \mathrm{deg}(X)\mathrm{deg}([V_{j-1}]).$$

Since the $[E_j]$'s and $[V_j]$'s are effective,

$$\mathrm{deg}([V_j]) \leq \mathrm{deg}([E_j]) = \mathrm{deg}(X)\mathrm{deg}([V_{j-1}]).$$

Thus for every $j \geq 1$

$$\deg([V_j]) \leq \deg(X)^j \deg(V) \leq \deg(X)^{r+1} \deg(V). \tag{2.7}$$

Thus (2.6) and (2.7) together imply that

$$|[V] \cdot [W]| \leq (r+2) \deg(X)^{r+1} \deg(V) \deg(W). \tag{2.8}$$

□

Remark 2.10. If V and W intersect properly, then the bound in (2.8) can be improved to $[V] \cdot [W] \leq \deg(V)\deg(W)$ [Hru12, Lemma 10.12]. Furthermore the bound in Proposition 2.9 can be generalized (in an appropriate sense) to Gysin pullbacks under regular embeddings of quasi-projective varieties and proved without recourse to the moving lemma [SV20, Appendix C].

3. Gromov algebra

Let $i : X \hookrightarrow \mathbb{P}^n_k$ be a smooth, projective variety over an algebraically closed field k of dimension r. Let $[H] \in A^1(X)$ be the class of a hyperplane section. Let ω be the cohomology class of $[H]$ in $H^2(X, \mathbb{Q}_\ell)$.

For $j \geq 1$, let $[H]^j$ (resp. ω^j) denote the j^{th} self intersection product (resp. self cup product) of $[H]$ (resp. ω) in $A^*(X)$ (resp. $H^*(X, \mathbb{Q}_\ell)$). Let $f \colon X \to X$ be a self map and $[\Gamma_f] \in A^r(X \times_k X)$ be the graph correspondence. For integers $0 \leq j \leq r$ let

$$\delta_j(f) := [H]^{r-j} \cdot f^*([H]^j) = f^*([H]^j) \cdot [H]^{r-j}. \tag{3.1}$$

Note that we have an equality

$$\delta_j(f) = \mathrm{Tr}_X(\omega^{r-j} \cup f^*(\omega^j)) = \mathrm{Tr}_X(f^*(\omega^j) \cup \omega^{r-j}), \tag{3.2}$$

where $\cup$ is the cup product on $H^*(X, \mathbb{Q}_\ell)$ and Tr_X is the usual trace map on top degree cohomology.

Lemma 3.1. Using the above notations,

$$(i \times i)_*[\Gamma_f] = \sum_{j=0}^{r} \delta_{r-j}(f)([\mathbb{P}^{r-j}_k] \times [\mathbb{P}^j_k]) \tag{3.3}$$

Proof. The exterior product map $A^*(\mathbb{P}^n_k) \otimes_{\mathbb{Z}} A^*(\mathbb{P}^n_k) \to A^*(\mathbb{P}^n_k \times_k \mathbb{P}^n_k)$ is an isomorphism of graded rings [Ful98, Example 8.3] and hence

$$(i \times i)_*[\Gamma_f] = \sum_{j=0}^{r} n_j([\mathbb{P}^{r-j}_k] \times [\mathbb{P}^j_k]) \tag{3.4}$$

where for any $j \geq 0$, $n_j = \Big((i \times i)_* [\Gamma_f] \cdot \Big([\mathbb{P}^{n-r+j}_k] \times [\mathbb{P}^{n-j}_k]\Big)\Big)$. The projection formula thus implies that

$$n_j = [\Gamma_f] \cdot \left([H]^{r-j} \times [H]^j\right), \tag{3.5}$$

and the result follows from Lefschetz trace formula [And04, Section 3.3.3]. □

Definition 3.2. The *homological Gromov algebra* $\mathrm{A}^{Gr}_{\text{hom}}(f, \omega)_{\mathbb{Q}}$ [3] (resp. the *numerical Gromov algebra*, $\mathrm{A}^{Gr}_{\text{num}}(f, [H])_{\mathbb{Q}}$) is the smallest f^*-stable (unital) sub-algebra of $A^*_{\text{hom}}(X)_{\mathbb{Q}}$ (resp. $A^*_{\text{num}}(X)_{\mathbb{Q}}$) containing ω (resp. $[H]$).

The numerical Gromov algebra with real coefficients $\mathrm{A}^{Gr}_{\text{num}}(f, [H])_{\mathbb{R}}$ is the $\mathbb{R}$-algebra $\mathrm{A}^{Gr}_{\text{num}}(f, [H])_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$.

Let λ_i (resp. χ_i) be the spectral radius [4] of f^* acting on $A^i_{\text{num}}(X)_{\mathbb{Q}}$, $0 \leq i \leq \dim(X)$ (resp. $A^i_{\text{hom}}(X)_{\mathbb{Q}}$, $0 \leq i \leq \dim(X)$). Let λ^{Gr} and χ^{Gr} be the spectral radii of f^* acting on $\mathrm{A}^{Gr}_{\text{num}}(f, [H])_{\mathbb{Q}}$ and $\mathrm{A}^{Gr}_{\text{hom}}(f, \omega)_{\mathbb{Q}}$ respectively. Note that λ^{Gr} is also the spectral radius of f^* acting on $\mathrm{A}^{Gr}_{\text{num}}(f, [H])_{\mathbb{R}}$.

Finally let μ_j be the spectral radius (with respect to τ in (2.1)) of f^* acting on $H^j(X, \mathbb{Q}_\ell), 0 \leq j \leq 2\dim(X)$. Following lemma is obvious.

Lemma 3.3. Using the above notations we have inequalities

$$\lambda^{Gr} \leq \max_{0 \leq i \leq \dim(X)} \lambda_i \leq \max_{0 \leq i \leq \dim(X)} \chi_i \leq \max_{0 \leq j \leq 2\dim(X)} \mu_j.$$

Further

$$\lambda^{\mathrm{Gr}} \leq \chi^{Gr} \leq \max_{0 \leq i \leq \dim(X)} \chi_i.$$

[3] In what follows when $k = \mathbb{C}$, we use Betti cohomology instead of ℓ-adic cohomology to define homological equivalence and hence to define $\mathrm{A}^{Gr}_{\text{hom}}(f, \omega)_{\mathbb{Q}}$. Thus when $k = \mathbb{C}$ there is no dependence on an auxillary prime ℓ.

[4] In what follows the spectral radius of a linear endomorphism of a vector space over a sub field of $\mathbb{C}$ is defined to be the maximum in absolute value of its complex eigenvalues.

We will also need the following lemma.

Lemma 3.4. Let $\{a_{m,i}\}_{m\geq 1}, 1 \leq i \leq s$ be a collection of sequences of complex number. Let $b_i,\ i \leq i \leq s$ be arbitrary complex numbers. Then

$$\limsup_m |\textstyle\sum_{i=1}^s a_{m,i}b_i|^{1/m} \leq \max_{1\leq i\leq s} \limsup_m |a_{m,i}|^{1/m}.$$

Proof. [5] The radius of absolute convergence of $\sum_{m\geq 0}\sum_{i=1}^s b_i a_{m,i} z^m$ is at least as large as the minimum of the radius of absolute convergence of $\sum_{m\geq 0} a_{m,i} z^m$. The desired result is now an immediate consequence of the Cauchy-Hadamard formula for the radius of absolute convergence. $\square$

Let V be any finite dimensional vector space over $\mathbb{R}$ (or $\mathbb{C}$) and $T : V \to V$ a linear map. Let $||\cdot||$ be any matrix norm. In what follows we will make use of the following theorem due to Gelfand [Rud87, Theorem 18.9].

Theorem 3.5. $\limsup_m ||T^m||^{1/m} = \rho(T)$, where $\rho(T)$ is the spectral radius of T.

Let K be a normed field such that there exists an embedding $\tau : K \hookrightarrow \mathbb{C}$ of normed fields. Let V be any finite dimensional vector space over K and $T : V \to V$ a linear map. We shall need the following standard result which we state without a proof.

Proposition 3.6. $\limsup_m |\mathrm{Tr}(T^m)|^{1/m} = \rho(T)$, where $\rho(T)$ is the spectral radius of T.

Now we prove the principal result of this article.

Theorem 3.7. Let X be a smooth, projective variety over an arbitrary algebraically closed field k of dimension r. Let $[H] \in A^1(X)$ (respectively $\omega \in H^2(X, \mathbb{Q}_\ell)$) be the class of an hyperplane section in the Chow group (respectively ℓ-adic cohomology). Let $f\colon X \to X$ be a self map of X. Then all the inequalities in Lemma 3.3 are in fact equalities.

Thus the spectral radius of f^* acting on $H^*(X, \mathbb{Q}_\ell)$ (with respect to $\tau : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$) is independent of τ, and coincides with the spectral radius of f^* on the numerical Gromov algebra.

[5] The short and elegant proof here was suggested to us by the referee.

Proof. Let $i\colon X \hookrightarrow \mathbb{P}^n_k$ be an embedding with $[H]$ being the class of the hyperplane section under i. Clearly it suffices to show that

$$\lambda^{\mathrm{Gr}} \geq \mu_i,\ 0 \leq i \leq 2r. \tag{3.6}$$

Let $[\Gamma_{f^m}] \in A^r(X \times_k X)$ be the graph of the m^{th} iterate of f. As before we denote by $\delta_j(f^m) = [H]^{r-j} \cdot (f^{m*}[H]^j) = (f^{m*}[H]^j) \cdot [H]^{r-j}$ (see Definition (3.1)). We shall first show that for any integer $i \in [0, 2r]$,

$$\mu_i \leq \max_{0 \leq j \leq r} \limsup_m |\delta_j(f^m)|^{1/m}. \tag{3.7}$$

It is clear from the definition of μ_i and $\delta_j(f^m)$ (see Equation 3.2) that they specialise well, and thus it suffices to prove the bound (3.7), when k is an algebraic closure of a finite field. Note that this is not the case with the bound in (3.6). Hence we need this intermediate step. Hence we now assume that k is an algebraic closure of a finite field, and that X and the self map f are defined over this finite field.

The work of Katz-Messing [KM74, Theorem 2.1] and the Lefschetz trace formula imply that for every integer $i \in [0, 2r]$, there exist an algebraic cycle $\pi^i_X \in Z^r(X \times X)_{\mathbb{Q}}$ (the i^{th} 'Kunneth component', see [And04, Section 3.3.3]) such that

$$\mathrm{Tr}(f^{m*}; H^i(X, \mathbb{Q}_\ell)) = (-1)^i [\Gamma_{f^m}] \cdot \pi^{2r-i}_X, \tag{3.8}$$

representing the trace as an intersection product (on the product variety $X \times_k X$). Recall that we have fixed an embedding $\tau : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ (see (2.1)). Thus $\mathbb{Q}_\ell$ is a normed field via this embedding. Proposition 3.6 and (3.8) together imply that,

$$\mu_i = \limsup_m |[\Gamma_{f^m}] \cdot \pi^{2r-i}_X|^{1/m},\ 0 \leq i \leq 2r. \tag{3.9}$$

There exist finitely many subvarieties $W^{2r-i}_j \subseteq X \times_k X$ of codimension r (the components of the 'Kunneth components') and a constant C' such that for every $m \geq 1$,

$$|[\Gamma_{f^m}] \cdot \pi^{2r-i}_X| \leq C' \sum_j |[\Gamma_{f^m}] \cdot [W^{2r-i}_j]|,\ 0 \leq i \leq 2r. \tag{3.10}$$

The estimate in Proposition 2.9 (applied to the smooth projective variety $X \times_k X \subseteq \mathbb{P}^{n^2+2n}_k$) and (3.10) imply that there exists a constant C''

(depending only on $i : X \hookrightarrow \mathbb{P}^n_k$ and the choice of Kunneth components) such that for every $m \geq 1$,

$$|[\Gamma_{f^m}] \cdot \pi_X^{2r-i}| \leq C'' \deg([\Gamma_{f^m}])(\sum_j \deg(W_j^{2r-1})),\ 0 \leq i \leq 2r. \tag{3.11}$$

The degree in (3.11) is with respect to the embedding $X \times_k X \subseteq \mathbb{P}_k^{n^2+2n}$. Moreover Lemma 3.3 implies that

$$\deg(\Gamma_{f^m}) = \textstyle\sum_{j=0}^r \delta_{r-j}(f^m) \deg([\mathbb{P}_k^{r-j}] \times [\mathbb{P}_k^j]).$$

Hence (3.9) and (3.11) together with Lemma 3.4 imply that for any integer $i \in [0, 2r]$,

$$\mu_i \leq \max_{0 \leq j \leq r} \limsup_m |\delta_j(f^m)|^{1/m}. \tag{3.12}$$

Thus we have obtained the bound (3.7) over an arbitrary algebraically closed field.

For the rest of the proof we work over the algebraically closed field k we started with. Let $\mathrm{A}_{\text{num}}^{Gr}(f, [H])_{\mathbb{R}}$ be the numerical Gromov algebra with $\mathbb{R}$-coefficients. Let $||.||$ be any norm on the finite dimensional $\mathbb{R}$-vector space $\mathrm{A}_{\text{num}}^{Gr}(f, [H])_{\mathbb{R}}$. Note that f^* is a graded linear transformation of $\mathrm{A}_{\text{num}}^{Gr}(f, [H])_{\mathbb{R}}$. For every integer $m \geq 1$, we denote the norm of the linear map f^{m*} acting on $\mathrm{A}_{\text{num}}^{Gr}(f, [H])_{\mathbb{R}}$ by $||f^{m*}||$.

Recall that $\delta_j(f^m) = f^{m*}([H]^j) \cdot [H]^{r-j}$. Since the intersection product is bilinear, the map from the j^{th} graded part of $\mathrm{A}_{\text{num}}^{Gr}(f, [H])_{\mathbb{R}}$ to $\mathbb{R}$, obtained by taking intersection product with $[H]^{r-j}$ is linear. Consequently there exists a constant $\widetilde{C'}$ independent of m, such that for any $m \geq 1$,

$$|\delta_j(f^m)| \leq \widetilde{C'} ||f^{m*}(H^j)||,\ 0 \leq j \leq r. \tag{3.13}$$

Since f^* is a linear map, (3.13) implies that there exists a constant $\widetilde{C}$ independent of m such that, for any $m \geq 1$,

$$|\delta_j(f^m)|^{1/m} \leq \widetilde{C}^{1/m} ||f^{m*}||^{1/m},\ 0 \leq j \leq r. \tag{3.14}$$

Thus Theorem 3.5, (3.12) and (3.14) together imply

$$\mu_i \leq \lambda^{Gr},\ 0 \leq i \leq 2r.$$

□

Corollary 3.8. Let X be a smooth, projective variety over an arbitrary algebraically closed field k of dimension r. Let $f\colon X \to X$ be a self map such that $f^*(\omega) = \lambda\omega$ for some ample class $\omega \in H^2(X, \mathbb{Q}_\ell)$ and integer λ. Then the spectral radius of f^* on $H^*(X, \mathbb{Q}_\ell)$ is λ^r.

When k is of characteristic 0, Corollary 3.8 is a consequence of Serre's result [Ser60, Théorème 1]. Also note that the assumption of Corollary 3.8 is automatic when $\mathrm{Pic}(X) = \mathbb{Z}$.

4. Gromov algebra as a Lefschetz Module

The aim of this section is to give a representation theoretic perspective to the (homological) Gromov algebra by relating it to the work of Looijenga-Lunts [LL97] and Verbitsky [Ver96]. In doing so we hope that this picture will give the right generalization of Esnault-Srinivas's result (Theorem 1.2, (2)) to higher dimensional varieties over $\mathbb{C}$.

In the language of Gromov algebra, the proof of Esnault-Srinivas (over $\mathbb{C}$) uses polarized Hodge structure on $H^2(X, \mathbb{Q})$ [ES13, Proposition 5.1] to obtain restrictions on eigenvalues of an automorphism of a surfaces on the orthogonal complement (with respect to cup-product) of the Gromov algebra with respect to any ample class. A natural question then is how do we decompose the cohomology of a higher dimensional variety into f^*-stable subspaces such that the Gromov algebra is a 'natural' component?

We now give a plausible answer to this question.

4.1. The LLV Lie algebra

Now we summarize the key construction from [LL97] and [Ver96].

Let K be a field of characteristic 0. Let $M_\bullet$ be a $\mathbb{Z}$-graded finite dimensional K-vector space. Let $h \in \mathrm{End}(M)$ be such that h acts by i in degree i. Thus the eigenspaces of h determine the grading of $M_\bullet$, and $u \in \mathrm{End}(M)$ has degree i iff $[h, u] = iu$.

An endomorphism $e \in \mathrm{End}(M)$ of degree 2 is said to have the *Lefschetz property*, if for every $i \in \mathbb{Z}$, $e^i\colon M_{-i} \to M_i$ is an isomorphism. Equivalently by the Jacobson-Morozov lemma this is equivalent to existence of a linear transformation $f \in \mathrm{End}(M)$ such that f is of degree -2 such that $[e, f] = h$.

Let $\mathfrak{a}$ be a finite dimensional K-vector space. We regard $\mathfrak{a}$ as a graded abelian Lie algebra which is homogeneous of degree 2. We say that a graded Lie homomorphism $e\colon \mathfrak{a} \to \mathfrak{gl}(M)$ has the *Lefschetz property* if for some $a \in \mathfrak{a}$, $e(a)$ has the Lefschetz property. Clearly having Lefschetz property is a Zariski open condition on $\mathfrak{gl}(M)$, and by Jacobson-Morozov we have a rational morphism $f\colon \mathfrak{a} \to \mathfrak{gl}(M)$ defined on this open subset. The Lie subalgebra of $\mathfrak{gl}(M)$ generated by $e(a), f(a)$ for all possible choices of a (denoted by $\mathfrak{g}(\mathfrak{a}, M)$) is the LLV lie algebra. Note that $\mathrm{ad}(h)$ induces a grading on $\mathfrak{g}(\mathfrak{a}, M)$ and that $\mathfrak{g}(\mathfrak{a}, M)$ is evenly graded under this action.

Definition 4.1. [Lefschetz Modules] We say that the pair $(\mathfrak{a}, M)$ is a *Lefschetz* $\mathfrak{a}$*-Module* if $\mathfrak{g}(\mathfrak{a}, M)$ is semisimple.

Basic properties of the Lefschetz Module. In what follows we fix $\mathfrak{a}$ as above.

(i) The collection of Lefschetz $\mathfrak{a}$-modules is closed under direct sums, taking tensor products and taking duals.

(ii) The category of Lefschetz $\mathfrak{a}$-modules is a semi-simple, Artinian and Noetherian category.

(iii) Given a Lefschetz module M, any representation of $\mathfrak{g}(\mathfrak{a}, M)$ is also a Lefshcetz module (under the natural $\mathfrak{a}$-action). Moreover this correspondence preserves irreducibility.

(iv) For any Lefschetz module M, $\mathfrak{g}(\mathfrak{a}, M)$ is naturally graded and compatible with the action on M.

4.2. Connections with the Gromov algebra

Let X be a smooth projective variety over $\mathbb{C}$ of dimension r. Let $f\colon X \to X$ be a finite self-map. Then f^* is an isomorphism on the top Betti cohomology and hence by non-degeneracy of the cup-product pairing it is so on all of $H^*(X, \mathbb{Q})$. Let $\omega \in H^2(X, \mathbb{Q})$ be an ample class.

Let $\mathfrak{a}$ be the f^* stable subspace generated by ω, considered as a graded Lie algebra in degree 2. Note that $\mathfrak{a}$ is by definition the degree 2 summand of $\mathrm{A}^{Gr}_{\mathrm{hom}}(f, \omega)_{\mathbb{Q}}$. Let $M := H^*(X, \mathbb{Q})[r]$ be the shifted (by r) total Betti cohomology of X. Note that there is a natural graded map from $\mathfrak{a} \to \mathfrak{gl}(M)$ given by cup product, and we denote the associated Lie algebra $\mathfrak{g}(\mathfrak{a}, M)$ by $\mathfrak{g}(f, \omega)$. Following result is an immediate consequence of [LL97, Proposition 1.6].

Proposition 4.2. $\mathfrak{g}(f, \omega)$ is semi-simple and hence $H^*(X, \mathbb{Q})[r]$ is a Lefschetz module over the degree 2 summand of $\mathrm{A}^{Gr}_{\mathrm{hom}}(f, \omega)_{\mathbb{Q}}$.

Remark 4.3. To be precise [LL97, Proposition 1.6] would show that $\mathfrak{g}(f,\omega) \otimes_{\mathbb{Q}} \mathbb{C}$ is semi-simple, which then implies the same for $\mathfrak{g}(f,\omega)$ since non-degeneracy of the Killing form can be checked after an extension of base field.

We end this brief section by showing that $\mathrm{A}^{Gr}_{\mathrm{hom}}(f,\omega)_{\mathbb{Q}}$ is an irreducible summand of $H^*(X,\mathbb{Q})[r]$ under the action of $\mathfrak{g}(f,\omega)$.

Proposition 4.4. The homological Gromov algebra $\mathrm{A}^{Gr}_{\mathrm{hom}}(f,\omega)_{\mathbb{Q}}[r]$ is an irreducible representation of $\mathfrak{g}(f,\omega)$ and hence equivalently also an irreducible Lefschetz module.

Proof. We will at once show that $\mathrm{A}^{Gr}_{\mathrm{hom}}(f,\omega)_{\mathbb{Q}}[r]$ is a representation of $\mathfrak{g}(f,\omega)$ and that it is an irreducible one. First note that the lowest graded piece of $\mathrm{A}^{Gr}_{\mathrm{hom}}(f,\omega)_{\mathbb{Q}}[r]$ lies in the kernel of $\mathfrak{g}(f,\omega)_{<0}$, the negatively graded part of $\mathfrak{g}(f,\omega)$. Thus [LL97, Proposition 1.12] implies that the $\mathfrak{g}(f,\omega)$ stable subspace generated by the lowest graded piece of $\mathrm{A}^{Gr}_{\mathrm{hom}}(f,\omega)_{\mathbb{Q}}[r]$ is generated by the action of Lefschetz operators and hence is equal to $\mathrm{A}^{Gr}_{\mathrm{hom}}(f,\omega)_{\mathbb{Q}}[r]$. Irreducibility is a consequence of [LL97, Corollary 1.13], since the lowest graded piece of $\mathrm{A}^{Gr}_{\mathrm{hom}}(f,\omega)_{\mathbb{Q}}[r]$ is a one dimensional space. $\square$

Proposition 4.4 seems to suggest that the Gromov algebra is one piece of a natural decomposition of the cohomology into f^* stable subspaces. Thus it would be interesting to study constraints analogous to Theorem 1.2 and Theorem 3.7 on the other pieces of this decomposition. A natural starting point would be the cohomology of compact hyper-Kählerian manifolds, where both the LLV lie algebra and questions of entropy are rather well studied (see for example [GLR19], [Obe19], [Ogu07]). We hope to come back to this question in the future.

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