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Symplectic realizations of holomorphic Poisson manifolds

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Symplectic realization is a longstanding problem which can be traced back to Sophus Lie. In this paper, we present an explicit solution to this problem for an arbitrary holomorphic Poisson manifold. More precisely, for any holomorphic Poisson manifold $(\mathscr{X}, \pi)$ with underlying real smooth manifold X, we prove that there exists a holomorphic symplectic structure in a neighborhood Y of the zero section of T^*X such that the projection map is a holomorphic symplectic realization of the given holomorphic Poisson manifold, and moreover the zero section is a holomorphic Lagrangian submanifold. We describe an explicit construction for such a new holomorphic symplectic structure on $Y \subseteq T^*X$.

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1. Introduction

The notion of "symplectic realizations" can be traced back to Sophus Lie who used the name "function group". In [22], Lie defined a "function group" as a collection of functions of the canonical variables $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ which is a subalgebra under the canonical Poisson bracket and generated by a finite number of independent functions $\phi_1, \cdots, \phi_r$. In modern language, this means that $\mathbb{R}^r$ has a Poisson structure induced from the canonical symplectic structure $\mathbb{R}^{2n}$ in the sense that $\Phi = (\phi_1, \cdots, \phi_r) : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^r$ is a Poisson map. In the C^∞-context, a symplectic realization of a Poisson manifold M, as defined by Weinstein [34] (called a full symplectic realization), is a Poisson map from a symplectic manifold V to M which is a surjective submersion. Since Sophus Lie's treasure work on the theory of transformation group [22], the following has become a central question:

Problem A. *Does a symplectic realization always exist for a given Poisson manifold?*

In fact, this question is closely related to Lie's theory on Lie groups. To get a flavor of this, consider the Lie-Poisson manifold $\mathfrak{g}^*$ corresponding to a finite dimensional Lie algebra $\mathfrak{g}$. A natural choice of a symplectic realization is $\Phi : T^*G \to \mathfrak{g}^*$ with the canonical cotangent symplectic structure on T^*G and Φ being the left translation, where G is a Lie group with Lie algebra $\mathfrak{g}$, and $\mathfrak{g}^* \cong T^*_e G$. Lie himself proved that a symplectic realization always exists locally for any smooth Poisson manifold of constant rank [22]. A local existence theorem for symplectic realizations of general smooth Poisson manifolds, was proved by Weinstein in 1983 [34]. Subsequently, Karasev [13] and Weinstein [35] independently proved the global existence theorem by gluing methods. Indeed, they proved a stronger result: for any C^∞-Poisson manifold, there exists an essentially unique, distinguished, symplectic realization which possesses a compatible local *groupoid* structure [35], a device which is now altogether called a symplectic local groupoid [5]. Furthermore, the infinitesimal object corresponding to this local Lie groupoid – its so-called Lie algebroid, as introduced by Pradines [29] – can be proved [5] to be isomorphic to the cotangent Lie algebroid $(T^*M)_\pi$ canonically associated to the Poisson manifold (M, π). The bracket of this Lie algebroid essentially extends the natural Lie bracket relation on exact forms: $[df, dg]_* = d\{f, g\}$ in an obvious way. For more details on symplectic local groupoids, see [5, 14, 35].

For a given Poisson manifold (M, π), the pair of Lie algebroids $((T^*M)_\pi, TM)$, where TM is the standard tangent Lie algebroid of M, constitute an example of the so-called *Lie bialgebroids* [24]. From the theory of integration of Lie bialgebroids of Mackenzie–Xu [25] (which extends the classical theory of Drinfeld [10, 11] for integrating Lie bialgebras), it follows that, under some mild topological assumption, a Lie groupoid with Lie algebroid $(T^*M)_\pi$ automatically carries a compatible symplectic structure, and is therefore a symplectic groupoid. As a consequence, any *local* Lie groupoid with Lie algebroid $(T^*M)_\pi$ – the existence of which is guaranteed [29] – gives *automatically* a symplectic realization of the underlying Poisson manifold. In this way, the Mackenzie–Xu integration method provided an alternative proof of the existence of global symplectic realizations [25]. However, all these results are existence results and are not constructive.

In 2001, while investigating Poisson sigma models, Cattaneo–Felder [4] discovered an explicit construction for the symplectic groupoid of an integrable Poisson manifold. Over the past 20 years, this construction — a certain quotient space of the Banach manifold of what would eventually be recognized in [6] as A-paths in the cotangent Lie algebroid of M — inspired many important works in Poisson geometry, among which the solution to the problem of integrability of Lie algebroids [6]. Although local symplectic groupoids are not mentioned explicitly in [4], an explicit construction of local symplectic groupoids is essentially given in [4, Theorem 4.7] since the hypothesis [4, Assumption 4.6] actually holds in a neighborhood of the unit space. We refer the interested reader to [2] for more details on the relation between the approach to the integration of general Lie algebroids developed in [6] and the results on the integration of Poisson manifolds exposed in [4]. In particular, it is shown in [2] that, provided integrability is assumed, the integration construction for Lie algebroids can be seen as a particular case of the integration construction for Poisson manifolds. Furthermore, around the same time, Ševera observed independently that the approach of [4] can be generalized to all Lie algebroids [30]. The main novelty in [6] is the precise integrability criterion guaranteeing the existence of global smooth Lie groupoids.

Although a lot of works focus on symplectic realizations in the C^∞-context, very little exists and is known in the holomorphic context. A holomorphic Poisson manifold is a complex manifold $\mathscr{X}$ whose sheaf of holomorphic functions $\mathcal{O}_{\mathscr{X}}$ is a sheaf of Poisson algebras. Symplectic realizations can be defined in a similar fashion as in the C^∞-case. Thus a natural question is

Problem B. *Does a symplectic realization always exist for a given holomorphic Poisson manifold? And, if so, is it possible to describe an explicit construction of a certain class of distinguished ones?*

To any holomorphic Poisson manifold $(\mathscr{X}, \pi)$ with underlying real smooth manifold X, one associates two C^∞-Poisson bivector fields. To see this, write the holomorphic Poisson tensor $\pi \in \Gamma(\wedge^2 T^{1,0}X)$ as $\pi_R + i\pi_I$, where π_R and $\pi_I \in \Gamma(\wedge^2 TX)$ are bivector fields. Then both π_R and π_I are indeed C^∞-Poisson bivector fields [19]. In 2009, Laurent-Gengoux, Stiénon and Xu proved that a holomorphic Poisson manifold is integrable if and only if either (X, π_R), or (X, π_I) are integrable as a real C^∞-Poisson manifold (Theorem 3.22 [20]). Since any C^∞-Poisson manifold admits a symplectic local groupoid, as a consequence, this result of Laurent-Gengoux, Stiénon and Xu implies that symplectic realizations do exist for any holomorphic Poisson manifolds. However, the conclusion is again not constructive. The purpose of the present paper is to describe an explicit construction of such a holomorphic symplectic local groupoid based on the Cattaneo-Felder's Poisson sigma model approach [4], and therefore to give an explicit affirmative answer to Problem B.

Our approach is based on the observation that a holomorphic Poisson manifold $(\mathscr{X}, \pi)$, where $\pi = \pi_R + i\pi_I$, gives rise to a Poisson–Nijenhuis [17, 26] structure (X, π_I, J) on the underlying real manifold X such that $\pi_R^\sharp = \pi_I^\sharp \circ J^T$ [19], where $J : TX \to TX$ is the underlying almost complex structure. Indeed, holomorphic Poisson manifolds are equivalent to a special class of Poisson–Nijenhuis manifolds, namely those where the Nijenhuis tensor is almost complex. Therefore, holomorphic symplectic local groupoids are equivalent to a special class of symplectic-Nijenhuis local groupoids in the sense of Stiénon–Xu [31]. Our goal is to describe an explicit construction of such a symplectic-Nijenhuis local groupoid. For this purpose, it suffices to construct explicitly two compatible symplectic structures on the local groupoid.

At this point, we must also mention the recent work of Crainic-Mărcuţ [8], where they present a very simple explicit construction of a symplectic realization of an arbitrary C^∞-Poisson manifold (M, π) on an open neighborhood of T^*M. In fact, another goal of our paper is to present a conceptual proof of their theorem. The idea is quite simple indeed. Given a local Lie groupoid Σ with Lie algebroid A, it is well known that, by choosing an A-connection on A, one can construct a local diffeomorphism – the exponential map [27] — from an open neighborhood of the zero section of A onto an open neighborhood of the unit space in Σ. Now, if Σ is a local symplectic groupoid, its Lie algebroid A is known to be isomorphic to $(T^*M)_\pi$ — see [5]. Pulling back the symplectic form on Σ, which Cattaneo-Felder described explicitly using Poisson sigma models [4, Equation (3.1), Theorem 3.3 and Section 4.3], via such an exponential map, one obtains a symplectic form on an open neighborhood of the zero section of the cotangent bundle T^*M. One can then verify directly that this symplectic form coincides with the one obtained in [8] (see also [3] for some related results). By applying a combination of techniques developed in the study of symplectic-Nijenhuis local groupoids [31] and the theory of Lie bialgebroids and Poisson groupoids [24, 25], we are able to describe explicitly the two compatible symplectic structures on the local groupoid, and thus obtain the following main result of the paper.

Theorem A. *Let $\mathscr{X}$ be a holomorphic Poisson manifold with underlying real smooth manifold X, almost complex structure J, and holomorphic Poisson bivector field $\pi \in \Gamma(\wedge^2 T^{1,0}X)$. Choose an affine connection ∇ on X. Let $\xi \in \mathfrak{X}(T^*X)$ be the Poisson geodesic vector field of ∇ as in Example 4.2. Denote by φ_t^ξ the flow of ξ on T^*X, and ω_{can} the canonical symplectic form on T^*X. The following then holds.*

(i) *There is an open neighborhood $Y \subset T^*X$ of the zero section such that $\underline{\omega}_R$ and $\underline{\omega}_I \in \Omega^2(Y)$ given, respectively, by*

$$\underline{\omega}_I = \int_0^1 (\varphi_t^\xi)^* \omega_{can}\, dt, \quad \text{and}$$
$$\underline{\omega}_R = -\int_0^1 \left(J^T \circ \varphi_t^\xi\right)^* \omega_{can}\, dt$$

are well-defined symplectic forms, and the $(1,1)$-tensor

$$\underline{J} = (\underline{\omega}_R^\flat)^{-1} \circ \underline{\omega}_I^\flat : TY \to TY$$

is an integrable almost complex structure on Y. In particular, Y endowed with $\underline{J}$ defines a complex manifold $\mathscr{Y}$.

(ii) *The 2-form $\underline{\omega} \in \Omega^2(Y) \otimes \mathbb{C}$ defined by*

$$\underline{\omega} := \frac{1}{4}\left(\underline{\omega}_R - i\,\underline{\omega}_I\right)$$

is holomorphic symplectic on $\mathscr{Y}$ and the natural projection $\mathrm{pr}|_Y : \mathscr{Y} \to \mathscr{X}$ is a holomorphic symplectic realization of $(\mathscr{X}, \pi)$.

(iii) *The zero section is a Lagrangian submanifold of $(\mathscr{Y}, \underline{\omega})$.*

Moreover, different choices of the affine connection ∇ give rise to isomorphic holomorphic symplectic realizations.

Note that if the Poisson structure is trivial (i.e. $\pi = 0$), then $(\mathscr{Y}, \underline{\omega})$ reduces to the canonical holomorphic symplectic manifold $T^*\mathscr{X}$. Therefore, the holomorphic symplectic manifold $(\mathscr{Y}, \underline{\omega})$ can be considered as a deformation of the canonical holomorphic symplectic manifold $T^*\mathscr{X}$ parameterized by the holomorphic Poisson structure π. It would be interesting to investigate how our result is related to Kodaira theory of deformation of complex structures [15].

The present paper was influenced in large measure by Petalidou's splendid work [28] on symplectic realizations of non-degenerate Poisson–Nijenhuis manifolds. Making use of the computational approach of [8], Petalidou discovered an explicit expression for the 2-forms on the symplectic realization. However, her proof of their compatibility is, to the best of our understanding, not entirely sound. In our approach, which is more conceptual, tracing the hidden underlying groupoid structures reveals crucial for proving the compatibility.

Finally, we would like to point out that our approach draws from various integration results valid only in the context of smooth manifolds. It is not clear whether this method will be of any use in the context of algebraic varieties. So the analogue of Problem B for algebraic Poisson varieties remains open.

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2. Holomorphic Poisson manifolds and symplectic realizations

In this section, we briefly recall, for the sake of completeness, some standard definitions on holomorphic Poisson structures. As most of those elementary notions closely parallel the real smooth Poisson case, we simply point the reader to the appropriate references for further details.

In what follows, let $\mathscr{X}$ be a complex manifold and X its underlying real manifold. We will denote the structure sheaf of $\mathscr{X}$ by $\mathcal{O}_{\mathscr{X}}$. Recall that a complex structure on $\mathscr{X}$ is equivalent to an integrable almost complex structure J on X, i.e. an endomorphism $J : TX \to TX$ of the underlying real tangent bundle TX with $J^2 = -1$ and with the vanishing Nijenhuis torsion. Furthermore, the holomorphic tangent bundle $T\mathscr{X}$ is isomorphic (as a complex vector bundle) to $T^{1,0}X \subset TX \otimes \mathbb{C}$.

Definition 2.1. By a *holomorphic Poisson structure* on a complex manifold $\mathscr{X}$, we mean that its structure sheaf $\mathcal{O}_{\mathscr{X}}$ is endowed with a bracket

$$\{\cdot,\cdot\}_U : \mathcal{O}_{\mathscr{X}}(U) \times \mathcal{O}_{\mathscr{X}}(U) \to \mathcal{O}_{\mathscr{X}}(U), \quad \forall\, U \subset \mathscr{X}$$

such that $(\mathcal{O}_{\mathscr{X}}, \{\cdot,\cdot\})$ is a sheaf of Poisson algebras.

A *holomorphic Poisson manifold* is a complex manifold $\mathscr{X}$ endowed with a holomorphic Poisson structure. As in the smooth case, Definition 2.1 is equivalent to a holomorphic Poisson bivector field on $\mathscr{X}$.

Proposition 2.1 ([19, 20]). *Let $\mathscr{X}$ be a complex manifold with a holomorphic Poisson structure $\{\cdot,\cdot\}$. There is a unique bivector field $\pi \in \Gamma(\wedge^2 T^{1,0}X)$ satisfying*

$$\bar{\partial}\pi = 0 \quad \text{and} \quad [\pi,\pi] = 0 \tag{1}$$

such that for any open subset $U \subset X$ and any $f, g \in \mathcal{O}_{\mathscr{X}}(U)$,

$$\{f, g\}_U = \langle \pi, \partial f \wedge \partial g \rangle.$$

Conversely, any bivector field $\pi \in \Gamma(\wedge^2 T^{1,0}X)$ satisfying (1) defines a unique holomorphic Poisson structure on $\mathscr{X}$.

In particular, π is called a *holomorphic Poisson bivector field* on $\mathscr{X}$ and $(\mathscr{X}, \pi)$ a holomorphic Poisson manifold. Note that π induces a morphism of holomorphic vector bundles $\pi^{\#} : T^*\mathscr{X} \to T\mathscr{X}$.

The next lemma, which connects holomorphic Poisson structures on $\mathscr{X}$ with Poisson-Nijenhuis structures on X (see Appendix B), will be needed in the proof of a slightly more general version of Theorem A.

Lemma 2.1 ([19]). *Let $\mathscr{X}$ be a complex manifold with almost complex structure J. Assume that $\pi = \pi_R + i\pi_I \in \Gamma(\wedge^2 T^{1,0}X)$, where π_R and $\pi_I \in \Gamma(\wedge^2 TX)$. Then π is a holomorphic Poisson tensor if and only if*

(i) *(π_I, J) defines a Poisson–Nijenhuis structure on X, and*

(ii) *$\pi_R^\sharp = \pi_I^\sharp \circ J^\tau : \ T^*X \to TX$, where $J^\tau : T^*X \to T^*X$ denotes the dual of J.*

A complex manifold $\mathscr{X}$ endowed with a holomorphic Poisson bivector field $\pi \in \Gamma(\wedge^2 T\mathscr{X})$ is called *holomorphic symplectic* if the associated morphism $\pi^\# : T^*\mathscr{X} \to T\mathscr{X}$ is invertible. In that case, we also say that π is *non-degenerate.*

Remark 2.1. If π is a non-degenerate holomorphic Poisson bivector field, for any $k > 0$, $\pi^\#$ extends to an isomorphism

$$\wedge^k \pi^\# : \wedge^k T^*\mathscr{X} \to \wedge^k T\mathscr{X},$$

of holomorphic vector bundles. Then $\omega = (\wedge^2 \pi^\#)^{-1}(\pi)$ is a holomorphic symplectic 2-form.

Assume $(\mathscr{X}, \pi_{\mathscr{X}})$ and $(\mathscr{Y}, \pi_{\mathscr{Y}})$ are two holomorphic Poisson manifolds. A holomorphic map $f : \mathscr{X} \to \mathscr{Y}$ is said to be *Poisson* if the pushforward $f_*(\pi_{\mathscr{X}})$ is well defined and $f_*(\pi_{\mathscr{X}}) = \pi_{\mathscr{Y}}$.

Definition 2.2. Let $\mathscr{X}$ be a holomorphic Poisson manifold. A *holomorphic symplectic realization* of $\mathscr{X}$ is a holomorphic symplectic manifold $\mathscr{Y}$ together with a holomorphic map $q : \mathscr{Y} \to \mathscr{X}$ such that:

1) $q : \mathscr{Y} \to \mathscr{X}$ is a surjective submersion, and

2) q is a Poisson map.

Example 2.1. Let $\mathscr{X}$ be a complex manifold. Let $\pi = 0$ be the zero bivector field on $\mathscr{X}$. Then π is a Poisson bivector field and $(\mathscr{X}, \pi)$ is a holomorphic Poisson manifold. The holomorphic cotangent bundle $T^*\mathscr{X}$, endowed with the canonical symplectic structure and the natural projection map $q : T^*\mathscr{X} \to \mathscr{X}$ gives a holomorphic symplectic realization of $\mathscr{X}$.

Example 2.2. Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra. Its complex dual $\mathfrak{g}^*$ admits a canonical linear holomorphic Poisson structure, called *Lie-Poisson structure*. Let G be a complex Lie group with Lie algebra $\mathfrak{g}$. Then G is a complex manifold, and T^*G, equipped with the canonical holomorphic symplectic structure and the left translation $q : T^*G \to T^*_eG \cong \mathfrak{g}^*$, defines a holomorphic symplectic realization of $\mathfrak{g}^*$.

3. Symplectic local groupoids: The Cattaneo-Felder construction

Let M be a real smooth manifold endowed with a Poisson bivector field $\pi \in \Gamma(\wedge^2 TM)$. In this section, we recall an explicit construction, due to Cattaneo–Felder [4], for the symplectic local groupoid associated with (M, π). The fundamental idea is to construct it as a quotient of the space of all paths of a certain type in the cotangent Lie algebroid of M. Cattaneo–Felder refer to these paths as *solutions of the constraint equation ("Gauss law")* [4, Equation (3.2)][1]. These paths characterize those whose values under the momentum map of an infinite dimensional Hamiltonian action vanish. The construction can be conceptually separated in two parts. The first is valid for an arbitrary Lie algebroid and constructs a local Lie groupoid out of a Lie algebroid (Theorem 3.1). The second explicitly deals with the symplectic structure by inducing a symplectic form on the local groupoid constructed in the first part (Theorem 3.2).

Let $I = [0, 1]$ be the closed unit interval, and n the dimension of M. For a smooth vector bundle $E \to M$ of rank k, consider the space $\widetilde{P}^p(E) = \mathcal{C}^p(I, E)$ of C^p-paths valued in E. It can be endowed with the structure of a smooth Banach manifold [18] by choosing a trivializing $\mathcal{C}^\infty$-atlas $(\varphi_i : E|_{U_i} \to \mathbb{R}^n \times \mathbb{R}^k)_{i\in J}$ for E and defining a family $(\widetilde{\varphi}_i)_{i\in J}$ by

$$\widetilde{\varphi}_i : \mathcal{C}^p(I, E|_{U_i}) \to \mathcal{C}^p(I, \mathbb{R}^n \times \mathbb{R}^k) : f \mapsto \varphi_i \circ f.$$

It is easily checked that the change of charts $\widetilde{\varphi}_i \circ \widetilde{\varphi}_j^{-1}$ are indefinitely Frechet-differentiable with respect to the $\mathcal{C}^p$-norm on $\mathcal{C}^p(I, \mathbb{R}^{n+e})$, and therefore the family $\{\widetilde{\varphi}_i\}$ induces an atlas for paths that fit in a single trivializing local chart for E. It is straightforward to extend it to an atlas for *all* paths and

[1]This construction was subsequently extended to arbitrary Lie algebroids in [6] and the paths became known as A-paths. Note that in [4], Cattaneo–Felder did not use the terminology 'local symplectic groupoids.' However, [4, Theorem 4.7] essentially gave an explicit construction of the *local* symplectic groupoid since [4, Assumtion 4.6] is always satisfied in a neighborhood of the unit space M.

thus $\widetilde{P}^p(E)$ is an infinite dimensional smooth (i.e. a C^∞-) Banach manifold.

Let A be a Lie algebroid over M with projection $p : A \to M$ and anchor $\rho : A \to TM$. In what follows, we will mostly be concerned with the space $\widetilde{P}^1(A)$ of $\mathcal{C}^1$-paths valued in A. We will abbreviate the notation by letting $\widetilde{P}(A) = \widetilde{P}^1(A)$. Recall that an element $a : I \to A$ in $\widetilde{P}(A)$ is called an A-*path* if

$$\rho\big(a(t)\big) = \frac{d\gamma(t)}{dt}, \tag{2}$$

where $\gamma(t) = (p \circ a)(t)$ is the base path. We will denote by $P(A)$ the set of all A-paths. It is easy to see that $P(A)$ is a closed infinite dimensional Banach submanifold of $\widetilde{P}(A)$.

In a way that closely parallels the case of finite dimensional manifolds, one can define [18] the tangent bundle $T\widetilde{P}(A)$ of $\widetilde{P}(A)$ as a certain collection of derivations. However, for what we will need, it is enough to recall that there exists a natural isomorphism

$$\tau : T\widetilde{P}(A) \to \widetilde{P}(TA)$$

of the tangent bundle of $\widetilde{P}(A)$ with $\mathcal{C}^1$-paths valued in TA. Explicitly, for a given $v \in T\widetilde{P}(A)$, choose a path $\theta : I \to \widetilde{P}(A)$ such that $v = \frac{d}{ds}\big|_{s=0} \theta_s$. Then

$$(\tau v)(t) \equiv \frac{d}{ds}\bigg|_{s=0} (\theta_s(t)) \in T_{\theta_0(t)}A. \tag{3}$$

Fibrewise, τ then gives an isomorphism

$$\tau : T_a\widetilde{P}(A) \to \{X \in \widetilde{P}(TA) \mid X(t) \in T_{a(t)}A\}$$

for all $a : I \to A$ in $\widetilde{P}(A)$.

Now let $\Sigma \rightrightarrows M$ be a local Lie groupoid with Lie algebroid A, source and target maps $\alpha, \beta : \Sigma \to M$, and unit map $\varepsilon : M \to \Sigma$. Let $\exp : \Gamma(A) \to \mathrm{Bis}(\Sigma \rightrightarrows M)$ be the usual exponential map, where $\mathrm{Bis}(\Sigma \rightrightarrows M)$ is the set of local bisections of $\Sigma \rightrightarrows M$ [23]. Recall that $\mathrm{Bis}(\Sigma \rightrightarrows M)$ acts on A by the differential of the conjugation map. Let us denote this action by

$$\mathrm{Ad} : \mathrm{Bis}(\Sigma \rightrightarrows M) \to \mathrm{Aut}(A).$$

Set

$$\widehat{\mathrm{ad}}(X)\Big|_{a_0} := \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp(tX)}(a_0)$$

for all $X \in \Gamma(A)$ and all $a_0 \in A$. In particular, we have a map $\widehat{\mathrm{ad}} : \Gamma(A) \to \mathfrak{X}(A)$, where $\mathfrak{X}(A)$ denotes the space of all vector fields on A.

It is well known [2, 4, 6, 12] that $\Sigma \rightrightarrows M$ can be reconstructed as a quotient of $P(A)$ by a certain integrable distribution $\mathfrak{D}(PA) \subset TP(A)$. More explicitly, for any $a \in P(A)$, denote

$$H_a := \left\{ \left[t \mapsto \widehat{\mathrm{ad}}(\xi(t))|_{a(t)} + \frac{d\xi(t)}{dt}|_{\gamma(t)} \right] \in \widetilde{P}(TA) \;\middle|\; \forall \xi : I \to \Gamma(A),\ \xi(0) = \xi(1) = 0 \right\},$$

where $\gamma(t)$ is the base path of $a(t)$, and

$$\left.\frac{d\xi(t)}{dt}\right|_{\gamma(t)} \in A_{\gamma(t)}$$

is naturally identified with a vertical tangent vector in $T_{a(t)}A$. Define, $\forall a \in P(A)$,

$$\mathfrak{D}_a(PA) := \tau^{-1} H_a,$$

where $\tau : T\widetilde{P}(A) \to \widetilde{P}(TA)$ is the isomorphism (3). The most important facts we will need are summarized in the following theorem. For details see [6].

Theorem 3.1. *The following statements hold.*

(i). *$\mathfrak{D}(PA)$ is a finite codimensional integrable distribution on $P(A)$.*

(ii). *Let $\mathcal{F}(A)$ be the foliation integrating $\mathfrak{D}(PA)$. Then there is an open neighborhood $P_{loc}(A) \subset P(A)$ of the natural embedding of M into $P(A)$ as constant paths, where the space of leaves*

$$\bar{P}_{loc}(A) := P_{loc}(A)/(\mathcal{F}(A) \cap P_{loc}(A))$$

is a finite dimensional smooth manifold. By

$$q : P_{loc}(A) \to \bar{P}_{loc}(A) \tag{4}$$

we denote the quotient map.

(iii). *The maps*

$$\begin{aligned} &\alpha : P(A) \to M : a \mapsto a(0), \\ &\beta : P(A) \to M : a \mapsto a(1), \\ &\varepsilon : M \to P(A) : m \mapsto a(t) \equiv \mathbf{0}_m, \end{aligned}$$

descend to smooth maps $\bar{\alpha} : \bar{P}_{loc}(A) \to M, \bar{\beta} : \bar{P}_{loc}(A) \to M$, *and* $\bar{\varepsilon} : M \to \bar{P}_{loc}(A)$ *on the quotient. Furthermore, there is an open neighborhood of the constant diagonal embedding*

$$M \hookrightarrow M \times M \hookrightarrow P_{loc}(A) \times_{\beta,M,\alpha} P_{loc}(A)$$

where the concatenation operation on paths induces a well defined local multiplication

$$\bar{\mu} : \bar{P}_{loc}(A) \times_{\bar{\beta},M,\bar{\alpha}} \bar{P}_{loc}(A) \to \bar{P}_{loc}(A)$$

on $\bar{P}_{loc}(A)$. *Finally, with* $\bar{\mu}$ *as multiplication, and* $\bar{\alpha}, \bar{\beta}$ *and* $\bar{\varepsilon}$ *as, respectively, source, target and unit maps,* $\bar{P}_{loc}(A) \rightrightarrows M$ *has the structure of a local Lie groupoid with Lie algebroid* A.

When A is the cotangent Lie algebroid $(T^*M)_\pi$ of a smooth Poisson manifold (M, π), one obtains the following theorem.

Theorem 3.2 ([4, 7]). *Let* (M, π) *be a Poisson manifold, and let* A *denote its corresponding cotangent Lie algebroid* $(T^*M)_\pi$. *The following statements hold.*

(i). *For all* $a \in \widetilde{P}(A)$, *and all* $u, v \in T_a\widetilde{P}(A)$, *define*

$$\widetilde{\omega}_{can}(u, v) = \int_0^1 \omega_{can}((\tau u)(t), (\tau v)(t))dt. \tag{5}$$

Then $\widetilde{\omega}_{can}$ *is a symplectic form on* $\widetilde{P}(A)$. *Here* $\omega_{can} \in \Omega^2(T^*M)$ *denotes the canonical symplectic form on* T^*M.

(ii). *There exists a symplectic form* $\overline{\omega}$ *on* $\bar{P}_{loc}(A)$, *with which the local groupoid* $\bar{P}_{loc}(A) \rightrightarrows M$ *from Theorem 3.1 (iii) becomes a symplectic* *local groupoid. Moreover, we have*

$$q^*\overline{\omega} = \iota^*\widetilde{\omega}_{can}, \tag{6}$$

where $q : P_{loc}(A) \to \bar{P}_{loc}(A)$ *is the quotient map, and* $\iota : P_{loc}(A) \hookrightarrow \widetilde{P}(A)$ *is the natural inclusion.*

Remark 3.1. Some historical remarks are in order. Part (i) of Theorem 3.2 was proved in [4] — see [4, Equation (3.1), Theorem 3.3 and Section 4.3]. Part (ii) was explicitly proved in [4] in the case of an integrable Poisson manifold — see [4, Theorem 4.7]. However, by restricting to a neighborhood of the unit space, one can adapt the argument to prove the existence of a local symplectic groupoid integrating a given Poisson manifold, since [4, Assumption 4.6] holds automatically. This was done in full details in [7].

Before we close this section, let us record the following proposition, which will be needed later on. Its proof is straightforward and follows immediately from the standard construction of $\bar{P}_{\mathrm{loc}}(A)$.

Proposition 3.1. *Let A and B be Lie algebroids over the same base manifold M, and let $\psi : A \to B$ be a Lie algebroid morphism over the identity map.*

(i) *The induced map on path spaces*

$$\widetilde{P}(\psi) : \widetilde{P}(A) \to \widetilde{P}(B) : [t \mapsto a(t)] \mapsto [t \mapsto \psi(a(t))]$$

preserves A-paths, and descends to a morphism of local Lie groupoids

$$\bar{P}(\psi) : \bar{P}_{loc}(A) \to \bar{P}_{loc}(B)$$

making the diagram

$$\begin{array}{ccc} P_{loc}(A) & \xrightarrow{\widetilde{P}(\psi)} & P_{loc}(B) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ \bar{P}_{loc}(A) & \xrightarrow[\bar{P}(\psi)]{} & \bar{P}_{loc}(B) \end{array} \tag{7}$$

commute. Here q and q' are the respective quotient maps as in (4).

(ii) *The diagram*

$$
\begin{array}{ccc}
TP(A) & \xrightarrow{\widetilde{P}(\psi)_*} & TP(B) \\
{\scriptstyle \tau\circ\iota_*}\downarrow & & \downarrow{\scriptstyle \tau\circ\iota_*} \\
\widetilde{P}(TA) & \xrightarrow[\widetilde{P}(\psi_*)]{} & \widetilde{P}(TB)
\end{array} \tag{8}
$$

commutes. Here, by abuse of notations, ι denotes both embeddings $P(A) \to \widetilde{P}(A)$ and $P(B) \to \widetilde{P}(B)$, and $\widetilde{P}(\psi_*) : \widetilde{P}(TA) \to \widetilde{P}(TB)$ is the map induced, as in part (i), from the tangent map $\psi_* : TA \to TB$.

4. Exponential maps

In Lie theory, the classical exponential map establishes a local diffeomorphism from an open neighborhood of zero in a Lie algebra to the corresponding local Lie group. This construction extends to Lie algebroids and local Lie groupoids. Unlike the Lie algebra case, however, one needs to choose some geometrical structure, namely an A-connection on A. In this section, we recall some basic facts about the exponential map for Lie groupoids, and describe the latter explicitly in the case of the local Lie groupoid of Theorem 3.1 (iii).

Let A be, as before, a Lie algebroid over M. By an A-connection on A we mean an $\mathbb{R}$-bilinear map

$$\nabla : \Gamma(A) \times \Gamma(A) \to \Gamma(A) : (X, Y) \mapsto \nabla_X Y$$

satisfying the conditions

$$
\begin{aligned}
&\nabla_{fX} Y = f \nabla_X Y, \quad \text{and} \\
&\nabla_X (fY) = (\rho(X) f) Y + f \nabla_X Y,
\end{aligned}
$$

for all $X, Y \in \Gamma(A)$ and $f \in \mathcal{C}^\infty(M)$.

Example 4.1. Any linear connection $\widetilde{\nabla}$ on the vector bundle A induces an associated A-connection on A by the formula $\nabla_X Y = \widetilde{\nabla}_{\rho(X)} Y$. However, not every A-connection on A is of this form.

Definition 4.1. An *A-geodesic* (or a *geodesic A-path*) is an A-path $a : I \to A$ satisfying the geodesic equation:

$$\nabla_{a(t)} a(t) = 0$$

for any $t \in I$.

An A-connection on A also defines a map $h : A \times_M A \to TA$, called a *horizontal lifting* [21]:

$$h(a,b) = \bar{b}_*(\rho(a)) - \tau_b\big((\nabla_a \bar{b})|_x\big) \in T_b A,$$

for any $x \in M$ and $a, b \in A_x$. Here $\bar{b} \in \Gamma(A)$ is any section satisfying $\bar{b}(x) = b$, and τ_b denotes the canonical linear isomorphism between the fiber A_x and the vertical tangent space of A at the point b. It is not hard to check that $h(a,b)$ does not depend on the choice of the extension $\bar{b}$.

Definition 4.2. The *geodesic vector field of* ∇ is the vector field $\xi \in \mathfrak{X}(A)$ defined by

$$\xi_a = h(a,a)$$

for any $a \in A$.

In what follows, for a given A-connection ∇ on A, we will denote by φ_t^∇ the flow of its geodesic vector field.

Proposition 4.1. *Let A be a Lie algebroid, and ∇ an A-connection on A. The following holds.*

(i). *There is a neighborhood $U \subset A$ of the zero section such that φ_t^∇ is defined for all $t \in I$ and,*

(ii). *for all $a_0 \in U$, the path $[t \in I \mapsto a(t) = \varphi_t^\nabla(a_0)]$ is A-geodesic.*

Proof. (i) Denote by $m_s : A \to A$ the fibrewise scalar multiplication by $s \in \mathbb{R}$. Let $\xi \in \mathfrak{X}(A)$ be the geodesic vector field of ∇. It is easily checked that $s\xi_a = (m_s)_*^{-1} \xi_{sa}$ for all $s > 0$ and all $a \in A$. It then follows that

$$s\varphi_{ts}^\nabla(a) = \varphi_t^\nabla(sa),$$

where one side is defined exactly when the other is. Rescaling locally, this yields the claim.

(ii) Fix any $a_0 \in U \subset A$, and let $a(t) = \varphi_t^{\nabla}(a_0)$. Denote by $\gamma(t) = p(a(t))$ the underlying base path. We have

$$p_*(\dot{a}(t)) = p_*(\xi(a(t))) = p_*(h(a(t), a(t))) = \rho(a(t)).$$

Hence $a(t)$ is indeed an A-path. Choose any time-dependent section $\bar{a} : I \times M \to A$ such that $\bar{a}(t, p(a(t))) = a(t)$. Then

$$\begin{aligned}\nabla_{a(t)} a(t) &= \frac{\partial}{\partial t}\bar{a}(t, \gamma(t)) + \nabla_{a(t)}\bar{a}(t, \gamma(t)),\\ &= \left[\dot{a}(t) - \bar{a}^t_*(\dot{\gamma}(t))\right] + \nabla_{a(t)}\bar{a}(t, \gamma(t))\\ &= \dot{a}(t) - \xi(a(t))\\ &= 0.\end{aligned}$$

Thus the conclusion follows. □

Example 4.2. Let (M, π) be a Poisson manifold, and $(T^*M)_\pi$ its cotangent Lie algebroid. Choose an affine connection $\nabla^{TM} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ on M. Let $\nabla^{T^*M} : \mathfrak{X}(M) \times \Gamma(T^*M) \to \Gamma(T^*M)$ be the corresponding linear connection on T^*M–the dual bundle of TM. Introduce a Lie algebroid $(T^*M)_\pi$-connection $\nabla : \Gamma(T^*M) \times \Gamma(T^*M) \to \Gamma(T^*M)$ on $(T^*M)_\pi$ by

$$\nabla_\lambda \nu = \nabla^{T^*M}_{\pi^\sharp(\lambda)}\, \nu, \qquad \forall \lambda, \nu \in \Gamma(T^*M).$$

In local coordinates $\{q^i\}$ on M, assume that

$$\nabla^{TM}_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial q^k} \quad \text{and} \quad \pi = \sum_{ij} \pi^{ij} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial q^j}.$$

Then the corresponding geodesic vector field $\xi \in \mathfrak{X}(T^*M)$ has the local expression:

$$\xi = \sum_{ij} p_i \pi^{ij} \frac{\partial}{\partial q^j} + \sum_{ijkl} p_k p_l \pi^{ki} \Gamma^l_{ij} \frac{\partial}{\partial p_j},$$

where $\{q^i, p_i\}$ are the induced local coordinates on T^*M. We call ξ the *Poisson geodesic vector field* of ∇^{TM} (it was called Poisson spray in [8]).

Let ∇ be an A-connection on A, and $\Sigma \rightrightarrows M$ a local Lie groupoid integrating A with source and target maps α and β, respectively. For any

$x \in M$, there is an affine connection on the source fiber $\Sigma_x = \alpha^{-1}(x)$, which we will denote by $\bar{\nabla}^x$. It is defined [27] uniquely by

$$\bar{\nabla}^x{}_{X^L|_{\alpha^{-1}(x)}} (Y^L|_{\alpha^{-1}(x)}) = (\nabla_X Y)^L|_{\alpha^{-1}(x)}, \tag{9}$$

for any $X, Y \in \Gamma(A)$. Here X^L denotes the left-invariant vector field on Σ associated to X.

Definition 4.3 ([27]). Let ∇ be an A-connection on A, and $\Sigma \rightrightarrows M$ a local Lie groupoid with Lie algebroid A. The *groupoid exponential map* defined by ∇ is the map $\exp^\nabla : A \to \Sigma$, defined on a neighborhood of the zero section in A, and which, on each fiber A_x, is given by the ordinary exponential map of the affine connection $\bar{\nabla}^x$ on Σ_x.

When no risk of ambiguity exists, we shall simply denote "$\exp^\nabla$" by "exp", hiding the dependency on the A-connection ∇.

It can be proved that exp is smooth [27]. Also note that, by definition, $\alpha \circ \exp = p$, where $p : A \to M$ is the base point projection. In particular, for any $a_0 \in A$, the exponential path $t \mapsto \exp(ta_0)$ is a source-path in Σ.

Letting $U \subset A$ as in Proposition 4.1 (i), we define

$$\Phi : U \to P(A) : a_0 \in U \mapsto [t \in I \mapsto \varphi_t^\nabla(a_0)], \tag{10}$$

i.e. $\Phi(a_0)$ is the A-geodesic stemming from a_0. One should think of Φ as a kind of exponential map at the level of A-paths [6]. Formally, the relation between Φ and the groupoid exponential map of Definition 4.3 is summarized in Proposition 4.2. Its proof is a consequence of the following simple lemma, which relates, for a given Lie algebroid element $a_0 \in A$, the groupoid exponential path $\exp(ta_0)$ to the A-geodesic $\varphi_t^\nabla(a_0)$ stemming from a_0.

Lemma 4.1. *Let $U \subset A$ be as in Proposition 4.1 (i) and fix any $a_0 \in U$. Let $a = \Phi(a_0)$ be the associated geodesic A-path, i.e. $a(t) = \varphi_t^\nabla(a_0)$. Also let $r(t) = \exp(ta_0)$ be the associated exponential path in Σ. Then $r(t)$ is a source-path that satisfies the conditions:*

$$\begin{cases} [L_{r^{-1}(t)}]_* \dot{r}(t) = a(t) \quad \forall t \in I, \textit{and} \\ r(0) = \varepsilon(p(a_0)),\ \dot{r}(0) = a_0. \end{cases}$$

Here, for any $g \in \Sigma$, the map $L_g : \Sigma_{\beta(g)} \to \Sigma_{\alpha(g)}$ denotes the left multiplication by g.

Proof. From Eq. (9), it follows that

$$0 = \bar{\nabla}^x{}_{\dot{r}(t)}\, \dot{r}(t) = [L_{r(t)}]_* \left(\nabla_{[L_{r^{-1}(t)}]_* \dot{r}(t)}\, [L_{r^{-1}(t)}]_* \dot{r}(t) \right).$$

Hence the A-path $[L_{r^{-1}(t)}]_* \dot{r}(t)$ is A-geodesic. Since we also have

$$[L_{r^{-1}(0)}]_* \dot{r}(0) = \varepsilon(p(a_0)) \cdot r(0) = a_0,$$

the result follows from the unicity of geodesics. □

The proof of the following proposition is a straightforward consequence of the construction of the local groupoid $\bar{P}_{\mathrm{loc}}(A) \rightrightarrows M$ of Theorem 3.1 combined with Proposition 4.1 and Lemma 4.1 (see [6, 12]).

Proposition 4.2. *Let ∇ be an A-connection on A and $U \subset A$ be as in Proposition 4.1. Then, up to choosing a sufficiently small open subset $P_{loc}(A) \subset P(A)$ as in Theorem 3.1 (ii), the restriction of the groupoid exponential map $\exp|_U : U \to \bar{P}_{loc}(A)$ is a diffeomorphism onto its image. Moreover, the diagram*

$$\begin{array}{ccc} U & \xrightarrow{\Phi} & P_{loc}(A) \\ & \searrow^{\exp} & \downarrow q \\ & & \bar{P}_{loc}(A) \end{array} \tag{11}$$

commutes. Here $q : P_{loc}(A) \to \bar{P}_{loc}(A)$ is the quotient map as in (4).

The following simple technical lemma will be useful in our subsequent discussions.

Lemma 4.2. *Let ∇ be an A-connection on A, $U \subset A$ and $P_{loc}(A) \subset P(A)$ as in Proposition 4.2. Let $\psi : A \to A$ be a morphism of Lie algebroids over the identity map, and $\Phi : U \to P_{loc}(A)$ be the map as in* (10).

(i) *For any $a \in U$ and $v \in T_a A$, we have*

$$\mathrm{ev}_t(\tau(\iota_*(\Phi_*(v)))) = (\varphi_t^{\nabla})_*(v), \quad \forall t \in I. \tag{12}$$

Here ev_t denotes the evaluation map of a path at time t. Also recall that ι denotes the embedding $P(A) \hookrightarrow \widetilde{P}(A)$.

(ii) *With the notation of Proposition 3.1, we have, for any $a \in U$ and any $v \in T_a A$:*

$$\mathrm{ev}_t(\tau(\iota_*(\widetilde{P}(\psi)_*(\Phi_*(v))))) = (\psi_* \circ (\varphi_t^\nabla)_*)(v), \quad \forall t \in I. \tag{13}$$

Proof.

(i) This follows immediately from Eq. (3).

(ii) By commutative diagram (8), we have $\tau \circ \iota_* \circ \widetilde{P}(\psi)_* = \widetilde{P}(\psi_*) \circ \tau \circ \iota_*$. Hence

$$\mathrm{ev}_t(\tau(\iota_*(\widetilde{P}(\psi)_*(\Phi_*(v))))) = \mathrm{ev}_t(\widetilde{P}(\psi_*)(\tau(\iota_*(\Phi_*(v))))) = (\psi_* \circ (\varphi_t^\nabla)_*)(v),$$

as claimed. □

5. Symplectic realizations of Poisson manifolds

Let (M, π) be a Poisson manifold, and A its cotangent Lie algebroid $(T^*M)_\pi$. Consider the symplectic local groupoid $(\bar{P}_{\mathrm{loc}}(A) \rightrightarrows M, \overline{\omega})$ as in Theorem 3.2 (ii).

Now, fix ∇ an A-connection on A, and let $U \subset A$ be a sufficiently small open neighborhood of the zero section as in Proposition 4.2. Set

$$\underline{\omega} := \exp^* \overline{\omega} \tag{14}$$

to be the pullback of $\overline{\omega}$ by the groupoid exponential map. Then $\underline{\omega}$ is a symplectic form on U.

Proposition 5.1. *The symplectic form $\underline{\omega}$ can be explicitly expressed as follows:*

$$\underline{\omega} = \int_0^1 (\varphi_t^\nabla)^* \omega_{can}\, dt, \tag{15}$$

where $\omega_{can} \in \Omega^2(T^*M)$ is the canonical symplectic form on T^*M, and φ_t^∇ is the flow of the geodesic vector field $\xi \in \mathfrak{X}(A)$ corresponding to ∇.

Proof. According to the commuting diagram (11), we have $\exp^* = \Phi^* \circ q^*$, and by Eq. (6), we have $q^*\bar{\omega} = \iota^* \widetilde{\omega}_{\mathrm{can}}$. Thus

$$\underline{\omega} = \Phi^* \iota^* \, \widetilde{\omega}_{\mathrm{can}}.$$

On the other hand, $\forall a \in U$ and $\forall u, v \in T_a A$, we have

$$\begin{aligned}\big(\Phi^* \iota^* \widetilde{\omega}_{\text{can}}\big)(u,v) &= \int_0^1 \omega_{\text{can}}([\tau(\iota_*(\Phi_*(u)))](t), [\tau(\iota_*(\Phi_*(v)))](t))\, dt \\ &= \int_0^1 \omega_{\text{can}}((\varphi_t^\nabla)_*(u), (\varphi_t^\nabla)_*(v))\, dt \\ &= \int_0^1 \big((\varphi_t^\nabla)^* \omega_{\text{can}}\big)(u,v)\, dt,\end{aligned}$$

where we used Eq. (5) for the first equality, and Eq. (12) for the second equality. The conclusion thus follows. □

As an immediate consequence of Eq. (15), we recover the following theorem, part (i) of which was proved by Crainic-Mărcuţ by a direct computation [8]. See also [3] for related results.

Theorem 5.1. *Let (M, π) be a Poisson manifold and $A = (T^*M)_\pi$ its cotangent Lie algebroid. Fix ∇ an A-connection on A and let $\xi \in \mathfrak{X}(A)$ be the associated geodesic vector field. Also let $U \subset A$ be, as in Proposition 4.2, a sufficiently small open neighborhood of the zero section in A so that, in particular, the flow $\varphi_t^\nabla(a_0)$ of ξ is defined for all $t \in I$, and all $a_0 \in U$. Then,*

(i) *the projection $\mathrm{pr}|_U : U \subset T^*M \to M$ together with the symplectic form $\underline{\omega} \in \Omega^2(U)$, as defined by Eq. (15), is a symplectic realization of (M, π); and*

(ii) *the zero section of T^*M is a Lagrangian submanifold of U.*

The geodesic vector field $\xi \in \mathfrak{X}(A)$ is called a *Poisson spray* in [8, 28].

6. Symplectic-Nijenhuis local groupoids

There is a one-to-one correspondence between Poisson manifolds and symplectic local groupoids. This is in fact a special case of the Mackenzie-Xu correspondence (Theorem A.1) recalled in the appendix below. Such a correspondence can also be extended to a one-to-one correspondence between Poisson-*Nijenhuis* manifolds and symplectic-*Nijenhuis* local groupoids. This result is due to Stiénon–Xu [31], which we recall in Theorem 6.1. In this section, we briefly go over the main idea of its proof.

Let $\Sigma \rightrightarrows M$ be a local Lie groupoid with source and target maps $\alpha : \Sigma \to M$ and $\beta : \Sigma \to M$, respectively, and with unit map $\varepsilon : M \hookrightarrow \Sigma$. Recall that a $(1,1)$-tensor $\bar{N} : T\Sigma \to T\Sigma$ on Σ is said to be *multiplicative* if it defines a morphism of local Lie groupoids

$$\begin{array}{ccc} T\Sigma & \xrightarrow{\quad \bar{N} \quad} & T\Sigma \\ \alpha_* \Big\downarrow \Big\downarrow \beta_* & & \alpha_* \Big\downarrow \Big\downarrow \beta_* \\ TM & \xrightarrow[\bar{N}|_{\varepsilon_*(TM)}]{} & TM \end{array} \tag{16}$$

Here $T\Sigma \rightrightarrows TM$ is the tangent local groupoid. Note that it is implicitly assumed, as part of the condition, that $\bar{N}(\varepsilon_*(TM)) \subset \varepsilon_*(TM)$.

Definition 6.1. A *symplectic-Nijenhuis local groupoid* is a symplectic local groupoid $(\Sigma \rightrightarrows M, \overline{\omega})$ equipped with a multiplicative $(1,1)$-tensor $\bar{N} : T\Sigma \to T\Sigma$ such that the triple $(\Sigma, \overline{\omega}, \bar{N})$ is a symplectic-Nijenhuis manifold.

Remark 6.1. Any symplectic-Nijenhuis local groupoid defines two Poisson local groupoid structures on the same underlying local groupoid $\Sigma \rightrightarrows M$. Indeed, let $(\Sigma \rightrightarrows M, \overline{\omega}, \bar{N})$ be a symplectic-Nijenhuis local groupoid and denote by $\bar{\pi} \in \Gamma(\wedge^2 T\Sigma)$ the Poisson bivector field given by inverting $\overline{\omega}$. Then the pair $(\Sigma \rightrightarrows M, \bar{\pi})$ is a Poisson local groupoid. Moreover, from Proposition B.1, it follows that the bivector field $\bar{\pi}_{\bar{N}}$ defined by

$$\bar{\pi}^{\sharp}_{\bar{N}} = \bar{N} \circ \bar{\pi}^{\sharp} \tag{17}$$

is another multiplicative Poisson structure on Σ, and thus in particular gives another Poisson local groupoid $(\Sigma \rightrightarrows M, \bar{\pi}_{\bar{N}})$. Note that, in general, the Nijenhuis tensor $\bar{N} : T\Sigma \to T\Sigma$ may not be invertible, and therefore the Poisson bivector field $\bar{\pi}_{\bar{N}}$ is not necessarily non-degenerate. In particular, we do not automatically have two symplectic groupoid structures on $\Sigma \rightrightarrows M$.

The following theorem is due to Stiénon–Xu [31].

Theorem 6.1.

(i) *The unit space of a symplectic-Nijenhuis local groupoid inherits an induced Poisson–Nijenhuis manifold structure.*

(ii) *Given a Poisson–Nijenhuis manifold (M, π, N), there is a unique, up to isomorphisms, symplectic–Nijenhuis local groupoid whose induced Poisson–Nijenhuis structure on the unit space is (M, π, N).*

In other words, there is a one-to-one correspondence between Poisson–Nijenhuis manifolds and symplectic-Nijenhuis local groupoids.

We will sketch a proof of this theorem since we will need some intermediate results for our argument later on (see Proposition 6.1), which seem to not have appeared in the literature.

Proof of Theorem 6.1. To prove (i), let $(\Sigma \rightrightarrows M, \overline{\omega}, \bar{N})$ be a symplectic-Nijenhuis local groupoid. Let $\bar{\pi}$ be the Poisson bivector field on Σ which is the inverse of $\overline{\omega}$. The pair $(\Sigma \rightrightarrows M, \overline{\omega})$ is a symplectic local groupoid. It is standard [5, 35] that the pushforward

$$\pi := \alpha_* \bar{\pi} \tag{18}$$

is a well defined Poisson bivector field on M, and that the Lie algebroid of $\Sigma \rightrightarrows M$ is isomorphic to the cotangent Lie algebroid $(T^*M)_\pi$ of (M, π).

Now, let $\bar{\pi}_{\bar{N}} \in \Gamma(\wedge^2 T\Sigma)$ be the bivector field on Σ as defined by Eq. (17). Then the pair $(\Sigma \rightrightarrows M, \bar{\pi}_{\bar{N}})$ is a Poisson local groupoid. Analogous to Eq. (18), set

$$\pi' := \alpha_* \bar{\pi}_{\bar{N}}. \tag{19}$$

Then π' is a well defined Poisson bivector field on M as well [36]. Finally, from Proposition B.1, it follows that the Schouten bracket $[\bar{\pi}, \bar{\pi}_{\bar{N}}]$ vanishes, and thus we have $[\pi, \pi'] = 0$.

On the other hand, the Lie groupoid morphism $\bar{N} : T\Sigma \to T\Sigma$ as in (16) induces a map $N = \bar{N}|_{\varepsilon_*(TM)} : TM \to TM$ on its unit space, which is, clearly, a $(1,1)$-tensor. The Nijenhuis torsion free condition for N then follows from that of $\bar{N}$. Moreover, it is clear that

$$\pi'^\sharp = N \circ \pi^\sharp. \tag{20}$$

Thus (M, π, N) is indeed a Poisson–Nijenhuis manifold, as desired.

Conversely, to see (ii), let (M, π, N) be a Poisson–Nijenhuis manifold. Let $A = (T^*M)_\pi$ be the cotangent Lie algebroid of (M, π), and $\bar{P}_{\mathrm{loc}}(A) \rightrightarrows M$ be the corresponding local Lie groupoid as in Theorem 3.1. Let $\overline{\omega}$ be the multiplicative symplectic form on $\bar{P}_{\mathrm{loc}}(A)$ as in Theorem 3.2 (ii), and $\bar{\pi}$ be its associated Poisson bivector field. Then, under the correspondence between Lie bialgebroids and Poisson local groupoids spelled out in Theorem A.1, the Poisson groupoid $(\bar{P}_{\mathrm{loc}}(A) \rightrightarrows M, \bar{\pi})$ is associated to the Lie bialgebroid $((T^*M)_\pi, TM)$.

On the other hand, out of the same Poisson–Nijenhuis structure (M,π,N), we can construct another natural Lie bialgebroid $((T^*M)_\pi,(TM)_N)$ [16]. Here, the Lie algebroid $(T^*M)_\pi$ is, as usual, the cotangent Lie algebroid of (M,π). The Lie algebroid $(TM)_N$ consists of the triple $(TM,\rho_N,[\cdot,\cdot]_N)$ defined as follows. The underlying vector bundle is the tangent bundle TM of M, while the anchor ρ_N and bracket $[\cdot,\cdot]_N$ are given, respectively, by

$$\begin{aligned}&\rho_N(X)=NX, \quad \text{and}\\&[X,Y]_N=[NX,Y]+[X,NY]-N[X,Y], \quad \forall X,Y\in\Gamma(TM).\end{aligned}$$

One should think of $(TM)_N$ as a twisted version of the tangent Lie algebroid TM, whose twist is given by the Nijenhuis tensor N.

To this Lie bialgebroid $((T^*M)_\pi,(TM)_N)$, we can apply Theorem A.1 to obtain a second natural multiplicative Poisson bivector field $\bar{\pi}'$ on $\bar{P}_{\mathrm{loc}}(A)$, which makes the pair $(\bar{P}_{\mathrm{loc}}(A)\rightrightarrows M,\bar{\pi}')$ into a Poisson local groupoid.

To complete the proof of part *(ii)*, it remains to prove that the two multiplicative Poisson structures $\bar{\pi}$ and $\bar{\pi}'$ on $\bar{P}_{\mathrm{loc}}(A)$ satisfy the condition:

$$[\bar{\pi},\bar{\pi}']=0. \tag{21}$$

Indeed, assuming Eq. (21) holds, let

$$\bar{N}=(\bar{\pi}')^\sharp\circ\overline{\omega}^\flat:T\bar{P}_{\mathrm{loc}}(A)\to T\bar{P}_{\mathrm{loc}}(A). \tag{22}$$

From Proposition B.2, it follows that $\bar{N}$ is indeed a Nijenhuis tensor, and therefore $(\bar{P}_{\mathrm{loc}}(A),\bar{\pi},\bar{N})$ is a Poisson-Nijenhuis manifold. Moreover $\bar{N}$ is a multiplicative $(1,1)$-tensor. Thus it follows that $(\bar{P}_{\mathrm{loc}}(A)\rightrightarrows M,\overline{\omega},\bar{N})$ is a symplectic-Nijenhuis local groupoid.

In order to prove Eq. (21), let $\delta:\Gamma(\wedge^\bullet T^*M)\to\Gamma(\wedge^{\bullet+1}T^*M)$ be the Chevalley–Eilenberg differential of the Lie algebroid $(TM)_N$ (see Eq. (A.1)). Then one can easily check that

$$[\delta,d_{\mathrm{DR}}]=0, \tag{23}$$

where d_{DR} is the De Rham differential operator on $\Gamma(\wedge^\bullet T^*M)=\Omega^\bullet(M)$ [31, Lemma 5.3]. It is well known that when $A=TM$ is the tangent Lie algebroid of a manifold M, its Chevalley–Eilenberg differential is the De Rham differential operator d_{DR}. According to the Universal Lifting Theorem [12], we see that Eq. (23) implies Eq. (21).

Finally, it is simple to check that the two constructions we spelled out in showing parts (*i*) and (*ii*) are indeed inverse to each other. This concludes the proof of Theorem 6.1. □

Let us single out the following important fact that we will need later on.

Proposition 6.1. *Let* $(\Sigma \rightrightarrows M, \overline{\omega}, \bar{N})$ *be a symplectic-Nijenhuis local groupoid with the induced Poisson–Nijenhuis structure* (M, π, N) *on its unit space as in Theorem 6.1. Then the source map* $\alpha : \Sigma \to M$ *is a Poisson–Nijenhuis map. In particular, we have*

$$\alpha_* \bar{\pi} = \pi, \ \alpha_* \bar{\pi}_{\bar{N}} = \pi_N, \ \alpha_* \circ \bar{N} = N \circ \alpha_*,$$

where $\bar{\pi}$ *denotes the bivector field on* Σ *inverse to* $\overline{\omega}$*.*

Proof. The first identity is exactly Eq. (18). The second identity follows from Eqs. (19)-(20). Finally, the last identity is a consequence of the fact that $\bar{N} : T\Sigma \to T\Sigma$ is a groupoid morphism and therefore commutes with the source map $\alpha_* : T\Sigma \to TM$ —see (16). □

7. The complete lift to the cotangent bundle

We start by recalling the definition of the complete lift of $(1,1)$-tensors to the cotangent bundle and some related standard facts. For details, we refer the readers to [9] and references there in.

Let $N : TM \to TM$ be a $(1,1)$-tensor on a manifold M. Denote by

$$\langle \cdot, \cdot \rangle : T^*M \times_M TM \to M \times \mathbb{R}$$

the canonical pairing. There is a natural 1-form $\theta_N \in \Omega^1(T^*M)$ defined by

$$\theta_N(u) = \langle \lambda, N(p_*(u)) \rangle$$

$\forall u \in T_\lambda(T^*M)$, where $p : T^*M \to M$ is the canonical projection. In particular, if $N = \mathrm{Id}$, then θ_N is just the Liouville form on T^*M.

Definition 7.1. The *complete lift of* N *to the cotangent bundle* is the $(1,1)$-tensor

$$N^c : TT^*M \to TT^*M$$

on T^*M defined by the property that

$$\omega_{\mathrm{can}}(N^c u, v) = (d\theta_N)(u, v), \tag{24}$$

for any $\lambda \in T^*M$ and any $u, v \in T_\lambda(T^*M)$. Here $\omega_{\text{can}} \in \Omega^2(T^*M)$ is the canonical symplectic form on T^*M.

It can be checked, by a direct computation, that

$$\omega_{\text{can}}(N^c u, v) = \omega_{\text{can}}((N^T)_* u, (N^T)_* v), \tag{25}$$

$\forall \lambda \in T^*M,\ u, v \in T_\lambda(T^*M)$. Here $N^T : T^*M \to T^*M$ is the dual of N, and $(N^T)_* : TT^*M \to TT^*M$ denotes its tangent map.

Lemma 7.1. *Let $N : TM \to TM$ be a Nijenhuis tensor on M. Denote by $\pi' \in \mathfrak{X}^2(T^*M)$ the Poisson bivector field on T^*M of the Lie–Poisson structure corresponding to the Lie algebroid $(TM)_N$. Then*

$$(\pi')^\sharp \circ \omega^\flat_{can} = N^c. \tag{26}$$

Proof. For any $X \in \mathfrak{X}(M)$, let $\ell_X \in C^\infty(T^*M)$ be the fibrewise linear function on T^*M defined by

$$\ell_X(\lambda) = \langle \lambda, X_x \rangle,$$

for any $\lambda \in T^*_x M$ $(x \in M)$. By definition, for any $X, Y \in \mathfrak{X}(M)$ and any $f, g \in \mathcal{C}^\infty(M)$, we have:

$$\{\ell_X, \ell_Y\}_{\pi'} = \ell_{[N(X),Y]+[X,N(Y)]-N([X,Y])}, \tag{27}$$

$$\{\ell_X, p^* f\}_{\pi'} = p^* \langle df, N(X) \rangle, \tag{28}$$

$$\{p^* f, p^* g\}_{\pi'} = 0. \tag{29}$$

For any given $\psi \in \mathcal{C}^\infty(T^*M)$, we denote by $\mathcal{H}(\psi) \in \mathfrak{X}(T^*M)$ the Hamiltonian vector field of ψ with respect to the canonical symplectic structure on T^*M, i.e. $\mathcal{H}(\psi) = \pi^\sharp_{\text{can}}(d\psi)$, where $\pi^\sharp_{\text{can}} = (\omega^\flat_{\text{can}})^{-1}$. Note that Eq. (26) is equivalent to

$$N^c \circ \pi^\sharp_{\text{can}} = (\pi')^\sharp. \tag{30}$$

The latter is equivalent to

$$N^c(\mathcal{H}(F))(G) = (\pi')^\sharp(dF)(G) \tag{31}$$

for any $F, G \in C^\infty(T^*M)$.

From Eq. (24), it follows that $\omega^\flat_{\text{can}} \circ N^c$ is skew-symmetric. Since $\pi^\sharp_{\text{can}} = (\omega^\flat_{\text{can}})^{-1}$, a simple linear algebra argument implies that $N^c \circ \pi^\sharp_{\text{can}} =$

$N^c \circ (\omega^\flat_{\text{can}})^{-1}$ is also skew-symmetric. Note that $C^\infty(T^*M)$ is spanned locally by two types of functions of the form ℓ_X and p^*f, $\forall X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. In order to prove Eq. (31), it thus suffices to prove the following identities:

$$N^c(\mathcal{H}(p^*f))(p^*g) = (\pi')^\sharp(d(p^*f))(p^*g), \quad \forall f, g \in C^\infty(M) \tag{32}$$

$$N^c(\mathcal{H}(\ell_X))(\ell_Y) = (\pi')^\sharp(d\ell_X)(\ell_Y) \quad \forall X, Y \in \mathfrak{X}(M), \tag{33}$$

$$N^c(\mathcal{H}(\ell_X))(p^*f) = (\pi')^\sharp(d\ell_X)(p^*f), \quad \forall f \in C^\infty(M),\ X \in \mathfrak{X}(M). \tag{34}$$

Since $p_*\big(\mathcal{H}(p^*f)\big) = 0$, it follows from the definition of the complete lift N^c that $p_*\big(N^c(\mathcal{H}(p^*f)\big) = 0$. Hence both sides of Eq. (32) vanish.

Now we prove Eq. (33). Note that the following relation is standard [9, Proposition 5.4.3]:

$$N^c(\mathcal{H}(\ell_X)) - \mathcal{H}(\ell_{N(X)}) = \pi^\sharp_{\text{can}}(\theta_{L_X N}), \tag{35}$$

where $L_X N$ denotes the usual Lie derivative of the $(1,1)$-tensor N given by $(L_X N)(Y) = [X, N(Y)] - N([X,Y])$, $\forall X, Y \in \mathfrak{X}(M)$. Also, the following identity can be verified directly:

$$[\pi^\sharp_{\text{can}}(\theta_{L_X N})](\ell_Y) = \ell_{(L_X N)(Y)}. \tag{36}$$

From Eq. (35)-Eq. (36), it thus follows that

$$\begin{aligned} [N^c(\mathcal{H}(\ell_X))](\ell_Y) &= \mathcal{H}(\ell_{N(X)})(\ell_Y) + (\pi^\sharp_{\text{can}}(\theta_{L_X N}))(\ell_Y) \\ &= \ell_{[N(X),Y]} + \ell_{(L_X N)(Y)} \\ &= \ell_{[N(X),Y]+[X,N(Y)]-N([X,Y])}. \end{aligned} \tag{37}$$

Now Eq. (33) follows from combining Eq. (37) with Eq. (27).

Finally, we prove Eq. (34). First, we have $[\pi^\sharp_{\text{can}}(\theta_{L_X N})](p^*f) = 0$. Now, applying Eq. (35) to the function p^*f, we have

$$\begin{aligned} [N^c(\mathcal{H}(\ell_X))](p^*f) &= \mathcal{H}(\ell_{N(X)})(p^*f) + (\pi^\sharp_{\text{can}}(\theta_{L_X N}))(p^*f) \\ &= p^*\big((NX)(f)\big) \\ &= (\pi')^\sharp(d\ell_X)(p^*f). \end{aligned} \tag{38}$$

This concludes the proof of the lemma. □

Remark 7.1. According to a theorem of Vaisman [32], the Poisson bivector field π' is compatible with the canonical Poisson structure π_{can} on T^*M

in the sense that the Schouten bracket $[\pi', \pi_{\text{can}}]$ vanishes. Therefore, from Lemma 7.1, it follows that $(T^*M, \pi_{\text{can}}, N^c)$ is a Poisson–Nijenhuis structure on T^*M, and moreover its *second* Poisson tensor π_{N^c} defined, as usual, by

$$\pi^\sharp_{N^c} = N^c \circ \pi^\sharp_{\text{can}} \tag{39}$$

coincides with the Lie–Poisson structure of the Lie algebroid $(TM)_N$.

We now recall the following well-known fact from the general theory of Poisson groupoids [24, 25]. For any Poisson local groupoid $(\Sigma \rightrightarrows M, \bar{\pi})$ with Lie bialgebroid (A, A^*), the following diagram of vector bundle morphisms:

$$\begin{array}{ccc} \mathrm{Lie}(T^*\Sigma) & \xrightarrow{\mathrm{Lie}(\bar{\pi}^\sharp)} & \mathrm{Lie}(T\Sigma) \\ {\scriptstyle j'_\Sigma}\downarrow & & \downarrow{\scriptstyle j_\Sigma} \\ T^*A & \xrightarrow[\pi^\sharp_A]{} & TA \end{array} \tag{40}$$

commutes. Here π_A denotes the Lie–Poisson structure on A induced by the Lie algebroid structure on A^*, and j'_Σ, j_Σ are natural vector bundle isomorphisms which we shall not make explicit for the sake of brevity. See [24, 25] for more details.

Let $\bar{N} : T\Sigma \to T\Sigma$ be a multiplicative $(1,1)$-tensor on Σ. There is an associated tensor $TA \to TA$ on A, which we will denote by $\mathbb{L}\mathrm{ie}(\bar{N})$ and call the *infinitesimal of* $\bar{N}$ following [20]. The relationship between the infinitesimal of $\bar{N}$ and the image of N under the usual Lie functor is given by the simple identity

$$\mathbb{L}\mathrm{ie}(\bar{N}) = j_\Sigma \circ \mathrm{Lie}(\bar{N}) \circ j_\Sigma^{-1}. \tag{41}$$

Let $(\Sigma \rightrightarrows M, \overline{\omega}, \bar{N})$ be a symplectic-Nijenhuis local groupoid with induced Poisson–Nijenhuis structure (M, π, N) on the unit space M. We can now show that the infinitesimal of the Nijenhuis tensor $\bar{N}$ coincides with the complete lift $N^c : TT^*M \to TT^*M$ of N.

Proposition 7.1. *Let $(\Sigma \rightrightarrows M, \overline{\omega}, \bar{N})$ be a symplectic-Nijenhuis local groupoid, and (M, π, N) the corresponding Poisson–Nijenhuis structure on M, as in Theorem 6.1. Then*

$$\mathbb{L}\mathrm{ie}(\bar{N}) = N^c. \tag{42}$$

Proof. By definition, $\bar{N} = \bar{\pi}^\sharp_{\bar{N}} \circ \overline{\omega}^\flat$, and hence

$$\mathbb{L}\mathrm{ie}(\bar{N}) = j_\Sigma \circ \mathrm{Lie}(\bar{\pi}^\sharp_{\bar{N}}) \circ \mathrm{Lie}(\overline{\omega}^\flat) \circ j_\Sigma^{-1}.$$

We now check that

$$\mathrm{Lie}(\overline{\omega}^\flat) = (j'_\Sigma)^{-1} \circ \omega^\flat_{\mathrm{can}} \circ j_\Sigma. \tag{43}$$

Let $\bar{\pi}$ be the Poisson bivector field on Σ inverse to $\overline{\omega}$. Then $(\Sigma \rightrightarrows M, \bar{\pi})$ is a Poisson groupoid, and, by Theorem A.1, its Lie bialgebroid is $((T^*M)_\pi, TM)$. Since the Lie–Poisson structure induced by the tangent bundle Lie algebroid TM coincides with the canonical symplectic structure on T^*M, the commutativity of diagram (40) implies that

$$\mathrm{Lie}(\bar{\pi}^\sharp) = j_\Sigma^{-1} \circ \pi^\sharp_{\mathrm{can}} \circ j'_\Sigma.$$

The latter is equivalent to Eq. (43), as claimed.

Finally, recall that the Lie bialgebroid of the Poisson groupoid $(\Sigma \rightrightarrows M, \bar{\pi}_{\bar{N}})$ is isomorphic to $((T^*M)_\pi, (TM)_N)$ according to the proof of Theorem 6.1. From commutative diagram (40), we thus have

$$\mathrm{Lie}(\bar{\pi}^\sharp_{\bar{N}}) = j_\Sigma^{-1} \circ (\pi')^\sharp \circ j'_\Sigma.$$

Here π' is the Lie–Poisson structure on T^*M corresponding to the Lie algebroid $(TM)_N$ as in Lemma 7.1. In particular, we have

$$\mathbb{L}\mathrm{ie}(\bar{N}) = j_\Sigma \circ \mathrm{Lie}(\bar{\pi}^\sharp_{\bar{N}}) \circ \mathrm{Lie}(\overline{\omega}^\flat) \circ j_\Sigma^{-1} = (\pi')^\sharp \circ \omega^\flat_{\mathrm{can}} = N^c.$$

Here we used Eq. (43) for the second equality, and Lemma 7.1 for the last equality. This concludes the proof. □

8. Symplectic realizations of non-degenerate Poisson–Nijenhuis manifolds

In this section, we conclude the proof of a more general version of Theorem A that holds in the case of non-degenerate Poisson–Nijenhuis structures. The main result here is Theorem 8.1. According to Lemma 2.1, it is clear that this includes the case of holomorphic Poisson manifolds.

The following standard lemma is crucial to our proof.

Lemma 8.1 ([16]). *Let (M, π, N) be a Poisson-Nijenhuis manifold and π_N its second Poisson structure, as in Proposition B.1, defined by Eq.* (43)*. The pair of maps*

$$N^T : (T^*M)_{\pi_N} \to (T^*M)_\pi,\ N : (TM)_N \to TM$$

defines a morphism of Lie bialgebroids

$$(N^T, N) : ((T^*M)_{\pi_N}, TM) \to ((T^*M)_\pi, (TM)_N).$$

In particular, if N is invertible, the pair (N^T, N) is an isomorphism of Lie bialgebroids.

Let us assume, throughout the remainder of this section, that we are given a Poisson–Nijenhuis manifold (M, π, N) whose Nijenhuis tensor $N : TM \to TM$ is invertible. In this case, we also say that (M, π, N) is a *non-degenerate Poisson-Nijenhuis manifold.* Also, in order to simplify notation, we denote by A (resp. A_N) the cotangent Lie algebroid $(T^*M)_\pi$ (resp. $(T^*M)_{\pi_N}$) of π (resp. π_N).

The pair $(\bar{P}_{\mathrm{loc}}(A_N) \rightrightarrows M, \overline{\omega}')$ is a symplectic local groupoid, where $\overline{\omega}'$ is the symplectic form as in Theorem 3.2. Let $\bar{\pi}'$ be the corresponding Poisson structure on $\bar{P}_{\mathrm{loc}}(A_N)$, which is the inverse of $\overline{\omega}'$. Then the Lie bialgebroid of the Poisson local groupoid $(\bar{P}_{\mathrm{loc}}(A_N) \rightrightarrows M, \bar{\pi}')$ is (A_N, TM).

On the other hand, we can construct another Poisson local groupoid $(\bar{P}_{\mathrm{loc}}(A) \rightrightarrows M, \bar{\pi}_{\bar{N}})$ as follows. According to Theorem 6.1, there is a symplectic-Nijenhuis local groupoid $(\bar{P}_{\mathrm{loc}}(A) \rightrightarrows M, \overline{\omega}, \bar{N})$, which induces the Poisson–Nijenhuis structure (M, π, N) on the unit space M. Let $\bar{\pi}$ be the Poisson bivector field associated to $\overline{\omega}$, and let $\bar{\pi}_{\bar{N}}$ be defined, as before, by the relation $\bar{\pi}^\sharp_{\bar{N}} = \bar{N} \circ \bar{\pi}^\sharp$. Then the Poisson local groupoid $(\bar{P}_{\mathrm{loc}}(A) \rightrightarrows M, \bar{\pi}_{\bar{N}})$ has Lie bialgebroid $((T^*M)_\pi, (TM)_N)$.

Now, according to Lemma 8.1, we have a Lie bialgebroid morphism

$$(N^T, N) : ((T^*M)_{\pi_N}, TM) \to ((T^*M)_\pi, (TM)_N). \tag{44}$$

Thus, from Theorem A.1, it follows that the induced morphism of local Lie groupoids

$$\bar{P}(N^T) : \bar{P}_{\mathrm{loc}}(A_N) \to \bar{P}(A), \tag{45}$$

as in Proposition 3.1 (i), is a Poisson map. Hence we have

$$\bar{P}(N^T)_* \bar{\pi}' = \bar{\pi}_{\bar{N}}. \tag{46}$$

Since N is invertible by assumption, the map in (44) is an isomorphism of Lie bialgebroids. Therefore the map in (45) is indeed an isomorphism of Poisson local groupoids. In particular, the bivector field $\bar{\pi}_{\bar{N}}$ is non-degenerate, since $\bar{\pi}'$ is non-degenerate.

Let $\overline{\omega}_{\bar{N}}$ be the (necessarily multiplicative) symplectic form on $\bar{P}_{\text{loc}}(A)$ whose Poisson bivector field is $\bar{\pi}_{\bar{N}}$. Then $(\bar{P}_{\text{loc}}(A) \rightrightarrows M, \overline{\omega}_{\bar{N}})$ is a symplectic local groupoid. Since $\bar{P}(N^T)$ is a Poisson isomorphism, we must have

$$\overline{\omega}_{\bar{N}} = (\bar{P}(N^T)^{-1})^* \overline{\omega}'. \tag{47}$$

Summarizing, we have proved the following

Proposition 8.1. *The pair $(\bar{P}_{loc}(A) \rightrightarrows M, \bar{\omega}_{\bar{N}})$ is a symplectic local groupoid which, as a Poisson groupoid, has Lie bialgebroid $((T^*M)_\pi, (TM)_N)$.*

Now fix ∇ an A-connection on A, and let $U \subset T^*M$ be a sufficiently small open neighborhood around the zero section, as in Proposition 4.2. Define

$$\begin{aligned} \underline{\omega} &= \exp^* \overline{\omega}, \quad \text{and} \\ \underline{\omega}_N &= \exp^* \overline{\omega}_{\bar{N}}, \end{aligned} \tag{48}$$

where $\exp : U \to \bar{P}_{\text{loc}}(A)$ is the groupoid exponential map associated to ∇. The formula (15) still holds for $\underline{\omega}$, since $(\bar{P}_{\text{loc}}(A) \rightrightarrows M, \overline{\omega})$ is exactly the same symplectic local groupoid as in Theorem 3.2. On the other hand, we also have

Proposition 8.2. *The symplectic form $\underline{\omega}_N \in \Omega^2(U)$ can be explicitly expressed as follows:*

$$\underline{\omega}_N = \int_0^1 ((N^T)^{-1} \circ \varphi_t^\nabla)^* \omega_{can}\, dt, \tag{49}$$

where $\omega_{can} \in \Omega^2(T^*M)$ is the canonical symplectic form on T^*M, and φ_t^∇ is the flow of the geodesic vector field of ∇.

Proof. By Proposition 3.1 (i), we have

$$P((N^{-1})^T)^* \circ q'^* = q^* \circ \bar{P}((N^T)^{-1})^*$$

where $q : P_{\text{loc}}(A) \to \bar{P}_{\text{loc}}(A)$ and $q' : P_{\text{loc}}(A_N) \to \bar{P}_{\text{loc}}(A_N)$ are the quotient maps as in (4). It is also simple to see that $P((N^T)^{-1}) = (P(N^T))^{-1}$, and

$\bar{P}((N^T)^{-1}) = (\bar{P}(N^T))^{-1}$. Since $q'^*\overline{\omega}' = \iota'^*\widetilde{\omega}_{\mathrm{can}}$, we have

$$q^*(\bar{P}(N^T)^{-1})^*\overline{\omega}' = P((N^{-1})^T)^* q'^*\overline{\omega}' = P((N^{-1})^T)^*\iota'^*\widetilde{\omega}_{\mathrm{can}}.$$

By commutative diagram (11), we have $\exp^* = \Phi^* q^*$. Thus it follows that

$$\underline{\omega}_N = \exp^*(\bar{P}(N^T)^{-1})^*\overline{\omega}' = \Phi^* q^*(\bar{P}(N^T)^{-1})^*\overline{\omega}' = \Phi^* P((N^{-1})^T)^*\iota'^*\widetilde{\omega}_{\mathrm{can}}.$$

Now $\forall\ a_0 \in U$ and $u, v \in T_{a_0}A$, we have

$$\begin{aligned}
\underline{\omega}_N(u,v) &= \widetilde{\omega}_{\mathrm{can}}(\iota'_* P((N^T)^{-1})_*\Phi_* u,\quad \iota'_* P((N^T)^{-1})_*\Phi_* v)\\
&= \int_0^1 \omega_{\mathrm{can}}((\tau[\iota'_* P((N^T)^{-1})_*\Phi_* u])(t), (\tau[\iota'_* P((N^T)^{-1})_*\Phi_* v])(t))dt\\
&= \int_0^1 \omega_{\mathrm{can}}(((N^T)^{-1})_*(\varphi_t^\nabla)_* u, ((N^T)^{-1})_*(\varphi_t^\nabla)_* v)dt\\
&= \left(\int_0^1 ((N^T)^{-1}\circ\varphi_t^\nabla)^*\omega_{\mathrm{can}} dt\right)(u,v),
\end{aligned}$$

where the second to last equality follows from Eq. (13). □

Combining Proposition 5.1, Proposition 6.1 and Proposition 8.2, we are finally led to the following main theorem of this section.

Theorem 8.1. *Let (M,π,N) be a non-degenerate Poisson–Nijenhuis manifold, and $A = (T^*M)_\pi$ the cotangent Lie algebroid of the Poisson manifold (M,π). Fix ∇ an A-connection on A and let φ_t^∇ be the flow of the geodesic vector field of ∇. Also let $U \subset A$ be a sufficiently small open neighborhood of the zero section of A as in Proposition 4.2. Then the following assertions hold.*

(i) *The projection $\mathrm{pr}|_U : U \to M$, together with the symplectic form $\underline{\omega}$ (resp. $\underline{\omega}_N$), defined by Eq. (15) (resp. Eq. (49)), is a symplectic realization of π (resp. π_N).*

(ii) *The $(1,1)$-tensor*

$$\underline{N} := (\underline{\omega}_N^\flat)^{-1}\circ\underline{\omega}^\flat : TU \to TU \tag{50}$$

is a Nijenhuis tensor on U. Furthermore, the triple $(U,\underline{\omega},\underline{N})$ is a symplectic-Nijenhuis manifold.

(iii) *Denote by* $\underline{\pi}$ *the Poisson bivector field inverse to* $\underline{\omega}$. *Then the canonical projection* $\mathrm{pr}|_U : U \to M$ *is a Poisson–Nijenhuis map with respect to* $(U, \underline{\pi}, \underline{N})$ *and* (M, π, N).

(iv) *The zero section is a Lagrangian submanifold of* U *with respect to both* $\underline{\omega}$ *and* $\underline{\omega}_N$.

Following Petalidou [28], we will call the symplectic-Nijenhuis manifold $(U, \underline{\omega}, \underline{N})$, endowed with the projection $\mathrm{pr}|_U : U \to M$, a *symplectic realization of the non-degenerate Poisson–Nijenhuis manifold* (M, π, N). Note that a symplectic realization of a Poisson–Nijenhuis manifold can only exist when the Nijenhuis tensor is invertible.

Remark 8.1. As an immediate consequence of Eq. (25), Eq. (49) can be rewritten as

$$\underline{\omega}_N(u, v) = \int_0^1 \big((\varphi_t^\nabla)^* \omega_{\mathrm{can}}\big)\big((N^c)^{-1} u, v\big)\, dt \tag{51}$$

$\forall \xi \in U$ and $u, v \in T_\xi(T^*M)$. The explicit formula of Eq. (51) is due to Petalidou [28]. In fact, Theorem 8.1 (i)-(ii) essentially recovers a theorem claimed by Petalidou [28], which was obtained by following closely the computational approach in [8]. From the discussion of this section, we see that both symplectic forms $\underline{\omega}$ and $\underline{\omega}_N$ are in fact conceptually parts of the data involved in constructing a (*a priori* hidden) symplectic-Nijenhuis local groupoid.

Remark 8.2. Theorem 8.1 was reproved using a different method in the preprint [1], which appeared after the present paper was posted on arXiv. See [1, Section 3.2] for details. We also refer the reader to [3] for results closely related to those in [1].

9. Holomorphic symplectic realizations of holomorphic Poisson manifolds

It remains to explain how Theorem A follows from Theorem 8.1. First, recall the following standard fact.

Proposition 9.1 ([19]). *Let* (X, ω), *where* $\omega = \omega_R + i\omega_I \in \Omega^{2,0}(X)$, *be a holomorphic symplectic manifold. Denote by* $\pi = \pi_R + i\pi_I \in \Gamma(\wedge^2 T^{1,0} X)$ *the associated holomorphic Poisson bivector field. Then the real differential* 2-*forms* ω_R *and* $\omega_I \in \Omega^2(X)$ *are symplectic, and their corresponding Poisson bivector fields are* $4\pi_R$ *and* $-4\pi_I$, *respectively.*

We are now ready to conclude the proof of the main theorem of this paper.

Proof of Theorem A. Let $(\mathscr{X}, \pi = \pi_R + i\pi_I)$ be a holomorphic Poisson manifold with almost complex structure J and the underlying real manifold X. By Lemma 2.1, (X, π_I, J) is a Poisson–Nijenhuis manifold. Let $A = (T^*X)_{\pi_I}$ be the cotangent Lie algebroid of the real Poisson manifold (X, π_I). Also fix an affine connection ∇^{TX} on X, and denote by ∇^{T^*X} the induced linear connection on T^*X. Finally, let ∇ be the A-connection on A as in Example 4.2, which is defined by

$$\nabla_a b = \nabla^{T^*X}_{\rho(a)} b \quad \forall a, b \in \Gamma(A),$$

where $\rho : A \to TX$ is the anchor of A.

From Theorem 8.1 (i), it follows that there is an open neighborhood $Y \subset T^*X$ of the zero section where the symplectic forms $\underline{\omega}_R$ and $\underline{\omega}_I$, given, respectively, by Eq. (49) and Eq. (15), together with the projection $\mathrm{pr}|_Y : Y \to X$, give symplectic realizations of π_R and π_I, respectively. Furthermore, by Theorem 8.1 (ii), the $(1,1)$-tensor

$$\underline{J} := (\underline{\omega}_R^\flat)^{-1} \circ \underline{\omega}_I^\flat : TY \to TY \tag{52}$$

is Nijenhuis, and $(Y, \underline{\omega}_I, \underline{J})$ is a symplectic-Nijenhuis manifold. Moreover, the canonical projection $\mathrm{pr}|_Y : Y \to X$ is a Poisson-Nijenhuis map according to Proposition 6.1.

We have the following lemma.

Lemma 9.1. *The* $(1,1)$*-tensor* $\underline{J}$ *in* (52) *is an almost complex structure on* Y*, i.e.* $\underline{J}^2 = -1$.

Proof. Recall that the $(1,1)$-tensor $\bar{J} : T\bar{P}_{\mathrm{loc}}(A) \to T\bar{P}_{\mathrm{loc}}(A)$, defined as in Eq. (22), is a local groupoid morphism with respect to the groupoid structure $\bar{P}_{\mathrm{loc}}(A) \rightrightarrows M$ of Theorem 3.1. Also note that, by definition,

$$\underline{J} = \exp_*^{-1} \circ \bar{J} \circ \exp_* .$$

On the other hand, we have $\mathbb{L}\mathrm{ie}(\bar{J}) = J^c$ by Proposition 7.1. Furthermore, since J is an almost complex structure, it follows (see [9] for example) that $(J^c)^2 = (J^2)^c = -1$. Thus $\mathbb{L}\mathrm{ie}(\bar{J}^2) = (\mathbb{L}\mathrm{ie}(\bar{J}))^2 = -1$. Since $\bar{J}$ is multiplicative on a local Lie groupoid, it follows that $\bar{J}^2 = -1$. Therefore we have $\underline{J}^2 = -1$ as well. This concludes the proof. □

Returning to the proof of Theorem A: since $\underline{J}$ is already a Nijenhuis tensor, from Lemma 9.1, it follows that $\underline{J}$ indeed induces a complex structure on the manifold Y, whose underlying complex manifold is denoted by $\mathscr{Y}$. Moreover, since $(Y, \underline{\omega}_I, \underline{J})$ is a symplectic-Nijenhuis manifold and its induced second Poisson structure is the Poisson structure corresponding to $\underline{\omega}_R$, it follows that $\underline{\omega} := \frac{1}{4}(\underline{\omega}_R - i\underline{\omega}_I) \in \Omega^2(Y) \otimes \mathbb{C}$ yields a holomorphic symplectic form on Y with respect to the new complex structure $\underline{J}$ [19]. In particular, $(Y, \underline{\omega}, \underline{J})$ is a holomorphic symplectic manifold. The triple $(Y, \underline{\omega}, \underline{J})$ is indeed the underlying holomorphic symplectic manifold of the holomorphic symplectic local groupoid integrating the given holomorphic Poisson structure π (see [20, Theorem 3.22] for an explanation of the factor $\frac{1}{4}$). Denote by $\underline{\pi}$ the associated holomorphic Poisson bivector field on Y. From Theorem 8.1 (ii), Proposition 9.1 and Proposition B.3, it follows that the projection $\mathrm{pr}|_Y : Y \to X$ is indeed a holomorphic Poisson map with respect to the holomorphic Poisson structures $(Y, \underline{\pi}, \underline{J})$ and (X, π, J). This concludes the proof. □

Appendix A. Lie bialgebroids and Poisson groupoids

Definition A.1. A *Poisson local groupoid* $(\Sigma \rightrightarrows M, \bar{\pi})$ is a local Lie groupoid $\Sigma \rightrightarrows M$ endowed with a Poisson bivector field $\bar{\pi} \in \mathfrak{X}^2(\Sigma)$ such that the Poisson graph Λ of multiplication in Σ:

$$\Lambda \equiv \{(x, y, x \cdot y) \mid (x, y) \in \Sigma \times \Sigma \text{ composable } \} \subset \Sigma \times \Sigma \times \bar{\Sigma}$$

is a coisotropic submanifold. Here $\bar{\Sigma}$ denotes Σ endowed with the Poisson bivector field $-\bar{\pi}$.

A bivector field $\bar{\pi} \in \mathfrak{X}^2(\Sigma)$ as in Definition A.1 is also called multiplicative. In this context, the following is standard [24].

Proposition A.1. *Let A be the Lie algebroid of $\Sigma \rightrightarrows M$. The bivector field $\bar{\pi}$ is multiplicative if and only if the map $\bar{\pi}^\sharp : T^*\Sigma \to T\Sigma$ is a local Lie groupoid morphism. Here $T^*\Sigma \rightrightarrows A^*$ is the cotangent local Lie groupoid [5] and $T\Sigma \rightrightarrows TM$ is the tangent local Lie groupoid of $\Sigma \rightrightarrows M$.*

Definition A.2. A *symplectic local groupoid* is a Poisson local groupoid $(\Sigma \rightrightarrows M, \bar{\pi})$ such that $\bar{\pi}$ is non-degenerate.

We now recall some fundamental facts regarding Lie bialgebroids. In the rest of this section, let A be a Lie algebroid with anchor ρ and Lie bracket $[\cdot,\cdot]$.

The Lie bracket, $[\cdot,\cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$, can be extended to a bilinear bracket of multisections $\Gamma(\wedge^k A) \times \Gamma(\wedge^l A) \to \Gamma(\wedge^{k+l-1} A)$. We will denote both the initial bracket and its extension by $[\cdot,\cdot]$. The triple $(\Gamma(\wedge^\bullet A), \wedge, [\cdot,\cdot])$ then forms a Gerstenhaber algebra [37].

Recall that Lie bialgebroids are a certain class of Lie algebroids A for which the dual vector bundle A^* also admits a compatible Lie algebroid structure. In order to define the compatibility condition, recall that the *Chevalley–Eilenberg differential* of the Lie algebroid A is the operator $d : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$ defined by

$$(d\lambda)(a_1,\dots,a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \rho(a_i) \cdot \lambda(a_1,\dots,\hat{a}_i,\dots,a_{k+1}) \tag{A.1}$$

$$+ \sum_{i<j} (-1)^{i+j} \lambda([a_i,a_j], a_1,\dots,\hat{a}_i,\dots,\hat{a}_j,\dots,a_{k+1}). \tag{A.2}$$

for any $\lambda \in \Gamma(\wedge^k A^*)$, and any $a_1,\dots,a_{k+1} \in \Gamma(A)$.

Example A.1. When $A = TM$ is the tangent Lie algebroid of a manifold M, the Chevalley–Eilenberg differential d coincides with the De Rham differential operator d_{DR} on $\Gamma(\wedge^\bullet T^*M) = \Omega^\bullet(M)$.

When A^* happens to be a Lie algebroid as well, we denote by $d_* : \Gamma(\wedge^k A) \to \Gamma(\wedge^{k+1} A)$ the associated Chevalley–Eilenberg differential (acting on sections of $A \cong (A^*)^*$).

Definition A.3 ([16, 24]). Let A be a Lie algebroid such that A^* also carries a Lie algebroid structure. Then (A, A^*) is a *Lie bialgebroid* if the Lie algebroid structures on A and A^* are compatible in the following sense. For any $a, a' \in \Gamma(A)$, one has

$$d_*[a,a'] = [d_* a, a'] + [a, d_* a']. \tag{A.3}$$

The compatibility condition (A.3) is equivalent to asking that d_* is a derivation of the Gerstenhaber algebra $(\Gamma(\wedge^\bullet A), \wedge, [\cdot,\cdot])$ [37].

Example A.2. Let $A = TM$ be the tangent Lie algebroid and $A^* = (T^*M)_\pi$ the cotangent Lie algebroid of a Poisson manifold (M,π). It is easy to see that (A, A^*) is a Lie bialgebroid. In fact, the graded Lie bracket $[\cdot,\cdot] : \Gamma(\wedge^k A) \times \Gamma(\wedge^l A) \to \Gamma(\wedge^{k+1-1} A)$ coincides with the Schouten bracket $[\cdot,\cdot]_S$ on $\Gamma(\wedge^\bullet TM)$, and $d_* = [\pi, \cdot]_S$. Thus (A.3) follows from the graded Jacobi identity of the Schouten brackets.

Definition A.4. Let (A, A^*) and (B, B^*) be Lie bialgebroids over the same base manifold M. A *Lie bialgebroid morphism* $(\psi, \psi^T) : (A, A^*) \to (B, B^*)$ is a vector bundle map $\psi : A \to B$ over the identity map such that

i.) $\psi : A \to B$ is a Lie algebroid morphism, and

ii.) its dual $\psi^T : B^* \to A^*$ is also a Lie algebroid morphism.

One can prove that the definition of a Lie bialgebroid (A, A^*) is symmetric in A and A^* [24, Theorem 3.10]. In particular, according to Example A.2, we have

Proposition A.2. *Let M be a Poisson manifold with Poisson bivector field $\pi \in \mathfrak{X}^2(M)$. Then $((T^*M)_\pi, TM)$ is a Lie bialgebroid.*

The "d_*" operator of the Lie bialgebroid in Proposition A.2 is simply the De Rham differential operator. The following theorem is standard [12, 24, 25][2], which extends a well-known classical result of Drinfeld concerning Poisson Lie groups [10, 11].

Theorem A.1.

(i). *Lie bialgebroids (A, A^*) are in one-to-one correspondence with Poisson local groupoids $(\Sigma \rightrightarrows M, \bar{\pi})$.*

(ii). *The correspondence in (i) is functorial. More precisely, let (A, A^*) and (B, B^*) be Lie bialgebroids over M. Then morphisms $(\psi, \psi^T) : (A, A^*) \to (B, B^*)$ of Lie bialgebroids are in one-to-one correspondence with morphisms of the associated Poisson local groupoids*

$$\bar{\psi} : (\Sigma_A \rightrightarrows M, \bar{\pi}_A) \to (\Sigma_B \rightrightarrows M, \bar{\pi}_B).$$

(iii). *Let $(\Sigma \rightrightarrows M, \overline{\omega})$ be a symplectic local groupoid, and let $\bar{\pi}$ be the multiplicative Poisson bivector field on Σ inverse to $\overline{\omega}$. Then, as a Poisson*

[2] In literature, this theorem is normally stated for global Lie groupoids, for which one needs to assume source connectedness and source simply connectedness. The conclusion (as well as the proof) holds for local Lie groupoids without such topological assumptions.

groupoid, $(\Sigma \rightrightarrows M, \bar{\pi})$ *has Lie bialgebroid* $((T^*M)_\pi, TM)$*. Here* TM *is the tangent bundle Lie algebroid of* M*.*

For completeness, let us recall that, by a morphism of local Lie groupoids, we mean a smooth map $\bar{\psi}_1 : \Sigma \to \Sigma'$, defined on neighborhoods of the unit spaces of Σ and Σ', together with a smooth map $\bar{\psi}_0 : M \to M'$ such that

$$\begin{array}{ccc} \Sigma & \xrightarrow{\bar{\psi}_1} & \Sigma' \\ \downarrow\downarrow & & \downarrow\downarrow \\ M & \xrightarrow[\bar{\psi}_0]{} & M' \end{array}$$

satisfies the usual axioms of a Lie groupoid morphism.

Appendix B. Poisson–Nijenhuis manifolds

Recall that a $(1,1)$-tensor $N : TM \to TM$ on a smooth manifold M is called *Nijenhuis* if its Nijenhuis torsion $T_N : \wedge^2 TM \to TM$ vanishes, where

$$T_N(X,Y) = [NX,NY] - N([NX,Y] + [X,NY]) + N^2[X,Y], \quad \forall X,Y \in \mathfrak{X}(M). \tag{B.4}$$

Definition B.1. Let π be a Poisson bivector field on M and N be a Nijenhuis $(1,1)$-tensor. We say that the triple (M,π,N) is a *Poisson–Nijenhuis manifold* [17, 26] if π and N satisfy the following compatibility relations for all $\xi,\eta \in \Omega^1(M)$:

$$N \circ \pi^\sharp = \pi^\sharp \circ N^T, \tag{B.5}$$

$$[\xi,\eta]_{\pi_N} = [N^T\xi,\eta]_\pi + [\xi,N^T\eta]_\pi - N^T[\xi,\eta]_\pi. \tag{B.6}$$

Here π_N is the bivector field on M defined by $\pi_N^\sharp = N \circ \pi^\sharp$ and $[\cdot,\cdot]_{\pi_N}$ is the associated bracket on $\Omega^1(M)$.

The following is standard in the theory of Poisson–Nijenhuis manifolds [17, 26, 33].

Proposition B.1. *Let (M, π, N) be a Poisson–Nijenhuis manifold. Then the bivector field $\pi_N \in \mathfrak{X}^2(M)$ defined by the property that*

$$\pi_N^\sharp = N \circ \pi^\sharp \tag{B.7}$$

is a Poisson bivector field.

An alternative description of various compatibility relations between π and N is summarized in the following well-known result.

Theorem B.1 ([17, 32]). *Let $\pi \in \mathfrak{X}^2(M)$ be a Poisson bivector field on a manifold M and let $N : TM \to TM$ be a $(1,1)$-tensor. Then the tensor π_N defined by*

$$\pi_N(\xi, \eta) = \eta\big(N\pi^\sharp\xi\big), \qquad \forall \xi,\ \eta \in \Omega^1(M)$$

is skew-symmetric if and only if Eq. (B.5) holds. In this case, we also have the following assertions:

(i) $[\pi, \pi_N] = 0$ *if Eq. (B.6) holds, and the converse holds if π is non-degenerate;*

(ii) $[\pi_N, \pi_N] = 0$ *if N is Nijenhuis.*

Definition B.2. A *symplectic-Nijenhuis manifold* is a Poisson–Nijenhuis manifold (M, π, N) whose Poisson bivector field π is non-degenerate.

A symplectic-Nijenhuis manifold is also denoted by (M, ω, N), where ω is the symplectic form corresponding to π.

The following theorem, due to Vaisman [32], essentially asserts that symplectic-Nijenhuis manifolds are equivalent to biHamiltonian systems with one Poisson structure being non-degenerate.

Proposition B.2. *[32, Corollary 1.5] Let π and π' be compatible Poisson structures on a smooth manifold M, i.e.,*

$$[\pi, \pi] = [\pi', \pi'] = [\pi, \pi'] = 0.$$

Assume that π is non-degenerate. Then (M, π, N) is a symplectic-Nijenhuis manifold such that $\pi_N = \pi'$, where N is the $(1,1)$-tensor on M defined by $N = (\pi')^\sharp \circ (\pi^\sharp)^{-1} : TM \to TM$.

Definition B.3. Let (X, π_X, N_X) and (Y, π_Y, N_Y) be Poisson–Nijenhuis manifolds. A *Poisson–Nijenhuis map* is a smooth map $f : X \to Y$ such that

$$f_* \circ N_X = N_Y \circ f_*, \quad \text{and} \quad f_* \pi_X = \pi_Y.$$

If $f : X \to Y$ is a Poisson–Nijenhuis map, then $f_* \pi_{N_X} = \pi_{N_Y}$ as well. The following is easily seen.

Proposition B.3. *Let $(X, \pi = \pi_R + i\pi_I)$ and $(Y, \pi' = \pi'_R + i\pi'_I)$ be holomorphic Poisson manifolds with almost complex structures J_X and J_Y, respectively. Let $f : X \to Y$ be a smooth map. Then*

(i) *the map f is holomorphic Poisson if and only if it is a Poisson–Nijenhuis map from (X, π_I, J_X) to (Y, π'_I, J_Y).*

(ii) *In particular, if f is a holomorphic map, then f is holomorphic Poisson if and only if $f_* \pi_I = \pi'_I$.*

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