Math. Res. Lett.
Volume 29, Number 4, 945–960, 2022

# Uniqueness of higher integrable solution to the Landau equation with Coulomb interactions

Jann-Long Chern and Maria Gualdani

We are concerned with the uniqueness of weak solution to the spatially homogeneous Landau equation with Coulomb interactions under the assumption that the solution is bounded in the space $L^\infty(0,T,L^p(\mathbb{R}^3))$ for some $p > 3/2$. The proof uses a weighted Poincaré-Sobolev inequality recently introduced in [11].

## 1. Introduction

The Landau equation was introduced in 1936 by Lev Landau as a correction of the Boltzmann equation to describe collision of particles interacting under a potential of Coulomb type. Collisions of such kind are predominant in hot plasma. In its homogeneous form the Landau equation reads as

$$\partial_t f = Q(f,f), \tag{1}$$

where $f = f(v,t)$ for $v \in \mathbb{R}^3,\ t > 0$ is a nonnegative function describing the evolution of the particle density and

$$Q(f,f) := \frac{1}{8\pi}\mathrm{div}\left(\int_{\mathbb{R}^3} \frac{1}{|v-w|}\left(\Pi(v-w)(f(w)\nabla_v f(v) - f(v)\nabla_w f(w)\right)\, dw\right), \tag{2}$$

with $\Pi(z)$ the projection onto the orthogonal subspace of $z$,

$$\Pi(z) := \mathbb{I} - \frac{z\otimes z}{|z|^2}, \quad z \neq 0.$$

JLC is supported by NSTC 110-2115-M-003-019-MY3 and NSTC 111-2218-E-008-004-MBK. MPG is supported by NSF DMS-2019335. JLC and MPG would like to thank KTH Royal Institute of Technology and then NCTS Mathematics Division Taipei for their kind hospitality.

Equation (1)-(2) has been extensively studied in the literature but the main question whether or not after a certain time solutions could become unbounded is still open. The possible blow-up in the $L^\infty$-norm could be caused by the quadratic nonlinearity in (2): assuming that $f$ is smooth enough, one can rewrite (2) as

$$Q(f,f) = \operatorname{div}\left(A[f]\nabla f - f\nabla a[f]\right),$$

where $A[f]$ is the diffusion matrix defined as

$$A[f](v,t) = \{a_{i,j}\}_{i,j} := \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{1}{|w|}\left(\mathbb{I} - \frac{w\otimes w}{|w|^2}\right) f(v-w,t)\, dw,$$

and

$$a[f](v,t) := \operatorname{tr}(A[f]) = (-\Delta)^{-1} f,$$

or in non-divergence form

$$Q(f,f) = \operatorname{tr}(A[f]D^2(f)) + f^2.$$

In the last formulation the quadratic nonlinearity is explicit.

Before we state the main result of this manuscript we briefly review the literature for (1)-(2), omitting the rather large literature on non-Coulomb potentials and spatially inhomogeneous case. Existing literature for (1)-(2) includes results on (i) local in time well-posedness of solutions, (ii) global in time existence and uniqueness of smooth solution for initial data close to equilibrium [14], (iii) global in time existence of (very) weak solutions [1, 5, 19] , and (iv) convergence of weak solutions towards the equilibrium function (Maxwellian) in the $L^1$-norm [2]. Very recently the second author and collaborators studied the partial regularity of weak solutions to (1)-(2) and showed in [7] that the Hausdorff measure of the set of singular times (i.e. times at which the function could be unbounded) is at most $\frac{1}{2}$. Partial regularity in space and velocity has been recently studied in [8]. We also mention an important result from [10]; there the authors study an *isotropic* version of the Landau equation, previously introduced by Krieger and Strain in [15],

$$\partial_t f = \operatorname{div}\left(a[f]\nabla f - f\nabla a[f]\right), \tag{3}$$

and show that (3) with spherically symmetric and radially decreasing initial data (but not small neither near equilibrium!) has smooth solutions which remain bounded for all times. Global existence of solutions for the isotropic Landau equation for a range of soft-potentials is shown in [12].

Since the main question of global well-posedness for general initial data for (1)-(2) is still open, in the most recent years there have been several conditional proofs of existence of bounded solutions and their regularity. In this directions we mention [9–11, 17].

In the current manuscript we are concerned with uniqueness of weak solutions in the class of higher integrable solutions, namely we assume that weak solutions belong to $L^\infty(0,T,L^p(\mathbb{R}^3))$ for some $p>\frac{3}{2}$ and have high enough bounded moments. Conditional uniqueness of bounded weak solutions for Landau-Coulomb has been previously studied in [6]; via a probabilistic approach using a stochastic representation of (1)-(2) the author shows uniqueness in the class of solutions $L^1(0,T,L^\infty(\mathbb{R}^3))$. A similar approach was recently used in [18] for the relativistic Landau-Coulomb equation.

Here is our main result:

**Theorem 1.** *The homogeneous Landau-Coulomb equation with initial data such that*

$$f_{in}\geq 0,\quad \int_{\mathbb{R}^3} f_{in}^2(1+|v|)^5\,dv\leq C,\quad \int_{\mathbb{R}^3} f_{in}(1+|v|)^q\,dv\leq C, \tag{4}$$

*for* $q=\frac{46(p-1)}{p-3/2}$*, has at most one solution in the time interval* $[0,T]$*,* $T>0$*, in the class of functions*

$$f\in L^\infty(0,T,L^p(\mathbb{R}^3)),\quad p>3/2. \tag{5}$$

The proof of Theorem 1 differs from the one in [6] in several aspects. We only require our solution to belong to some $L^p(\mathbb{R}^3)$ space with $p>3/2$, uniformly in time. Our method uses the weak representation of (1)-(2) provided in [13] and a new weighted Poincaré inequality (10) recently introduced in [11]. This inequality is shown to be valid for any solution $f$ to the Landau equation that is uniformly in time $L^p(\mathbb{R}^3)$-integrable, for some $p>3/2$ (and has high enough bounded moments). The question whether (10) holds without the extra integrability assumption is still open and very interesting. In [11] the authors showed that (10) *nearly* holds if we only assume uniformly in time $L^1(\mathbb{R}^3)$- integrability for $f$; this means that the diffusion $\operatorname{div}(A[f]\nabla f)$ and the reaction $f^2$ are of the same order. In this regard we should think of (1)-(2) as a *critical* equation.

The rest of the manuscript is organized as follows: in Section 2 we recall some useful well-known results, in Section 3 we present the weighted Poincaré inequality. In Section 4 we show integrability and weighted estimates for the gradient. Section 5 contains the proof of Theorem 1.

## 2. Well-known results

The following quantities will be frequently used throughout the paper. We respectively define the mass, momentum and entropy of a nonnegative function $h(v)$ the quantities

$$\int_{\mathbb{R}^3} h(v)\, dv, \quad \int_{\mathbb{R}^3} h(v)|v|^2\, dv, \quad \int_{\mathbb{R}^3} h(v) \ln h(v)\, dv.$$

We start by recalling the definition of weak solution [5]: given initial data $f_{in}$ with finite mass, first, second moment and entropy, a weak solution to the Landau is a nonnegative function $f$ such that

$$(1+|v|^2)^{-3/2} f \in L^1(0,T,L^3(\mathbb{R}^3)),$$

has finite mass, first, second momentum and entropy and for all $\varphi \in C_c^2([0,T]\times\mathbb{R}^3)$

$$\begin{aligned}
(6)\qquad & -\int_{\mathbb{R}^3} f_{in}(v)\varphi(v,0)\, dv - \int_0^T \int_{\mathbb{R}^3} f(v,t)\partial_t\varphi(v,t)\, dvdt \\
&= \frac{1}{2}\sum_{i=1}^3\sum_{j=1}^3 \int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v,t)f(w,t)a_{ij}(v-w) \\
&\qquad \times (\partial_{ij}\varphi(v,t) + \partial_{ij}\varphi(w,t))\, dvdwdt \\
&\quad + \sum_{i=1}^3 \int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v,t)f(w,t)(\mathrm{div}_v A[f])_i(v-w) \\
&\qquad \times (\partial_i\varphi(v,t) - \partial_j\varphi(w,t))\, dvdwdt.
\end{aligned}$$

Recently the authors in [13] improved the regularity of the weak solutions: let $f$ be a weak solution to the Landau equation as in (6); then

$$A[f] \in L^\infty(0,T;L^3_{loc}(\mathbb{R}^3)),\ \nabla a[f] \in L^\infty(0,T;L^{3/2}_{loc}(\mathbb{R}^3)),$$

and for all $\phi \in L^\infty(0,T;W_c^{1,\infty}(\mathbb{R}^3))$ the function $f$ satisfies

$$\int_0^T \langle \partial_t f\,,\phi\,\rangle dt + \int_0^T \int_{\mathbb{R}^3} (A[f]\nabla f - f\nabla a[f])\cdot\nabla\phi\,dvdt = 0. \tag{7}$$

Next we recall some well-known results used later in the manuscript. The first one concerns lower bounds for $a[f]$ and $A[f]$.

**Lemma 1.** *(Bound from below) There is a constant $c$ only determined by the mass, energy, and entropy of $f$, such that for all $v \in \mathbb{R}^3$*

$$\begin{aligned} a[f](v) &\geq c\langle v\rangle^{-1},\\ A[f](v) &\geq a^*(v)\mathbb{I} \geq c\langle v\rangle^{-3}\mathbb{I}, \end{aligned}$$

*where $\langle v\rangle := (1+|v|^2)^{1/2}$ and $a^*(v)$ is the smallest eigenvalue of $A[f]$ defined as*

$$a^*(v) = \inf_{e\in\mathbb{S}^2}(A[f](v)e,e).$$

**Lemma 2.** *(Propagation of moments, [5] Proposition 4.) Let $f$ be a weak solution to the Landau equation with initial datum $f_{in}$. Assume also that $f$ satisfies the conservation of mass, momentum and energy. For all $k \geq 0$ such that*

$$\int_{\mathbb{R}^3} f_{in}(1+|v|^2)^k\,dv < +\infty,$$

*we have that*

$$\sup_{[0,T]}\int_{\mathbb{R}^3} f(1+|v|^2)^k\,dv \leq C(1+T),$$

*where $C$ depends on the energy, mass, entropy and $k$-moments of the initial data.*

We also recall the Boltzmann H-Theorem: let $\rho_{in}$ denote the Maxwellian with same mass, center of mass, and energy as $f_{in}$. We have

$$\int_0^T \int_{\mathbb{R}^3} 4(A[f]\nabla f^{1/2},\nabla f^{1/2}) - f^2\,dvdt \leq H(f_{in}) - H(\rho_{in}). \tag{8}$$

## 3. The $\varepsilon$-Poincaré inequality

In this section we present a weighted Poincaré inequality; this inequality plays a key role in the proof of Theorem 1. It is an adaptation of another inequality proven in [11] and is based on the general weighted Poincaré-Sobolev inequality shown in [16].

**Theorem 2.** *Let $N \geq 1$ and assume there exist $s > 1$, a nonnegative function $f$ and a modulus of continuity $\eta(\cdot)$ such that for any cube $Q \subset \mathbb{R}^3$ with side length $r \in (0,1)$ the following inequality holds:*

$$|Q|^{\frac{1}{3}} \left( \frac{1}{|Q|} \int_Q (1+|v|)^{Ns/2} f^s \, dv \right)^{\frac{1}{2s}} \left( \frac{1}{|Q|} \int_Q (1+|v|)^{3s} \, dv \right)^{\frac{1}{2s}} \leq \eta(r). \tag{9}$$

*Then, given any $\varepsilon \in (0,1)$, for any smooth functions $\phi$ we have the the following $\varepsilon$-Poincaré inequality:*

$$\int_{\mathbb{R}^3} (1+|v|)^{N/2} f \phi^2 \, dv \leq \varepsilon \int_{\mathbb{R}^3} (1+|v|)^{-3} |\nabla \phi|^2 \, dv + \tilde{\eta}(\varepsilon) \int_{\mathbb{R}^3} \phi^2 \, dv, \tag{10}$$

*where $\tilde{\eta} : (0,1) \mapsto \mathbb{R}$ is a decreasing function with $\tilde{\eta}(0+) = \infty$ determined by $\eta$.*

*Proof.* The proof can be found in Theorem 2.7 in [11]. □

The validity of (10) depends on certain properties of the function $f$, see (9). The next proposition shows that (9) is satisfied if $f \in L^\infty(0,T, L^p \cap L^1(\mathbb{R}^3))$ for some $p > \frac{3}{2}$ and has high enough moments.

**Proposition 1.** *Let $f$ be a nonnegative function with $f \in L^\infty(0,T, L^p \cap L^1(\mathbb{R}^3))$ for some $p > \frac{3}{2}$. Assume also that $f$ has bounded moments of order $\frac{(N+6)(p-1)}{p-3/2}$. Then there exists a number $s \leq 2$ with $\frac{3}{2} < s < p$ and a modulus of continuity $\eta(r)$ such that for any $Q$ cube in $\mathbb{R}^3$ with length $r$ inequality (9) holds.*

*Proof.* Let $Q$ a cube of length $r$ and center $v_0$. Hölder inequality yields

$$\begin{aligned} \int_Q (1+|v|)^{Ns/2} f^s \, dv &\leq \left( \int_Q (1+|v|)^{Ns\alpha/2} f \, dv \right)^{\frac{1}{\alpha}} \left( \int_Q f^{(s-1/\alpha)\alpha'} \, dv \right)^{\frac{1}{\alpha'}} \\ &\leq \|f\|_{L^p}^{\alpha' p} \left( \int_Q (1+|v|)^{Ns\alpha/2} f \, dv \right)^{\frac{1}{\alpha}}, \end{aligned}$$

by choosing $\alpha = \frac{p-1}{p-s}$, $p > s$, so that

$$(s - 1/\alpha)\alpha' = p.$$

Then

$$\begin{aligned}
&|Q|^{\frac{1}{3}}\left(\frac{1}{|Q|}\int_Q(1+|v|)^{Ns/2}f^s\,dv\right)^{\frac{1}{2s}}\left(\frac{1}{|Q|}\int_Q(1+|v|)^{3s}\,dv\right)^{\frac{1}{2s}}\\
&\quad\le C(\|f\|_{L^p})|Q|^{\frac{1}{3}-\frac{1}{2s}}(1+|v_0|)^{3/2}\left(\int_Q(1+|v|)^{Ns\alpha/2}f\,dv\right)^{\frac{1}{2s\alpha}}\\
&\quad\le C(\|f\|_{L^p})|Q|^{\frac{1}{3}-\frac{1}{2s}}\left(\int_Q(1+|v|)^{(N+6)s\alpha/2}f\,dv\right)^{\frac{1}{2s\alpha}}\\
&\quad\le C(\|f\|_{L^p})|Q|^{\frac{1}{3}-\frac{1}{2s}}\|f\langle v\rangle^{(N+6)s\alpha/2}\|_{L^1}^{\frac{1}{2s\alpha}}.
\end{aligned}$$

The modulus of continuity $\eta(r)$ is proportional to $C(T)r^{1-\frac{3}{2s}}$, where $C(T)$ depends on the $\frac{(N+6)s\alpha}{2}$-moments of $f$ and on the $L^p$ norm of $f$. □

## 4. Higher integrability and weighted gradient estimates

The first immediate consequence of Proposition 1, Theorem 2 and Boltzmann's H Theorem is a $L^2((0,T),L^2(\mathbb{R}^3))$ integrability estimate for $f$.

**Theorem 3.** *Let $f$ be a solution to the Landau equation with initial datum $f_{in}$ such that $f\in L^\infty((0,T),L^p(\mathbb{R}^3))$ for some $p>3/2$. Assume moreover that*

$$\int_{\mathbb{R}^3} f_{in}(1+|v|)^k\,dv<+\infty,$$

*for any $1\le k\le\frac{6(p-1)}{p-3/2}$. Then $f\in L^2((0,T),L^2(\mathbb{R}^3))$ and*

$$\|f\|_{L^2((0,T),L^2(\mathbb{R}^3))}\le C(f_{in},T,\|f\|_{L^\infty(L^p)}).$$

*Proof.* The function $f$ satisfies the assumptions for (9), following Proposition 1. Then, combining (10) with $\phi=\sqrt{f}$, $N=0$ and (8), we get:

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^3}f^2\,dvdt&\le\varepsilon\int_0^T\int_{\mathbb{R}^3}(A[f]\nabla\sqrt{f},\nabla\sqrt{f})\,dvdt+C(f_{in})\tilde\eta(\varepsilon)T\\
&\le\varepsilon\left(H(f_{in})-H(\rho_{f_{in}})+\int_0^T\int_{\mathbb{R}^3}f^2\,dvdt\right)+C(f_{in})\tilde\eta(\varepsilon)T.
\end{aligned}$$

The thesis follows by choosing $\varepsilon<1$. □

Once we have the bound $L^2((0,T),L^2(\mathbb{R}^3))$, we can get an estimate for $f$ in the space $L^\infty((0,T),L^2(\mathbb{R}^3))$, as shown in the following theorem:

**Theorem 4.** *Let $f$ and $f_{in}$ as is Theorem 3. Assume moreover that $f_{in} \in L^2(\mathbb{R}^3)$. Then $f \in L^\infty((0,T), L^2(\mathbb{R}^3))$ and $\|f\|_{L^\infty(L^2)} \leq C(f_{in}, T, \|f\|_{L^\infty(L^p)})$.*

*Proof.* The proof is a simple consequence of Gronwall's lemma. Take $f$ as test function in (7) and integrate by parts; this gives

$$\begin{aligned}(11)\qquad \int_{\mathbb{R}^3} f^2(T)\,dv = \int_{\mathbb{R}^3} f_{in}^2\,dv - \int_0^T \int_{\mathbb{R}^3} (A[f]\nabla f, \nabla f)\,dvdt \\ + \int_0^T \int_{\mathbb{R}^3} f^3\,dvdt.\end{aligned}$$

Since $f \in L^2((0,T), L^2(\mathbb{R}^3))$ by Theorem 3, we use (10) with $\phi = f$, $N = 0$, and $\varepsilon < 1$ and get

$$\int_{\mathbb{R}^3} f^2(T)\,dv \leq \int_{\mathbb{R}^3} f_{in}^2\,dv + \frac{1}{\varepsilon}\int_0^T \int_{\mathbb{R}^3} f^2\,dvdt.$$

Gronwall's inequality yields

$$\int_{\mathbb{R}^3} f^2(T)\,dv \leq e^{\frac{1}{\varepsilon}T} \int_{\mathbb{R}^3} f_{in}^2\,dv.$$

Note that the above computations are formal. To make them rigorous one first considers a truncation of $f$ of the form $f\eta_R(v)$ where $\eta_R(v) = \eta(v/R)$ and $\eta(v) = 1$ inside a ball of center 0 and radius 1, $\eta(v) = 0$ outside the ball of center 0 and radius 2 and smooth in between. Thanks to the condition that $f \in L^\infty((0,T), L^p \cap L^1(\mathbb{R}^3))$ for some $p > 3/2$ both $A[f]$ and $a[f]$ are uniformly bounded and one can take $f\eta_R(v)$ as test function in (7). Since $\nabla \eta_R \to 0$ as $R \to +\infty$ and both $A[f]$ and $a[f]$ are uniformly bounded one can pass to the limit $R \to +\infty$ and obtain (11). □

For proving our uniqueness result, we also need the following weighted gradient bound.

**Proposition 2.** *Let $N \geq 0$ and $f \in L^\infty(0,T,L^p)$ with $p > 3/2$ be a weak solution to the Landau equation with initial data $f_{in} \in L^1 \cap L^2(\mathbb{R}^3)$ and $\frac{(4N+6)(p-1)}{p-3/2}$-moments bounded. Let moreover $\int_{\mathbb{R}^3} f_{in}^2 (1+|v|)^N\,dv < +\infty$. For* *any $T > 0$ we have*

$$\int_{\mathbb{R}^3} f^2(1+|v|)^N \, dv + \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} (1+|v|)^{N-3}|\nabla f|^2 \, dvdt \\ \leq C(T, f_{in}, \|f\|_{L^\infty(0,T,L^p)}).$$

*Proof.* There exists a universal constant $C$ such that

(12) $$\|a[f]\|_{L^\infty(\mathbb{R}^3)} \leq C\|f\|_{L^1}^{1-q/3}\|f\|_{L^{q/(q-1)}}^{q/3}, \quad \forall\, 1 \leq q < 3.$$

In particular

(13) $$\|a[f]\|_{L^\infty(\mathbb{R}^3)} \leq C\|f\|_{L^1}^{1/3}\|f\|_{L^2}^{2/3} \leq C\|f\langle v\rangle^{m/2}\|_{L^2},$$

for $m > 3$ and $\langle v\rangle := (1+|v|^2)^{1/2}$. For large $v$, one can obtain a sharper estimate:

(14) $$a[f](v,t) \leq \frac{C(\|f\|_{L^{3/2+}}, f_{in}, T)}{1+|v|}, \quad \forall v \in \mathbb{R}^3 \; t \in [0,T].$$

Let $|v|$ be large enough; for $2 \geq s > 3/2$ Hölder inequality yields:

$$|a[f]| \leq |v|^{\frac{3-s'}{s'}} \left(\int_{B_{\frac{|v|}{2}}(|v|)} f^s \, dy\right)^{1/s} + \frac{1}{|v|}\|f\|_{L^1(\mathbb{R}^3)} \\ \leq c\frac{|v|^{\frac{3-s'}{s'}}}{(1+|v|)^{\lambda/s}} \left(\int_{\mathbb{R}^3} f^s(1+|y|)^\lambda \, dy\right)^{1/s} + \frac{1}{|v|}\|f\|_{L^1(\mathbb{R}^3)},$$

with $\frac{1}{s} + \frac{1}{s'} = 1$ and $s' < 3$. Chose $\lambda = 3(s-1)$ so that $\frac{3-s'}{s'} - \frac{\lambda}{s} = -1$ and get

$$|a[f](v)| \leq \frac{1}{(1+|v|)} \left(\int_{\mathbb{R}^3} f^s(1+|y|)^{3(s-1)} \, dy\right)^{1/s} + \frac{1}{|v|}\|f\|_{L^1(\mathbb{R}^3)}.$$

Hölder's inequality yields

$$\int_{\mathbb{R}^3} f^s(1+|y|)^{3(s-1)} \, dy \leq \left(\int_{\mathbb{R}^3} f^p dy\right)^{1/\alpha'} \left(\int_{\mathbb{R}^3} f(1+|y|)^{\frac{3(p-1)}{(p-3/2)}} dy\right)^{1/\alpha},$$

with $\alpha = (p-1)/(p-3/2)$. We use Lemma 2 to bound the last integral and get (14).

Take now $\phi := f(1+|v|)^N$ as test function in (7):

$$\begin{aligned}\int_{\mathbb{R}^3} f_t f(1+|v|)^N\,dv \le &-\int_{\mathbb{R}^3}\langle A[f](1+|v|)^N\nabla f,\nabla f\rangle\,dv\\ &+N\int_{\mathbb{R}^3} a[f](1+|v|)^{N-1} f|\nabla f|\,dv\\ &+\int_{\mathbb{R}^3} f(1+|v|)^N\nabla f\cdot\nabla a[f]\,dv\\ &+N\int_{\mathbb{R}^3} f^2(1+|v|)^{N-1}|\nabla a[f]|\,dv\\ =&\ I_1+I_2+I_3+I_4.\end{aligned}$$

By Lemma 1 we have

$$I_1 \le -c_1\int_{\mathbb{R}^3}(1+|v|)^{N-3}|\nabla f|^2\,dv.$$

Using (12) to bound the $L^\infty$-norm of $a[f]$, Young's inequality yields

$$I_2 \le \omega\int_{\mathbb{R}^3}(1+|v|)^{N-3}|\nabla f|^2\,dv+\frac{C^2}{\omega}\int_{\mathbb{R}^3}(1+|v|)^{N-2}f^2\,dv,$$

where $C$ only depends on the $L^\infty(0,T,L^p)$ and on the $L^\infty(0,T,L^1)$-norm of $f$. From integration by parts one obtains

$$\begin{aligned}I_3+I_4 &\le c_1\int_{\mathbb{R}^3}(1+|v|)^{N-1}f^2|\nabla a[f]|\,dv + c_2\int_{\mathbb{R}^3}(1+|v|)^N f^3\,dv\\ &\lesssim \int_{\mathbb{R}^3}|\nabla a[f]|^3\,dv+\int_{\mathbb{R}^3}(1+|v|)^{2N}f^3\,dv\\ &\le C\|f\|^3_{L^{3/2}}+\varepsilon\int_{\mathbb{R}^3}(1+|v|)^{-3}|\nabla f|^2\,dv+\tilde\eta(\varepsilon)\int_{\mathbb{R}^3}f^2\,dv,\end{aligned}$$

using Hardy-Littlewood-Sobolev inequality

$$\|\nabla a[f]\|_{L^{3p/(3-p)}(\mathbb{R}^3)} \le C\|f\|_{L^p(\mathbb{R}^3)} \quad \forall\, p\in(1,2], \tag{15}$$

and (10) with weight $(1+|v|)^{2N}f$ to bound the weighted cubic norm of $f$. Summarizing we have

$$\begin{aligned}\partial_t\int_{\mathbb{R}^3} f^2(1+|v|)^N\,dv \le &-(c-\varepsilon)\int_{\mathbb{R}^3}(1+|v|)^{N-3}|\nabla f|^2\,dv\\ &+C\int_{\mathbb{R}^3}(1+|v|)^{N-2}f^2\,dv + C(\|f\|_{L^p}).\end{aligned}$$

Taking $\varepsilon$ sufficiently small we get the desired estimate. $\square$

## 5. The contraction argument

We have the following uniqueness result.

**Theorem 5.** *Let $u, \phi \in L^\infty(0, T, L^p)$ for some $p > 3/2$ be two solutions to the Landau equation with nonnegative initial data $f_{in}$ such that*

$$\int_{\mathbb{R}^3} f_{in} \langle v \rangle^k \, dv < +\infty, \quad \int_{\mathbb{R}^3} f_{in}^2 \langle v \rangle^{10} \, dv < +\infty,$$

*for any $0 \le k \le \frac{46(p-1)}{p-3/2}$. Then $u = \phi$ .*

*Proof.* Define $w = u - \phi$. Take $w\langle v \rangle^m$ with $m = 4$ as test function in the resulting equation for $w$, After integration by parts one obtains

$$\begin{aligned}
\int_{\mathbb{R}^3} w^2(T) \langle v \rangle^m \, dv = & -\int_0^T \int_{\mathbb{R}^3} A[u] \nabla w \cdot \nabla (w \langle v \rangle^m) \, dv dt \\
& - \int_0^T \int_{\mathbb{R}^3} A[w] \nabla \phi \cdot \nabla (w \langle v \rangle^m) \, dv dt \\
& + \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} w \nabla a[u] \cdot \nabla (w \langle v \rangle^m) \, dv dt \\
& + \int_0^T \int_{\mathbb{R}^3} \phi \nabla a[w] \cdot \nabla (w \langle v \rangle^m) \, dv dt \\
& + \int_{\mathbb{R}^3} w_{in}^2 \langle v \rangle^m \, dv \\
=: & \, I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Using Lemma 1 one gets

$$\begin{aligned}
I_1 \le & -(1-\varepsilon) \int_0^T \int_{\mathbb{R}^3} \frac{\langle v \rangle^m}{(1+|v|)^3} |\nabla w|^2 \, dv dt \\
& + \frac{m^2}{\varepsilon} \int_0^T \|A[u]\|_{L^\infty} \int_{\mathbb{R}^3} w^2 \langle v \rangle^m \, dv dt.
\end{aligned}$$

To estimate $\|A[u]\|_{L^\infty}$ we use (12). For $I_2$, we use (13) with $m = 4$ and Young's inequality:

$$
\begin{aligned}
I_2 \le{}& \varepsilon \int_0^T \int_{\mathbb{R}^3} \frac{\langle v\rangle^m}{(1+|v|)^3} |\nabla w|^2 \, dvdt \\
&+ \frac{1}{\varepsilon} \int_0^T \int_{\mathbb{R}^3} A^2[w] \langle v\rangle^m (1+|v|)^3 |\nabla \phi|^2 \, dvdt \\
&+ m \int_0^T \int_{\mathbb{R}^3} w A[w] \langle v\rangle^{m-2} \nabla\phi \cdot v \, dvdt \\
\le{}& \varepsilon \int_0^T \int_{\mathbb{R}^3} \frac{\langle v\rangle^m}{(1+|v|)^3} |\nabla w|^2 \, dvdt \\
&+ \frac{1}{\varepsilon} \int_0^T \int_{\mathbb{R}^3} A^2[w] \langle v\rangle^m (1+|v|)^3 |\nabla \phi|^2 \, dvdt \\
&+ m \int_0^T \int_{\mathbb{R}^3} \langle v\rangle^m w^2 \, dvdt + m \int_0^T \int_{\mathbb{R}^3} A^2[w] \langle v\rangle^{m-2} |\nabla\phi|^2 \, dvdt \\
\le{}& \varepsilon \int_0^T \int_{\mathbb{R}^3} \frac{\langle v\rangle^m}{(1+|v|)^3} |\nabla w|^2 \, dvdt + \int_0^T B(t) \int_{\mathbb{R}^3} \langle v\rangle^m w^2 \, dvdt
\end{aligned}
$$

with $B(t) := \int_{\mathbb{R}^3} \langle v\rangle^{m+3} |\nabla\phi|^2 \, dv + 1$. Note that $B(t)$ is integrable, as shown in Proposition 2 for $N = m + 6 = 10$. We rewrite $I_3$ as

$$
\begin{aligned}
I_3 ={}& \frac{1}{4} \int_0^T \int_{\mathbb{R}^3} \langle v\rangle^{-m} \nabla a[u] \cdot \nabla (w^2 \langle v\rangle^{2m}) \, dvdt \\
={}& \frac{1}{4} \int_0^T \int_{\mathbb{R}^3} u \langle v\rangle^m w^2 \, dvdt + m \int_0^T \int_{\mathbb{R}^3} w \nabla w \cdot v \langle v\rangle^{m-2} a[u] \, dvdt \\
&+ \frac{m}{2} \int_0^T \int_{\mathbb{R}^3} w^2 v \cdot \nabla \langle v\rangle^{m-2} a[u] \, dvdt + \frac{3m}{2} \int_0^T \int_{\mathbb{R}^3} w^2 \langle v\rangle^{m-2} a[u] \, dvdt \\
\le{}& \frac{1}{4} \int_0^T \int_{\mathbb{R}^3} u \langle v\rangle^m w^2 \, dvdt + c_m \|a[u]\|_{L^\infty(\mathbb{R}^3)} \int_0^T \int_{\mathbb{R}^3} w^2 \langle v\rangle^{m-2} \, dvdt \\
&+ \varepsilon \int_0^T \int_{\mathbb{R}^3} \frac{\langle v\rangle^m}{(1+|v|)^3} |\nabla w|^2 \, dvdt + \frac{1}{\varepsilon} \int_0^T \int_{\mathbb{R}^3} w^2 a^2[u] \langle v\rangle^{m+1} \, dvdt.
\end{aligned}
$$

To bound the first integral we use (10) with $N = 2m$ and get

$$
\int_{\mathbb{R}^3} u \langle v\rangle^m w^2 \, dv \le \varepsilon \int_{\mathbb{R}^3} \frac{|\nabla w|^2}{(1+|v|)^3} \, dv + C \int_{\mathbb{R}^3} w^2 \, dv.
$$

This yields

$$I_3 \leq \varepsilon \int_0^T \int_{\mathbb{R}^3} \frac{\langle v\rangle^m}{(1+|v|)^3} |\nabla w|^2 \, dvdt + \int_0^T B_1(t) \int_{\mathbb{R}^3} \langle v\rangle^m w^2 \, dvdt,$$

with $B_1(t) := C(\|a[u]\|_{L^\infty(\mathbb{R}^3)} + \|a[u]\langle v\rangle\|_{L^\infty(\mathbb{R}^3)} + 1)$. Note that $B_1(t)$ is integrable, thanks to (14). Finally,

$$\begin{aligned}
I_4 \leq& \int_0^T \|\nabla a[w]\|_{L^6(\mathbb{R}^3)} \left\| \frac{\langle v\rangle^{m/2} \nabla w}{(1+|v|)^{3/2}} \right\|_{L^2(\mathbb{R}^3)} \left\| \phi \langle v\rangle^{m/2+3/2} \right\|_{L^3(\mathbb{R}^3)} \, dt \\
&+ m \int_0^T \|\nabla a[w]\|_{L^6(\mathbb{R}^3)} \left\| w \langle v\rangle^{m/2} \right\|_{L^2(\mathbb{R}^3)} \left\| \phi \langle v\rangle^{m/2-1} \right\|_{L^3(\mathbb{R}^3)} \, dt \\
\leq& \frac{1}{\varepsilon} \int_0^T \|w\|^2_{L^2(\mathbb{R}^3)} \left( \int_{\mathbb{R}^3} \phi^3 (1+|v|)^{3m/2+9/2} \, dv \right)^{2/3} \, dt \\
&+ \varepsilon \int_0^T \int_{\mathbb{R}^3} \frac{|\nabla w|^2 \langle v\rangle^m}{(1+|v|)^3} \, dvdt \\
&+ m \int_0^T \left\| w \langle v\rangle^{m/2} \right\|^2_{L^2(\mathbb{R}^3)} \left( \int_{\mathbb{R}^3} \phi^3 (1+|v|)^{3m/2-3} \, dv \right)^{1/3} \, dt.
\end{aligned}$$

Thanks again to (10), we get

$$\int_{\mathbb{R}^3} \phi^3 (1+|v|)^{3m/2+9/2} \, dv \leq C_{\varepsilon,m,\phi_{in}} \int_{\mathbb{R}^3} \frac{|\nabla \phi|^2}{(1+|v|)^3} \, dv + \int_{\mathbb{R}^3} \phi^2 \, dv,$$

and conclude that

$$I_4 \leq C_{\varepsilon,m,\phi_{in}} \int_0^T B_2(t) \left\| w \langle v\rangle^{m/2} \right\|^2_{L^2(\mathbb{R}^3)} \, dt + \varepsilon \int_0^T \int_{\mathbb{R}^3} \frac{|\nabla w|^2 \langle v\rangle^m}{(1+|v|)^3} \, dvdt$$

with

$$B_2(t) := \left( \int_{\mathbb{R}^3} \frac{|\nabla \phi|^2}{(1+|v|)^3} \, dv + \int_{\mathbb{R}^3} \phi^2 \, dv \right)^{2/3} + \left( \int_{\mathbb{R}^3} \frac{|\nabla \phi|^2}{(1+|v|)^3} \, dv + \int_{\mathbb{R}^3} \phi^2 \, dv \right)^{1/3}.$$

The function $B_2(t)$ is integrable thanks to Proposition (2) with $N = 0$.

Summarizing the estimates for $I_1, .., I_4$, for $\varepsilon$ small enough we get

$$\int_{\mathbb{R}^3} w^2(T)\langle v\rangle^m \, dv \le \int_{\mathbb{R}^3} w_{in}^2 \langle v\rangle^m \, dv + C\int_0^T (B(t) + B_1(t) + B_2(t)) \int_{\mathbb{R}^3} w^2 \langle v\rangle^m \, dvdt,$$

with $\int_0^T B(t) + B_1(t) + B_2(t) \, dt < +\infty$.

Since $w_{in}(\cdot) = 0$, Gronwall's inequality yields

$$\int_{\mathbb{R}^3} w^2(T)\langle v\rangle^m \, dv \le 0,$$

and this concludes the proof. □

## Acknowledgements

JLC is supported by NSTC 110-2115-M-003-019-MY3 and NSTC 111-2218-E-008-004-MBK. MPG is supported by NSF DMS-2019335. JLC and MPG would like to thank KTH Royal Institute of Technology and then NCTS Mathematics Division Taipei for their kind hospitality. We would like to express our thanks to the referee for his/her careful reading and valuable comments for this paper.

## References

[1] R. Alexandre and C. Villani. *On the Landau approximation in plasma physics.* Ann. Inst. Henri Poincaré, C Anal. Non Linéaire 21 (1) (2004) 61–95.

[2] K. Carrapatoso, L. Desvillettes, and L. He. *Estimates for the large time behavior of the Landau equation in the Coulomb case.* Arch. Ration. Mech. Anal., (2):381–420, 2017.

[3] S. Chanillo and R. Wheeden. *L-p estimates for fractional integrals and Sobolev inequalities with applications to Schrödinger operators.* Communications in partial differential equations, 10(9):1077–1116, 1985.

[4] S. Chanillo and R. Wheeden. *Weighted Poincaré and Sobolev inequalities and estimates for weighted Peano maximal functions.* American Journal of Mathematics, 107(5):1191–1226, 1985.

[5] L. Desvillettes. *Entropy dissipation estimates for the Landau equation in the Coulomb case and applications.* Journal of Functional Analysis 269 (2015) 1359–1403.

[6] N. Fournier. *Uniqueness of bounded solutions for the homogeneous Landau equation with a Coulomb potential.* Comm. Math. Phys. 299 (2010), no. 3, 765–782.

[7] F. Golse, M. Gualdani, C. Imbert, and A. Vasseur. *Partial regularity in time for the space homogeneous Landau equation with Coulomb potential.* Ann. Sci. Ec. Norm. Super. (4) 55 (2022), no. 6, 1575–1611.

[8] F. Golse, C. Imbert, and A. Vasseur. *Local regularity for the space-homogenous Landau equation with very soft potentials.* Preprin, `arXiv:2206.05155`.

[9] F. Golse, C. Imbert, C. Mouhot, and A. Vasseur. *Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 19 (2019), no. 1, 253–295.

[10] M. Gualdani and N. Guillen. *Estimates for radial solutions of the homogeneous Landau equation with Coulomb potential.* Analysis and PDE, 9(8):1772–1809, 2016.

[11] M. Gualdani and N. Guillen. *On $A_p$ weights and the homogeneous Landau equation.* Calc. Var. Partial Differential Equations 58 (2019), no. 1, Art. 17, 55 pp.

[12] M. Gualdani and N. Guillen. *Hardy's inequality and the isotropic Landau equation.* Funct. Anal. 283 (2022), no. 6, Paper No. 109559.

[13] M. Gualdani and N. Zamponi. *Global existence of weak even solutions for an isotropic Landau equation with Coulomb potential.* SIAM J. Math. Anal. 50 (2018), no. 4, 3676–3714.

[14] Y. Guo. *The Landau equation in a periodic box.* Communications in mathematical physics, 231(3):391–434, 2002.

[15] J. Krieger and R.Strain. *Global solutions to a non-local diffusion equation with quadratic nonlinearity.* Comm. Partial Differential Equations, 37(4):647–689, 2012.

[16] E. Sawyer and R. Wheeden. *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces.* American Journal of Mathematics, 114(4):813–874, 1992.

[17] L. Silvestre. *Upper bounds for parabolic equations and the Landau equation.* J. Differential Equations, 262 (2017), no. 3, 3034–3055.

[18] R. Strain and Z. Wang. *Uniqueness of bounded solutions for the homogeneous relativistic Landau equations with Coulomb interactions.* Quart. Appl. Math., 2019.

[19] C. Villani. *On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations.* Arch. Ration. Mech. Anal. 143 (3) (1998) 273–307.

[20] C. Villani. *A review of mathematical topics in collisional kinetic theory. Handbook of mathematical fluid dynamics.* 1:71–74, 2002.

Department of Mathematics, National Taiwan Normal University
Taipei 11677, Taiwan
*E-mail address*: chern@gapps.ntnu.edu.tw

Department of Mathematics, The University of Texas at Austin
Austin, TX 78712, USA
*E-mail address*: gualdani@math.utexas.edu

Received March 23, 2020
Accepted July 21, 2020