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On the semi-simplicity conjecture for $\mathbb{Q}^{\mathrm{ab}}$

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We show that the semi-simplicity conjecture for finitely generated fields follows from the conjunction of the semi-simplicity conjecture for finite fields and for the maximal abelian extension of the field of rational numbers.

Notation

Let k be a field, $\bar{k}$ an algebraic closure, and ℓ a prime number different from the characteristic of k. Write Γ_k for the Galois group $\mathrm{Gal}(\bar{k}/k)$. We denote by $\mathbf{Rep}_{\mathbb{Q}_\ell}(\Gamma_k)$ the category of finite-dimensional ℓ-adic representations of Γ_k. If X is a smooth and projective variety over k, the profinite group Γ_k acts continuously on $H^\bullet_{\text{ét}}(X \otimes_k \bar{k}, \mathbb{Q}_\ell)$. We write $\mathbf{Rep}_{\mathbb{Q}_\ell}(\Gamma_k)^{\mathrm{geo}} \subseteq \mathbf{Rep}_{\mathbb{Q}_\ell}(\Gamma_k)$ for the smallest strictly full abelian $\otimes$-subcategory of $\mathbf{Rep}_{\mathbb{Q}_\ell}(\Gamma_k)$ closed under duals and subobjects, which contains all the ℓ-adic representations of Γ_k of this form. We say that an object in $\mathbf{Rep}_{\mathbb{Q}_\ell}(\Gamma_k)^{\mathrm{geo}}$ is an *ℓ-adic representation of Γ_k coming from geometry.*

1. Introduction

1.1.

We consider the following statement.

> $S(k)$: For every prime number ℓ different from the characteristic of k, an ℓ-adic representation of Γ_k coming from geometry is semi-simple.

Grothendieck and Serre conjectured that for every finitely generated field k, the assertion $S(k)$ is true. This conjecture is commonly known as the *semi-simplicity conjecture* (see [Fu99] and [Kah04] for an overview). Note that the conjecture predicts that $S(k)$ is true even if k is an infinite Galois extension of a finitely generated field (see Lemma 2.3). For this reason, Grothendieck–Serre semi-simplicity conjecture predicts, for example, that if $\mathbb{Q}^{\mathrm{ab}}$ is the maximal abelian extension of $\mathbb{Q}$, then $S(\mathbb{Q}^{\mathrm{ab}})$ is true. In this article we prove the following implication.

Theorem 1.2. *Let k be a Galois extension of a finitely generated field. The conjunction of $S(\mathbb{F}_p)$ for every prime number p and $S(\mathbb{Q}^{\mathrm{ab}})$ implies $S(k)$.*

Let us briefly sketch the idea of the proof. Deligne proved in [Del80, Théorème 3.4.1.(iii)] that $S(k_\infty)$ is true when $k_\infty/\bar{\mathbb{F}}_p$ is a finitely generated field extension. As a consequence, Fu proved in [Fu99, Corollary] that for every finitely generated field k of positive characteristic p, the assertion $S(\mathbb{F}_p)$ implies $S(k)$ using the exact sequence

$$1 \to \Gamma_{k\bar{\mathbb{F}}_p} \to \Gamma_k \to \widehat{\mathbb{Z}} \to 1.$$

More precisely, Fu combined the semi-simplicity of the restriction to $\Gamma_{k\bar{\mathbb{F}}_p}$ of a representation coming from geometry, provided by [Del80, Théorème 3.4.1.(iii)], and the semi-simplicity of one Frobenius at closed point of a model of k. To prove Theorem 1.2 we use the analogy between $\mathbb{Q}$ and $\mathbb{F}_p(t)$, or, more precisely, between the Galois extensions $\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}$ and $\bar{\mathbb{F}}_p(t)/\mathbb{F}_p(t)$. We have the exact sequence

$$1 \to \Gamma_{\mathbb{Q}^{\mathrm{ab}}} \to \Gamma_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^\times \to 1,$$

where $\widehat{\mathbb{Z}}^\times = \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ by Kronecker–Weber theorem. Since $\widehat{\mathbb{Z}}^\times$ is not pro-cyclic, to readapt Fu's proof in this case we fix a quotient $\delta_\ell : \Gamma_{\mathbb{Q}} \twoheadrightarrow \mathbb{Z}_\ell$ (unique up to $\mathbb{Z}_\ell^\times$-multiplication) which should be thought as an ℓ-adic degree function. We prove the following result.

Proposition 1.3. *Let ρ be an ℓ-adic representation of $\Gamma_{\mathbb{Q}}$ which is semi-simple when restricted to $\Gamma_{\mathbb{Q}^{\mathrm{ab}}}$. If there exists an element $g \in \Gamma_{\mathbb{Q}}$ such that $\delta_\ell(g) \neq 0$ and $\rho(g)$ is semi-simple then ρ is a semi-simple representation of $\Gamma_{\mathbb{Q}}$.*

Thanks to Proposition 1.3 and some classical specialisation arguments we are then able to prove Theorem 1.2. In §3 we propose a possible approach to prove $S(\mathbb{Q}^{\mathrm{ab}})$ which mimics the strategy of [Del80, Théorème 3.4.1.(iii)].

2. Our main results

To prove Theorem 1.2 we work with an *ad hoc* notion of a Weil group of $\mathbb{Q}$.

Definition 2.1. For $g \in \Gamma_{\mathbb{Q}}$, let $W_{\mathbb{Q},g}$ be the semi-direct product $\Gamma_{\mathbb{Q}^{\mathrm{ab}}} \rtimes \mathbb{Z}$, where $\mathbb{Z}$ is endowed with the discrete topology and $1 \in \mathbb{Z}$ acts on $\Gamma_{\mathbb{Q}^{\mathrm{ab}}}$ as g acts on $\Gamma_{\mathbb{Q}^{\mathrm{ab}}}$ by conjugation. The group $W_{\mathbb{Q},g}$ is naturally endowed with a continuous morphism $W_{\mathbb{Q},g} \to \Gamma_{\mathbb{Q}}$ sending $1 \in \mathbb{Z}$ to g.

For an ℓ-adic representation ρ of $\Gamma_{\mathbb{Q}}$, let Π be the image of $\Gamma_{\mathbb{Q}}$ and let Π^0 be the image of $\Gamma_{\mathbb{Q}^{\mathrm{ab}}}$. Write $\bar{\Pi}$ for the quotient Π/Π^0 and π for the natural projection $\pi : \Pi \twoheadrightarrow \bar{\Pi}$. We have the following commutative diagram of profinite groups with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_{\mathbb{Q}^{\mathrm{ab}}} & \longrightarrow & \Gamma_{\mathbb{Q}} & \longrightarrow & \widehat{\mathbb{Z}}^{\times} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \rho & & \downarrow & & \\ 1 & \longrightarrow & \Pi^0 & \longrightarrow & \Pi & \xrightarrow{\pi} & \bar{\Pi} & \longrightarrow & 1. \end{array}$$

The lower row is an exact sequence of compact ℓ-adic Lie groups.

Proof of Proposition 1.3. If ρ is a representation satisfying the hypothesis in Proposition 1.3, arguing as in the proof of [Fu99, Theorem], we deduce that the restriction $\rho|_{W_{\mathbb{Q},g}}$ is semi-simple. To end the proof we need the following lemma. □

Lemma 2.2. *Let ρ be an ℓ-adic representation of $\Gamma_{\mathbb{Q}}$. If for some $g \in \Gamma_{\mathbb{Q}}$ satisfying $\delta_\ell(g) \neq 0$ the restriction of ρ to $W_{\mathbb{Q},g}$ is semi-simple, then ρ is a semi-simple representation of $\Gamma_{\mathbb{Q}}$.*

Proof. Let Π_W be the closure of the image of $W_{\mathbb{Q},g}$ in Π. In light of [Fu99, Lemma 1], it is enough to show that Π_W has finite index in Π. Since Π_W contains Π^0, if we set $\bar{\Pi}_W := \pi(\Pi_W)$, we have to show that $\bar{\Pi}_W$ has finite index in $\bar{\Pi}$. Note that $\bar{\Pi}$, being a commutative compact ℓ-adic Lie group, it is isomorphic to $\mathbb{Z}_\ell^{\oplus n} \oplus M$ where $n \in \mathbb{Z}_{\geq 0}$ and M is a finite commutative group. In addition, since $\widehat{\mathbb{Z}}^{\times}$ surjects to $\mathbb{Z}_\ell^{\oplus n}$, the exponent n can be 0 or 1. We distinguish the two cases. If $n = 0$, then $\bar{\Pi}$ is finite and the result holds trivially. If $n = 1$, the element $\pi(\rho(g)) \in \bar{\Pi}_W$ has infinite order because $\delta_\ell(g) \neq 0$ (we are using that the quotient $\delta_\ell : \Gamma_{\mathbb{Q}} \twoheadrightarrow \mathbb{Z}_\ell$ is unique up to $\mathbb{Z}_\ell^{\times}$-multiplication). This implies that the profinite subgroup $\bar{\Pi}_W \subseteq \bar{\Pi} \simeq \mathbb{Z}_\ell \oplus M$ has finite index, as we wanted. □

Besides Proposition 1.3, in order to prove Theorem 1.2 we also need the following lemma.

Lemma 2.3. *If k'/k is a Galois extension, then $S(k)$ implies $S(k')$. On the other hand, if k'/k is finite (not necessarily Galois), then $S(k')$ implies $S(k)$.*

Proof. Suppose first that k'/k is Galois and $S(k)$ is true. If X' is a smooth projective variety over k', there exists a finite field extension k''/k such that X'/k' descends to X''/k''. We have that $H^\bullet_{\text{ét}}(X' \otimes_{k'} \bar{k}, \mathbb{Q}_\ell) = H^\bullet_{\text{ét}}(X'' \otimes_{k''} \bar{k}, \mathbb{Q}_\ell)$ is a direct summand of $H^\bullet_{\text{ét}}(X'' \otimes_k \bar{k}, \mathbb{Q}_\ell)$. Since the representation of Γ_k on $H^\bullet_{\text{ét}}(X'' \otimes_k \bar{k}, \mathbb{Q}_\ell)$ is semi-simple and $\Gamma_{k'}$ is a normal subgroup of Γ_k, we deduce that the representation of $\Gamma_{k'}$ on $H^\bullet_{\text{ét}}(X' \otimes_{k'} \bar{k}, \mathbb{Q}_\ell)$ is semi-simple. This shows that $S(k')$ is true. Conversely, by [Fu99, Lemma 1], if k'/k is finite then $S(k')$ implies $S(k)$. □

Proof of Theorem 1.2. If k is of positive characteristic p, combining [Fu99, Corollary] and Lemma 2.3, we deduce that $S(\mathbb{F}_p)$ implies $S(k)$. In characteristic 0, thanks to Serre's specialisation argument in [Ser81, 1st Letter to Ribet] and Lemma 2.3, we may assume $k = \mathbb{Q}$. Let X be a smooth projective variety over $\mathbb{Q}$. We choose a prime number p where X admits a smooth projective reduction $\tilde{X}/\mathbb{F}_p$. Since we are assuming that $S(\mathbb{F}_p)$ is true, the Frobenius acting on $H^\bullet_{\text{ét}}(\tilde{X} \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p, \mathbb{Q}_\ell)$ is semi-simple. If $F_p \in \Gamma_{\mathbb{Q}}$ is a lift of the Frobenius at p, by the smooth and proper base-change theorem, the action of F_p on $H^\bullet_{\text{ét}}(X \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_\ell)$ is semi-simple as well. Note that $\delta_\ell(F_p) \neq 0$, since the cyclotomic character $\Gamma_{\mathbb{Q}} \twoheadrightarrow \mathbb{Z}_\ell^\times$ sends F_p to p, which has infinite order. Thanks to Proposition 1.3 applied with $g = F_p$, we deduce that the representation of $\Gamma_{\mathbb{Q}}$ on $H^\bullet_{\text{ét}}(X \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_\ell)$ is semi-simple. This yields the desired result. □

Remark 2.4. In [Kah04, Remark 8.2] the author states a variant of Theorem 1.2, but in the proof he sketches he treats K^{ab} as it was $K(\zeta_{\ell^\infty})$. In a private communication he agreed that his proof of the stated result had a gap and to prove Theorem 1.2 one needs an additional argument as the one proposed here.

3. Final comments on $S(\mathbb{Q}^{\text{ab}})$

We would like to speculate a bit more on $S(\mathbb{Q}^{\text{ab}})$. Continuing the previous analogy, we wonder whether is it possible to prove $S(\mathbb{Q}^{\text{ab}})$ via a suitable theory of weights. In this case one cannot hope that every pure ℓ-adic representation of $\Gamma_{\mathbb{Q}}$ is semi-simple when restricted to $\Gamma_{\mathbb{Q}^{\text{ab}}}$, as we illustrate in the following example.

Example 3.1. Let $K/\mathbb{Q}$ be an imaginary quadratic extension and let K_∞^-/K be the anti-cyclotomic $\mathbb{Z}_\ell$-extension of K. The Galois group $\mathrm{Gal}(K_\infty^-/\mathbb{Q})$ is isomorphic to $\mathbb{Z}_\ell \rtimes \mathbb{Z}/2$ where $\mathbb{Z}/2$ acts on $\mathbb{Z}_\ell$ via multiplication by -1. Write χ for the non-trivial character of $\mathbb{Z}/2$. There exists a non-trivial extension of ℓ-adic representations of $\mathrm{Gal}(K_\infty^-/\mathbb{Q})$

$$0 \to \chi \to V \to \mathbb{Q}_\ell \to 0$$

constructed by sending

$$(1,0) \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } (0,1) \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This extension is non-trivial when restricted to $\Gamma_{\mathbb{Q}^{ab}}$ because K_∞^- is not in $\mathbb{Q}^{ab}$. On the other hand, V is pure of weight 0.

Remark 3.2. Following the philosophy of Fontaine–Mazur conjecture, to exclude extensions as the one presented in the previous example, one should add conditions coming from *ℓ-adic Hodge theory*. Indeed, in Example 3.1, if λ is a finite place of K dividing ℓ, the restriction $V|_{\Gamma_{K_\lambda}}$ is a unipotent non-trivial representation. Therefore, $V|_{\Gamma_{K_\lambda}}$ is not Hodge–Tate by [BC09, §2.4.5]. On the other hand, we have that the ℓ-adic representations coming from geometry are de Rham at the finite places over ℓ by [Fal89, Theorem 8.1].

3.3.

We would like to end this section presenting a possible approach to $S(\mathbb{Q}^{ab})$ which takes into account Example 3.1. In order to prove $S(\mathbb{Q}^{ab})$, one has to show that pure extensions $0 \to W_1 \to W \to W_2 \to 0$ of representations of $\Gamma_{\mathbb{Q}^{ab}}$ coming from geometry are all trivial. Since $\mathrm{Ext}^1_{\Gamma_k}(W_2, W_1) = \mathrm{Ext}^1_{\Gamma_k}(\mathbb{Q}_\ell, W_2^\vee \otimes W_1)$, one can simply focus on the extensions of the trivial representation by a pure representation V of weight 0. Let $N \neq 0$ be a multiple of the product of all the prime numbers where V is ramified and let $\mathbb{Z}[\zeta_\infty, N^{-1}] \subseteq \mathbb{Q}^{ab}$ be the ring generated over $\mathbb{Z}$ by all the roots of unity and N^{-1}. We consider the vector space

$$H := H^1_{\text{ét}}(\mathrm{Spec}(\mathbb{Z}[\zeta_\infty, N^{-1}]), V)$$

endowed with the left action of the group $\mathrm{Aut}(\mathbb{Z}[\zeta_\infty, N^{-1}]) = \widehat{\mathbb{Z}}^\times$ acting by pushforward. The group H parametrises all the extensions of $\mathbb{Q}_\ell$ by $V|_{\Gamma_{\mathbb{Q}^{ab}}}$

which are unramified away of N. We choose a prime number $p \nmid N\ell$ and an element $\widetilde{p} \in \widehat{\mathbb{Z}}^\times$ which is mapped to p via the quotient map $\widehat{\mathbb{Z}}^\times \twoheadrightarrow (\widehat{\mathbb{Z}}/\mathbb{Z}_p)^\times$. Write $\varphi_{\tilde{p}}$ for the endomorphism of H induced by $\tilde{p}$. In analogy with [BK90, (3.7.2)], let $H_g \subseteq H$ be the subspace of those extensions whose restriction to $\Gamma_{\mathbb{Q}_\ell^{\mathrm{ab}}}$ comes from geometry. Consider the following statement.

$P(V,N)$: The restriction of $\varphi_{\tilde{p}}$ to H_g has no fixed-points.

If $P(V,N)$ is true, every extension of $\mathbb{Q}_\ell$ by V over $\mathrm{Spec}(\mathbb{Z}[\zeta_\infty, N^{-1}])$ which descends to $\mathrm{Spec}(\mathbb{Z}[N^{-1}])$ and comes from geometry is trivial. We have proven the following result.

Proposition 3.4. *If $P(V,N)$ is true for every V and N as above, then $S(\mathbb{Q}^{\mathrm{ab}})$ is true.*

This translates $S(\mathbb{Q}^{\mathrm{ab}})$ to a problem concerning the eigenvalues of $\varphi_{\tilde{p}}$. We hope that this shift in perspective may suggest new developments for Grothendieck–Serre semi-simplicity conjecture.

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References

[BK90] S. Bloch and K. Kato, L-functions and Tamagawa numbers of motives, in *The Grothendieck Festschrift, Vol. I*, Progr. Math. Birkhäuser, 1990, 333–400.

[BC09] O. Brinon and B. Conrad, *CMI Summer school notes on p-adic Hodge theory*, preprint available at `http://math.stanford.edu/~conrad/papers/notes.pdf` (2009).

[Del80] P. Deligne, La Conjecture de Weil, II, *Publ. Math. IHES* **52** (1980), 137–252.

[Fal89] G. Faltings, Crystalline cohomology and p-adic Galois representations, in *Algebraic analysis, geometry and number theory (J. I. Igusa ed.)*, Johns Hopkins University Press, 1989, 25–80.

[Fu99] L. Fu, On the semisimplicity of pure sheaves, *Proc. Amer. Math. Soc.* **127** (1999), 2529–2533.

[Kah04] B. Kahn, On the semi-simplicity of Galois actions, *Rendiconti del Seminario Matematico della Università di Padova* **112** (2004), 97–102.

[Ser81] J.-P. Serre, Lettres à Ken Ribet du 1/1/1981 et du 29/1/1981, in *Œuvres - Collected Papers IV*, Springer-Verlag, 2000.

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