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Donaldson-Thomas invariants, linear systems and punctual Hilbert schemes

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We study certain DT invariants arising from stable coherent sheaves in a nonsingular projective threefold supported on the members of a linear system of a fixed line bundle. When the canonical bundle of the threefold satisfies certain positivity conditions, we relate the DT invariants to Carlsson-Okounkov formulas for the "twisted Euler number" of the punctual Hilbert schemes of nonsingular surfaces, and conclude they have a modular property.

1. Introduction

1.1. Overview

S-duality predicts that certain generating functions of DT invariants of semistable 2-dimensional sheaves inside a Calabi-Yau threefold are modular (see for example [OSV, DM, GST]). In [GS] we studied these DT invariants for K3 fibration over curves (which are not necessarily Calabi-Yau) and still got a modular answer. In this paper, we study a certain type of these DT invariants in some other special cases of non-Calabi-Yau geometries and show that they have modular properties. To do this, we express them in terms of integrals over the Hilbert scheme of points on nonsingular surfaces. Carlsson-Okounkov [CO] found an explicit formula for the generating series of the integrals that arise this way extending Göttsche's formula for the generating series of Euler numbers of the Hilbert schemes. We give several examples of threefolds for which our required conditions are all satisfied.

1.2. Statement of the result

Let $(X, \mathcal{O}(1))$ be a nonsingular polarized threefold over $\mathbb{C}$ with

$$H^1(\mathcal{O}_X) = 0 = H^2(\mathcal{O}_X)$$

and L be a fixed line bundle on X generated by its global sections (equivalently, L is base point free) and satisfying the following further conditions:

Assumption 1. *Assume that* $H^0(L \otimes K_X) = 0 = H^1(L \otimes K_X)$ *and*

$$-K_X \cdot L^2 > L^3, \qquad -K_X \cdot L \cdot \mathcal{O}(1) > 0.$$

We think of the condition $-K_X \cdot L^2 > L^3$ as saying that $-K_X$ is sufficiently positive with respect to L. The condition $-K_X \cdot L \cdot \mathcal{O}(1) > 0$ is immediate for example if $-K_X$ and L are ample. In Lemma 2 we prove some consequences of this assumption.

Consider the moduli space of coherent sheaves in X, which are supported on the members of the linear system $|L|$. For this, fix a Chern character vector

$$\text{(1)} \qquad \mathrm{ch} = \big(\mathrm{ch}_0 = 0,\ \mathrm{ch}_1 = L,\ \mathrm{ch}_2 = \gamma,\ \mathrm{ch}_3 = \xi\big) \in \oplus_{i=0}^3 H^{2i}(X, \mathbb{Q}).$$

We denote the moduli space of Gieseker semistable sheaves (with respect to $\mathcal{O}(1)$) with Chern character ch by $\mathcal{M}(X, \mathrm{ch})$. It is a projective scheme. The Hilbert polynomial of coherent sheaves with Chern character ch is of degree 2, and the coefficients of degree 2 and degree 1 terms are respectively given by

$$a_2 = L \cdot \mathcal{O}(1)^2/2, \qquad a_1 = \gamma \cdot \mathcal{O}(1) + b/2,$$

where $b = -K_X \cdot L \cdot \mathcal{O}(1)$.

We make the following assumption on $\mathrm{ch}_1, \mathrm{ch}_2$ to ensure that for any choice of ch_3 the semistability implies stability (Lemma 3) and that $\mathcal{M}(X, \mathrm{ch})$ carries a perfect obstruction theory (Lemma 4):

Assumption 2. *Assume that L and γ satisfy the following condition: for any decomposition $L = L_1 + L_2$, where L_1 and L_2 are the classes of nonzero effective divisors, and for any $m \in \mathbb{Z}$, one has*[1]

$$\frac{2m + (L_1 - K_X) \cdot L_1 \cdot \mathcal{O}(1)}{L_1 \cdot \mathcal{O}(1)^2} \neq \frac{a_1}{a_2}, \qquad \frac{-2K_X \cdot L_1 \cdot \mathcal{O}(1)}{L_1 \cdot \mathcal{O}(1)^2} \leq \frac{b}{a_2}.$$

For example, if L is an irreducible class the conditions above are immediate (as in Example 1). If $\mathcal{O}(1) = -kK_X$ for a $k > 0$ then the right side condition is satisfied (as in Example 2).

[1]These conditions are stated for L_1, and can be stated with the same right hand sides for L_2.

The moduli space $\mathcal{M}(X, \mathrm{ch})$ then carries a virtual cycle (Lemma 4)

$$(2) \qquad [\mathcal{M}(X, \mathrm{ch})]^{vir} \in A_v(\mathcal{M}(X, \mathrm{ch})), \qquad v := 1 - K_X \cdot L^2/2.$$

An interesting feature for us is that the virtual dimension v only depends on $\mathrm{ch}_1 = L$ (and not on $\mathrm{ch}_2, \mathrm{ch}_3$) and is equal to the dimension of the linear system $|L|$ (Lemma 5). We define the Donaldson-Thomas invariant $\mathrm{DT}(X, \mathrm{ch})$ as follows. Let

$$\rho : \mathcal{M}(X, \mathrm{ch}) \to |L|, \qquad \rho(F) := \mathrm{Div}(F),$$

where $\mathrm{Div}(F)$ is the divisor associated to the coherent sheaf F in the sense of [F, KM]. To see ρ is a morphism of schemes one uses moduli space properties of its source and target and the fact that the construction of $\mathrm{Div}(-)$ is well-behaved under base change (see [G] for details). Define

$$\mathrm{DT}(X, \mathrm{ch}) = \rho_* \, [\mathcal{M}(X, \mathrm{ch})]^{vir} \in A_v(|L|) \cong \mathbb{Z}.$$

We put these invariants into a generating series in which $\mathrm{ch}_1 = L$ and $\mathrm{ch}_2 = \gamma$ are kept fixed and $\mathrm{ch}_3 = \xi$ is allowed to vary:

$$(3) \qquad \mathrm{DT}_{L,\gamma}(X, q) = q^{-\gamma \cdot L/2 - L^3/12} \sum_{\xi} \mathrm{DT}(X, \mathrm{ch}) \, q^{-\xi}.$$

We will shortly motivate this choice of prefactor in (3).

To state the main result of the paper, let S be a general member of $|L|$, and

$$(4) \qquad \beta \in H^2(S, \mathbb{Z}) \quad \text{such that} \quad (i_* \beta^{PD})^{PD} = \gamma + L^2/2,$$

where PD denotes the Poincaré dual, and $i : S \hookrightarrow X$ is the inclusion[2].

Given β and ch as above define

$$(5) \qquad n(\beta, \mathrm{ch}) = \beta^2/2 - \gamma \cdot L/2 - L^3/12 - \xi.$$

In the right hand side of (5), the first term is an intersection number on S, and the second and third terms are intersection numbers on X. In Lemma 6 we will show that provided $\mathcal{M}(X, \mathrm{ch})$ is nonempty $n(\beta, \mathrm{ch})$ is a nonnegative integer. By Proposition 7 the converse is also true. In particular, for fixed

[2] The term $L^2/2$ is a correction term that comes out of the Grothendieck-Riemann-Roch formula as in the proof of Lemma 6.

L, γ, summing over ξ as in (3) is equivalent to summing over nonnegative integers and β satisfying (4). This explains the choice of powers of q in (3).

Finally, define an invariant of the linear system $|L|$

$$\delta(L) = e(S) - K_S \cdot L|_S + L^3. \tag{6}$$

The main result of the paper is a closed formula for the generating series (3):

Theorem 1. *Suppose Assumptions 1 and 2 hold. Then*

$$\mathrm{DT}_{L,\gamma}(X,q) = q^{\delta(L)/24} \sum_{\beta \text{ as in } (4)} q^{-\beta^2/2}\, \eta(q)^{-\delta(L)},$$

where $\eta(q) = q^{1/24} \prod_{k>0}(1-q^k)$ *is the eta function.*

2. Proof of Theorem 1

Conventions: Given a map $f\colon X \to Y$, we often use the same letter f to denote its base change by any map $Z \to Y$, i.e. $f\colon X \times_Y Z \to Z$. We also sometimes suppress pullback maps f^* on sheaves. In the following, we will apply base change theorems for the perfect complexes of sheaves and their cohomologies. See [StP] for a comprehensive reference.

Let X, L and ch be as in Section 1.2, and also suppose Assumptions 1, 2 in there are satisfied.

Lemma 2. *Let* $S \in |L|$ *be a general member. Then* S *is a nonsingular surface and*

(i) $H^1(\mathcal{O}_S) = 0 = H^2(\mathcal{O}_S)$,

(ii) $2 \leq \dim |L| = h^0(L|_S)$,

(iii) $H^1(L|_S) = 0 = H^2(L|_S)$.

Proof. Since $|L|$ is base point free by Bertini's theorem S is nonsingular. The natural short exact sequence $0 \to L^* \to \mathcal{O}_X \to \mathcal{O}_S \to 0$ gives the exact sequence for $i = 1, 2$

$$H^i(\mathcal{O}_X) \to H^i(\mathcal{O}_S) \to H^{i+1}(L^*) \cong H^{2-i}(L \otimes K_X)^*.$$

By our assumptions the first and the last terms vanish and hence so does the middle term.

Again since $|L|$ is base point free $\dim |L| \neq 0$ and if $\dim |L| = 1$ any two distinct members of the linear system do not intersect and so $L^2 = 0$, which contradicts the first inequality in Assumption 1. Therefore, we must have $\dim |L| \geq 2$. Since $H^1(\mathcal{O}_X) = 0$, the natural short exact sequence $0 \to \mathcal{O}_X \to L \to L|_S \to 0$ gives the exact sequence

$$0 \to H^0(\mathcal{O}_X) \to H^0(L) \to H^0(L|_S) \to 0,$$

which proves that $\dim |L| = h^0(L) - 1 = h^0(L|_S)$.

Next, consider the linear system $|L|_S|$ on S. It has to be base point free because $|L|$ is. Let $C \in |L|_S|$ be a general member, which must be smooth by Bertini's theorem. By Serre duality and adjunction formula

$$H^1(L|_C) \cong H^0(L^*|_C \otimes K_C)^* \cong H^0(K_S|_C)^* \cong H^0(L|_C \otimes K_X|_C) = 0,$$

where the last vanishing is because $\deg(L|_C \otimes K_X|_C) = K_X \cdot L^2 + L^3 < 0$. Also, $H^2(L|_C) = 0$ for dimension reason. So applying cohomology to the natural short exact sequence $0 \to \mathcal{O}_S \to L|_S \to L|_C \to 0$, and using the vanishings $H^{i\geq 1}(\mathcal{O}_S) = 0$ proven above, we see that $H^{i\geq 1}(L|_S) = 0$ as claimed. □

Lemma 3. $\mathcal{M}(X, \mathrm{ch})$ *contains no strictly semistable sheaves.*

Proof. Suppose F is a strictly semistable sheaf with Chern character ch. Then there exists a nonzero quotient G of F such that $\mathrm{ch}_1(G) = L_1$, $\mathrm{ch}_2(G) = \gamma_1$, $L - L_1$ is nonzero and effective, and

$$\frac{\gamma_1 \cdot \mathcal{O}(1) - L_1 \cdot K_X \cdot \mathcal{O}(1)/2}{L_1 \cdot \mathcal{O}(1)^2/2} = \frac{a_1}{a_2}.$$

Here, the left hand side is the ratio of coefficients of degree 1 and 2 terms of Hilbert polynomial of G. But $\gamma_1 = -c_2(G) + L_1^2/2$ and $m := -c_2(G) \cdot \mathcal{O}(1) \in \mathbb{Z}$, so we get

$$\frac{2m + (L_1 - K_X) \cdot L_1 \cdot \mathcal{O}(1)}{L_1 \cdot \mathcal{O}(1)^2} = \frac{a_1}{a_2}$$

contradicting the left side condition in Assumption 2. □

Lemma 4. $\mathcal{M}(X, \mathrm{ch})$ *carries a perfect obstruction theory and hence a virtual class as in* (2).

Proof. Let $F \in \mathcal{M}(X, \mathrm{ch})$ be a closed point. By Lemma 3, F must be stable. We show that $F \otimes K_X$ is semistable. If not, there exists a quotient sheaf G of it such that the degree 1 term of the reduced Hilbert polynomial $F \otimes K_X$ is bigger than or equal to that G. By the right side condition in Assumption 2 this will imply that the degree 1 term of the reduced Hilbert polynomial F is bigger than or equal to that of $G \otimes K_X^*$, which is a quotient of F, contradicting the stability of F.

By Serre duality

$$\mathrm{Ext}^3(F, F) \cong \mathrm{Hom}(F, F \otimes K_X)^* = 0$$

for any closed point $F \in \mathcal{M}(X, \mathrm{ch})$. This is because by the inequality $K_X \cdot L \cdot \mathcal{O}(1) < 0$ the Hilbert polynomial of F is greater than that of $F \otimes K_X$ and so by the semistability of F and $F \otimes K_X$ there is no nontrivial homomorphism between them ([HL, Proposition 1.2.7]).

Therefore, by [T, Theorem 3.30] $\mathcal{M}(X, \mathrm{ch})$ carries a perfect obstruction theory. Its virtual dimension can be calculated by the Hirzebruch-Riemann-Roch formula:

$$\begin{aligned} v =& \mathrm{ext}^1(F, F) - \mathrm{ext}^2(F, F) = 1 - \chi(F, F) \\ =& 1 - (-L + \gamma - \xi) \cdot (L + \gamma + \xi) \cdot \mathrm{td}(X) = 1 - K_X \cdot L^2/2. \end{aligned}$$

□

Lemma 5. $\dim |L| = 1 - K_X \cdot L^2/2$.

Proof. The right hand side is the virtual dimension v (2) and it does not depend on ch_2 and ch_3. So by choosing ch_2 and ch_3 suitably we may assume that $\mathcal{O}_S$ corresponds to a closed point of $\mathcal{M}(X, \mathrm{ch})$ for some general member $S \in |L|$. We can write

$$v = \mathrm{ext}^1(\mathcal{O}_S, \mathcal{O}_S) - \mathrm{ext}^2(\mathcal{O}_S, \mathcal{O}_S) = 1 - \chi(\mathcal{O}_S, \mathcal{O}_S).$$

Applying $\mathrm{Hom}(-, L|_S)$ to the natural short exact sequence $0 \to \mathcal{O}_X \to L \to L|_S \to 0$ and taking Euler characteristics, we get

$$\chi(\mathcal{O}_S, \mathcal{O}_S) = \chi(\mathcal{O}_S) - \chi(L|_S) = h^0(\mathcal{O}_S) - h^0(L|_S) = 1 - h^0(L|_S).$$

Here the second equality is because of the vanishing

$$h^{i\geq 1}(\mathcal{O}_S) = 0 = h^{i\geq 1}(L|_S)$$

by Assumption 1 and Lemma 2. So we showed that $v = h^0(L|_S) = \dim |L|$ by Lemma 2. $\square$

Lemma 6. *(i) If $\mathcal{M}(X, \mathrm{ch}) \neq \emptyset$ then $n(\beta, \mathrm{ch})$ as defined in* (5) *is a non-negative integer for any choice of β satisfying* (4)*.*

(ii) For a given integer n and ch*, there are finitely many classes β satisfying* (4) *and* (5) *with $n(\beta, \mathrm{ch}) = n$.*

Proof. Let $F \in \mathcal{M}(X, \mathrm{ch})$ be a closed point corresponding to a coherent sheaf supported on the general member $S \in |L|$, such that $\mathrm{ch}(F) = \mathrm{ch}$. Then, S is nonsingular and F is rank 1 on its support, so $F \cong i_*(N \otimes I)$, where $i \colon S \hookrightarrow X$ is the inclusion, N is a line bundle, and I is an deal sheaf of points on S. By Grothendieck-Riemann-Roch,

$$\mathrm{ch}(i_*(N \otimes I)) = i_* \left(\mathrm{ch}(N \otimes I) \cdot (1 - L/2 + L^2/6)|_S\right).$$

Setting this equal to (1), we find that $i_* c_1(N) = \gamma + L^2/2$ and $c_2(I) = c_1(N)^2/2 - \gamma \cdot L/2 - L^3/12 - \xi$ are respectively given by (4) and (5) with $c_1(N) = \beta$ and $c_2(I) = n(\beta, \mathrm{ch})$. Since $c_2(I)$ is the colength of I it must be a nonnegative integer. This proves (i).

Suppose n and γ are fixed. Since $H^{i\geq 1}(\mathcal{O}_S) = 0$ we know that $\mathrm{Pic}(S) \cong H^2(S, \mathbb{Z})$ is a finitely generated abelian group. Clearly there are finitely many contributions from its torsion part. For the contributions of the free part, by Hodge index theorem we can find a basis $e_1, \ldots, e_s$ for $H^2(S, \mathbb{R})$ such that $e_1^2 = 1$ and $e_i^2 = -1$ for $i \geq 2$ and $e_i \cdot e_j = 0$ for $i \neq j$ and e_1 is a (real) multiple of the class of $\mathcal{O}(1)|_S$. Any $\beta \in H^2(S, \mathbb{Z})$ can be written as $\beta = \sum_{i=1}^s \alpha_i e_i$ where $(\alpha_1, \ldots, \alpha_s)$ varies in a lattice in $\mathbb{R}^s$. Since γ and L are fixed, (4) implies that α_1 must be fixed. (5) then implies that $\sum_{i=2}^s \alpha_i^2$ is fixed, and hence there are finitely many choices for α_is. This establishes (ii).

$\square$

Suppose that $S \in |L|$ is a general member, and $n = n(\beta, \mathrm{ch})$ is related to ch by (4) and (5). Let $S^{[n]}$ be the Hilbert scheme of n points on S. It is nonsingular of dimension $2n$. By Lemma 6 part (ii) and using $H^1(\mathcal{O}_S) = 0$, we see that $\rho^{-1}(S)$ is isomorphic to a finite disjoint union of these Hilbert schemes.

Proposition 7. *The morphism* $\rho\colon \mathcal{M}(X,\mathrm{ch}) \to |L|$ *is smooth of relative dimension* $2n$ *over an open neighborhood* $V \subset |L|$ *of* S *consisting of nonsingular divisors. Let* $\mathcal{M}_V := \rho^{-1}(V)$*. Then*

$$\mathcal{M}_V \cong \mathrm{Hilb}^n(\mathcal{D}/V) \times_V \mathrm{Pic}_\gamma(\mathcal{D}/V),$$

where $\mathcal{D} \subset X \times V$ *is the (restriction of the) universal divisor, and*

$$\mathrm{Hilb}^n(\mathcal{D}/V), \qquad \mathrm{Pic}(\mathcal{D}/V)$$

are respectively the relative Hilbert scheme of length n *subschemes and the relative Picard scheme of* $\mathcal{D}/V$[3], *and* $\mathrm{Pic}_\gamma(\mathcal{D}/V)$ *is the union of components of* $\mathrm{Pic}(\mathcal{D}/V)$*, whose* S*-fiber is one of* $\mathrm{Pic}_\beta(S)$ *with* β *is as in* (4).

Proof. Since $\mathcal{D}/V$ has nonsingular fibers of dimension 2, $\mathrm{Hilb}^n(\mathcal{D}/V)$ is nonsingular of dimension $2n+v$. Moreover, since by Lemma 2 we have

$$h^{0,1} = 0 = h^{0,2}$$

for the fibers of $\mathcal{D}/V$, by [K, Theorem 5.19] $\mathrm{Pic}(\mathcal{D}/V)$ is smooth of relative dimension 0 and locally of finite type over V. By Lemma 6, $\mathrm{Pic}_\gamma(\mathcal{D}/V)$ is of finite type. So the smoothness of ρ follows from the second statement in the Proposition.

To prove the statement about $\mathcal{M}_V$, let B be a scheme over V. If $\mathcal{F}$ is a flat family over $X \times B$ define

$$\mathcal{D}_B := \mathrm{Div}(\mathcal{F}) \overset{i}{\hookrightarrow} X \times B.$$

Since $\mathrm{Div}(-)$ is well-behaved with respect to base change [F, KM], $\mathcal{D}_B \cong \mathcal{D} \times_V B$. We can then regard $\mathcal{F}$ as $i_*\mathcal{G}$ for some rank 1 torsion free sheaf $\mathcal{G}$ over $\mathcal{D}_B/B$. By construction $\mathcal{G}$ is flat and $\mathcal{D}_B$ is smooth over B. Taking double duals $\mathcal{G} \subset \mathcal{G}^{**}$, by [Ko, Lemma 6.13] $\mathcal{N} := \mathcal{G}^{**}$ is a line bundle, and hence we get a family of ideal sheaves $\mathcal{G} \otimes \mathcal{N}^* \subset \mathcal{O}_{\mathcal{D}_B}$ flat over B, where each B-fiber is an ideal of colength n. Therefore, $\mathcal{G} \otimes \mathcal{N}^*$ and $\mathcal{N}$ determine a B-valued point of

$$\mathrm{Hilb}^n(\mathcal{D}/V) \times_V \mathrm{Pic}_\gamma(\mathcal{D}/V).$$

Conversely, if a pair $(\mathcal{I}, \mathcal{N})$, of B-flat families of ideals of colength n and line bundles of class γ on the fibers of $\mathcal{D}_B := \mathcal{D} \times_V B$ is given then $i_*(\mathcal{I} \boxtimes \mathcal{N})$

[3]By [K, Theorem 4.8], $\mathrm{Pic}(\mathcal{D}/V)$ represents the sheafification of the Picard functor in étale topology.

determines a B-valued point of $\mathcal{M}_V$. These assignments are evidently inverse of each other, and so the claim is proven. $\square$

Note that the Poincaré line bundles exist only after an étale base change. Therefore, by construction above there is an étale base change

$$\begin{array}{ccc} \mathcal{M}_{V^{\text{ét}}} & \longrightarrow & \mathcal{M}_V \\ \downarrow \rho & & \downarrow \rho \\ V^{\text{ét}} & \longrightarrow & V \end{array}$$

such that the universal sheaf exists over $X \times \mathcal{M}_{V^{\text{ét}}}$. Let us denote it by $\mathcal{F}$ again and let $\mathcal{D}_{V^{\text{ét}}} := \operatorname{Div}(\mathcal{F}) \stackrel{i}{\hookrightarrow} X \times V^{\text{ét}}$. Also, denote by $\mathcal{I}$ the universal ideal sheaf over $\mathcal{D}_{V^{\text{ét}}} \times_{V^{\text{ét}}} \operatorname{Hilb}^n(\mathcal{D}_{V^{\text{ét}}}/V^{\text{ét}})$.

Our goal is to write $\operatorname{DT}(X, \operatorname{ch})$ as an integral over $S^{[n]}$. For this, we find a relation between tangent/obstruction theory of $\mathcal{M}_V$ and $S^{[n]}$ using the identification above. Note that $\operatorname{Pic}_\gamma(\mathcal{D}/V)$ consists of finitely many nonsingular components each isomorphic to V. We can work over each connected component of $\mathcal{M}_V$ at a time, which is thus identified with $\operatorname{Hilb}^n(\mathcal{D}/V)$. This shows in particular each connected component of $\mathcal{M}_V$ is nonsingular of dimension $2n + v$.

We denote the tangent and the obstruction sheaves (of the fixed component) of $\mathcal{M}_V$ by Tan and Ob, respectively. If $\pi\colon X \times \mathcal{M}_{V^{\text{ét}}} \to \mathcal{M}_{V^{\text{ét}}}$ is the projection then

$$\mathcal{E}xt^1_\pi(\mathcal{F}, \mathcal{F}), \qquad \mathcal{E}xt^2_\pi(\mathcal{F}, \mathcal{F})$$

descend to Tan and Ob, respectively (see [HL, Section 10.2] or Remarks before the proof of Theorem 3.30 in [T]). These are the first and second cohomologies of the perfect complex $R\mathcal{H}om_\pi(\mathcal{F}, \mathcal{F})$. We claim Ob is locally free. Consider the Cartesian diagram

$$\begin{array}{ccc} \mathcal{D}_{\mathcal{M}_{V^{\text{ét}}}} & \stackrel{i}{\hookrightarrow} & \mathcal{M}_{V^{\text{ét}}} \times X \\ \downarrow & & \downarrow (\rho, id) \\ \mathcal{D}_{V^{\text{ét}}} & \stackrel{i}{\hookrightarrow} & V^{\text{ét}} \times X \end{array}$$

and let $\pi' = i \circ \pi$. By the proof of Proposition 7 $\mathcal{F} = i_*(\mathcal{G})$, where $\mathcal{G} = \mathcal{I} \otimes \mathcal{N}$ for a line bundle $\mathcal{N}$. Applying the functor $R\mathcal{H}om_\pi(-, \mathcal{G})$ to the exact triangle

[H, Corollary 11.4]

$$\mathcal{G}(-\mathcal{D}_{\mathcal{M}_{V^{\text{ét}}}})[1] \to Li^*i_*\mathcal{G} \to \mathcal{G},$$

and using adjoint functors $Li^* \vdash i_*$, we get an exact triangle

$$R\mathcal{H}om_{\pi'}(\mathcal{I},\mathcal{I}) \to R\mathcal{H}om_{\pi}(\mathcal{F},\mathcal{F}) \to R\mathcal{H}om_{\pi'}(\mathcal{I},\mathcal{I}(\mathcal{D}_{\mathcal{M}_{V^{\text{ét}}}}))[-1].$$

Taking cohomology, we get the exact sequence

$$0 = \mathcal{E}xt^2_{\pi'}(\mathcal{I},\mathcal{I}) \to \mathcal{E}xt^2_{\pi}(\mathcal{F},\mathcal{F}) \to \mathcal{E}xt^1_{\pi'}(\mathcal{I},\mathcal{I}(\mathcal{D}_{\mathcal{M}_{V^{\text{ét}}}})) \to 0$$

in which the first vanishing is by base change and Nakayama Lemma and the vanishing $H^2(\mathcal{O}_S) = 0$ for all closed $S \in V$ (Lemma 2) that in turn implies $\operatorname{Ext}^2_S(I, I) = 0$ for any colength n ideal $I \subset \mathcal{O}_S$.

Next, we show that $\mathcal{E}xt^1_{\pi'}(\mathcal{I},\mathcal{I}(\mathcal{D}_{\mathcal{M}_{V^{\text{ét}}}}))$ is locally free of rank $2n$ and hence the same will be true for $\mathcal{E}xt^2_{\pi}(\mathcal{F},\mathcal{F})$ by the exact sequence above. To do this we use base change again and show that over any closed point $S \in V$ the cohomologies of $R\mathcal{H}om_{\pi'}(\mathcal{I},\mathcal{I}(\mathcal{D}_{\mathcal{M}_{V^{\text{ét}}}}))$, given by $\operatorname{Ext}^i_S(I, I \otimes L|_S)$ for colength n ideals $I \subset \mathcal{O}_S$ have constant dimensions. For $i > 2$ they are 0 because S is nonsingular of dimension 2. For $i = 2$, it is 0 because of $H^2(L|_S) = 0$ (Lemma 2). Finally, $\operatorname{Hom}_S(I, I \otimes L|_S) \cong H^0(L|_S)$ has dimension v. Therefore, by Riemann-Roch $\operatorname{Ext}^1_S(I, I \otimes L|_S)$ has dimension $2n$. This proves the claim.

Now (any connected component of) $\mathcal{M}_V$ is nonsingular of dimension $2n + v$ and the obstruction sheaf Ob is locally free of rank $2n$. By [BF, Proposition 5.6],

$$[\mathcal{M}_V]^{vir} = c_{2n}(\mathrm{Ob}) \cap [\mathcal{M}_V]. \tag{7}$$

The exact sequence above together with Lemma 2 also show that the restriction of the obstruction bundle Ob to (any component of) the fiber $\rho^{-1}(S)$ is equivalent to the Carlson-Okounkov K-theory element

$$Rp_*(L|_S) - R\mathcal{H}om_p(\mathcal{I},\mathcal{I} \otimes L|_S)$$

where $p\colon S \times S^{[n]} \to S^{[n]}$ is the projection.

Now let $j\colon \operatorname{Spec}\mathbb{C} \hookrightarrow |L|$ be the inclusion corresponding to the general member $S \in |L|$, and form the fibered square

$$\begin{array}{ccc} \rho^{-1}(S) & \hookrightarrow & \mathcal{M}(X,\mathrm{ch}) \\ \downarrow \rho & & \downarrow \rho \\ \operatorname{Spec}\mathbb{C} & \overset{j}{\hookrightarrow} & |L|. \end{array}$$

By the definition of $\mathrm{DT}(X,\mathrm{ch})$ and [Fu, Theorems 6.2], we can write

$$\mathrm{DT}(X,\mathrm{ch}) = j^*\rho_*[\mathcal{M}(X,\mathrm{ch})]^{vir} = \rho_* j^![\mathcal{M}(X,\mathrm{ch})]^{vir}.$$

Since j factors as $\operatorname{Spec}\mathbb{C} \overset{j'}{\hookrightarrow} V \subset |L|$ by [Fu, Theorems 6.5]

$$\rho_* j^![\mathcal{M}(X,\mathrm{ch})]^{vir} = \rho_* j'^![\mathcal{M}_V]^{vir}.$$

Using (7) we therefore have

$$\begin{aligned} \mathrm{DT}(X,\mathrm{ch}) &= \rho_* j'^! e(\mathrm{Ob}) \cap [\mathcal{M}_V] \\ &= \sum_{\beta \text{ as in (4)}} \int_{S^{[n]}} c_{2n}\big(Rp_*(L|_S) - R\mathcal{H}om_p(\mathcal{I},\mathcal{I}\otimes L|_S)\big), \end{aligned}$$

where $n = n(\beta,\mathrm{ch})$ and the last equality is because of the discussion after (7) and that the components of $\rho^{-1}(S)$ correspond to the classes β as in (4).

According to (5), $n(\beta,\mathrm{ch}) - \beta^2/2 = -\gamma\cdot L/2 - L^3/12 - \xi$, so the generating series (3) gives

$$\begin{aligned} &\mathrm{DT}_{L,\gamma}(X,q) \\ &\quad = \sum_{\beta \text{ as in (4)}} q^{-\beta^2/2} \sum_{n=0}^{\infty} q^n \int_{S^{[n]}} c_{2n}\big(Rp_*(L|_S) - R\mathcal{H}om_p(\mathcal{I},\mathcal{I}\otimes L|_S)\big). \end{aligned}$$

By [CO, Corollary 1]

$$\sum_{n=0}^{\infty} q^n \int_{S^{[n]}} c_{2n}\big(Rp_*(L|_S) - R\mathcal{H}om_p(\mathcal{I},\mathcal{I}\otimes L|_S)\big) = \prod_{k>0}(1-q^k)^{-\delta(L)},$$

where $\delta(L) = e(S) - K_S\cdot L|_S + L^3$ as introduced in (6). This completes the proof of Theorem 1.

3. Examples

Example 1 (hypersurfaces in $\mathbb{P}^4$). Let $X \subset \mathbb{P}^4$ be a nonsingular hypersurface of degree $d \leq 3$ with the choices

$$\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}^4}(1)|_X =: L.$$

Then $H^{i\geq 1}(X, \mathcal{O}_X) = 0$, and we have $K_X \cong \mathcal{O}(d-5)$, so $-K_X \cdot L \cdot \mathcal{O}(1) = 5 - d > 0$ and $-K_X \cdot L^2 = (5-d)d > d = L^3$ are satisfied for $d \leq 3$. Moreover, by the Lefschetz hyperplane theorem L is an irreducible class. Therefore, both Assumption 1 and Assumption 2 (for any γ) are satisfied for these three geometries. Note that if $d = 4$, the first inequality in Assumption 1 is not satisfied but the second one is, and if $d \geq 5$ none of the inequalities are satisfied. For $d = 4, 5$ the DT invariants of X are still defined but they don't fit into the framework of this paper. When $d \geq 6$ even the DT invariants of X are not defined. Here, by DT invariants we mean $\mathrm{DT}(X, \mathrm{ch})$ for $\mathrm{ch} = (0, L, \gamma, \xi)$.

In the rest of this example, we consider the case $d = 2$. By Lefschetz hyperplane theorem

$$H^2(X, \mathbb{Z}) \cong \mathbb{Z} \cong \mathrm{Pic}(X), \quad H^4(X, \mathbb{Z}) \cong \mathbb{Z}$$

are respectively generated by the class of L and the class of line $\ell = L^2/2$. Tensoring by $\otimes L^{\pm 1}$ induces an isomorphism $\mathcal{M}(X, \mathrm{ch}) \cong \mathcal{M}(X, \mathrm{ch}')$, where $\mathrm{ch}_1 = L = \mathrm{ch}'_1$ and $\mathrm{ch}_2 = \mathrm{ch}'_2 \mp 2\ell$. Because of this identification, we have only two different generating series of DT invariants

$$\mathrm{DT}_{L,\ell}(X, q), \quad \mathrm{DT}_{L,2\ell}(X, q).$$

A general member of $|L|$ is a nonsingular quadratic surface in $\mathbb{P}^3$ and so is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Let e_1, e_2 be the generators of $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$. According to (4) for any $k \in \mathbb{Z}$, the classes $\beta = k(e_1 - e_2)$ correspond to $\gamma = \ell$ and the classes $\beta = e_1 + k(e_1 - e_2)$ correspond to $\gamma = 2\ell$. By Theorem 1

$$\mathrm{DT}_{L,\ell}(X, q) = \sum_{k \in \mathbb{Z}} q^{k^2} \, \eta(q)^{-10},$$
$$\mathrm{DT}_{L,2\ell}(X, q) = \sum_{k \in \mathbb{Z}} q^{k^2+k} \, \eta(q)^{-10}.$$

Example 2 (Blow ups of $\mathbb{P}^3$). Let X be the blow up of $\mathbb{P}^3$ at a disjoint union of m points and n lines, with $n < 16$. Let E be the exceptional divisor and L be the (pullback of) hyperplane class. Take $\mathcal{O}(1) = -kK_X$ for some $k \gg 0$ for the polarization. X is birational to $\mathbb{P}^3$ and hence $H^1(X, \mathcal{O}) = H^2(X, \mathcal{O}) = 0$.

$K_X = -4L + E$, so $-K_X \cdot L^2 = 4 > 1 = L^3$ and $-K_X \cdot L \cdot \mathcal{O}(1) = (16 - n)k > 0$. Suppose $Q \subset X$ is the proper transform of a general cubic surface. Let $0 \to \mathcal{O}(-3L + E) \to \mathcal{O}(E) \to \mathcal{O}_Q(E) \to 0$ be the natural short exact sequence. Then $H^0(\mathcal{O}(E)) \cong H^0(\mathcal{O}_Q(E)) \cong \mathbb{C}$, and hence

$$\begin{aligned} H^0(L \otimes K_X) &\cong H^0(\mathcal{O}(-3L + E)) = 0, \\ H^1(L \otimes K_X) &\cong H^1(\mathcal{O}(-3L + E)) \cong H^1(\mathcal{O}(E)) = 0. \end{aligned}$$

To see the last vanishing, note that E is a disjoint union of finitely many copies of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, and over each copy $\mathcal{O}_E(E)$ is either $\mathcal{O}_{\mathbb{P}^2}(-1)$ or $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-\Delta)$, so $H^1(\mathcal{O}_E(E)) = 0$. Together with the natural short exact sequence $0 \to \mathcal{O} \to \mathcal{O}(E) \to \mathcal{O}_E(E) \to 0$ we conclude that $H^1(\mathcal{O}(E)) = 0$. We have checked that the conditions of Assumption 1 are satisfied for this geometry.

The right side condition on Assumption 2 trivially holds. For simplicity, we only verify the left side condition in Assumption 2 in the case X is the blow up of $\mathbb{P}^3$ at one point or one line. In the case of blow up of a point, $a_2 = L \cdot k^2(4L - E)^2/2 = 8k^2$ and

$$a_1 = \gamma \cdot k(4L - E) + L \cdot k(4L - E)^2/2 = (2r - s + 8)k,$$

where $\gamma = rL^2/2 + sE^2$ for some $r, s \in \mathbb{Z}$. In the case of blow up of a line, $a_2 = L \cdot k^2(4L - E)^2/2 = 15k^2/2$ and

$$a_1 = \gamma \cdot k(4L - E) + L \cdot k(4L - E)^2/2 = (2r + 4s_1 + s_2 + 15/2)k,$$

where $\gamma = rL^2/2 + s_1 e_1 + s_2 e_2$ for some $r, s_1, s_2 \in \mathbb{Z}$, and e_1, e_2 the generators of $H^2(E)$.

Now if $L_1 = E$ the condition is for any $m \in \mathbb{Z}$

$$\begin{aligned} 2m &\neq \frac{2r - s + 8}{k} \quad \text{(for blow up at a point)}, \\ m + 2 &\neq \frac{(4r + 8s_1 + 2s_2 + 15)}{k} \quad \text{(for blow up at a line)}. \end{aligned}$$

So for a given γ, it suffices to pick $k \gg 0$ such that the right hand side is not an integer.

For an example of the generating series of DT invariants let us try the case $\gamma = L^2/2$. A general member of $S \in |L|$ is isomorphic to $\mathbb{P}^2$ blown up at n points, and only the class $\beta = 0$ on S corresponds to the class $\gamma = L^2/2$ on X under (4). So by Theorem 1 we have

$$\mathrm{DT}_{L,L^2/2}(X,q) = \eta(q)^{-n-7}.$$

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