Math. Res. Lett.
Volume 29, Number 4, 1165–1196, 2022

Classification of homogeneous strictly pseudoconvex hypersurfaces in $\mathbb{C}^3$

Ilya Kossovskiy and Alexander Loboda

Locally homogeneous strictly pseudoconvex hypersurfaces in $\mathbb{C}^2$ were classified by E. Cartan in 1932. In this work, we complete the classification of locally homogeneous strictly pseudoconvex hypersurfaces in $\mathbb{C}^3$.

1. Introduction

1.1. Homogeneous CR-manifolds

Homogeneous CR-manifolds form an important class of real submanifolds in a complex manifold. Usually, one distinguishes between the *global* and the *local* homogeneities, respectively. Globally homogeneous (compact) CR-manifolds are, in turn, a rare find in CR-geometry (see, e.g., [MN63],[BS78]), that is why most of the work on homogeneous CR-manifolds is dedicated to the locally homogeneous ones. Recall that a CR-manifolds $M \subset \mathbb{C}^N$ is called *locally homogeneous* if the CR-geometries at any two points in it are isomorphic to each other, that is for every $p, q \in M$, there exists a CR-diffeomorphism $H : (M,p) \longrightarrow (M,q)$ between the germs of M at p and q respectively. A useful modern exposition of different notions of homogeneity in CR-geometry is given in the paper [Za07] of Zaitsev. According to [Za07], every locally homogeneous CR-submanifold can be already assumed to be *real-analytic*, and the local homogeneity near one point in M propagates analytically along M. Furthermore, the local homogeneity is equivalent to each of the following three conditions:

(1) For every $p, q \in M$, there exists a local biholomorphism $H : (M,p) \longrightarrow (M,q)$ (i.e., the germs of M at every two points on it are biholomorphic).

(2) Near every point $p \in M$, there exists a locally transitive (real) Lie group action on M by biholomorphic transformation.

(3) For each $p \in M$, the *infinitesimal automorphism algebra* $\mathfrak{hol}(M,p)$ of M at p (i.e., the Lie algebra of holomorphic vector fields

$$X = \left\{ f_1(z)\frac{\partial}{\partial z_1} + \cdots + f_N(z)\frac{\partial}{\partial z_n} \right\}$$

such that their coefficients f_j are holomorphic near p and $\operatorname{Re} X$ is tangent to M pointwise) is transitive on M near p (i.e. the values at p of the vector fields $X \in \mathfrak{hol}(M,p)$ span the entire tangent space T_pM).

We note here that $\mathfrak{hol}(M,p)$ is precisely the algebra of all vector fields in $\mathbb{C}^N$ the flow of which consists of holomorphic transformations and preserves M, locally near p.

Since the pioneering 1932 work of E. Cartan [Ca32] who classified all locally homogeneous strictly pseudoconvex hypersurfaces in $\mathbb{C}^2$ (which implies the classification of *all* locally homogeneous 3-dimensional CR-manifolds), a lot of work has been dedicated to the general project of holomorphic classification of locally homogeneous CR-manifolds. For some complete classifications, we shall mention the work [BK11] of Beloshapka-Kossovskiy who classified locally homogeneous 4-dimensional CR-manifolds, and the work of Fels-Kaup (*Acta Math. 2008*) who classified all Levi-degenerate locally homogeneous 5-dimensional CR-manifolds. However, somewhat surprisingly, the classification of locally homogeneous *strictly pseudoconvex* hypersurfaces in $\mathbb{C}^3$ (which seems to be the most natural development of Cartan's 1932 work) has been open till present in its full generality. It is the main goal of this paper to provide finally such a classification. We do so by treating the remaining open case when a locally homogeneous hypersurface under consideration is *simply homogeneous*, that is, it admits a *free* local Lie group action (in other words, its *isotropy algebra*

$$\mathfrak{aut}\,(M,p) := \{X \in \mathfrak{hol}\,(M,p):\ X|_p = 0\}$$

at the reference point p is trivial).

We note that the classification of locally homogeneous CR-manifolds is important in that it is closely related to the classification of homogeneous domains in complex manifolds, in particular, that of Hermitian Symmetric Domains (see in this regards e.g. Cartan [Ca32], Isaev [I06], Kaup and Zaitsev [KZ06], and also the work of Huang and Yuan [HY15] and references therein).

Before stating our classification theorem, we outline below the progress in the classification of locally homogeneous strictly pseudoconvex hypersurfaces in $\mathbb{C}^3$ in the case when the isotropy algebra $\mathfrak{aut}\,(M,p)$ has a positive dimension. In the latter case, a powerfull tool for the classification is the normal form theory for Levi-nondegenerate hypersurfaces due to Chern-Moser [CM74]. For a locally homogeneous hypersurface, its complete normal form is simply *constant* along a hypersurface. Furthermore, the presence of a non-trivial stability group of a hypersurface at a point p puts a lot of restrictions on the normal form at p. These two aspects put together make it possible to detect a few coefficients of the normal form competely determining a homogeneous hypersurface, and classify subsequently the hypersurfaces under consideration. This approach was realized mainly by the school of A. Vitushkin: see e.g. Ezhov-Loboda-Schmalz [ELS99] and Loboda [Lo00, Lo03, Lo01]. In particular, it was shown that a possible dimension of the isotropy algebra at a point is either 10 (the spherical case), or otherwise $2, 1$, or 0. Strictly pseudoconvex locally homogeneous hypersurfaces with stability algebras of dimensions 2 and 1 were classified in the work [Lo03, Lo01], respectively. An alternative approach in the case $\dim \mathfrak{aut}\,(M,p) > 0$, employing already the Cartan moving frame method and representation theory for Lie algebras, was suggested by Doubrov-Medvedev-The in [DMT17]. In the latter work, the authors were able (among other significant results) to revisit Loboda's classification and supplement it by one missing hypersurface in the case of 1-dimensional isotropy. The approach in [DMT17] shares certain traits with the approach of Fels-Kaup in [FK08] used in the Levi-degenerate case (see in this regard also the work [AMN06] of Altomani-Medori-Nacinovic and references therein).

We shall emphasize, however, that both mentioned approaches (the one based on normal forms and the one employing the moving frame method) strictly rely on the existence of a non-trivial isotropy algebra, and are not able to provide any information on the classification when $\mathfrak{aut}\,(M,p)=0$. In this way, the simply homogeneous case treated in the present paper remained open, as discussed above. Our treatment of this case is rather close to the original Cartan's approach in [Ca32]. That is, we use the existence of a (locally) transitively acting 5-dimensional real Lie algebra on a simply homogeneous hypersurface M, and then use subsequently the strict pseudoconvexity for providing certain normal forms already for the algebras of holomorphic vector fields acting on M. When doing so, we rely on the classification of 5-dimensional real Lie algebras due to Mubarakzjanov [Mu63] (this classification is also given in the Appendix for this paper, for reader's convenience).

1.2. Main results

We now provide our results in detail. Let us recall first the construction of a natural class of locally homogeneous strictly pseudoconvex hypersurfaces in $\mathbb{C}^3$, which is the class of *tubes over affinely homogeneous surfaces.* Let us take an affinely homogeneous strictly geometrically convex (resp. concave) surface $B\subset\mathbb{R}^3$ (the *base* of the tube), and then consider the tubular CR-hypersurface

$$M=B+i\mathbb{R}^3=\{z\in\mathbb{C}^3:\ \mathrm{Re}\,z\in B\}.$$

The hypersurface $M\subset\mathbb{C}^3$ is clearly strictly pseudoconvex. If now $\mathfrak{a}$ is the Lie algebra of (real) affine vector fields of the kind

$$L_j(x)\frac{\partial}{\partial x},\quad x\in\mathbb{R}^3,\ j=1,2,...,k$$

acting transitively on B, and $\mathfrak{b}$ is the 3-dimensional abelian algebra spanned by the vector fields $i\frac{\partial}{\partial z_j}$ (generating the real shifts $z\mapsto z+ib$, $b\in\mathbb{R}^3$), then the Lie algebra $\mathfrak{g}$ spanned by $\mathfrak{b}$ and the vector fields

$$L_j(z)\frac{\partial}{\partial z},\quad z\in\mathbb{C}^3,\ j=1,2,...,k$$

clearly acts transitively already on M. In this way, M is locally homogeneous. A substantial part of the final classification list in Theorem 2 below (but not the entire list!) is obtained precisely in this way.

Our main result below shows that, somewhat surprisingly, *all* locally homogeneous strictly pseudoconvex hypersurfaces in $\mathbb{C}^3$ can be reduced to the above tubular CR-hypersurfaces, under the simple homogeneity assumption.

Theorem 1. *Let $M \subset \mathbb{C}^3$ be a simply homogeneous strictly pseudoconvex hypersurface. Then M is locally biholomorphic, near any point p in it, to the tube over an affinely homogeneous strictly geometrically convex surface $B \subset \mathbb{R}^3$.*

Putting together Theorem 1 with earlier classifications in the case of positive-dimensional stabilizer and existing classifications of affinely homogeneous surfaces (see Section 4 below), we finally obtain the complete classification of *all* locally homogeneous strictly pseudoconvex hypersurfaces in $\mathbb{C}^3$.

Theorem 2. *Let $M \subset \mathbb{C}^3$ be a locally homogeneous strictly pseudoconvex hypersurface. Then M is locally biholomorphic, near every point p in it, to one of the following hypersurfaces (here $(z_1, z_2, z_3) \in \mathbb{C}^3$, $x_j = \operatorname{Re} z_j$):*

1) $x_3 = |z_1|^2 + |z_2|^2$ *(the hyperquadric)*

2) $x_3 = \ln(1 + |z_1|^2) + b\ln(1 + |z_2|^2), \quad 0 < b \leq 1$

3) $x_3 = \ln(1 + |z_1|^2) - b\ln(1 - |z_2|^2), \quad b > 0,\, b \neq 1$

4) $x_3 = \ln(1 - |z_1|^2) + b\ln(1 - |z_2|^2), \quad 0 < b \leq 1$

5) $x_3 = \varepsilon \ln(1 + \varepsilon|z_1|^2) + |z_2|^2, \quad \varepsilon = \pm 1$

6) $x_3 = \pm x_1^\alpha + x_2^2, \quad \pm\alpha(\alpha - 1) > 0,\ \alpha \neq 2$

7) $x_3 = x_1 \cdot \ln x_1 + x_2^2$

8) $x_1 x_3 = -x_1 \cdot \ln x_1 + x_2^2$

9) $\pm x_1^2 \pm x_2^2 + x_3^2 = 1$

10) $1 \pm (|z_1|^2 + |z_2|^2) + |z_3|^2 = a|z_1^2 + z_2^2 + z_3^2|, \quad a > 1$

11) $1 \pm (|z_1|^2 + |z_2|^2) - |z_3|^2 = a|z_1^2 + z_2^2 - z_3^2|, \quad 0 < a < 1$

12) $x_3 = x_1^\alpha x_2^\beta, \quad \alpha\beta(1 - \alpha - \beta) > 0,\ |\alpha|, |\beta| \leq 1,\ |\alpha| \leq |\beta|$

13) $x_3 = \left(x_1^2 + x_2^2\right)^\beta \cdot \exp\left(\alpha \arctan \frac{x_2}{x_1}\right), \quad \alpha \geq 0,\ \beta > \frac{1}{2},\ (\alpha, \beta) \neq (0, 1)$

14) $x_1 x_3 = x_1^2 \ln x_1 + x_2^2$

15) $x_1 x_3 = \pm x_1^\alpha + x_2^2, \quad \pm(\alpha - 1)(\alpha - 2) > 0$

16) $\left(x_3 - x_1 x_2 + \frac{1}{3}x_1^3\right)^2 = \alpha\left(x_2 - \frac{1}{2}x_1^2\right)^3, \quad \alpha < -\frac{8}{9}$

17) $x_3 = x_1(\alpha \ln x_1 - \ln x_2), \quad \alpha > 1$

Here each of the hypersurfaces **1)**– **17)** *shall be considered near an arbitrary strictly pseudoconvex point q in it. Furthermore, the dimensions of the stability algebras for the hypersurfaces* **1)**– **17)** *are as follows:*

- *dim* $\mathfrak{aut}\,(M,q) = 10$ *for the hyperquadric* **1)**,
- *dim* $\mathfrak{aut}\,(M,q) = 2$ *for hypersurfaces* **2)**– **5)**,
- *dim* $\mathfrak{aut}\,(M,q) = 1$ *for hypersurfaces* **6)**– **11)**, *and*
- *dim* $\mathfrak{aut}\,(M,q) = 0$ *for hypersurfaces* **12)**– **17)**.

Finally, any two hypersurfaces in the list **1)**– **17)** *are pairwise locally holomorphically inequivalent.*

Acknowledments

The first author was supported by the GACR (Czech Grant Agency, grant 17-19427S) and the FWF (Austrian Science Fund, grant P29468) during the preparation of this paper. The second author was supported by the RFBR (Russian Foundation for Basic Research) grant 170100592-a.

2. Principal approach

As discussed above, we are concerned with the case of a 5-dimensional Lie algebra transitively acting on a strictly pseudoconvex real hypersurface by CR-transformations. Our approach to the classification then is based on realizing (abstract) 5-dimensional Lie algebras acting transitively on a real hypersurface by holomorphic vector fields, and finding subsequently appropriate normal forms for such realizations. In accordance with that, we will make extensive use of the classification of (abstract) 5-dimensional Lie algebras up to an isomorphism. The latter was obtained by Mubarakzjanov in [Mu63], in the sense that any 5-dimensional real Lie algebra is equivalent to a one in Mubarakzjanov's list (we provide the entire list in Appendix A, for convenience of the reader).

In what follows, we fix the following notations and conventions: (z_1, z_2, z_3) denote the coordinates in $\mathbb{C}^3$, M denotes a (real-analytic) strictly pseudoconvex simply homogeneous near a point $p \in M$ hypersurface in $\mathbb{C}^3$, and $\mathfrak{g}$ a 5-dimensional Lie algebra of holomorphic vector fields acting on M locally transitively near the point p. We have, accordingly,

$$\mathfrak{hol}\,(M,p) = \mathfrak{g}, \quad \mathfrak{aut}\,(M,p) = 0. \tag{2.1}$$

We also denote by X_j, $j = 1, ..., 5$ a collection of holomorphic vector fields from $\mathfrak{g}$ defined in some neighborhood U of the point p and pointwice linearly independent in U (over $\mathbb{R}$). Thus, we have $\mathfrak{g} = \operatorname{span}\{X_1, ..., X_5\}$ pointwice in U. We call such a collection *a basis* for $\mathfrak{g}$.

We also make use of the following

Convention. Solely for the purposes of the proof of Theorem 1, we assume tubular CR-hypersurfaces to be invariant under the *real* shifts

$$z \mapsto z + a, \quad a \in \mathbb{R}^3$$

(unlike the set up in the Introduction). Accordingly, a tubular real hypersurface looks as

$$M = \mathbb{R}^3 + iB,$$

where B is a surface in $\mathbb{R}^3$, and $\mathfrak{hol}(M, 0)$ contains the abelian subalgebra spanned by $\frac{\partial}{\partial z_j}$, $j = 1, 2, 3$, which fits better our normalization procedure for the Lie algebras of holomorphic vector fields.

Our goal is, based on properties of $\mathfrak{g}$ as an abstract Lie algebra, bring the basis vector fields $X_1, ...X_5$ (and hence $\mathfrak{g}$ itself) by a series of biholomorphic transformations to a certain normal form, in which $\mathfrak{g}$ is "maximally simplified". The latter makes it possible to either obtain a contradiction with the strong pseudoconvexity of M, or to recognize M (up to a local biholomorphic equivalence) as a tube over an affinely homogeneous hypersurface in $\mathbb{R}^3$.

Let us make the following useful

Observation 2.1. In order to prove the assertion of Theorem 2 at a reference point $p \in M$, it is obviously sufficient to prove the same assertion at any other point $s \in M$ close by p (in view of the local homogeneity of M). In view of that, we may change during the proof the reference point under consideration.

We make use of the following two important propositions.

Proposition 2.2. *Let $X, Y \in \mathfrak{g}$ be two vector fields such that*

(i) X, Y linearly independent over $\mathbb{R}$ at a point $q \in M$;

(ii) the real span of X, Y is a subalgebra in $\mathfrak{g}$ (that is, $[X, Y] \in span_{\mathbb{R}}\{X, Y\}$ at every point).

Then X, Y are also linearly independent over $\mathbb{C}$ at q.

Proof. Assume, by contradiction, that X, Y span a 1-dimensional complex plane at q. Consider then the orbit at q of the action of the above 2-dimensional subalgebra spanned by X, Y (denote the latter by S). We have $S \subset M$ and $T_qS = \text{span}_{\mathbb{R}}\{X_q, Y_q\}$, so that by assumption the plane T_qS is a 1-dimensional complex plane and thus S is a complex curve (since S is homogeneous). Since $S \subset M$, this gives a contradiction with the strict pseudoconvexity of M, and proves the proposition. □

Proposition 2.3. *Let $X, Y, Z \in \mathfrak{g}$ be three commuting vector fields which are linearly independent over $\mathbb{R}$ at a point $q \in M$. Then X, Y, Z are also linearly independent over $\mathbb{C}$ at q.*

Proof. Assume, by contradiction, that $\dim \operatorname{span}_{\mathbb{C}} \{X_q, Y_q, Z_q\} = 2$ (complex dimension 1 is excluded since the real span has dimension 3). Then the orbit of the 3-dimensional abelian algebra $\mathfrak{a}$ spanned by X, Y, Z is 3-dimensional real manifold N contained in a complex hypersurface S (which is the orbit of the complex action of X, Y, Z near q). This implies that N has a 1-dimensional complex tangent at q, which means that there exists $\mathbb{R}$-linearly independent vector fields $U, V \in \mathfrak{a}$ with $V_q = iU_q$. Since U, V form a 2-dimensional abelian algebra, the latter contradicts Proposition 2.2. This proves the proposition. □

3. Proof of Theorem 1

In this section, we apply Proposition 2.2 and Proposition 2.3 (read together with Observation 2.1) to prove Theorem 1. More precisely, we show that any Lie algebra from Mubarakzjanov's list (see Appendix A) realized as a 5-dimensional Lie algebra can act locally transitively by holomorphic transformations on a strictly pseudoconvex hypersurface $M \subset \mathbb{C}^3$ only if the latter is (up to a local biholomorphic equivalence) a tube over an affinely homogeneous strictly geometrically convex (resp. concave) hypersurface $S \subset \mathbb{R}^3$.

We use the set-up and notations of Section 2. Let us first observe the following important fact.

Proposition 3.1. *In the notations and setting of Section 2, if the algebra $\mathfrak{g}$ contains a 3-dimensional abelian ideal $\mathfrak{a}$, then M is biholomorphically equivalent (locally near p) to the tube over an affinely homogeneous strictly geometrically convex hypersurface $S \subset \mathbb{R}^3$.*

Proof. Let us choose a basis for $\mathfrak{g}$ in such a way that $\mathfrak{a}$ is spanned by X_1, X_2, X_3. According to Proposition 2.3 (and Observation 2.1), the vector fields X_1, X_2, X_3 can be assumed to have compelx rank 3 at p. Hence there exists a biholomorphic coordinate change near p mapping p into the origin and X_1, X_2, X_3 onto $\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}$ respectively. Since $\mathfrak{a}$ is an ideal in $\mathfrak{g}$, we have

$$\left[\frac{\partial}{\partial z_l}, X\right] = \sum_{j=1}^{3} \alpha_{lj} \frac{\partial}{\partial z_j}$$

for any $X \in \mathfrak{g}$, where α_{lj} are real coefficients. Thus all the derivatives in z_l of the components of Y are real constants, and we conclude that vector fields X_4, X_5 completing X_1, X_2, X_3 to a basis have the form:

$$X_l = \sum_{j=1}^{3} a_{lj} \frac{\partial}{\partial z_j} + Z \cdot B_l \cdot \frac{\partial}{\partial Z}, \quad l = 4, 5. \tag{3.1}$$

Here a_{lj} are complex constants, $Z = (z_1, z_2, z_3)$, $\partial Z = \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}\right)^T$, and B_l are constant *real* 3×3 matrices (in particular, X_4, X_5 are affine vector fields). By adding to X_4, X_5 appropriate real linear combinations of X_1, X_2, X_3, we further achieve $a_{lj} \in i\mathbb{R}$. Moreover, the fact that $\operatorname{rank}_{\mathbb{R}} \{X_1, ..., X_5\} = 5$ at 0 implies that the real rank of the matrix $\{a_{lj}\}$ equals 2. All the latter precisely means that the orbit of $\mathfrak{g}$ at 0 (which coincides with M) is the tube over an affinely homogeneous surface S, which is in turn the orbit at 0 of

$$\sum_{j=1}^{3} \operatorname{Im} a_{lj} \frac{\partial}{\partial y_j} + Y \cdot B_l \cdot \partial Y, \quad l = 4, 5, \quad Y = \operatorname{Im} Z.$$

This proves the proposition. □

We now refer directly to Mubarakzjanov's list described in Appendix A.

Case 1: decomposable solvable algebras.

It is possible to see from Mubarakzjanov's classification (see the first table in Appendix A) that all the decomposable solvable Lie algebras, with the exeption of $\mathfrak{m}_{26}$, contain a 3-dimensional abelian ideal. Applying now Proposition 3.1, we conclude that any possible strongly pseudoconvex orbit in Case 1 is biholomorphic to the tube over an affinely homogeneous strictly geometrically convex hypersurface $S \subset \mathbb{R}^3$, with possibly the only exception of

$$\mathfrak{g} = \mathfrak{m}_{26}.$$

The latter exceptional algebra shall be be treated separately. We claim that for this algebra there are no simply homogeneous strictly pseudoconvex orbits. Indeed, the nontrivial commuting relations for $\mathfrak{g}$ are:

$$[X_2, X_3] = X_1,\ [X_1, X_4] = 2qX_1,\ [X_2, X_4] = qX_2 - X_3,$$
$$[X_3, X_4] = X_2 + qX_3,\ q \geq 0.$$

We introduce the vector fields

$$X_2' := X_2 + iX_3, \quad X_3' := X_2 - iX_3.$$

Then the nontrivial commuting relations involving X_2, X_3 turn into

$$[X_2', X_3'] = -2iX_1, \quad [X_2', X_4] = (q+i)X_2', \quad [X_3', X_4] = (q-i)X_3'.$$

Note that both triples X_1, X_2, X_5 and X_1, X_3, X_5 form abelian subalgebras, thus both triples have complex rank 3 at the reference point p by Proposition 2.3. This implies that at least one of the triples X_1, X_2', X_5 and X_1, X_3', X_5 (say the first one) has complex rank 3 at p. We then straighten the commuting vector fields near p and get:

$$X_1 = \frac{\partial}{\partial z_1}, \quad X_2' = \frac{\partial}{\partial z_2}, \quad X_5 = \frac{\partial}{\partial z_3}.$$

Now from the commuting relations of the remaining fields with X_1, X_2', X_5 we easily get:

$$X_3' = (-2iz_2 + c)\frac{\partial}{\partial z_1} + a\frac{\partial}{\partial z_2} + b\frac{\partial}{\partial z_3},$$
$$X_4 = 2qz_1\frac{\partial}{\partial z_1} + (q+i)z_2\frac{\partial}{\partial z_2} + d\frac{\partial}{\partial z_3}$$

(a, b, c, d - constants). Further, the commuting relation for X_3', X_4 gives (by considering the components $\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_2}$ respectively):

$$c = 0, \quad a = 0, \quad b = 0.$$

We finally get:

$$X_2 = -iz_2\frac{\partial}{\partial z_1} + \frac{1}{2}\frac{\partial}{\partial z_2}, \quad X_3 = z_2\frac{\partial}{\partial z_1} + \frac{1}{2i}\frac{\partial}{\partial z_2}.$$

We claim now that all orbit of the 5-dimensional algebra $\mathfrak{g}$ obtained above have an additional holomorphic symmetry and thus are not simply homogeneous. Indeed, consider the vector field

$$Y := iz_2 \frac{\partial}{\partial z_2}.$$

It is easy to check that

$$[Y, X_1] = [Y, X_4] = [Y, X_5] = 0, \quad [Y, X_2] = X_3, \quad [Y, X_3] = -X_2.$$

The latter means $[Y, \mathfrak{g}] \subset \mathfrak{g}$. At the same time, at points on the hypersurface $\Sigma := \{z_2 = 0\}$ the algebra $\mathfrak{g}$ has the full rank 5, while the vector field Y vanishes. This means that the orbits through these points of the algebras $\mathfrak{g}$ and $\mathfrak{g} \oplus \mathbb{C}Y$ coincide. Since a generic orbit must intersect Σ, this proves that all the orbits of $\mathfrak{g}$ are invariant under the action of $Y \notin \mathfrak{g}$, as required.

Case 2: Nondecomposable solvable algebras. These algebras form the largest subset in Mubarakzjanov's list, and we step-by-step go through the list of algebras.

First, all the algebras in the ranges $g_1 - g_{18}$, $g_{30} - g_{35}$, and $g_{38} - g_{39}$ contain the 3-dimensional abelian ideal spanned by e_1, e_2, e_3. Next, all the algebras in the ranges g_{19} - $-g_{24}$ and $g_{27} - g_{29}$ contain the 3-dimensional abelian ideal spanned by e_1, e_3, e_4. According to Proposition 3.1, the orbits of all the mentioned algebras then appear to be locally biholomorphic to tubes over affinely homogeneous hypersurfaces. We are left with the four exceptional algebras: $g_{25}, g_{26}, g_{36}, g_{37}$.

Subcase $\mathfrak{g} = g_{36}$. In the latter case, $\mathfrak{g}$ is realized by holomorphic in a neighborhood of p vector fields X_i, $i = 1, .., 5$ with the following only nontrivial commuting relations:

$$[X_2, X_3] = X_1, \; [X_1, X_4] = X_1, \; [X_2, X_4] = X_2,$$
$$[X_2, X_5] = -X_2, \; [X_3, X_5] = X_3.$$

According to Proposition 2.2, the complex rank at p of X_1, X_2 equals 2, thus these two vector fields can be simultaneously straightened near p:

$$X_1 = \frac{\partial}{\partial z_1}, \; X_2 = \frac{\partial}{\partial z_2}.$$

Taking into account the commuting relations of X_1, X_2 with X_3, we conclude that in such coordinates X_3 has the form:

$$X_3 = (z_2 + f(z_3))\frac{\partial}{\partial z_1} + g(z_3)\frac{\partial}{\partial z_2} + h(z_3)\frac{\partial}{\partial z_3}. \tag{3.2}$$

First, consider the situation $h \equiv 0$. Then we do the variable change $z_2^* = z_2 + f(z_3)$ and X_3 becomes

$$X_3 = z_2\frac{\partial}{\partial z_1} + A(z_3)\frac{\partial}{\partial z_2}.$$

We then work out X_4. The commuting relations with X_1, X_2 give:

$$X_4 = (z_1 + f(z_3))\frac{\partial}{\partial z_1} + (z_2 + g(z_3))\frac{\partial}{\partial z_2} + h(z_3)\frac{\partial}{\partial z_3}. \tag{3.3}$$

Note that $h \equiv 0$ is not possible for the latter identity, since otherwice X_1, X_2, X_3, X_4 span a subalgebra of vector fields non of which has the $\frac{\partial}{\partial z_3}$ component, thus their orbit at p lies in $z_3 = const$, hence it coincides with $z_3 = const$ and we obtain a contradiction with the strict pseudoconvexity. In view of that, after possibly changing the base point p, we may assume $h(z_3) \neq 0$ at p in (3.3) and straighten the vector field $h(z_3)\frac{\partial}{\partial z_3}$. This means

$$X_4 = (z_1 + f(z_3))\frac{\partial}{\partial z_1} + (z_2 + g(z_3))\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}.$$

A variable change $z_1^* = z_1 + \psi(z_3)$ for appropriate ψ allows to further make $f = 0$ (this is accomplished by choosing ψ such that $\psi' - \psi + f = 0$). Using now $[X_3, X_4] = 0$, we get first $g = 0$ by considering the $\frac{\partial}{\partial z_1}$ component, and then $A - A' = 0$ by considering the $\frac{\partial}{\partial z_3}$ component. This finally gives

$$X_3 = z_2\frac{\partial}{\partial z_1} + \alpha e^{z_3}\frac{\partial}{\partial z_2}, \quad X_4 = z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}$$

(α here is a constant). Finally, consider the vector field

$$X_5' := 2X_4 + X_5.$$

We have

$$[X_1, X_5'] = 2X_1,\ [X_2, X_5'] = X_2,\ [X_3, X_5'] = X_3.$$

The commuting relations with X_1, X_2 give

$$X_5' = (2z_1 + f(z_3))\frac{\partial}{\partial z_1} + (z_2 + g(z_3))\frac{\partial}{\partial z_2} + h(z_3)\frac{\partial}{\partial z_3}.$$

Then the commuting relation with X_3 gives, by considering the $\frac{\partial}{\partial z_2}$ component: $\alpha e^{z_3} h = 0$, so either:

(i) $h = 0$, so the vector fields X_1, X_2, X_3, X_5' span a subalgebra of vector fields non of which has the $\frac{\partial}{\partial z_3}$ component, and repeating the argument above, we obtain a contradiction with the strict pseudoconvexity,

or

(ii) $\alpha = 0$, and the vector fields X_1, X_3 provide a contradiction with Proposition 2.2.

This means that $h \not\equiv 0$ in (3.2). Shifting if necessary the base point p, we may assume $h(z_3) \neq 0$ at p in (3.2) and straighten the vector field $h(z_3)\frac{\partial}{\partial z_3}$. Thus X_3 becomes:

$$X_3 = (z_2 + f(z_3))\frac{\partial}{\partial z_1} + g(z_3)\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}.$$

Acting as above, we further find functions $\phi(z_3), \psi(z_3)$ such that the variable change

$$z_1^* = z_1 + \phi(z_3), \quad z_2^* = z_2 + \psi(z_3), \quad z_3^* = z_3$$

annihilates f, g, i.e. X_3 becomes

$$X_3 = z_2\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_3}.$$

Next, for X_4 we use commuting relations with X_1, X_2 and get:

$$X_4 = (z_1 + f(z_3))\frac{\partial}{\partial z_1} + (z_2 + g(z_3))\frac{\partial}{\partial z_2} + h(z_3)\frac{\partial}{\partial z_3}.$$

The commuting relation with X_3 implies (by considering the $\frac{\partial}{\partial z_3}$ component): $h' = 0$, so $h = c$ (c is a constant). Similarly, for X_5 we get:

$$X_5 = F(z_3)\frac{\partial}{\partial z_1} + (-z_2 + G(z_3))\frac{\partial}{\partial z_2} + H(z_3)\frac{\partial}{\partial z_3}.$$

Now the commuting relation with X_3 implies (by considering the $\frac{\partial}{\partial z_3}$ component): $H' = 1$, so $H = z_3 + C$ (C is a constant). Finally, the commuting relation of X_4 and X_5 implies (by considering the $\frac{\partial}{\partial z_3}$ component) $c = 0$.

We now consider the subalgebra spanned by X_1, X_2, X_4. Recall that

$$X_1 = \frac{\partial}{\partial z_1}, \quad X_2 = \frac{\partial}{\partial z_2}, \quad X_4 = (z_1 + f(z_3))\frac{\partial}{\partial z_1} + (z_2 + g(z_3))\frac{\partial}{\partial z_2}. \tag{3.4}$$

Integrating the action of (3.4) near p gives the *flat* orbit

$$z_3 = const, \quad \mathrm{Im}\,(z_1 - z_2) = const,$$

which contains, in particular, complex lines. This gives a contradiction with the strict pseudoconvexity.

We finally conclude that there are no strictly pseudoconvex orbits in the case $\mathfrak{g} = g_{36}$.

Subcase $\mathfrak{g} = g_{25}$. Here the nontrivial commuting relations are

$$[X_2, X_3] = X_1,\ [X_1, X_5] = 2qX_1,\ [X_2, X_5] = qX_2 - X_3,$$
$$[X_3, X_5] = X_2 + qX_3,\ [X_4, X_5] = pX_4,\ p \neq 0.$$

Arguing very similarly to the case $\mathfrak{g} = g_{4.9} \oplus g_1$, we conclude that in appropriate local holomorphic coordinates $\mathfrak{g}$ can be represented as:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial z_1},\ X_2 = -iz_2\frac{\partial}{\partial z_1} + \frac{1}{2}\frac{\partial}{\partial z_2},\ X_3 = z_2\frac{\partial}{\partial z_1} + \frac{1}{2i}\frac{\partial}{\partial z_2}, \\ X_4 &= \frac{\partial}{\partial z_3},\ X_5 = 2qz_1\frac{\partial}{\partial z_1} + (q+i)z_2\frac{\partial}{\partial z_2} + qz_3\frac{\partial}{\partial z_3}. \end{aligned} \tag{3.5}$$

Introducing as above the vector field

$$Y := iz_2\frac{\partial}{\partial z_2},$$

we check that

$$[Y, X_1] = [Y, X_4] = [Y, X_5] = 0, \quad [Y, X_2] = X_3, \quad [Y, X_3] = -X_2.$$

The latter means $[Y, \mathfrak{g}] \subset \mathfrak{g}$, and arguing as in Case 1 we conclude that all the orbits of $\mathfrak{g}$ are invariant under the action of $Y \notin \mathfrak{g}$, so that the orbits of $\mathfrak{g}$ are not simply homogeneous in the case $\mathfrak{g} = g_{25}$.

Subcase $\mathfrak{g} = g_{26}$. Here the nontrivial commuting relations are

$$[X_2, X_3] = X_1,\ [X_1, X_5] = 2qX_1,\ [X_2, X_5] = qX_2 - X_3,$$
$$[X_3, X_5] = X_2 + qX_3,\ [X_4, X_5] = \epsilon X_1 + 2qX_4,$$

where $q \in \mathbb{R},\ \epsilon = \pm 1$. Arguing, again, very similarly to the case $\mathfrak{g} = g_{4.9} \oplus g_1$, we conclude that in appropriate local holomorphic coordinates $\mathfrak{g}$ can be represented as:

$$X_1 = \frac{\partial}{\partial z_1},\ X_2 = -iz_2\frac{\partial}{\partial z_1} + \frac{1}{2}\frac{\partial}{\partial z_2},\ X_3 = z_2\frac{\partial}{\partial z_1} + \frac{1}{2i}\frac{\partial}{\partial z_2}, \tag{3.6}$$
$$X_4 = \frac{\partial}{\partial z_3},\ X_5 = (2qz_1 + \epsilon z_3)\frac{\partial}{\partial z_1} + (q+i)z_2\frac{\partial}{\partial z_2} + 2qz_3\frac{\partial}{\partial z_3}.$$

Introducing as above the vector field

$$Y := iz_2\frac{\partial}{\partial z_2},$$

we check that

$$[Y, X_1] = [Y, X_4] = [Y, X_5] = 0, \quad [Y, X_2] = X_3, \quad [Y, X_3] = -X_2.$$

The latter means $[Y, \mathfrak{g}] \subset \mathfrak{g}$, and arguing as in Case 1 we conclude that all the orbits of $\mathfrak{g}$ are invariant under the action of $Y \notin \mathfrak{g}$, so that the orbits of $\mathfrak{g}$ are not simply homogeneous in the case $\mathfrak{g} = g_{26}$ as well.

Subcase $\mathfrak{g} = g_{37}$. Here we have the following nontrivial commutation relations:

$$[X_2, X_3] = X_1,\ [X_1, X_4] = 2X_1,\ [X_2, X_4] = X_2,$$
$$[X_3, X_4] = X_3,\ [X_2, X_5] = -X_3,\ [X_3, X_5] = X_2.$$

Using (2.2), we straighten X_1, X_2 so that:

$$X_1 = \frac{\partial}{\partial z_1},\ \ X_2 = \frac{\partial}{\partial z_2}.$$

Taking into account the commuting relations of X_1, X_2 with X_3, we conclude that in such coordinates X_3 has the form:

$$X_3 = (z_2 + f(z_3))\frac{\partial}{\partial z_1} + g(z_3)\frac{\partial}{\partial z_2} + h(z_3)\frac{\partial}{\partial z_3}. \tag{3.7}$$

Under the assumption $h \not\equiv 0$ in (3.7), it is not difficult by arguing as above to further simplify X_3 to become:

$$X_3 = z_2\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_3}.$$

After that, by using the commuting relations for X_4, X_5 in a straightforward manner as shown above, we compute that X_4, X_5 have the form:

$$X_4 = (2z_1 + Az_3)\frac{\partial}{\partial z_1} + (z_2 + A)\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3},$$
$$X_5 = \left(\frac{1}{2}(z_3^2 - z_2^2) + \frac{1}{2}A^2\right)\frac{\partial}{\partial z_1} + z_3\frac{\partial}{\partial z_2} - (z_2 + A)\frac{\partial}{\partial z_3}$$

(after a shift in z_1). Here $A = a + bi$ is a complex constant. Further, it is convenient to shift z_2 by A which finally gives:

$$X_3 = (z_2 - A)\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_3},$$
$$X_4 = (2z_1 + Az_3)\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3},$$
$$X_5 = \left(\frac{1}{2}(z_3^2 - z_2^2) + Az_2\right)\frac{\partial}{\partial z_1} + z_3\frac{\partial}{\partial z_2} - z_2\frac{\partial}{\partial z_3},$$

and X_1, X_2 are as above. It is straightforward to check then that the real parts of all these five vector fields are tangent to the 1-parameter family of real hyperquadrics

$$y_1 = x_3(y_2 - b) - ay_2 + N(y_2^2 + y_3^2).$$

The latter means that all the strictly pseudoconvex orbits of the algebra $\mathfrak{g}$ are automatically *spherical* under the assumption $h \not\equiv 0$ in (3.7).

If, otherwise, $h \equiv 0$ in (3.7), we can claim that for the vector field

$$X_4 = u\frac{\partial}{\partial z_1} + v\frac{\partial}{\partial z_2} + w\frac{\partial}{\partial z_3}$$

we have $w(p) \neq 0$ (since otherwise the orbit at p of the algebra spanned by X_1, X_2, X_3, X_4 has a 2-dimensional complex tangent at p and hence is a complex hypersurface itself). This allows us, by arguing as above, to simplify X_4 to be

$$X_4 = 2z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}$$

(while X_1, X_2 stay the same). It is convenient for us now to do the substitution $z_3^* = e^{z_3}$ which turns X_4 into

$$X_4 = 2z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3}.$$

Having X_1, X_2, X_3 normalized as above, it is straightforward to compute by using the commuting relations for X_3, X_5 that the latter two vector fields look as

$$X_3 = z_2 \frac{\partial}{\partial z_1} + A \frac{\partial}{\partial z_2},$$
$$X_5 = (B z_3^2 - \frac{1}{2} z_2^2) \frac{\partial}{\partial z_1} + C z_2 \frac{\partial}{\partial z_2} + D z_3 \frac{\partial}{\partial z_3}.$$

Finally, by quadratic changes of variables we are able to simplify X_5 to be

$$X_5 = C z_2 \frac{\partial}{\partial z_2} + D z_3 \frac{\partial}{\partial z_3}$$

keeping the other vector fields unchanged.

We now use a similar idea to that for the algebras g_{25}, g_{26} and notice that the vector field

$$Y := z_2 \frac{\partial}{\partial z_2}$$

satisfies

$$[\mathfrak{g}^{\mathbb{C}}, Y] \subset \mathfrak{g}^{\mathbb{C}}$$

(where $\mathfrak{g}^{\mathbb{C}}$ is the complexified algebra of holomorphic vector fields). By considering orbits through points with $z_2 = 0$ and generic z_1, z_3 (where the algebra $\mathfrak{g}$ has the full rank 5), we see that the external complexification $M^{\mathbb{C}} \subset \mathbb{C}^3 \times \overline{\mathbb{C}}$ of M has an infinitesimal automorphism algebra of dimension at least 6. Since $\mathfrak{g}$ is a real form of $\mathfrak{g}^{\mathbb{C}}$, we conclude finally that M is not simply homogeneous.

In view of that, the algebra g_{37} does not have any simply homogeneous strictly pseudoconvex orbits.

Case 3: decomposable nonsolvable algebras. The latter case occurs when the minimal decomposition of $\mathfrak{g}$ contains a 3-dimensional simple term.

We start with the situation when the minimal decomposition of $\mathfrak{g}$ contains 3 terms, i.e. $\mathfrak{g}$ is the sum of a 3-dimensional simple term $\mathfrak{a}$ and a 2-dimensional abelian term $\mathfrak{b}$. Fix a reference point p on the orbit and consider the intersection of $\mathfrak{a}$ evaluated at p with the complexification of $\mathfrak{b}$ evaluated at p. In view of dimension reasons, such intersection is nonempty, hence there exists $W \in \mathfrak{a}$ such that the complex rank of $\mathbb{C}W \oplus \mathfrak{b}$ at p equals 2. On the other hand, $\mathbb{C}W \oplus \mathfrak{b}$ is abelian, and we obtain a contradiction with Proposition 2.3.

We conclude finally that the only possibility of obtaining a strictly pseudoconvex orbit is the following: $\mathfrak{g}$ is the sum of a 3-dimensional simple term

$\mathfrak{a}$ and the 2-dimensional nonabelian term $\mathfrak{g}_2$. This leads to the following two subcases.

Subcase $\mathfrak{g} = \mathfrak{m}_{16} = \mathfrak{su}(1,1) \oplus \mathfrak{g}_2$. In this case, we have the following commutation relations:

$$[X_1, X_2] = X_1,\ [X_1, X_3] = 2X_2,\ [X_2, X_3] = X_3,\ [X_4, X_5] = X_4.$$

Using (2.2), we straighten X_3, X_4 so that:

$$X_3 = \frac{\partial}{\partial z_1},\ \ X_4 = \frac{\partial}{\partial z_2}.$$

Taking into account the commuting relations of X_3, X_4 with X_5, we conclude that X_5 has the form:

$$X_5 = f(z_3)\frac{\partial}{\partial z_1} + (z_2 + g(z_3))\frac{\partial}{\partial z_2} + h(z_3)\frac{\partial}{\partial z_3}. \tag{3.8}$$

We claim that $h \not\equiv 0$ in (3.8). Indeed, arguing by contradiction, we consider first $[X_1, X_4] = 0$ and conclude that components of X_1 do not depend on z_2. Next, we consider the last component of the identity $[X_1, X_5] = 0$ and conclude that $f \cdot H_{z_1} = 0$ (if $X_1 = F\frac{\partial}{\partial z_1} + G\frac{\partial}{\partial z_2} + H\frac{\partial}{\partial z_3}$), so that $H_{z_1} \equiv 0$ ($f \equiv$ is not possible, since then X_5 is a product of X_4 with a holomorphic function, which implies the holomorphic degeneracy of the orbit). Finally, considering the last component in $[X_1, X_3] = 2X_2$, we conclude that the last component of X_2 vanishes identically. All together, this implies that for the four vector fields X_2, X_3, X_4, X_5 (forming a subalgebra in $\mathfrak{g}$) their last component is zero, thus the orbit of the subalgebra at every point is a complex plane, which is a contradiction with the Levi-degeneracy of the orbit. This proves the claim.

In this way, we assume $h \not\equiv 0$ in (3.8), and then move to a nearby point in M in order to have $h(p) \neq 0$. It is not difficult then, by arguing as above, to further simplify X_5 to become:

$$X_5 = z_2\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}.$$

Further, the substitution $z_3^* = e^{z_3}$ makes

$$X_5 = z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3}.$$

After that, considering the commutators of X_2 with X_3, X_4 and X_5, it is not difficult to get:

$$X_2 = -z_1\frac{\partial}{\partial z_1} + Az_3\frac{\partial}{\partial z_2} + Bz_3\frac{\partial}{\partial z_3}. \tag{3.9}$$

In case $B \neq 0$ in (3.9), by means of a linear substitution (the non-identical part of which has the form $z_2 \mapsto z_2 + \alpha z_3$), it is possible to further achieve

$$X_2 = -z_1\frac{\partial}{\partial z_1} + Bz_3\frac{\partial}{\partial z_3}$$

(preserving the form of the other vector fields). Finally, we figure out the form of X_1. Commutation relations of X_1 with X_4, X_3, X_5 and X_2 respectively yield (after a straightforward calculation):

$$X_1 = z_1^2\frac{\partial}{\partial z_1} + Cz_3\frac{\partial}{\partial z_2} - 2Bz_1z_3\frac{\partial}{\partial z_3},$$

where

$$C(B+1) = 0.$$

In case $C = 0$, we observe that all the vector fields

$$Y_\lambda := \lambda z_3\frac{\partial}{\partial z_3}, \quad \lambda \in \mathbb{C}$$

commute with $\mathfrak{g}$. For an appropriate choice of λ, the value of Y_λ at p must lie in T_pM (recall that $z_3 \neq 0$ at p). This means that M is invariant under the action of $\mathfrak{g} \oplus \mathbb{C}Y_\lambda$, so that M is not simply homogeneous.

In case $C \neq 0$, $B = -1$, we observe the following. The linear map

$$\sigma(z_1, z_2, z_3) := (z_2, z_1, z_3)$$

preserves the subalgebra spanned by X_1, X_2, X_3, X_4, while the vector field X_5 becomes

$$X_1' := Cz_3\frac{\partial}{\partial z_1} + z_2^2\frac{\partial}{\partial z_2} + 2z_2z_3\frac{\partial}{\partial z_3}.$$

It is straightforward to check then that we have $[X_1', \mathfrak{g}] \subset \mathfrak{g} \oplus \mathbb{C}X_1'$. Then we argue as above and note that, for example, at points with $z_2 = 0$, $z_1 \neq 0$ the algebra $\mathfrak{g}$ has full rank while the value of X_1' at p lies in the span of $X_1, .., X_5$, so that the orbits at such points are invariant under the action of $\mathfrak{g} \oplus \mathbb{C}X_1$ and X_1 is thus an additional infinitesimal automorphism. The latter applies, by uniqueness, to all the orbits. We again conclude that neither of the orbits is simply homegeneous.

Finally, in the case $B = 0$ in (3.9), we may replace X_2 by $X_2 + X_5$ and then argue identically to the case $B \neq 0$ to simplify X_2. Since $[X_1, X_2 + X_5] = [X_1, X_2]$, we get an identical representation to the above for X_1 and again conclude, that the orbits are not simply homogeneous.

Subcase $\mathfrak{g} = \mathfrak{m}_{17} = \mathfrak{su}(2) \oplus \mathfrak{g}_2$. In this case, we have the following nontrivial commutation relations:

$$[X_1, X_2] = X_3,\ [X_1, X_3] = -X_2,\ [X_2, X_3] = X_1,\ [X_4, X_5] = X_4.$$

Let us introduce the new vector fields

$$X_1' := X_1 + iX_3, \quad X_3' := X_1 - iX_3.$$

Then the respective modified (nontrivial) commutation relations are:

$$[X_1', X_3'] = 2iX_2,\ [X_1', X_2] = -iX_1',\ [X_2, X_3'] = -iX_3',\ [X_4, X_5] = X_4.$$

We observe that the modified commutation relations are very similar to that for $\mathfrak{g} = \mathfrak{su}(1,1) \oplus \mathfrak{g}_2$. In accordance with that, arguing identically to the case $\mathfrak{g} = \mathfrak{su}(1,1) \oplus \mathfrak{g}_2$, we can simplify $\mathfrak{g}$ in order that the (modified) basic vector fields look as:

$$X_1' = -z_1^2 \frac{\partial}{\partial z_1} + Cz_3 \frac{\partial}{\partial z_2} - 2iBz_1z_3 \frac{\partial}{\partial z_3}, \quad X_2 = iz_1 \frac{\partial}{\partial z_1} + Bz_3 \frac{\partial}{\partial z_3},$$
$$X_3' = \frac{\partial}{\partial z_1}, \quad X_4 = \frac{\partial}{\partial z_2}, \quad X_5 = z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3},$$

where the complex constants B, C, $B \neq 0$ satisfy:

$$C(B - i) = 0.$$

In case $C = 0$, the basic vector fields finally look as

$$2X_1 = (1 - z_1^2) \frac{\partial}{\partial z_1} - 2iBz_1z_3 \frac{\partial}{\partial z_3}, \quad X_2 = iz_1 \frac{\partial}{\partial z_1} + Bz_3 \frac{\partial}{\partial z_3},$$
$$2X_3 = i(1 + z_1^2) \frac{\partial}{\partial z_1} - 2Bz_1z_3 \frac{\partial}{\partial z_3}, \quad X_4 = \frac{\partial}{\partial z_2}, \quad X_5 = z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3}.$$

Arguing now again identically to the situation of $\mathfrak{g} = \mathfrak{su}(1,1) \oplus \mathfrak{g}_2$ and employing the vector field

$$Y_\lambda := \lambda z_3 \frac{\partial}{\partial z_3}, \quad \lambda \in \mathbb{C},$$

we similarly conclude that the orbits in this case are not simply homogeneous.

In case $C \neq 0$, $B = i$, the basic vector fields finally look as

$$\begin{aligned} 2X_1 &= (1 - z_1^2)\frac{\partial}{\partial z_1} + Cz_3\frac{\partial}{\partial z_2} + 2z_1z_3\frac{\partial}{\partial z_3}, \quad X_2 = iz_1\frac{\partial}{\partial z_1} + Bz_3\frac{\partial}{\partial z_3}, \\ 2X_3 &= i(1 + z_1^2)\frac{\partial}{\partial z_1} - iCz_3\frac{\partial}{\partial z_2} - 2iz_1z_3\frac{\partial}{\partial z_3}, \quad X_4 = \frac{\partial}{\partial z_2}, \\ X_5 &= z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3}. \end{aligned}$$

We claim that the Levi-nondegenerate orbits of the latter algebra are *not* strictly pseudoconvex. For that, we first note that at all points in $\mathbb{C}^3$ with $z_1 = z_2 = 0$, $z_3 \neq 0$, the real rank of the vector fields $X_1, ..., X_5$ is 5. This means that a generic orbit of $\mathfrak{g}$ intersects the complex line

$$L := \{z_1 = z_2 = 0\}.$$

At the same time, at all points in L the values of the commuting vector fields X_2 and X_5 are linearly dependent over $\mathbb{C}$. The latter contradicts Proposition 2.2 and proves that neither of the orbits is strictly pseudoconvex.

Case 4: nondecomposable nonsolvable algebras. According to Mubarakzjanov's classification, there is a unique nondecomposable nonsolvable 5-dimensional Lie algebra, namely $\mathfrak{g}_5$ (see Appendix A). This particular algebra and its orbits in $\mathbb{C}^3$ were considered in the recent paper [AL19] of Atanov-Loboda, and the outcome of Case 4 can be read from the latter paper. However, we provide, for completeness, an alternative proof here.

Nontrivial commutation relations in $\mathfrak{g} = \mathfrak{g}_5$ look as:

$$\begin{aligned} &[X_1, X_2] = 2X_1,\ [X_1, X_3] = -X_2,\ [X_2, X_3] = 2X_3,\ [X_1, X_4] = X_5, \\ &[X_2, X_4] = X_4,\ [X_2, X_5] = -X_5,\ [X_3, X_5] = X_4. \end{aligned} \tag{3.10}$$

According to (2.2), we can straighten the commuting vector fields X_1 and X_5, so that

$$X_1 = \frac{\partial}{\partial z_1},\ X_5 = \frac{\partial}{\partial z_3}.$$

Using the commutation relations for X_4 with X_1, X_5, we get:

$$X_4 = f(z_2)\frac{\partial}{\partial z_1} + g(z_2)\frac{\partial}{\partial z_2} + (z_1 + h(z_2))\frac{\partial}{\partial z_3}. \tag{3.11}$$

Similarly, for X_2 we get:

$$X_2 = (2z_1 + u(z_2))\frac{\partial}{\partial z_1} + v(z_2)\frac{\partial}{\partial z_2} + (z_3 + w(z_2))\frac{\partial}{\partial z_3} \tag{3.12}$$

(here f, g, h, u, v, w are all holomorphic functions depending on z_2 only near the reference point $p = (p_1, p_2, p_3)$). Obviously, the identity $g(p_2) = v(p_2) = 0$ is not possible, since then the real span of the subalgebra spanned by X_1, X_2, X_4, X_5 at p is the complex 2-plane $z_2 = const$, so that the orbit of this subalgebra is a complex surface, which is a contradiction. We now come to a case distinction.

Assume first that $g(p_2) \neq 0$, and then perform a transformation in z_2 only straightening the vector field $g(z_2)\frac{\partial}{\partial z_2}$. In this way, the form of X_1, X_5, X_2 remains the same, while X_4 simplifies to

$$X_4 = f(z_2)\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + (z_1 + h(z_2))\frac{\partial}{\partial z_3}$$

(the coefficient functions f, h, u, v, w possibly change). Next, we perform a variable change

$$z_1^* = z_1 + \phi(z_2), \quad z_2^* = z_2, \quad z_3^* = z_3 + \psi(z_2).$$

Then it is not difficult to compute that, choosing ϕ, ψ as solutions of the system of ODEs

$$f + \phi' = 0, \quad h + \psi' - \phi = 0,$$

we get finally:

$$X_4 = \frac{\partial}{\partial z_2} + z_1\frac{\partial}{\partial z_3}.$$

Using now $[X_2, X_4] = X_4$, it is not difficult to obtain:

$$X_2 = (2z_1 + A)\frac{\partial}{\partial z_1} + (B - z_2)\frac{\partial}{\partial z_2} + (z_3 + Az_2 + C)\frac{\partial}{\partial z_3}$$

for some constants A, B, C. A shift in z_2 allows us to assume further $B = 0$.

It remains finally to use the three nontrivial commutation relations containing X_3 in (3.10). A straightforward calculation (the details of which we leave to the reader) give then, first of all,

$$A = C = 0,$$

and second:

$$X_3 = -z_1^2 \frac{\partial}{\partial z_1} + (z_1 z_2 - z_3) \frac{\partial}{\partial z_2} - z_1 z_3 \frac{\partial}{\partial z_3}.$$

In this way, the initial algebra (3.10) of holomorphic vector field can be brought to the unique normal form given by the above formulas for $X_1, ...X_5$.

It remains to integrate the normal form. If M is the orbit of it at some point, then the tangency with X_1, X_5 gives that M is given by an equation

$$y_3 = F(y_1, x_2, y_2), \quad z_j = x_j + iy_j.$$

The tangency with the three remaining vector fields give the following system of PDEs for F:

$$(3.13) \qquad \begin{aligned} & 2y_1 \frac{\partial F}{\partial y_1} - x_2 \frac{\partial F}{\partial x_2} - y_2 \frac{\partial F}{\partial y_2} - F = 0, \\ & - 2x_1 y_1 \frac{\partial F}{\partial y_1} + (x_1 x_2 - y_1 y_2 - x_3) \frac{\partial F}{\partial x_2} \\ & \qquad + (x_1 y_2 + x_2 y_1 - F) \frac{\partial F}{\partial y_2} + (x_1 F + y_1 x_3) = 0, \\ & \frac{\partial F}{\partial x_2} - y_1 = 0. \end{aligned}$$

Using the first and the third equations in (3.13), we can simplify the second equation to

$$(3.14) \qquad -y_1^2 y_2 + (x_2 y_1 - F) \frac{\partial F}{\partial y_2} = 0.$$

The third equation in the system yields

$$F(y_1, x_2, y_2) = x_2 y_1 + G(y_1, y_2).$$

Substituting this into the third equation in (3.13), we get

$$G G_{y_2} + y_1^2 y_2 = 0,$$

so that

$$G^2 = -y_1^2 y_2^2 + H(y_1). \tag{3.15}$$

Finally, substituting the latter into the first equation in (3.13), we obtain:

$$2y_1 G_{y_1} - y_2 G_{y_2} - G = 0.$$

After multiplying by G, by using (3.15), we get $y_1 H' = H$ and so

$$H(y_1) = \alpha y_1,\ \alpha \in \mathbb{R}^*, \quad F = x_2 y_1 \pm \sqrt{\alpha y_1 - y_1^2 y_2^2}.$$

In view of that, the orbit M is an open subset of the real-analytic set

$$(y_3 - x_2 y_1)^2 + y_1^2 y_2^2 = \alpha y_1. \tag{3.16}$$

It is not difficult to compute that the smooth part of (3.16) is Levi-indefinite for $\alpha \neq 0$, and so is M.

If, otherwise, $g(p_2) = 0$ in (3.11), we either change the base point p and arrive to the previous case $g(p_2) \neq 0$, or have $g \equiv 0$. In the latter case we conclude, as discussed above, that $v(p_2) \neq 0$ in (3.12). Arguing as above, we normalize the vector field X_2 to become:

$$X_2 = 2z_1 \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3}.$$

Further, we make use of the substitution $z_2^* = e^{z_2}$ and get:

$$X_2 = 2z_1 \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3}.$$

Using now the commutation relations for X_4 and the fact that $g \equiv 0$ in (3.11), it is straightforward to compute that:

$$X_4 = A z_2^3 \frac{\partial}{\partial z_1} + (z_1 + C z_2^2) \frac{\partial}{\partial z_3}.$$

It remains to use the commutation relations for X_3. The commutators with X_1 and X_3 respectively give

$$\frac{\partial}{\partial z_1} X_3 = -X_2, \quad \frac{\partial}{\partial z_1} X_3 = -X_2 = -X_4. \tag{3.17}$$

Considering finally $[X_2, X_3] = 2X_3$, taking the first component of the latter identity and taking (3.17) into account, it is not difficult to obtain $A =$

0. The latter means that the algebra $\mathfrak{g}$ contain simultaneously the vector fields X_5 and the proportional to it vector field $X_4 = (z_1 + Cz_2^2)X_5$, which immediately implies the holomorphic degeneracy of the orbit M.

We summarize by concluding that there are *no* strictly pseudoconvex orbits in the case $\mathfrak{g} = g_5$.

We have gone through the entire list of algebras in Mubarakzjanov's classification. Putting together the outcomes in all of the cases above finally proves Theorem 1. □

4. The classification

Upon completing the proof of Theorem 1, we are finally able to provide the complete classification of locally homogeneous strictly pseudoconvex hypersurfaces in $\mathbb{C}^3$. We would need first of all the following proposition helping to distinguish between two tubular hypersurfaces in our list.

Proposition 4.1. *Let $M_1, M_2 \subset \mathbb{C}^3$ be two tubular hypersurfaces over affinely homogeneous bases B_1, B_2, respectively. Assume further that M_1, M_2 are simply homogeneous and that the abelian ideal I spanned by the real shifts $\left\{ i\frac{\partial}{\partial z_j} \right\}$, $j = 1, 2, 3$, is the unique 3-dimensional abelian ideal in both $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Then M_1, M_2 are biholomorphic at some points if and only if their bases are affinely equivalent.*

Proof. Assume first there exists a biholomorphism $H = (f_1, f_2, f_3) : (M_1, p_1) \longrightarrow (M_2, p_2)$. Then, in view of the simple homogeneity, $\mathfrak{g}_1$ is mapped onto $\mathfrak{g}_2$, and in view of the uniqueness I is mapped into itself. Writing down the fact that the derivations $\frac{\partial}{\partial z_j}$, $j = 1, 2, 3$ are mapped onto (constant) real linear combinations of themselves, we easily conclude that the partial derivatives $\frac{\partial f_k}{\partial z_j}$ are all real constants, so that H is an affine map with a real linear part. Combining with the shifts, we finally get that H is a real affine map. Such a map transforms the bases B_1, B_2 onto each other, as follows from the definition of tubular hypersurfaces.

On the other hand, (the complexification of) a real affine map between bases obviously performs an affine equivalence of the tubular manifolds. This proves the proposition. □

Proof of Theorem 2. Applying the results of [Lo03, Lo01, DMT17], we can conclude that any locally homogeneous strictly pseudoconvex hypersurface $M \subset \mathbb{C}^3$ with $\dim \mathfrak{aut}\,(M, p) > 0$ is locally biholomorphic to one of the hypersurfaces 1) – 11) considered near a strictly pseudoconvex point in it, and that any two hypersurfaces in the list 1) - 11) are locally biholomorphically inequivalent. In the case $\dim \mathfrak{aut}\,(M,p)=0$, we apply Theorem 1 and conclude that M is locally biholomorphic to the tube over an affinely homogeneous surface in $\mathbb{R}^3$. The latter surfaces are classified (locally) by Doubrov-Komrakov-Rabinovich in [DKR96] and independently by Ezhov-Eastwood in [EE99], up to an affine equivalence. Recall also that, according to Proposition 4.1, the holomorphic classification in the simply homogeneous case is reduced to the affine classification, provided the 3-dimensional abelian ideal $I \subset \mathfrak{g}$ is unique.

Next, note that a tube over a surface in $\mathbb{R}^3$ is strictly pseudoconvex iff its base is strictly affinely convex (resp. strictly affinely concave). Now a straightforward calculation of the second fundamental form for the surfaces in the list in [DKR96] allows to exclude from this list all the surfaces violating the strong convexity (resp. strong concavity) condition.

Further, for the resulting list of real hypersurfaces, we exclude those showing up in the lists of hypersurfaces with $\dim \mathfrak{aut}\,(M,p)>0$ obtained in [Lo03, Lo01, DMT17]. This finally gives the list of hypersurfaces 12) – 17) and proves that any locally homogeneous strcitly pseudoconvex hypersurfaces in $\mathbb{C}^3$ is locally equivalent to one of the hypersurfaces 1) – 17).

As the next step, we need to show that all the hypersurfaces 12) - 17) indeed have a trivial stability algebra. For doing so, we first note that the family 16) was studied by Beloshapka-Kossovskiy in [BK10] and it was proved there that all the Levi-nondegenerate hypersurfaces in the family have a trivial stabilizer. For hypersurfaces 12) – 15) and 17), we have to compute the coefficient tensors $N_{22}(0), N_{23}(0)$ in the Chern-Moser normal form at a strictly pseudoconvex point. As shown in [Lo03], a necessary condition for the triviality of the stabilizer is the fact that, in any Chern-Moser normal form, we have

$$N_{22}(0) \neq \pm E_0, \quad E_0 := |z_1|^4 - 4|z_1|^2|z_2|^2 + |z_1|^4.$$

A computation employing the MAPLE package shows that, for hypersurfaces 13), 14) and 17), we have $N_{22}(0) \neq E_0$ in any normal form, that is why the latter hypersurfaces have a trivial stabilizer. Next, for hypersurfaces 15) with $\alpha \neq 4$, we similarly have $N_{22}(0) \neq E_0$ in any normal form, so that the respective stabilizer is trivial. However, for $\alpha = 4$, we have $N_{22}(0) = E_0$ in the special normal form, and one has to analyze the tensor $N_{23}(0)$. Not going into further technical details, we again employ the MAPLE package and the results in [Lo03] and conclude that the tensor $N_{23}(0)$ in the case under discussion contains components contradicting the nontriviality of the stabilizer. Similar situation occurs for hypersurfaces 12) with $\alpha = \beta = -1$. Namely, we have $N_{22}(0) = E_0$ in some normal form, while further computations employing the MAPLE package show that the tensor $N_{23}(0)$ contains components contradicting the nontriviality of the stabilizer. In contrast, for $(\alpha, \beta) \neq (-1, -1)$, we have $N_{22}(0) \neq E_0$ in any normal form (by employing MAPLE computations). This finally proves that all hypersurfaces 12) – 17) have a trivial stibilizer.

It remains to prove that hypersurfaces 12) – 17) are all pairwise locally holomorphically inequivalent. Indeed, it follows directly from the explicit description in [DKR96] of the 2-dimensional affine Lie algebras $\mathfrak{a}$ acting on the bases of the surfaces 12) – 17) that, in each case, I is the unique 3-dimensional abelian ideal in the Lie algebra $\mathfrak{g}$ freely acting on a hypersurface (note that $\mathfrak{g}$ equals, as a linear space, to $I \oplus \mathfrak{a}$). Hence Proposition 4.1 is applicable, the equivalence problem is reduced to the affine equivalence problem, and it remains to finally show that the bases of the tubular hypersurfaces 12) – 17) are pairwise affinely inequivalent. The latter is accomplished by a (somewhat technical but elementary) computation, the details of which we leave to the reader.

The theorem is completely proved now. $\square$

5. Appendix A: Mubarakzjanov's classification of 5-dimensional real Lie algebras

We provide in the tables below Mubarakzjanov's list of 5-dimensional real Lie algebra (see the next two pages). As mentioned above, any 5-dimensional real Lie algebra is isomorphic to a one in Mubarakzjanov's list. We do not consider in these tables any restrictions on the parameters in the list, as well as possible equivalences between the algebras, since this information is not used in the paper.

Decomposable 5-dimensional real Lie algebras

	$[e_1,e_2]$	$[e_1,e_3]$	$[e_1,e_4]$	$[e_1,e_5]$	$[e_2,e_3]$	$[e_2,e_4]$	$[e_2,e_5]$	$[e_3,e_4]$	$[e_3,e_5]$	$[e_4,e_5]$
$\mathfrak{m}_1$										
$\mathfrak{m}_2$	e_1									
$\mathfrak{m}_3$	e_1							e_3		
$\mathfrak{m}_4$					e_1					
$\mathfrak{m}_5$		e_1			e_1+e_2					
$\mathfrak{m}_6$		e_1			e_2					
$\mathfrak{m}_7$		e_1			he_2					
$\mathfrak{m}_8$		pe_1-e_2			e_1+pe_2					
$\mathfrak{m}_9$	e_1	$2e_2$			e_3					
$\mathfrak{m}_{10}$	e_3	$-e_2$			e_1					
$\mathfrak{m}_{11}$					e_1					e_4
$\mathfrak{m}_{12}$		e_1			e_1+e_2					e_4
$\mathfrak{m}_{13}$		e_1			e_2					e_4
$\mathfrak{m}_{14}$		e_1			he_2					e_4
$\mathfrak{m}_{15}$		pe_1-e_2			e_1+pe_2					e_4
$\mathfrak{m}_{16}$	e_1	$2e_2$			e_3					e_4
$\mathfrak{m}_{17}$	e_3	$-e_2$			e_1					e_4
$\mathfrak{m}_{18}$						e_1		e_2		
$\mathfrak{m}_{19}$			αe_1			e_2		e_2+e_3		
$\mathfrak{m}_{20}$			e_1					e_2		
$\mathfrak{m}_{21}$			e_1			e_1+e_2		e_2+e_3		
$\mathfrak{m}_{22}$			e_1			βe_2		γe_3		
$\mathfrak{m}_{23}$			αe_1			pe_2-e_3		e_2+pe_3		
$\mathfrak{m}_{24}$			$2e_1$		e_1	e_2		e_2+e_3		
$\mathfrak{m}_{25}$			$(1+q)e_1$		e_1	e_2		qe_3		
$\mathfrak{m}_{26}$			$2pe_1$		e_1	pe_2-e_3		e_2+pe_3		
$\mathfrak{m}_{27}$		e_1	$-e_2$		e_2	e_1				

The algebras $\mathfrak{m}_9, \mathfrak{m}_{10}, \mathfrak{m}_{16}$ and $\mathfrak{m}_{17}$ are non-solvable, the others are solvable.

Non-decomposable solvable 5-dimensional real Lie algebras

	$[e_1,e_2]$	$[e_1,e_3]$	$[e_1,e_4]$	$[e_1,e_5]$	$[e_2,e_3]$	$[e_2,e_4]$	$[e_2,e_5]$	$[e_3,e_4]$	$[e_3,e_5]$	$[e_4,e_5]$
$\mathfrak{g}_{5,1}$									e_1	e_2
$\mathfrak{g}_{5,2}$							e_1		e_2	e_3
$\mathfrak{g}_{5,3}$						e_3	e_1			e_2
$\mathfrak{g}_{5,4}$						e_1			e_1	
$\mathfrak{g}_{5,5}$							e_1	e_1	e_2	
$\mathfrak{g}_{5,6}$							e_1	e_1	e_2	e_3
$\mathfrak{g}_{5,7}$				e_1			αe_2		βe_3	γe_4
$\mathfrak{g}_{5,8}$							e_1		e_3	γe_4
$\mathfrak{g}_{5,9}$				e_1			e_1+e_2		βe_3	γe_4
$\mathfrak{g}_{5,10}$							e_1		e_2	e_4
$\mathfrak{g}_{5,11}$				e_1			e_1+e_2		e_2+e_3	γe_4
$\mathfrak{g}_{5,12}$				e_1			e_1+e_2		e_2+e_3	e_3+e_4
$\mathfrak{g}_{5,13}$				e_1			γe_2		pe_3-se_4	se_3+pe_4
$\mathfrak{g}_{5,14}$							e_1		pe_3-e_4	e_3+pe_4
$\mathfrak{g}_{5,15}$				e_1			e_1+e_2		γe_3	$e_3+\gamma e_4$
$\mathfrak{g}_{5,16}$				e_1			e_1+e_2		pe_3-se_4	se_3+pe_4
$\mathfrak{g}_{5,17}$				pe_1-e_2			e_1+pe_2		qe_3-se_4	se_3+qe_4
$\mathfrak{g}_{5,18}$				pe_1-e_2			e_1+pe_2		$e_1+pe_3-e_4$	$e_2+e_3-pe_4$
$\mathfrak{g}_{5,19}$				$(1+\alpha)e_1$	e_1		e_2		αe_3	βe_4
$\mathfrak{g}_{5,20}$				$(1+\alpha)e_1$	e_1		e_2		αe_3	$e_1+(1+\alpha)e_4$
$\mathfrak{g}_{5,21}$				$2e_1$	e_1		e_2+e_3		e_3+e_4	e_4
$\mathfrak{g}_{5,22}$					e_1		e_3			e_4
$\mathfrak{g}_{5,23}$				$2e_1$	e_1		e_2+e_3		e_3	βe_4
$\mathfrak{g}_{5,24}$				$2e_1$	e_1		e_2+e_3		e_3	εe_1+2e_4
$\mathfrak{g}_{5,25}$				$2pe_1$	e_1		pe_2+e_3		$-e_2+pe_3$	βe_4
$\mathfrak{g}_{5,26}$				$2pe_1$	e_1		pe_2+e_3		$-e_2+pe_3$	εe_1+2pe_4
$\mathfrak{g}_{5,27}$				e_1	e_1				e_3+e_4	e_1+e_4
$\mathfrak{g}_{5,28}$				$(1+\alpha)e_1$	e_1		αe_2		e_3+e_4	e_4
$\mathfrak{g}_{5,29}$				e_1	e_1		e_2		e_4	
$\mathfrak{g}_{5,30}$				$(2+h)e_1$		e_1	$(1+h)e_2$	e_2	he_3	e_4
$\mathfrak{g}_{5,31}$				$3e_1$		e_1	$2e_2$	e_2	e_3	e_3+e_4
$\mathfrak{g}_{5,32}$				e_1		e_1	e_2	e_2	he_1+e_3	
$\mathfrak{g}_{5,33}$			e_1				e_2	βe_3	γe_3	
$\mathfrak{g}_{5,34}$			αe_1	e_1		e_2		e_3	e_2	
$\mathfrak{g}_{5,35}$			he_1	αe_1		e_2	$-e_3$	e_3	e_2	
$\mathfrak{g}_{5,36}$			e_1		e_1	e_2	$-e_2$		e_3	
$\mathfrak{g}_{5,37}$			$2e_1$		e_1	e_2	$-e_3$	e_3	e_2	
$\mathfrak{g}_{5,38}$			e_1				e_2			e_3
$\mathfrak{g}_{5,39}$			e_1	$-e_2$		e_2	e_1			e_3

Non-decomposable non-solvabale 5-dimensional real Lie algebra

	$[e_1,e_2]$	$[e_1,e_3]$	$[e_1,e_4]$	$[e_1,e_5]$	$[e_2,e_3]$	$[e_2,e_4]$	$[e_2,e_5]$	$[e_3,e_4]$	$[e_3,e_5]$	$[e_4,e_5]$
$\mathfrak{g}_5$	$2e_1$	$-e_2$	e_5		$2e_3$	e_4	$-e_5$		e_4	

References

[AMN06] Altomani, A.; Medori, C.; Nacinovich, M. *The CR structure of minimal orbits in complex flag manifolds.* J. Lie Theory **16** (2006), no. 3, 483–530.

[AL19] Atanov, A.; Loboda, A. *On the orbits of one non-solvable* 5-*dimensional Lie algebra.* Mathematical Physics and Computer Simulation (2019), Mathematics and Mechanics. V. 22, no. 2, 5–20.

[BK10] Beloshapka, V. K.; Kossovskiy, I. G. *Homogeneous hypersurfaces in* $\mathbb{C}^3$*, associated with a model CR-cubic.* J. Geom. Anal. **20** (2010), no. 3, 538–564.

[BK11] Beloshapka, V. K.; Kossovskiy, I. G. *Classification of homogeneous CR-manifolds in dimension 4.* J. Math. Anal. Appl. **374** (2011), no. 2, 655–672.

[BS78] Burns, D.; Shnider, S. *Spherical hypersurfaces in complex manifolds.* Invent. Math. **33** (1976), no. 3, 223–246.

[Ca32] Cartan, É. *Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes II.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2) **1** (1932), no. 4, 333–354.

[CM74] Chern, S. S.; Moser, J. K. *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271.

[DKR96] Doubrov, B.; Komrakov, B.; Rabinovich, M. *Homogeneous surfaces in the three-dimensional affine geometry.* Geometry and topology of submanifolds, VIII (Brussels, 1995/Nordfjordeid, 1995), 168-–178, World Sci. Publ., River Edge, NJ, 1996.

[DMT17] Doubrov, B.; Medvedev, A.; The, D. *Homogeneous Levi non-degenerate hypersurfaces in* $\mathbb{C}^3$. Available at `arXiv:1711.02389`.

[EE99] Eastwood, M.; Ezhov, V. *On affine normal forms and a classification of homogeneous surfaces in affine three-space.* Geom. Dedicata **77** (1999), no. 1, 11–69.

[ELS99] Ezhov, V. V.; Loboda, A. V.; Shmal'ts, G. *Canonical form of a fourth-degree polynomial in a normal equation of a real hypersurface in* $\mathbb{C}^3$ (Russian) Mat. Zametki **66** (1999), no. 4, 624–626; translation in Math. Notes **66** (1999), no. 3-4, 513–515 (2000).

[FK08] Fels, G.;, Kaup, W. *Classification of Levi degenerate homogeneous CR-manifolds in dimension 5.* Acta Math. **201** (2008), no. 1, 1–82.

[HY15] Huang, X.; Yuan, Y. *Submanifolds of Hermitian symmetric spaces.* Analysis and geometry, 197–206, Springer Proc. Math. Stat., **127**, Springer, Cham, 2015.

[I06] Isaev, A. *Analogues of Rossi's map and E. Cartan's classification of homogeneous strongly pseudoconvex 3-dimensional hypersurfaces.* J. Lie Theory, **16** (2006), no. 3, 407–426.

[KZ06] Kaup, W.; Zaitsev, D. *On local CR-transformation of Levi-degenerate group orbits in compact Hermitian symmetric spaces.* J. Eur. Math. Soc. (JEMS) **8** (2006), no. 3, 465–490.

[Lo00] Loboda, A. V. *A local description of homogeneous real hypersurfaces of the two-dimensional complex space in terms of their normal equations.* (Russian) Funktsional. Anal. i Prilozhen. **34** (2000), no. 2, 33–42, 95; translation in Funct. Anal. Appl. **34** (2000), no. 2, 106–113.

[Lo01] Loboda, A. V. *Homogeneous strictly pseudoconvex hypersurfaces in C3 with two-dimensional isotropy groups.* (Russian) Mat. Sb. **192** (2001), no. 12, 3–24; translation in Sb. Math. **192** (2001), no. 11-12, 1741–1761.

[Lo03] Loboda, A. V. *On the determination of a homogeneous strictly pseudoconvex hypersurface from the coefficients of its normal equation.* (Russian) Mat. Zametki **73** (2003), no. 3, 453–456; translation in Math. Notes **73** (2003), no. 3-4, 419–423.

[MN63] Morimoto, A.; Nagano, T. *On pseudo-conformal transformations of hypersurfaces*, J. Math. Soc. Japan **15** (1963), 289–300.

[Mu63] Mubarakzjanov, G. M. *Classification of real structures of Lie algebras of fifth order.* (Russian) Izv. Vysš. Učebn. Zaved. Matematika (1963), no. 3 (34), 99–106.

[Za07] Zaitsev, D. *On different notions of homogeneity for CR-manifolds.* Asian J. Math. **11** (2007), no. 2, 331–340.

Department of Mathematics, Masaryk University in Brno
Kotlarska 2, 620 00 Brno, Czech Republic
Faculty of Mathematics, University of Vienna
Oskar-Morgenstern-Platz-1, 1090 Vienna, Austria
E-mail address: kossovskiyi@math.muni.cz, ilya.kossovskiy@univie.ac.at

Department of Mathematics, Voronez State Technical University
Moskovskiy Prospekt, 14, Voronezh 394000, Russia
E-mail address: lobvgasu@yandex.ru

Received January 29, 2020
Accepted August 25, 2020