Non-algebraic deformations of flat Kähler manifolds

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Let X be a compact Kähler manifold with vanishing Riemann curvature. We prove that there exists a manifold X' deformation equivalent to X which is not an analytification of any projective variety, if and only if $H^0(X, \Omega_X^2) \neq 0$. Using this, we recover a recent theorem of Catanese and Demleitner, which states that a rigid smooth quotient of a complex torus is always projective.

We also produce many examples of non-algebraic flat Kähler manifolds with vanishing first Betti number.

1. Introduction and preliminaries

Any smooth projective variety over $\mathbb{C}$ can be viewed as a compact Kähler manifold.

However, the world of Kähler geometry is noticeably larger than the world of complex projective geometry. There exist holomorphic deformations of projective varieties which cannot be holomorphically embedded into any complex projective space. Moreover, Claire Voisin has constructed examples of compact Kähler manifolds, which are not homeomorphic to any projective variety ([Vois04]).

Let us say that a complex structure is *algebraic*, or, more generally, that a complex manifold is *algebraic*, if it can be obtained as analytification of a smooth projective variety. The list of manifolds that admit both algebraic and non-algebraic complex structures includes K3-surfaces, Hyperkähler manifolds, complex tori and others. For complex tori, it is well-known that in every dimension $n > 1$ a very general n-dimensional complex torus is non-algebraic ([BL99], Ch. 1, Corollary 6.3. See also [Sh], Ch. VII, Subsection 1.4.). As it follows from Barlet's theorems about the semi-continuity property for the algebraic dimension ([Bar]), if there exists at

least one non-algebraic complex structure in a given deformation class, then a very general member of this family is non-algebraic.

The aim of this article is to study the existence of non-algebraic deformations of complex manifolds admitting flat Kähler metric. Such manifolds are natural generalisations of complex tori, and, as follows from Bieberbach's theory of crystallographic groups (see subsection 1.1), every flat Kähler manifold X is isomorphic to a quotient T/G for a complex torus T and a finite group G acting on T holomorphically and freely.

Observe that in this case $H^\bullet(X, \mathbb{Q}) = H^\bullet(T, \mathbb{Q})^G$ (see e.g. [Hat], Prop. 3G.1 on p. 321) and the isomorphism is compatible with Hodge decomposition ([Vois02], Section 7.3.2).

In general, there might be no non-trivial holomorphic 2-forms on X. This gives an obvious obstruction for X to admit a non-algebraic deformation. If $H^0(X, \Omega_X^2) = H^{2,0}(X) = 0$, the same holds for any deformation X' of X, since Hodge numbers are constant in flat smooth families. The cone of Kähler classes is open inside

$$H^2(X', \mathbb{R}) = H^2(X', \mathbb{R}) \cap H^{1,1}(X')$$

and we are able to find a rational Kähler class. By Kodaira's embedding theorem this realises X' as a subvariety in $\mathbb{CP}^N$.

In Theorem 2.1 we show that this is indeed the only obstruction for a flat Kähler manifold to admit a non-algebraic deformation: if $H^0(X, \Omega^2) \neq 0$, there exists a family of flat Kähler manifolds over a disc (in fact, over a projective line) with special fibre being isomorphic to X and very general fibre being non-algebraic (see Section 2 for the precise statement).

We would also like to mention another motivation for our work, which comes from the Amerik-Rovinsky-Van de Ven conjecture. In their article ([ARVdV], Section 3) these authors conjectured that the second Betti number of any compact flat Kähler manifold is at least two. Observe that any compact Kähler manifold has $b_2 > 0$ and if $b_2 = 1$, this manifold automatically admits a Kähler form with integral cohomology class and therefore is algebraic. There is a recent preprint of R. Lutowski [L], claiming to prove a statement equivalent to ARVdV Conjecture. The work of Lutowski is sufficiently based on an earlier paper [HS], which uses the classification of simple finite groups. In spite of this, we are still hoping for some advances in understanding this problem from a more geometric point of view.

The paper is organised in the following way: first, we explain the main properties of compact flat Kähler manifolds and give some examples of those.

In Section 2 we give the precise formulation for our main theorem (Theorem 2.1). Then we recall some heuristics coming from Hyperkähler geometry, prove a pair of lemmas from linear algebra, and deduce the main theorem. As a corollary, we obtain a new proof of the Catanese-Demleitner theorem ([CD+C], Theorem 1).

Finally, in Section 3 we describe examples of non-algebraic flat Kähler manifolds and provide a construction for flat Kähler manifolds which admit holomorphic symplectic form and have vanishing first Betti number. Together with the main theorem, this gives many examples of non-algebraic flat Kähler manifolds with $b_1 = 0$. This is of independent interest since explicit examples of such manifolds do not seem to be present in the literature.

Acknowledgements

1.1. Flat Kähler manifolds

Recall that *a crystallographic group* is a discrete subgroup $\Gamma \subset \mathrm{Iso}(\mathbb{R}^n)$ of isometries of Euclidean space, such that $\mathbb{R}^n/\Gamma$ is compact. The study of such groups was initiated by Evgraf Fedorov [Fed]. The modern theory of crystallographic groups is based on the following theorem of Ludwig Bieberbach([Bieb]):

Theorem 1.1 (Bieberbach). *A finitely generated group Γ can be realised as a crystallographic group in $\mathrm{Iso}(\mathbb{R}^n)$ if and only if Γ contains a normal free abelian subgroup of finite index. This subgroup is of rank n and acts on $\mathbb{R}^n$ by translations. For a given n there exists only a finite number of such groups $\Gamma \subset \mathrm{Iso}(\mathbb{R}^n)$ up to conjugation.*

Remark. The finiteness result in fact gives a partial solution for Hilbert's 18th problem, which originally motivated Bieberbach in his studies. The finite group $G = \Gamma/\mathbb{Z}^n$ is often called *the rotation group* of Γ.

Lemma 1.2. *Let M be a compact manifold of dimension n. The following conditions are equivalent:*

(1) M *admits a Riemannian metric g, such that the Riemann curvature tensor R_g vanishes everywhere;*

(2) $M = \mathbb{R}^n/\Gamma$ *for some torsion-free crystallographic group $\Gamma \subset \mathrm{Iso}(\mathbb{R}^n)$;*

(3) $M = T/G$, *where $T = (S^1)^{\times n}$ is a torus and G is a finite group which acts on T freely.*

The proof can be founded e.g. in [W] (Theorem 3.3.1).

In this article, we are using the complex version of the same theory. Let $\mathrm{Iso}_{\mathbb{C}}(\mathbb{C}^n)$ be the group of biholomorphisms of $\mathbb{C}^n$ preserving the standard Hermitian metric. A *complex crystallographic group* is a discrete subgroup $\Gamma \subset \mathrm{Iso}_{\mathbb{C}}(\mathbb{C}^n)$, such that $\mathbb{C}^n/\Gamma$ is compact. From the Lemma 1.2 we easily deduce the following generalisation:

Lemma 1.3. *Let X be a compact complex manifold. The following conditions are equivalent:*

(1) X *admits a Kähler metric with everywhere vanishing curvature tensor;*

(2) $X = \mathbb{C}^n/\Gamma$ *for some torsion-free complex crystallographic group $\Gamma \subset \mathrm{Iso}_{\mathbb{C}}(\mathbb{C}^n)$;*

(3) $X = T/G$, *where T is a (compact) complex torus and G is a finite group which acts on T freely and holomorphically.*

Remark. The torsion-free condition can be omitted if we consider *(Kähler) orbifolds* instead of manifolds.

Proof. (1) $\implies$ (2). Let $\tilde{X} \to X$ be the universal covering. Lift a flat Kähler metric to $\tilde{X}$, choose local coordinates on a small ball $B \subset \tilde{X}$ and extend them globally, using the Levi-Civita connection. Since this connection is flat, it is well-defined and establishes a biholomorphism $\tilde{X} \to \mathbb{C}^n$. Moreover, since the connection is orthogonal, this is an isometry. Since $\pi_1(X)$ acts on $\tilde{X}$ by holomorphic isometries, it is complex crystallographic.

(2) $\implies$ (3) It follows from Bieberbach's theorem that X admits a finite Galois covering by a smooth compact manifold T. Moreover, T is isomorphic to $\mathbb{C}^n/\Lambda$ where Λ is a free abelian group acting on $\mathbb{C}^n$ by holomorphic isometries. It follows that T is a complex torus.

(3) $\implies$ (1) Take any flat Kähler metric on T and average its Kähler form by the action of G. This will descend to a Kähler form on X and the associated metric will be flat. $\qquad\square$

Remark. A manifold which satisfies one of the properties of Lemma 1.3 is sometimes called *Kähler-Bieberbach manifold*. Another frequent term for compact Kähler manifolds admitting flat Kähler metric, but not isomorphic to a complex torus, is *hyperelliptic manifolds*. In the context of algebraic geometry, one can also use the term *hyperelliptic varieties*.

Remark. If $X = \mathbb{C}^n/\Gamma = T/G$ is a flat Kähler manifold, then the monodromy of its flat Levi-Civita connection is isomorphic to G. Moreover, if $p\colon T \to X$ is the natural projection and $x = p(0)$, the monodromy representation of G on $T_x X \simeq T_0 T$ is isomorphic to the representation of G on $H_1(T,\mathbb{R})$ via the exponential isomorphism $T_0 T \simeq H_1(X,\mathbb{R})$.

Let X be a flat Kähler manifold isomorphic to T/G for a complex torus T and a finite group G, which acts on T freely. Modulo torsion the cohomology ring of X is determined by the representation $G \to \mathrm{GL}(H^1(T,\mathbb{Q}))$. Indeed,

$$H^\bullet(X,\mathbb{Q}) = (H^\bullet(T,\mathbb{Q}))^G = (\Lambda^\bullet H^1(T,\mathbb{Q}))^G.$$

1.2. Examples

We finish this section by giving a number of examples of flat Kähler manifolds:

(1) The obvious ones are just complex tori. The first non-trivial examples arise in complex dimension 2. These are the so-called *bielliptic surfaces*. Here is an explicit example of a bi-elliptic surface. Take two elliptic curves E_1 and E_2. The group of automorphisms of E_1 preserving the origin is isomorphic to $\mathbb{Z}/d\mathbb{Z}$, where $d \in \{2,4,6\}$. Pick a d-torsion element $\tau \in E_2$. The group $\mathbb{Z}/d\mathbb{Z}$ acts on $E_1 \times E_2$ by the formula

$$[m] \cdot (x,y) = ([m] \cdot x, y + m\tau)$$

(here m is an integer and $[m]$ denotes its residue class in $\mathbb{Z}/d\mathbb{Z}$). This action is free and the quotient surface $S = (E_1 \times E_2)/(\mathbb{Z}/d\mathbb{Z})$ admits a flat Kähler metric. See ([BHPV]. Ch. V. 5, Example BII) for further discussion.

(2) Observe that one can apply an analogue of the construction from the previous example for any pair of complex tori T_1 and T_2, and get a flat Kähler manifold

$$X := (T_1 \times T_2)/(\mathbb{Z}/2\mathbb{Z}),$$

where the group acts as multiplication by (-1) on T_1 and as a translation by a 2-torsion element on T_2.

Sometimes T_1 admits a (group) automorphism of order $d > 2$, and then one can use the same construction to obtain a free action of $\mathbb{Z}/d\mathbb{Z}$ on $T_1 \times T_2$. Such manifolds are particular cases of higher-dimensional analogues of bi-elliptic surfaces, known as *Bagnera- de Franchis manifolds*[1] see e.g. [Cat15], Subsection 4.1.

Of course, if T_1 and T_2 are non-algebraic, then $X = (T_1 \times T_2)/(\mathbb{Z}/d\mathbb{Z})$ is also non-algebraic. However, all constructions of this kind can lead us only to flat Kähler manifolds with $b_1 \neq 0$. Thus, for our purposes flat Kähler manifolds with trivial first cohomology are of the main interest.

(3) In [DHS] many examples of flat Kähler manifolds are given. There one can also find a complete list of deformation types of 3-dimensional flat Kähler manifolds with their Hodge and Betti numbers. For example, there exists a 3-dimensional flat Kähler manifold with trivial first cohomology and rotation group isomorphic to the dihedral group D_4 (i.e. the one of order 8; see [CD18] for the explicit construction). There is also a family of n-dimensional flat Kähler manifolds with vanishing first Betti number and rotation group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ the so-called *complex Hantzsche - Wendt manifolds*[2] ([H]; [DHS], Section 4). Both Hantzsche-Wendt manifolds and flat Kähler D_4-threefolds possess no non-zero holomorphic 2 - forms. Therefore, as we explained in Introduction, they are always algebraic.

A priori it is unclear if non-algebraic flat Kähler manifolds with vanishing first Betti number exist. At the end of this paper (Section 3) we produce several examples of such manifolds. We also give a general construction which starts with a flat Kähler manifold with $b_1 = 0$ and produces a new flat Kähler manifold, which also has $b_1 = 0$ and admits a (non-degenerate) holomorphic 2-form. As it follows from Theorem 2.1, such manifolds can be deformed to non-algebraic ones.

Finally, we would like to emphasise, that Bieberbach's theorem (Theorem 1.1) together with Lemma 1.3 implies that there exist only finitely many topological types of flat Kähler manifolds in every dimension.

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[1]The bi-elliptic surfaces were initially studied by Bagnera and de Franchis in 1908, [BdF]. See also an early work of Enriques and Severi [ES].

[2]By definition, the monodromy of the Levi-Civita connection on Hantzsche-Wendt manifold should also be contained in $\mathrm{SU}(n)$

2. Non-algebraic deformations of flat Kähler manifolds with $h^{2,0} \neq 0$

In this section we prove the following theorem:

Theorem 2.1. *Let X be a compact complex manifold admitting a flat Kähler metric. Assume that $H^0(X, \Omega^2) \neq 0$. Then there exists a flat smooth holomorphic family $\pi \colon \mathcal{X} \to \mathbb{CP}^1$, such that*

- *All the fibres $X_t := \pi^{-1}(t)$ also admit flat Kähler metric.*

- $\pi^{-1}(0) \simeq X$

- *The set $\mathcal{R} := \{t \in \mathbb{CP}^1 \mid X_t \text{ is algebraic}\}$ is at most countable.*

2.1. Twistor spaces: classical theory

Let us, first of all, explain the heuristics that stay behind our theorem. It comes from the theory of Hyperkähler manifolds. Recall that *a Hyperkähler manifold* is a quadruple (M, g, I, J), where (M, g) is a Riemannian manifold and I and J are integrable complex structures, which anti-commute and both are Kähler with respect to g. Together I and J generate an action of the algebra of (Hamiltonian) quaternions on the tangent bundle of M.

If ω_I and ω_J are the g-Hermitian forms of I and J, the form $\sigma_I := \omega_I + \sqrt{-1}\omega_J$ is a holomorphic symplectic form on (M, I). Vice versa, if (M, I) is a complex manifold of Kähler type which admits a holomorphic non-degenerate 2-form, there exist such g and J, that (M, g, I, J) is a Hyperkähler manifold (see e.g. [Beau], Prop. 4).[3]

A holomorphically symplectic manifold is said to be *irreducible holomorphic symplectic* if the space of holomorphic 2-forms on it is generated by a symplectic form. The Beauville-Bogomolov decomposition theorem ([Beau], Th. 2) implies that any Hyperkähler manifold, up to a finite covering, is isomorphic to a product of irreducible ones and a complex torus.

Consider the set of purely imaginary quaternions of unit norm. Every such quaternion q satisfies $q^2 = -1$. This naturally forms a 2-dimensional sphere S^2 inside $\mathbb{H}$ which we immediately identify with $\mathbb{CP}^1$. For each

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[3]In the literature (M, I) is often assumed to be irreducible holomorphic symplectic and simply connected. A more general version of this statement follows from Beaville-Bogomolov decomposition theorem and some equivariant techniques which we describe below.

$q \in \mathbb{CP}^1$ one has the corresponding almost complex structure I_q on the smooth manifold M.

It is straightforward to see that all I_q are integrable and Kähler with respect to g.

Consider the product $M \times \mathbb{CP}^1$ with the almost complex structure I_{tw} defined by the following rule: in a point $(m, q) \in M \times \mathbb{CP}^1$ the operator I_{tw} is equal to $I_q \oplus J_0$, where J_0 is the standard complex structure on $\mathbb{CP}^1$. The complex manifold $\text{Tw}(M) := (M \times \mathbb{CP}^1, I_{\text{tw}})$ is called *the twistor space* of M.

Theorem 2.2. *Let $M = (M, g, I, J)$ be a Hyperkähler manifold and I_{tw} the almost complex structure operator on $\text{Tw}(M) = M \times \mathbb{CP}^1$ as above.*

(1) *The complex structure I_{tw} is integrable and the natural projection $\text{Tw}(M) \xrightarrow{\pi} \mathbb{CP}^1$ is holomorphic. Each fibre $\pi^{-1}(q)$ is a complex manifold $M_q := (M, I_q)$ with the complex structure I_q as defined above.*

(2) *Assume that M is compact and irreducible holomorphic symplectic. The set*

$$R := \{q \in \mathbb{CP}^1 | M_q \text{ is algebraic}\}$$

is at most countable.

Proof. See [K] for the first statement and [Ver], Theorem 2.2. for the second. Originally the second statement appeared in [Fuj] (Theorem 4.8(2)).　□

One would like to say that Theorem 2.2 immediately implies Theorem 2.1 in case when X admits a holomorphic symplectic form. Unfortunately, even if a flat Kähler manifold is holomorphically symplectic, it is usually not irreducible holomorphic symplectic. This can be overruled by introducing equivariant versions of some classical linear algebraic arguments, which we do in the next subsection.

What seems to be a more serious problem is that X might carry only degenerate holomorphic 2-forms. Our plan is to take any non-zero holomorphic 2-form on X and, using the flatness assumption, imitate the twistor family in the direction, in which this form is non-degenerate, without changing the holomorphic structure on the kernel of this form.

2.2. Equivariant $\mathbb{C}$-symplectic linear algebra

Now we are going to describe some linear algebraic constructions relating complex symplectic structures to Hermitian quaternionic structures. This is rather classical, but we will also need an equivariant version of it.

Fix the following notation. Let V be a real vector space with a complex structure operator

$$I\colon V \to V,\ I^2 = -1.$$

Then the exterior powers of $V_{\mathbb{C}}^* := V^* \otimes \mathbb{C}$ carry the decomposition

$$\Lambda^k V_{\mathbb{C}}^* = \bigoplus_{p+q=k} \Lambda^{p,q} V_{\mathbb{C}}^*.$$

Let $\sigma \in \Lambda^{2,0} V_{\mathbb{C}}^*$ be a complex non-degenerate form of type $(2,0)$. A *compatible hyper-Hermitian structure* on V is a pair (g, J), where J is an automorphism of V satisfying $J^2 = -1$ and $JI = -IJ$, and g is a positive scalar product on V, Hermitian with respect to both I and J. Observe that in this case the algebra generated by I and J is isomorphic to the algebra of Hamiltonian quaternions.

Lemma 2.3. *Assume that a finite group G acts on V linearly, commuting with I and preserving σ. Then there exists a G-equivariant compatible hyper-Hermitian structure (that is, both g and J can be chosen to be G-invariant).*

Proof. **Step 1.** Denote by σ_1 the real part of σ and let h_1 be any G-invariant I-Hermitian metric on V. Consider the operator $A = h_1^{-1} \circ \sigma_1 \colon V \to V^* \to V$. In other words, A is defined by the property

$$h_1(Ax, -) = \sigma_1(x, -).$$

Of course, A is invertible and commutes with every element in G. Observe also, that A anti-commutes with I. Indeed,

$$h_1(IAx, y) = -h_1(Ax, Iy) = -\sqrt{-1}\sigma_1(x, y),$$

while

$$h_1(AIx, y) = \sigma_1(Ix, y) = \sqrt{-1}\sigma_1(x, y).$$

Step 2. Since σ_1 is skew-symmetric, we see that A is also skew-symmetric (with respect to h_1), thus A^2 is symmetric. Moreover, $h_1(A^2 x, x) =$

$-h_1(Ax, Ax)$, so A^2 is negative definite. Therefore A^2 can be diagonalized in an h_1-orthonormal basis:

$$A^2 = \begin{pmatrix} \alpha_1 & 0 & 0 & \dots & 0 \\ 0 & \alpha_2 & 0 & \dots & 0 \\ 0 & 0 & \alpha_3 & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \alpha_n \end{pmatrix}$$

with $\alpha_i \in \mathbb{R}_{<0}$. Consider a polynomial $P \in \mathbb{R}[t]$ which satisfies $P(\alpha_i) = \frac{1}{\sqrt{-\alpha_i}}$ for $1 \le i \le n$ and denote $S := P(A^2)$. By construction, this is a symmetric positive operator, which commutes both with A and with elements from G. We also have $(AS)^2 = -\operatorname{Id}_V$.

Since A anti-commutes with I, and S is a polynomial of even degree in A, the operator S commutes with I.

Step 3. Define $g(x, y) := h_1(Sx, y)$ and $J := AS$. It is clear that both g and J are G-invariant and that J is an operator of an almost complex structure.

Since S commutes with I, the metric g is I-Hermitian. What is left to check is that J is g-orthogonal and anti-commutes with I.

The first property follows from an explicit computation, which uses the fact that A is skew-symmetric and S is symmetric. The second follows from

$$g(IJx, y) = g(Jx, -Iy) = h_1(AS^2x, -Iy)$$
$$= \sigma_1(S^2x, -Iy) = -\sqrt{-1}\sigma_1(S^2x, y),$$

while

$$g(JIx, y) = g(Ix, -Jy) = h_1(AS^2y, -Ix)$$
$$= \sigma_1(S^2y, -Ix) = -\sigma_1(-Ix, S^2y) = \sqrt{-1}\sigma_1(S^2x, y)$$

Here we used the fact that $\sigma_1(S^2x, y) = h_1(AS^2x, y) = h_1(Ax, S^2y) = \sigma_1(x, S^2y)$, and, what is more important, the fact that σ_1 is a real part of $(2,0)$-symplectic form with respect to I. $\square$

Let us also prove the following linear algebraic proposition, which will play an important role in the proof of the main theorem:

Proposition 2.4. *Let U and W be finite-dimensional real vector spaces. Assume that I is a complex structure operator on U and W is endowed with a faithful action of the quaternionic algebra $\mathbb{H} \times W \to W$. Therefore, for*

every purely imaginary $q \in \mathbb{H}$ with $q^2 = -1$, we obtain a complex structure operator $I_q \colon W \to W$. Let $H \subset U \oplus W$ be a vector subspace of real codimension 2, which is a complex hyperplane in $(U \oplus W, J_q := I \oplus I_q)$ for every $q \in \mathbb{CP}^1 \subset \mathbb{H}$. Then $H = W$ (and consequently $\dim_{\mathbb{C}} U = 1$).

Proof. Consider $H' := H \cap W$. By the assumptions, this is an $\mathbb{H}$-submodule in W. Therefore, its real dimension is divisible by 4. But $\operatorname{codim}_W H' \leq \operatorname{codim}_{U \oplus W} H = 2$. Hence $\operatorname{codim}_W H' = 0$ and $H' = W$. Thus, H contains W. This is possible only if $H = W$ and $\dim_{\mathbb{C}} U = 1$. $\qquad\square$

2.3. Construction of the twistor family

Starting from now, let X be a compact flat Kähler manifold and $p \colon T \to X$ be a Galois covering by a complex torus T, so that $X = T/G$ for some finite group G. Assume that $\eta \in H^0(X, \Omega^2)$ is a non-zero holomorphic 2-form. It defines a morphism of holomorphic vector bundles $\eta \colon T_X \to T_X^*$. Consider its kernel $\mathcal{E} := \operatorname{Ker} \eta \subset TX$. A priori this is only a holomorphic subsheaf, but since $p^*\eta$ is a holomorphic 2-form on T and all holomorphic forms on a complex torus are invariant under parallel transport, it is of constant rank, hence a $\mathcal{E} \subset TX$ is a subbundle.

Moreover, since η is closed, $\mathcal{E}$ is involutive and defines a holomorphic foliation on X.

Proposition 2.5. *There exists a Riemannian metric g on X, a family of complex structures J_q on X, parametrised by $\mathbb{CP}^1$, and an integrable subbundle $\mathcal{F} \subset TX$, such that:*

(1) *The metric g is flat;*

(2) *all the complex structures J_q are integrable and Kähler with respect to g;*

(3) *the subbundles $\mathcal{E}$ and $\mathcal{F}$ are holomorphic with respect to J_q for any q;*

(4) *$TX = \mathcal{E} \oplus \mathcal{F}$ as a holomorphic vector bundle and $\mathcal{F} \perp_g \mathcal{E}$.*

Proof. Denote by V the (real) tangent space to the origin in T. The complex structure on T induces a complex structure operator I on V. As usual, we can identify V with $H_1(T, \mathbb{R})$. Therefore, V is naturally endowed with a complex-linear action of G. Every G-invariant tensor on V defines a homogeneous G-invariant tensor on T, which descends to $X = T/G$.

Let $\eta_0 := (p^*\eta)(0) \in \Lambda^{2,0} V_{\mathbb{C}}^*$ and $E = (p^*\mathcal{E})_0 = \operatorname{Ker} \eta_0$. Since η_0 is a G-invariant complex bilinear form, E is a G-invariant subspace of V, and is

preserved by I. Since G is finite, we can find a G-invariant I-complex complement $F \subset V$, so that $V = E \oplus F$ as a complex representation of G. The form η_0 restricts to a non-degenerate form on F. By Lemma 2.3 there exists a G-invariant compatible hyper-Hermitian structure on F. Let g_F be the corresponding (hyper-)Hermitian metric on F and $(I_q)_{q \in \mathbb{CP}^1}$ be the associated family of complex structure operators.

Take a G-invariant I-Hermitian metric g_E on E and put $g_0 := g_E \oplus g_F$ and

$$J_{q,0} := I|_E \oplus I_q.$$

As we have explained above, the metric g_0 induces a T-invariant metric $\tilde{g}$ on T, which descends to a metric g on X. Similarly, the family of operators $J_{q,0}$ induce a family of g-orthogonal almost complex structures J_q on X, which are obtained by descending the G-invariant family of homogeneous complex structures $\tilde{J}_q$ on T. Finally, F induces a homogeneous G-invariant subbundle $\widetilde{F}$ in the holomorphic tangent bundle of T, which descends to a subbundle $\mathcal{F}$ on X. This foliation is clearly preserved by every J_q and its fibres are g-orthogonal complements to the fibres of $\mathcal{E}$ in each point.

Now we need to check the properties listed above. Since all of them are local, it is sufficient to check them for the pre-images of these objects in T, that is for $\tilde{g}, \widetilde{J}_q. \widetilde{F}$ and $\widetilde{\mathcal{E}} = p^*\mathcal{E}$. There exists a flat torsion-free connection ∇ on T, which preserves all T-invariant tensors. Therefore $\nabla \tilde{g} = 0$ and ∇ is the Levi-Civita connection of $\tilde{g}$. In particular, $\tilde{g}$ is flat. Because ∇ is torsion-free and preserves $\widetilde{\mathcal{F}}$, this subbundle is integrable. From $\nabla \widetilde{J}_q = 0$ for every q, we deduce (2). The properties (3) and (4) follow from the construction. $\square$

Now we are able to define $\mathcal{X}$ as the "twistor space" of (X, g, J_q), that is, an almost complex manifold isomorphic to $\mathbb{CP}^1 \times X$ as a smooth manifold, and endowed with the almost complex structure

$$(I_{\text{tw}})_{(q,x)} = I_0 \oplus J_q,$$

where I_0 is the standard complex structure on $\mathbb{CP}^1$.

Proposition 2.6. *The almost complex structure I_{tw} is integrable.*

Proof. Since integrability of the complex structure is a local property, we can prove it for a universal cover $\widetilde{\mathcal{X}} \to \mathcal{X}$. It is isomorphic to $E \times \text{Tw}(F)$, where E and F are complex vector spaces as above and F is endowed with the Hyper-Hermitian structure (g, J_q). The classical twistor space $\text{Tw}(F)$ is a complex manifold by Theorem 2.2, [K]. $\square$

Remark. In [CC] Catanese and Corvaja gave a description of the Teichmüller space $\mathfrak{T}(X)$ of a flat Kähler manifold X. Every family of the form $\mathcal{X} = (X, g, J_q)$ defines a holomorphic map $\mathbb{C}\mathbf{P}^1 \to \mathfrak{T}(X)$. Studying the images of these curves might be useful for better understanding of the geometry of $\mathfrak{T}(X)$.

2.4. SU(2)-action

The group generated by unitary imaginary quaternions inside $\mathbb{H}^\times$ is isomorphic to SU(2). If X is a flat Kähler manifold endowed with a non-zero holomorphic 2-form, from Proposition 2.5 we obtain a faithful action of the quaternionic algebra $\mathbb{H}$ on the tangent bundle of X, which induces an action of SU(2) on smooth sections of the tangent bundle. In the notations of the previous subsection, it is generated by the operators $(\mathrm{Id}_\mathcal{E} \oplus I_q)_{q \in \mathbb{C}\mathbf{P}^1}$.

This action can be extended to a linear action of SU(2) on the space of differential forms on X.

Proposition 2.7. *The defined above* SU(2)*-action descends to action by algebra automorphisms on cohomology. This action preserves the intersection form.*

Proof. Each cohomology class on X can be uniquely represented by a harmonic form. The map $p \colon (T, \tilde{g}) \to (X, g)$ is a local isometry, thus harmonic forms on X are the same as G-invariant harmonic forms on T. These are the same as G-invariant forms preserved by the action of T on itself by translations. Here we use the fact that harmonic forms for a Kähler metric on a torus are the same as left-invariant forms, see [BL92], Prop. 1.4.7. Such forms clearly form a SU(2)-invariant subalgebra inside the de Rham algebra of T, hence this action descends to cohomology.

By construction, this action preserves the metric g, so it also preserves the associated volume form and the Poincaré pairing. $\qquad\square$

Proposition 2.8. *Let $\alpha \in H^{2p}(X, \mathbb{C})$. Let $\pi \colon \mathcal{X} \to \mathbb{C}\mathbf{P}^1$ be the 'twistor' family as defined above and let $X_q := (X, J_q) = \pi^{-1}(q)$.*

Then $\alpha \in H^{p,p}(X_q)$ for every $q \in \mathbb{C}\mathbf{P}^1$ if and only if it is SU(2)*-invariant.*

Proof. Each complex structure J_q generates an U(1)-action on the tangent bundle of X which descends to the cohomology (and in fact defines the Hodge decomposition on $H^\bullet(X, \mathbb{C})$). The condition $\alpha \in H^{p,p}(X_q)$ means that α is invariant under this U(1)-action and by assumption this holds for every

$q \in \mathbb{CP}^1$. But a vector is invariant under a representation of a compact Lie group if and only if it is invariant under every its 1-parameter subgroup. $\square$

2.5. Non-algebraic points in the twistor family

Let us now prove the following proposition:

Proposition 2.9. *Let $D \subset X$ be an immersed submanifold of (real) codimension 2. Assume that D is holomorphic for every complex structure J_q. Then the divisor $[D]$ is not ample in X_q for any q.*

Proof. Take any smooth point $x \in D$ and apply the Proposition 2.4 for $W := \mathcal{F}|_x$ and $U := \mathcal{E}|_x$ and $H := T_x D \subset U \oplus W = T_x X$. We see that $T_x D = \mathcal{F}|_x$, hence it D coincides with a leaf of the foliation, defined by $\mathcal{F}$.

Assume that $[D] \in \mathrm{Pic}(X_q)$ is ample for some q. Let g_E be the restriction of the metric g on $\mathcal{E}$ and ω_E be its Hermitian form on X_q. This is a non-negative closed $(1,1)$-form, which is positive on $\mathcal{E}$ and its kernel coincides with $\mathcal{F}$. In particular, ω_E restricts on D by 0 and

$$\int_{X_q} \omega_E \cdot [D]^{n-1} = 0$$

But the class of ω_E is nef and D cannot be ample. $\square$

Now we are ready to prove the main theorem:

Proof of the Theorem 2.1. Let $\mathcal{X}$ be a "twistor space" as above. By the construction, this is a complex manifold which admits a holomorphic topologically trivial fibration $\pi \colon \mathcal{X} \to \mathbb{CP}^1$. For every $q \in \mathbb{CP}^1$ the fibre $X_q = (X, J_q)$ admits a flat Kähler metric g. Moreover, $X_0 \simeq (X, I)$.

Now we want to prove, that for a very general $q \in \mathbb{CP}^1$ the fibre X_q is not algebraic.

For every $\alpha \in H^2(X, \mathbb{Q})$ define the set

$$R_\alpha := \{q \in \mathbb{CP}^1 | \alpha \in H^{1,1}(X_q)\}.$$

The subsets $R_\alpha \subseteq \mathbb{CP}^1$ are algebraic subvarieties since they are the Noether-Lefschetz loci for the (non-polarizable) variation of Hodge structures $R^2\pi_*\underline{\mathbb{Z}}_{\mathcal{X}}$. This means, that for each α either $R_\alpha = \mathbb{CP}^1$, or R_α is finite. In the former case, as follows from Proposition 2.8, α is SU(2)-invariant.

We claim that if $R_\alpha = \mathbb{CP}^1$, then for every $q \in \mathbb{CP}^1$ the class α is not a class of a very ample divisor on X_q.

Assume the opposite. Without loss of generality, we may assume that α is a class of a very ample divisor $D \subset X_0$. We may also assume that D is smooth and irreducible (by Bertini theorem).

Recall the Wirtinger-Federer theorem (see e.g. [St], Ch. I): If Z is a smooth compact submanifold of dimension $2k$ inside a compact Kähler manifold (X, g, ω), then

$$\int_Z \omega^k \leq \mathrm{Vol}_g(Z),$$

and the equality holds if and only if $Z \subset X$ is complex analytic.

Let $\omega_q := g(J_q \cdot, \cdot)$ be the Kähler form on X_q. Let $n = \dim X_q$. For every $q \in \mathbb{CP}^1$ there exists an element $B(q) \in \mathrm{SU}(2)$, such that $[\omega_q] = B(q) \cdot [\omega_0]$ in $H^2(X, \mathbb{C})$. Hence

$$\int_D \omega_q^{n-1} = \langle (\omega_q)^{n-1}, [D] \rangle = \langle B^{n-1}(q) \cdot \omega_0^{n-1}, \alpha \rangle$$

$$= \langle B^{n-1}(q) \cdot \omega_0^{n-1}, B^{n-1}(q) \cdot \alpha \rangle = \int_D \omega_0^{n-1} = \mathrm{Vol}_g(D)$$

We deduce that D is holomorphic with respect to every complex structure J_q in our family. But this leads to a contradiction by Proposition 2.9.

Let $\mathrm{Amp}(X)$ be the set of $\alpha \in H^2(X, \mathbb{Z})$ such that α is a class of a very ample divisor on X_q for some $q \in \mathbb{CP}^1$. As we have shown, for each such α the set R_α is a proper algebraic subset of $\mathbb{CP}^1$, hence finite. The set

$$\mathcal{R} := \bigcup_{\alpha \in \mathrm{Amp}(X)} R_\alpha$$

is a countable union of finite sets, hence at most countable.

Take any point $q \in \mathbb{CP}^1 \setminus \mathcal{R}$. The complex manifold X_q has no ample divisors and thus is non-algebraic. This finishes the proof. $\square$

Corollary 2.10. *Let X be a compact flat Kähler manifold. The following conditions are equivalent:*

- *There exists a smooth holomorphic family of complex manifolds over a marked complex analytic variety $\pi \colon \mathcal{X} \to (B, 0)$ with $\pi^{-1}(0) \simeq X$, such that every neighbourhood of 0 contains t with $\pi^{-1}(t) = X_t$ non-algebraic;*

- $H^0(X, \Omega_X^2) \neq 0$.

Proof. Assume that $H^0(X, \Omega_X^2) \neq 0$. One can take the family $\pi \colon \mathcal{X} \to \mathbb{CP}^1$ which is constructed in Theorem 2.1. By the result of the Theorem 2.1, the

set of points $t \in \mathbb{C}\mathbf{P}^1$ for which $\pi^{-1}(t)$ is algebraic is at most countable, therefore every neighbourhood of 0 contains a point from its complement.

Vice versa, assume that $H^0(X, \Omega_X^2) = 0$, but such family $\pi \colon \mathcal{X} \to (B, 0)$ exists. Choose a neighbourhood $B' \subset B$ of 0 such that all the fibres $\pi^{-1}(t)$, $t \in B'$ are Kähler (this can be done because $\pi^{-1}(0)$ is Kähler and being Kähler is an open property). Now, the argument, which we already mentioned in the Introduction, can be applied: take arbitrary $t \in B'$. The Kähler manifold $X_t := \pi^{-1}(t)$ has the same Hodge numbers as X, in particular, $H^0(X_t, \Omega_{X_t}^2) = 0$. Therefore every real cohomology class of degree 2 is of Hodge type $(1,1)$ and $H^2(X_t, \mathbb{Q})$ is dense inside $H^2(X_t, \mathbb{R}) = H^{1,1}(X_t) \cap H^2(X_t, \mathbb{R})$. But the set of classes of Kähler forms form an open cone $\mathcal{K} \subset H^{1,1}(X_t) \cap H^2(X_t, \mathbb{R})$. Therefore we can find a Kähler form with a rational cohomology class. By Kodaira embedding theorem this means that X_t is algebraic. $\qquad\square$

As a corollary we immediately obtain the following (global version of) theorem from [CD+C]:

Theorem 2.11 (Catanese, Demleitner). *Let X be a flat Kähler manifold. Assume that X is rigid (i.e. every holomorphic family of complex manifolds with special fibre isomorphic to X is constant). Then X is algebraic.*

Proof. Assume that X is non-algebraic. Then, of course, $h^{2,0}(X) \neq 0$. Consider the family $\mathcal{X} \to \mathbb{C}\mathbf{P}^1$ from the Theorem 2.1. It is non-constant because J_0 and J_1 anticommute and therefore are not conjugated by a diffeomorphism of X. Hence, X is not rigid. $\qquad\square$

Observe that in the paper [CD+C] only infinitesimal deformations were considered, so our result is slightly stronger.

3. Examples of non-algebraic flat Kähler manifolds with $b_1 = 0$

As we mentioned in the Introduction, many examples of non-algebraic flat Kähler manifolds are given by complex tori of (complex) dimension greater than one. Somehow opposite to complex tori are the flat Kähler manifolds with vanishing first Betti number: these are the manifolds of the form T/G, where T is a complex torus and the finite group G acts on T sufficiently non-trivially from the topological point of view (i.e. $H_1(T, \mathbb{Q})^G = 0$). As it was pointed out to us by the reviewer, one can construct non-algebraic flat Kähler

manifolds with $b_1 = 0$ by mimicing the construction of 3-dimensional flat Kähler manifolds with monodromy group D_4, as it is presented in [CD18].

We will explain this construction and then present yet another one, which allows finding many examples of holomorphically symplectic flat Kähler manifolds with $b_1 = 0$ and which we consider to be of independent interest.

First of all, let E be an elliptic curve and S a non-algebraic complex torus. Let τ_1, τ_2 be two different non-trivial 2-torsion elements of E and $\sigma \in S$ be an element of order 4. Let $T' := E \times E \times S$.

Proposition 3.1. *There exists a torus T isogenous to T' and a free holomorphic action of the dihedral group $G \simeq D_4$ on T such that $b_1(T/G) = 0$. The flat Kähler manifold $X = T/G$ is non-algebraic.*

Proof. Let G' be the subgroup of holomorphic automorphisms of T' generated by

$$s \colon (x_1, x_2, y) \mapsto (x_2 + \tau_1, x_1 + \tau_2, -y)$$

and

$$r \colon (x_1, x_2, y) \mapsto (x_2, -x_1, y + \sigma).$$

Here x_i are coordinates on the two copies of E and y is the coordinate on S. One sees that $s^2 = r^4 = \mathrm{Id}$, while

$$(sr)^2(x_1, x_2, y) = (x_1 + \tau_1 + \tau_2, x_2 + \tau_1 + \tau_2, y).$$

Therefore the action of this group descends to $T = T'/\langle \tau_1 + \tau_2 \rangle$ and the resulting group G acting on T is isomorphic to the dihedral group D_4. The quotient $X = T/G$ is a flat Kähler manifold. If we decompose $H_1(T, \mathbb{Q})$ as $H_1(E, \mathbb{Q}) \oplus H_1(E, \mathbb{Q}) \oplus H_1(S, \mathbb{Q})$, we see that s acts on this space through

$$H_1(s) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\mathrm{Id} \end{pmatrix},$$

while r acts through

$$H_1(r) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}.$$

One easily sees that $H_1(T, \mathbb{Q})^G = 0$.

If X were algebraic, then T' would be, as it is a finite cover of X. Although, T' contains a non-algebraic subtorus S. □

Now let us present more examples of non-algebraic flat Kähler manifolds with vanishing b_1.

We start with the following definition:

Definition 1. Let (X, I_X) be a smooth complex manifold. Assume that X admits a flat connection ∇ on TX, such that $\nabla I_X = 0$ and the monodromy action of ∇ preserves a lattice $\Lambda_x \subset T_x X$. Consider the union of all possible parallel transports of vectors in Λ_x as a subset $\Lambda \subset TX$ inside the total space of the tangent bundle. The intersection of Λ with each fibre of the tangent bundle is a lattice inside a real vector space, thus we can take a fibrewise quotient $TX/\Lambda =: X_+$. The resulting manifold is called *the quaternionic double* of X.

This definition appears, for example, in the work of Soldatenkov and Verbitsky ([SV], Section 3.2). The name is motivated by the following property: the flat connection ∇ splits the tangent bundle TX_+ into a direct sum of two copies of TX. One can consider the three complex structures on X_+ written with respect to this splitting as

$$I := \begin{pmatrix} I_X & 0 \\ 0 & -I_X \end{pmatrix}; \ J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \ K := \begin{pmatrix} 0 & -I_X \\ -I_X & 0 \end{pmatrix}.$$

In the same paper([SV]) Soldatenkov and Verbitsky proved (Section 3.2.) that these complex structures are integrable and satisfy quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -1.$$

Moreover, as they showed, this establishes a Hyperkähler structure on X_+ if and only if the initial flat connection ∇ on X was orthogonal with respect to some Hermitian metric (equivalently, X is flat Kähler and ∇ is its Levi-Civita connection).

Corollary 3.2. *Let X be a flat Kähler manifold. Then there exists a flat Kähler manifold X_+ endowed with a holomorphic projection $X_+ \to X$ such that*

- *X_+ admits a holomorphic symplectic form and the fibres of the projection are Lagrangian tori;*

- *$b_1(X_+) = 2b_1(X)$.*

In particular, if X is a flat Kähler manifold with $b_1 = 0$, then X_+ is a flat Kähler manifold with $b_1 = 0$, which admits a non-algebraic deformation.

Proof. Let T be a torus, which covers X, so that $X = T/G$. The group G acts on $H_1(T, \mathbb{R})$ preserving the lattice $H_1(T, \mathbb{Z})$, and this induces a G-action by automorphisms on the torus

$$\mathrm{Alb}(T) = H_1(T, \mathbb{R})/H_1(T, \mathbb{Z}).$$

Of course, $H_1(T, \mathbb{R})/H_1(T, \mathbb{Z})$ is isomorphic to T, however this new action of G is different from the initial one since it preserves the origin. Now the identification of $H_1(T, \mathbb{R})$ with a tangent space to T in the origin gives rise to an identification of $H_1(T, \mathbb{R})$ with the tangent space to a point on X as a G-module. One sees from the construction that the quaternionic double X_+ can be also described as

$$X_+ = \frac{T \times (H_1(T, \mathbb{R})/H_1(T, \mathbb{Z}))}{G}.$$

(where the action of G on $T \times (H_1(T, \mathbb{R})/H_1(T, \mathbb{Z}))$ is diagonal). Therefore,

$$b_1(X_+) = \dim(H_1(T, \mathbb{R}) \oplus H_1(T, \mathbb{R}))^G = 2b_1(X).$$

The existence of non-algebraic deformations follows from Theorem 2.1. $\quad\square$

Remark. Instead of taking the tangent bundle in the definition of quaternionic double one might take the **cotangent** bundle and obtain another flat Kähler manifold X^+ (the *co-quaternionic double*). It also naturally carries a holomorphic symplectic form, which descends from the total space of the cotangent bundle, and also has $b_1(X^+) = 2b_1(X)$. Moreover, manifolds X^+ and X_+ admit holomorphic Lagrangian fibrations over the same base X with fibres being dual complex tori.

The quaternionic and co-quaternionic doubles might serve as a "toy version" for several important examples where a similar configuration occurs, for example, a pair of Hitchin fibrations on the moduli space of G-Higgs bundles and the moduli space of the Langlands - dual $^L G$-Higgs bundles (see e.g. [Hit]).

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